Some remarks on the radius of spatial analyticity for the Euler equations

Published version:

DOI:10.3233/ASY-141260

SOME REMARKS ON THE RADIUS OF SPATIAL ANALYTICITY FOR THE EULER EQUATIONS

MARCO CAPPIELLO AND FABIO NICOLA

Abstract. We consider the Euler equations on $\mathbb{T}^d$ with analytic data and prove lower bounds for the radius of spatial analyticity $\varepsilon(t)$ of the solution using a new method based on inductive estimates in standard Sobolev spaces. Our results are consistent with similar previous results proved by Kukavica and Vicol, but give a more precise dependence of $\varepsilon(t)$ on the radius of analyticity of the initial datum.

1. Introduction

Consider, on the d-dimensional torus $\mathbb{T}^d$, the Euler equations

$$\frac{\partial u}{\partial t} + P(u \cdot \nabla_x u) = 0, \quad \operatorname{div} u = 0, \tag{1.1}$$

where P is the Leray projection in L^2 on the subspace of divergence free vector fields. It is well known that the corresponding Cauchy problem is locally well posed in H^k if $k > d/2 + 1$, see e.g. [14, Chapter 17, Section 2]. The analyticity of the solution in the space variables, for analytic initial data is also an important issue, investigated in [1, 2, 3, 4, 10, 11, 12], and one is specially interested in lower bounds for the radius of analiticity $\varepsilon(t)$ as t grows.

To be precise, if f is an analytic function on the torus, its radius of analyticity is the supremum of the constants $\varepsilon > 0$ such that $\|\partial^\alpha f\|_{L^\infty} \leq C\varepsilon^{-|\alpha|}|\alpha|!$ for some constant $C > 0$. Notice that we can also replace the L^∞ norm with a Sobolev norm H^k, $k \geq 0$.

Concerning the Euler equations on the torus, a recent result by Kukavica and Vicol [10] states that for the radius of analyticity $\varepsilon(t)$ of any analytic solution $u(t)$ we have the lower bound

$$\varepsilon(t) \geq C(1+t)^{-2} \exp\Big(- C_0 \int_0^t \|\nabla u(s)\|_{L^\infty}\, ds \Big) \tag{1.2}$$

for a constant $C_0 > 0$ depending on the dimension and $C > 0$ depending on the norm of the initial datum in some *infinite order* Sobolev space.

2000 *Mathematics Subject Classification.* Primary 76B03, Secondary 35L60.
Key words and phrases. Euler equations, radius of analyticity.

The same authors in [11] obtained a better lower bound for $\varepsilon(t)$ for the Euler equations in a half space replacing $(1+t)^{-2}$ by $(1+t)^{-1}$ in (1.2). Now, one suspects that the same bound should hold also on $\mathbb{T}^d$ and on $\mathbb{R}^d$. In this paper we give a new and more elementary proof of the results above on $\mathbb{T}^d$, which yields some improvements concerning the dependence on the initial data. In fact, it is natural to expect that the constant C in (1.2) should be comparable with the inverse of the radius of analyticity of the initial datum. The dependence found in [10, 11] is due to the fact that the proof given there relies on the energy method in infinite order Gevrey-Sobolev spaces (cf. also [5, 6, 8, 9, 12]). In our recent paper [7] we developed a method for the estimate of the radius of analyticity for semilinear symmetrizable hyperbolic systems based on inductive estimates in standard Sobolev spaces. The purpose of this note is to adapt this method to the Euler equations on $\mathbb{T}^d$ and to prove that

$$\varepsilon(t) \geq C(1+t)^{-1} \exp\Big(-C_0 \int_0^t \|\nabla u(s)\|_{L^\infty}\, ds\Big) \tag{1.3}$$

(as suggested in [11]) with a neat dependence of the constant C on the initial datum.

Namely, we have the following result.

Theorem 1.1. *Let $k > d/2+1$ be fixed. There exists constants $C_0, C_1 > 0$, depending only on k and d, such that the following is true.*

Let u_0 be analytic in $\mathbb{T}^d$, $\operatorname{div} u_0 = 0$, satisfying[1]

$$\|\partial^\alpha u_0\|_{H^k} \leq B A^{|\alpha|-1} |\alpha|!/(|\alpha|+1)^2, \quad \alpha \in \mathbb{N}^d, \tag{1.4}$$

for some $B \geq \frac{9}{4}\|u_0\|_{H^{2k+1}}$, $A \geq 1$.

Let $u(t,x)$ by the corresponding H^k maximal solution of the Euler equations (1.1), *with $u(0,\cdot) = u_0$. Then $u(t,\cdot)$ is analytic with radius of analyticity*

$$\varepsilon(t) \geq A^{-1}(1+C_1Bt)^{-1} \exp\Big(-C_0 \int_0^t \|\nabla u(s)\|_{L^\infty}\, ds\Big). \tag{1.5}$$

The proof is different and more elementary than the one in [10, 11], and it is in part inspired by the arguments in [1]. It proceeds by estimating by induction the growth of the spatial derivatives of u in *finite order* Sobolev spaces. Moreover, the same argument can be readily repeated replacing $\mathbb{T}^d$ by $\mathbb{R}^d$. Although in the case of the Euler equations it provides only minor improvements to the results in [10, 11], our method seems to be adaptable also to other types of quasilinear evolution equations and conservation laws. We shall treat these applications in a future paper.

[1] We use the sequence $|\alpha|!/(|\alpha|+1)^2$ in place of $|\alpha|!$ in (1.4) just for technical reasons; this does not change the radius of analyticity.

2. Notation and preliminary results

In the following we use the notation $X \lesssim Y$ if $X \le CY$ for some constant C *depending only on the dimension d and on the index k in Theorem 1.1.*

Moreover, as in [1, page 196] we consider the sequence

$$M_n = \frac{n!}{(n+1)^2}, \quad n \ge 0, \tag{2.1}$$

so that

$$\sum_{\beta<\alpha} \binom{\alpha}{\beta} M_{|\alpha-\beta|} M_{|\beta|+1} \le C|\alpha| M_{|\alpha|}. \tag{2.2}$$

for some constant $C > 0$.

We also recall from [14, Chapter 13, Proposition 3.6], for future reference, the following estimates

$$\|\partial^\alpha u \cdot \partial^\beta v\|_{L^2} \le C(\|u\|_{L^\infty}\|v\|_{H^m} + \|u\|_{H^m}\|v\|_{L^\infty}), \quad |\alpha| + |\beta| = m \tag{2.3}$$

for a constant $C > 0$ depending on m and on the dimension d.

We will use the following form of the Gronwall inequality.

Lemma 2.1. *Let $f(t) \ge 0$, $g(t) \ge 0$, $h(t) \ge 0$ be continuous functions on $[0,T]$ and $C \ge 0$, such that*

$$f(t) \le C + \int_0^t h(s)f(s)\,ds + \int_0^t g(s)\,ds, \quad t \in [0,T].$$

Then, with $H(t) := \int_0^t h(s)\,ds$, we have

$$f(t) \le e^{H(t)}\Big[C + \int_0^t e^{-H(s)} g(s)\,ds\Big], \quad t \in [0,T].$$

Proof. The result can be obtained for example by applying Gronwall lemma in [13, Lemma 2.1.3], and integrating by parts. ∎

Finally we recall from [14, Chapter 17, Section 2 and Exercise 1, page 485] that if u_0 is a smooth vector field (and $\operatorname{div} u_0 = 0$), then the maximal H^k-solution $u(t)$ of the Euler equations, with $u(0) = u_0$, $k > d/2+1$, is smooth as well and moreover the following estimates hold for its Sobolev norms: if $s \ge k > d/2+1$,

$$\|u(t)\|_{H^s} \le \|u(0)\|_{H^s} \exp\Big(C_0 \int_0^t \|\nabla u(s)\|_{L^\infty}\,ds\Big) \tag{2.4}$$

for some constant $C_0 > 0$ depending on the dimension and on s. Indeed, in the sequel we will use these estimates for some fixed s (depending only on d).

3. Proof of the main result (Theorem 1.1)

As observed in the previous section, we already know that the solution u is smooth, since u_0 is. Now, it is sufficient to prove that for $|\alpha| = N \geq 2$ we have

$$\frac{\|\partial^\alpha u(t)\|_k}{M_{|\alpha|}} \leq 2BA^{N-1} \exp\Big(C_0(N-1)\int_0^t \|\nabla u(s)\|_{L^\infty}\, ds\Big)(1+C_1Bt)^{N-2}, \tag{3.1}$$

where the sequence $M_{|\alpha|}$ is defined by (2.1) and C_0, C_1 are positive constants depending only on k and d.

We also set

$$\mathcal{E}_N[u(t)] = \sup_{|\alpha|=N} \frac{\|\partial^\alpha u(t)\|_k}{M_{|\alpha|}}. \tag{3.2}$$

We proceed by induction on N. The result is true for $N = 2$ by (2.4) with $s = k+2 \leq 2k+1$ and by the assumption $B \geq \frac{9}{4}\|u_0\|_{H^{2k+1}}$, $A \geq 1$. Hence, let $N \geq 3$ and assume (3.1) holds for multi-indices α of length $2 \leq |\alpha| \leq N-1$ and prove it for $|\alpha| = N$.

For $|\alpha| = N$, $|\gamma| \leq k$ we estimate $\|\partial^{\alpha+\gamma}u\|_{L^2}$ starting from the following formula, which is well-known (see e.g. [14, pag. 477]):

$$\frac{d}{dt}\|\partial^{\alpha+\gamma}u\|^2_{L^2} = -2([\partial^{\alpha+\gamma}, L]u, \partial^{\alpha+\gamma}u)_{L^2}, \tag{3.3}$$

with $Lw = L_u w := u \cdot \nabla w$. Now, we have

$$[\partial^{\alpha+\gamma}, L]u = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \sum_{\substack{\delta\leq\gamma \\ |\beta|+|\delta|<|\alpha|+|\gamma|}} \binom{\gamma}{\delta} \partial^{\alpha-\beta+\gamma-\delta}u \cdot \nabla\partial^{\beta+\delta}u. \tag{3.4}$$

We estimate the L^2 norm of each term, considering first the sum

$$\sum_{\beta\in\mathcal{A}_\alpha} \binom{\alpha}{\beta} \sum_{\substack{\delta\leq\gamma \\ |\beta|+|\delta|<|\alpha|+|\gamma|}} \binom{\gamma}{\delta} \|\partial^{\alpha-\beta+\gamma-\delta}u \cdot \nabla\partial^{\beta+\delta}u\|_{L^2},$$

where

$$\mathcal{A}_\alpha := \{\beta :\ \beta \leq \alpha, 0 \neq |\beta| \leq |\alpha| - 2\}.$$

Using (2.3) and the fact that $k > d/2 + 1$ we see that for $\beta \in \mathcal{A}_\alpha$ we have

$$\|\partial^{\alpha-\beta+\gamma-\delta}u \cdot \nabla\partial^{\beta+\delta}u\|_{L^2} \lesssim \|\partial^{\alpha-\beta}u\|_k \|\nabla\partial^\beta u\|_k.$$

By the inductive hypothesis (3.1) (note that $2 \leq |\alpha-\beta| \leq |\alpha|-1 = N-1$, $2 \leq |\beta|+1 \leq N-1$) we obtain

$$\sum_{\beta \in \mathcal{A}_{\gamma,\delta,\alpha}} \binom{\alpha}{\beta} \sum_{\delta \leq \gamma} \binom{\gamma}{\delta} \|\partial^{\alpha-\beta+\gamma-\delta} u \cdot \nabla \partial^{\beta+\delta} u\|_{L^2}$$

$$\lesssim \sum_{\beta<\alpha} \binom{\alpha}{\beta} M_{|\alpha-\beta|} M_{|\beta|+1} B^2 A^{N-1} \exp\Big(C_0(N-1)\int_0^t \|\nabla u(s)\|_{L^\infty}\,ds\Big)(1+C_1Bt)^{N-3}$$

$$\lesssim N M_N B^2 A^{N-1} \exp\Big(C_0(N-1)\int_0^t \|\nabla u(s)\|_{L^\infty}\,ds\Big)(1+C_1Bt)^{N-3},$$

where we used (2.2).

It remains to estimate the L^2 norms of the terms in (3.4) when $\beta \leq \alpha$, $\delta \leq \gamma$ and $|\beta|+|\delta| < |\alpha|+|\gamma|$, but the conditions

$$0 \neq |\beta| \leq |\alpha|-2$$

fail.

Consider first the terms where the highest order derivatives fall on a single factor, namely $|\beta|+|\delta| = |\alpha|+|\gamma|-1$ or $|\beta|+|\delta| = 0$. We distinguish three cases: for the terms with $\beta = \alpha$ and $|\delta| = |\gamma|-1$ we have

$$\binom{\alpha}{\alpha} \|\partial^{\gamma-\delta} u \cdot \nabla \partial^{\alpha+\delta} u\|_{L^2} \lesssim \|\nabla u\|_{L^\infty} \|\partial^\alpha u\|_{H^{|\delta|+1}} \leq \|\nabla u\|_{L^\infty} \|\partial^\alpha u\|_{H^k}$$

whereas for those with $|\beta| = |\alpha|-1$, $\delta = \gamma$ we use

$$\binom{\alpha}{\beta} \|\partial^{\alpha-\beta} u \cdot \nabla \partial^{\beta+\gamma} u\|_{L^2} \lesssim |\alpha| \|\nabla u\|_{L^\infty} \sum_{j:\alpha_j \geq 1} \|\partial^{\alpha-e_j} u\|_{H^{k+1}}$$

$$\lesssim |\alpha| \|\nabla u\|_{L^\infty} \sum_{\substack{j:\alpha_j \geq 1 \\ 1 \leq k \leq d}} \|\partial^{\alpha-e_j+e_k} u\|_{H^k} \lesssim N M_N \|\nabla u\|_{L^\infty} \mathcal{E}_N[u],$$

where $\mathcal{E}_N[u]$ is defined in (3.2).

Finally, for the terms with $|\beta|+|\delta| = 0$ we have

$$\binom{\alpha}{0} \|\partial^{\alpha+\gamma} u \cdot \nabla u\|_{L^2} \lesssim \|\nabla u\|_{L^\infty} \|\partial^\alpha u\|_{H^k}.$$

We now consider the terms with $0 \neq |\beta|+|\delta| \leq |\alpha|+|\gamma|-2$ but $\beta = \alpha$ or $|\beta| = |\alpha|-1$ or $\beta = 0$.

If $\beta=\alpha$ then $|\delta|\le|\gamma|-2\le k-2$ and we can write

$$\begin{aligned}\binom{\alpha}{\alpha}\|\partial^{\gamma-\delta}u\cdot\nabla\partial^{\alpha+\delta}u\|_{L^2}&\le\|\partial^{\gamma-\delta}u\|_{L^\infty}\|\nabla\partial^{\alpha+\delta}u\|_{L^2}\\&\lesssim\|\partial^{\gamma-\delta}u\|_{H^k}\|\partial^{\alpha}u\|_{H^{k-1}}\lesssim\|\partial^{\gamma-\delta}u\|_{H^k}\sup_{j:\alpha_j\ge1}\|\partial^{\alpha-e_j}u\|_{H^k}\\&\lesssim M_{N-1}\|u(0)\|_{H^{2k}}\exp\Big(C_0\int_0^t\|\nabla u(s)\|_{L^\infty}\,ds\Big)\\&\qquad\times BA^{N-2}\exp\Big(C_0(N-2)\int_0^t\|\nabla u(s)\|_{L^\infty}\,ds\Big)(1+C_1Bt)^{N-3}\\&\lesssim M_{N-1}B^2A^{N-2}\exp\Big(C_0(N-1)\int_0^t\|\nabla u(s)\|_{L^\infty}\,ds\Big)(1+C_1Bt)^{N-3},\end{aligned}$$

where we used (2.4) (with $s=|\gamma-\delta|+k\le 2k$), the inductive hypothesis (3.1), and the fact that $B>\|u_0\|_{H^{2k+1}}$.

If $|\beta|=|\alpha|-1$ then $|\delta|\le k-1$ and we have

$$\begin{aligned}\binom{\alpha}{\beta}\|\partial^{\gamma-\delta}u\cdot\nabla\partial^{\beta+\delta}u\|_{L^2}&\lesssim|\alpha|\cdot\|\partial^{\gamma-\delta}u\|_{L^\infty}\|\nabla\partial^{\beta+\delta}u\|_{L^2}\lesssim|\alpha|\cdot\|\partial^{\gamma-\delta}u\|_{H^k}\|\partial^{\beta}u\|_{H^k}\\&\lesssim NM_{N-1}\|u(0)\|_{H^{2k}}\exp\Big(C_0\int_0^t\|\nabla u(s)\|_{L^\infty}\,ds\Big)\\&\qquad\times BA^{N-2}\exp\Big(C_0(N-2)\int_0^t\|\nabla u(s)\|_{L^\infty}\,ds\Big)(1+C_1Bt)^{N-3}\\&\lesssim NM_{N-1}B^2A^{N-2}\exp\Big(C_0(N-1)\int_0^t\|\nabla u(s)\|_{L^\infty}\,ds\Big)(1+C_1Bt)^{N-3}.\end{aligned}$$

Finally, if $\beta=0$ then $\delta\ne0$ and we have

$$\begin{aligned}\binom{\alpha}{0}\|\partial^{\alpha+\gamma-\delta}u\cdot\nabla\partial^{\delta}u\|_{L^2}&\lesssim\|\partial^{\alpha}u\|_{H^{k-1}}\|\nabla\partial^{\delta}u\|_{H^k}\\&\lesssim\sup_{j:\alpha_j\ge1}\|\partial^{\alpha-e_j}u\|_{H^k}\|\nabla\partial^{\delta}u\|_{H^k}\\&\lesssim M_{N-1}B^2A^{N-2}\exp\Big(C_0(N-1)\int_0^t\|\nabla u(s)\|_{L^\infty}\,ds\Big)\\&\quad\times(1+C_1Bt)^{N-3}.\end{aligned}$$

Summing up, we have

$$\begin{aligned}\|[\partial^{\alpha+\gamma},L]u\|_{L^2}&\lesssim NM_N\|\nabla u\|_{L^\infty}\mathcal{E}_N[u]\\&\qquad+NM_NB^2A^{N-1}\exp\Big(C_0(N-1)\int_0^t\|\nabla u(s)\|_{L^\infty}\,ds\Big)(1+C_1Bt)^{N-3}.\end{aligned}$$

By applying the Cauchy-Schwarz inequality in L^2 in (3.3) and summing over $|\gamma| \le k$ we then obtain[2]

$$\begin{aligned}\frac{d}{dt}\frac{\|\partial^\alpha u(t)\|_{H^k}}{M_{|\alpha|}} &\le C(N-1)\|\nabla u(t)\|_{L^\infty}\mathcal{E}_N[u(t)] \\ &\quad + CNA^{N-1}B^2\exp\Big(C_0(N-1)\int_0^t \|\nabla u(s)\|_{L^\infty}\,ds\Big)(1+C_1Bt)^{N-3}\end{aligned}$$

for a constant $C > 0$ depending only on the dimension d and k.

Now we integrate from 0 to t and take the supremum on $|\alpha| = N$. We obtain

$$\begin{aligned}\mathcal{E}_N[u(t)] &\le \int_0^t C(N-1)\|\nabla u(s)\|_{L^\infty}\mathcal{E}_N[u(s)]\,ds + \mathcal{E}_N[u(0)] \\ &+ \int_0^t CNB^2A^{N-1}\exp\Big(C_0(N-1)\int_0^s \|\nabla u(\tau)\|_{L^\infty}\,d\tau\Big)(1+C_1Bs)^{N-3}\,ds.\end{aligned}$$

We can take $C_0 \ge C$, so that Gronwall inequality (Lemma 2.1) gives

$$\begin{aligned}\mathcal{E}_N[u(t)] &\le \exp\Big(C_0(N-1)\int_0^t \|\nabla u(s)\|_{L^\infty}\,ds\Big) \\ &\quad\times\Big[\mathcal{E}_N[u(0)] + CNB^2A^{N-1}\int_0^t (1+C_1Bs)^{N-3}\,ds\Big] \\ &\le \exp\Big(C_0(N-1)\int_0^t \|\nabla u(s)\|_{L^\infty}\,ds\Big) \\ &\quad\times\Big[\mathcal{E}_N[u(0)] + \frac{C}{C_1}\frac{N}{N-2}BA^{N-1}(1+C_1Bt)^{N-2}\Big]\end{aligned}$$

Now, we have $\mathcal{E}_N[u(0)] \le BA^{N-1}$ by the assumption (1.4). If we choose $C_1 = 3C$, so that $3C/C_1 = 1$, since $A \ge 1$, $N \ge 3$ we have

$$\begin{aligned}\mathcal{E}_N[u(0)] + &\frac{C}{C_1}\frac{N}{N-2}BA^{N-1}(1+C_1Bt)^{N-2} \\ &\le [BA^{N-1} + \frac{3C}{C_1}BA^{N-1}](1+C_1Bt)^{N-2} \\ &\le 2BA^{N-1}(1+C_1Bt)^{N-2},\end{aligned}$$

and we obtain exactly (3.1) for $|\alpha| = N$. The theorem is proved.

[2] We use $N/(N-1) \le 3/2$ if $N \ge 3$, and also $\frac{d}{dt}\|\partial^\alpha u(t)\|_{H^k}^2 = 2\|\partial^\alpha u(t)\|_{H^k}\frac{d}{dt}\|\partial^\alpha u(t)\|_{H^k}$, which in fact holds for $\|\partial^\alpha u(t)\|_{H^k} \ne 0$ but a standard argument, see e.g. [13, pag. 47-48], shows that the results below still hold when $\|\partial^\alpha u(t)\|_{H^k} = 0$.

References

[1] S. Alinhac and G. Métivier, *Propagation de l'analyticité des solutions de systèmes hyperboliques non-linéaires*, Invent. Math. **75** (1984), 189–204.

[2] S. Benachour, *Analyticité des solutions périodiques de l'équation d'Euler en trois dimensions*, C.R. Acad. Sci. Paris Sér. A-B**283** (1976), A107–A110.

[3] C. Bardos and S. Benachour, *Domaine d'analyticité des solutions de l'équation d'Euler dans un ouvert de* $\mathbb{R}^n$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **4** (1977), no. 4, 647-687.

[4] C. Bardos, S. Benachour and M. Zerner, *Analyticité des solutions périodiques de l'équation d'Euler en deux dimensions*, C. R. Acad. Sci. Paris Sér. A-B **282** (1976), no. 17, A995-A998.

[5] J. Bona and Z. Grujic', *Spatial analyticity properties of nonlinear waves.* Dedicated to Jim Douglas, Jr. on the occasion of his 75th birthday. Math. Models Methods Appl. Sci., **13** (2003), 345–360.

[6] J. Bona, Z. Grujic' and H. Kalisch *Algebraic lower bounds for the uniform radius of spatial analyticity for the generalized KdV equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **22** (2005), 783–797.

[7] M. Cappiello, P. D'Ancona and F. Nicola, *On the radius of spatial analyticity for semilinear symmetric hyperbolic systems*, J. Differential Equations **256** (2014), 2603–2618.

[8] C. Foias and R. Temam, *Gevrey class regularity for the solutions of the Navier-Stokes equations*, J. Funct. Anal. **87** (1989), no. 2, 359-369.

[9] Z. Grujic' and I. Kukavica, *Space analyticity for the nonlinear heat equation in a bounded domain*, J. Differential Equations **154** (1999), no. 1, 42-54.

[10] I. Kukavica and V. Vicol, *On the radius of analyticity of solutions to the three-dimensional Euler equations*, Proc. Amer. Math. Soc. **137** (2009) no. 2, 669–677.

[11] I. Kukavica and V. Vicol, *The domain of analyticity of solutions to the three dimensional Euler equations in half space*, Discrete Contin. Dyn. Syst. **29** (2011), no. 1, 285-303.

[12] C.D. Levermore and M. Oliver, *Analyticity of solutions for a generalized Euler equation*, J. Differential Equations **133** (1997), 321–339.

[13] J. Rauch, *Hyperbolic Partial Differential Equations and Geometric Optics*, GSM Series **133**, Amer. Math. Soc., 2012.

[14] M. Taylor, *Partial differential equations III. Nonlinear equations.* Springer-Verlag, New-York, 1996.

Dipartimento di Matematica, Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy

E-mail address: marco.cappiello@unito.it

Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

E-mail address: fabio.nicola@polito.it