MORSE INDEX AND BIFURCATION OF p-GEODESICS ON SEMI RIEMANNIAN MANIFOLDS.

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ABSTRACT. Given a one-parameter family $\{g_\lambda \colon \lambda \in [a,b]\}$ of semi Riemannian metrics on an n-dimensional manifold M, a family of time-dependent potentials $\{V_\lambda \colon \lambda \in [a,b]\}$ and a family $\{\sigma_\lambda \colon \lambda \in [a,b]\}$ of trajectories connecting two points of the mechanical system defined by (g_λ, V_λ), we show that there are trajectories bifurcating from the trivial branch σ_λ if the generalized Morse indices $\mu(\sigma_a)$ and $\mu(\sigma_b)$ are different. If the data are analytic we obtain estimates for the number of bifurcation points on the branch and, in particular, for the number of strictly conjugate points along a trajectory using an explicit computation of the Morse index in the case of locally symmetric spaces and a comparison principle of Morse Schöenberg type.

1. INTRODUCTION

Perturbed geodesics (p-geodesics) are trajectories of a simple mechanical system on a semi Riemannian manifold M of finite dimension n. They are critical points of the energy functional defined on the space of paths on M by

$$(1.1) \qquad E(\gamma) = \int_0^1 \frac{1}{2} g(\gamma'(x), \gamma'(x)) dx - \int_0^1 V(x, \gamma(x)) dx$$

where g is a semi Riemannian metric on M and V is a time-dependent potential. When the metric is indefinite, the functional E is of strongly indefinite type and the ordinary Morse index of its critical points is infinite. In [36], we associated to each non degenerate perturbed geodesic γ a generalized Morse index $\mu_{\mathrm{spec}}(\gamma) \in \mathbb{Z}$ and showed that it coincides with the conjugate index $\mu_{\mathrm{con}}(\gamma)$, an integer that counts algebraically the total number of conjugate points along γ. In the present paper we will use the generalized Morse index in order to study bifurcation of perturbed geodesics under various conditions including variation of metric and potential.

Recently, several kind of bifurcation phenomena have been studied in differential geometry and in mechanics. The change of qualitative properties of closed geodesics under one parameter variation of the metric was discussed in [30, Section 3.4], and [7]. How the crossing of 0 by a simple eigenvalue of the index form influences the topology of geodesics on a Lorentzian surface was studied in [6]. Bifurcation of relative equilibria of symmetric mechanical systems from a point of view close to ours was analyzed in [12]. A similar approach to bifurcation of minimal surfaces on a Riemannian manifold was taken in [27]. In this paper we will deal only with bifurcation of geodesics from a trivial branch. This is one of the typical bifurcation phenomena for solutions of differential equations studied in nonlinear functional analysis. Roughly speaking, given a smooth branch $\sigma \equiv \{\sigma_\lambda \colon \lambda \in [a,b]\}$ of perturbed geodesics, a *bifurcation point from the trivial branch* σ is a point λ_* in $[a,b]$ such that arbitrary close to the pair $(\lambda_*, \sigma_{\lambda_*})$ there are pairs $(\lambda, \gamma_\lambda)$ where γ_λ is a p-geodesic with the same end points as σ_λ but not belonging to the branch σ.

If we take as trivial branch σ_λ the restrictions of a given perturbed geodesic $\gamma \colon [0,1] \to M$ to subintervals of the form $[0, \lambda]$ it is easy to see that bifurcation can occur only at points which are conjugate to 0.

Jacobi found a geometric characterization of conjugate points along extremals of a variational integral $\int_0^1 L(x, y(x), y'(x))\, dx$. He proved that conjugate points arise as intersection

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points of a given extremal with the envelope of the family of extremals through a point. By the envelope theory, this fact has as a consequence that all conjugate points along extremals of variational integrals are bifurcation points from the trivial branch. This was known from the very beginning of the calculus of variation (cf. [8, page 63]), although it was never formulated in terms of bifurcation simply because this theory was inexistent at that time. Jacobi's characterization extends to vector-valued variational integrals whose Lagrangian L verifies the Legendre condition: $L_{\dot{q}_i \dot{q}_j}$ is positive definite. For example, in [23] this fact was used in order to give an alternate definition of conjugate point.

The energy functional of a Riemannian metric verifies the Legendre condition trivially. Therefore all conjugate points along perturbed geodesics on Riemannian manifold are bifurcation points (cf. Corollary 4.11, [30, Complement 2.1.13]).

On semi Riemannian manifolds with indefinite metric the above conclusion does not hold anymore and conjugate points split in two classes: those which are points of bifurcation and those which are not. The present paper is devoted to the analysis of this situation. One of the consequences of the theory developed here is that, in the framework of semi Riemannian manifolds, bifurcation invariants become useful in order to find among all conjugate points those at which bifurcation occurs. We will call them *strictly conjugate points* here.

In general, conjugate points of perturbed geodesics need not to be isolated and when the metric is indefinite the same holds even in the geodesic case [26]. Because of this, we will adopt an approach to bifurcation invented by Krasnoselskij and his collaborators in the sixties [31]. This approach is based on topological invariants that are stable under perturbations and allows to deal with non isolated potential bifurcation points. An alternative non perturbative approach in the case of geodesics was explored in [40, 25].

The methods are of functional analytic type. Although the book [30] is concerned with Riemannian manifolds only, it represents a main source for the analytic setting used here. The definition of the generalized Morse index in [36] is based on the concept of spectral flow of a family of Fredholm quadratic forms. It was shown in [19] that non vanishing of the spectral flow of the family of Hessians along the trivial branch leads to bifurcation of critical points of strongly indefinite functionals of Fredholm type. We will use an improved version of this result in order to prove our main theorem:

Let M, g, V, σ be as described above. Assume that p-geodesics σ_a and σ_b are non degenerate (and hence have a well defined generalized Morse index).

If $\mu(\sigma_a) \neq \mu(\sigma_b)$, then there must be at least one point $\lambda_* \in (a, b)$ which is a bifurcation point from the trivial branch.

If σ_λ is degenerate only at a finite number of points $\lambda \in I$, e.g. when all data are analytic, then there are at least $|\mu(\sigma_a) - \mu(\sigma_b)|/n$ distinct bifurcation points $\lambda \in (a, b)$.

In particular if the manifold and potential are analytic then along any non degenerate perturbed geodesic γ there are at least $|\mu(\gamma)|/n$ strictly conjugate points.

The precise statement is in Section 4, Theorem 4.2, where we also derive other consequences regarding the number of strictly conjugate points along a perturbed geodesic. The proof of the theorem is deferred to Section 8.

When the manifold is a locally symmetric space, the contribution of the curvature tensor to the generalized Morse index can be computed quite explicitly, since the matrix of the curvature operator in a parallel orthonormal frame is constant.

Let J be the canonical flat metric of index ν in $\mathbb{R}^n$. In Section 5, for each symmetric matrix S we define an index, analogous to the index for periodic solutions of linear autonomous Hamiltonian systems, considered in [24, 13, 20], but associated to the Dirichlet boundary value problem for second order system $Ju'' + Su = 0$. The index $\mathrm{ind}_J(S)$ can be computed from the eigenvalues of the matrix JS plus a correction term which can be easily estimated. The generalized Morse index of a geodesic on a locally symmetric space equals

the index of the system arising from the equation for Jacobi fields written in a parallel orthonormal frame. Therefore, in Section 6, we take $\mathrm{ind}_J(S)$ as our basic invariant and use it in Corollary 6.1 and Corollary 6.2 in order to obtain computable criteria for bifurcation and estimates for the number of bifurcation points along geodesics on locally symmetric spaces.

If the potential is time independent and the p-geodesic $\sigma_a \equiv p$ and $\sigma_b \equiv q$ are constant, then the number of bifurcation points along any trivial branch of perturbed geodesics with end points σ_a and σ_b can be estimated from below by the index of the Hessian of the potential V at p and q respectively. This is the content of Corollary 6.3. On symmetric spaces this can be used in order to study bifurcation of periodic p-geodesics from an equilibrium point (cf. Corollary 6.4).

In a more general setting we can still obtain bounds from below for the number of bifurcation points using the previous calculations and a comparison principle. For geodesics this principle is an extension of the classical Morse-Schöenberg comparison theorem to semi Riemannian manifolds. For lightlike and timelike geodesics on Lorentzian manifolds it was proved earlier by Beem, Ehrlich and Easley [5].

The comparison principle is formulated in Section 7, Theorem 7.1 and proved in Section 9. In Section 7, we use Theorem 7.1 in several ways both for geodesics and for perturbed geodesics. In Corollaries 7.3 and 7.4 we estimate the number of strictly conjugate points along a geodesic on analytic semi Riemannian manifolds by comparison with a locally symmetric space of the same dimension and index. A similar approach leads to an estimate for the number of strictly conjugate points along a horizontal geodesic on a warped product with locally symmetric base in Corollary 7.5.

Finally, in Theorem 7.6 we obtain an estimate for the number of bifurcation points from a trivial branch of p-geodesics on a locally symmetric space by comparing the Morse index of a non degenerate p-geodesic with the Morse index of a geodesic with the same starting point.

The remaining sections are as follows: Section 2 contains a discussion of Jacobi's geometric characterization of conjugate points mentioned above. Section 3 is a short review of the results in [36] with the purpose to introduce the notations. In the appendix we sketch the proof of the vector bundle neighborhood theorem needed for the reduction of our problem to the framework considered in [19].

2. Conjugate points and bifurcation of extremals in calculus of variation

As a motivation for the theory developed here, let us make a digression on the relation between the Jacobi's geometric characterization of conjugate points and bifurcation from the trivial branch.

Consider the one dimensional variational integral $\phi(y) = \int_0^1 L(x, y, y') \, dx$. Assume that L is smooth with non vanishing partial derivative $\frac{\partial L}{\partial z}$. Assume moreover that the initial value problem $y(0) = 0, y'(0) = z$ for the Euler Lagrange equation of the functional ϕ has a solution $y(x, z)$ defined on the whole interval $[0, 1]$.

The plane curve given by the graph γ_0 of the function $y_0(x) = y(x, 0)$ is naturally embedded into a one parameter family of curves γ_z, given by the graphs of $y_z(x) = y(x, z), 0 \leq x \leq 1$, passing through the point $(0, 0)$.

The *envelope* $\mathcal{E}$ of this family of plane curves is by definition the set $\mathcal{E} = \{(x, y) \mid y - y(x, z) = 0, \frac{\partial y}{\partial z}(x, z) = 0\}$. Jacobi characterized the points of the extremal γ_0 conjugate to $\gamma_0(0)$ as intersection points of γ_0 with $\mathcal{E}$. Since $\mathcal{E}$ can be also seen as the set of limit points of intersections of nearby curves in the family (the set of intersections points of two consecutive curves, in the pictoresque terminology of G. Peano [39]), Jacobi's characterization leads to the following conclusion (cf. [8, page 63]):

Proposition 2.1. *If the point $Q_* = \gamma_0(x_*)$ is conjugate to $\gamma_0(0)$ along γ_0, then for small enough z the extremal γ_z intersects γ_0 at a unique point $Q_z = \gamma_0(x_z)$ such that $Q_z \to Q_*$ as $z \to 0$.*

The proof of Proposition 2.1 is quite simple. It can be reduced to a very special case of one of the basic results in bifurcation theory, known as Crandal-Rabinowitz bifurcation theorem. For this, let us consider the variable x as a parameter. Let $f \colon [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined by $f(x,z) = y(x,z) - y(x,0)$. Then the equation $f(x,z) = 0$ has a trivial branch of solutions of the form $(x,0)$. Since $Q_* \in \mathcal{E}$, we have that $\frac{\partial f}{\partial z}(x_*,0) = \frac{\partial y}{\partial z}(x_*,0) = 0$. On the other hand, $\frac{\partial y}{\partial z}(x,0)$ is a Jacobi field along y_0 derived from a nontrivial deformation of y_0 by extremals. Since $\frac{\partial y}{\partial z}(x_*,0) = 0$, the partial $\frac{\partial^2 y}{\partial x \partial z}(x_*,0) = \frac{\partial^2 f}{\partial x \partial z}(x_*,0)$ cannot vanish. Thus, we are in the hypothesis of the Crandal-Rabinowitz theorem [16]. Under this hypothesis a nontrivial branch of solutions of $f(x,z) = 0$ of the form $(x(z),z) = 0$ with $x(0) = x_*$ can be found by developing in Taylor series the rescaled function $g(w) = f(x_* + zw, z)$ dividing by z^2 and solving in w using the implicit function theorem. Then the points $Q_z = (x(z),z)$ are intersection points of γ_z with γ_0 such that $Q_z \to Q_*$ as $z \to 0$.

Of course, the relation of the envelope of a family of curves with bifurcation, understood in the broad sense, is not limited to the variational setting. It is one of the main subjects of singularity theory.

The approach to bifurcation of perturbed geodesics which we will use in this paper is a reinterpretation of Proposition 2.1 from the point of view of nonlinear functional analysis. Let us explain this using our example.

We look for solutions of a parametrized family of boundary value problems for the Euler-Lagrange equations of the variational integral that are close to the restrictions of our distinguished solution y_0 to subintervals $[0,\lambda]$, but in order to work in a fixed space of functions, we reparametrize them. To be more precise, we choose a $\delta > 0$ such that on $\gamma_0([0,\delta])$ there are no conjugate points to $Q_0 = (0,0)$. Then, for $\lambda \in [\delta,1]$ we consider $\tilde{y}_\lambda(x) = y(\lambda \cdot x, 0)$. The path $\tilde{y} \colon [\delta,1] \to C^2[0,1]$ defined by $\tilde{y}(\lambda) = \tilde{y}_\lambda$ will be considered as a trivial branch of solutions of the family of boundary value problems

$$(2.1) \qquad \frac{d}{dx} L_{u'}(\lambda, x, u, u') - L_u(\lambda, x, u, u') = 0, \ u(0) = 0, \ u(1) = y_0(\lambda).$$

where $L(\lambda, x, y, z) = L(\lambda x, y, 1/\lambda z)$.

If in Proposition 2.1 we take $z_n = 1/n$, $\lambda_n = x_{z_n}$ and $y_n(x) = y(\lambda_n \cdot x, z_n)$, then we have that $\lambda_n \to x_*$, the sequence y_n converges to $\tilde{y}(x_*)$ in $C^2[0,1]$, and each y_n is a solution of (2.1) not in the branch $\tilde{y}$. Thus, by the very definition of bifurcation point, any conjugate instant $\lambda = x_*$ along y_0 is a bifurcation point for solutions of (2.1) from the trivial branch $\tilde{y}$.

In the particular case considered above a whole smooth curve of nontrivial solutions $y_z(x) = y(x_z \ x, z)$ branches of from $\tilde{y}$ at $\tilde{y}(x_*)$. However under the general topological assumptions of our main theorem we will only obtain a sequence of nontrivial solutions converging to the trivial branch.

3. Perturbed geodesics and the energy functional

Let M be a smooth connected manifold of dimension n. Let g be a semi Riemannian metric on M and let $V \colon I \times M \to \mathbb{R}$ be a smooth time-dependent potential. We will denote by D the associated Levi-Civita connection and by $\frac{D}{dx}$ the covariant derivative of a vector field along a smooth curve γ. Here we use the notation introduced by Milnor, in [35].

A *perturbed geodesic* (*p-geodesic*) is a solution $\gamma \colon [0,1] \to M$ of the second order differential equation

$$(3.1) \qquad\qquad \frac{D}{dx}\,\gamma'(x) \,+\, \nabla V(x, \gamma(x)) \,=\, 0$$

where ∇ is the gradient of a function on M with respect to the metric g.

Let Ω be the manifold of all H^1-paths in M. It is well known that Ω is a smooth Hilbert manifold modelled by $H^1([0,1];\mathbb{R}^n)$. The tangent space $T_\gamma\Omega$ at the point γ can be identified in a natural way with the Hilbert space $H^1(\gamma) = \{\xi \in H^1([0,1];TM): \tau\xi = \gamma\}$ of all H^1-vector fields along γ. Here $\tau: TM \to M$ is the projection of the tangent bundle of M to its base.

Any choice of a Riemannian (positive definite) metric on M endows Ω with an associated Riemannian structure and hence with a distance which makes Ω a metric space. The end-point map

$$\pi: \Omega \to M \times M; \quad \pi(\gamma) = (\gamma(0), \gamma(1))$$

is known to be a submersion and therefore for each $(p,q) \in M \times M$ the fiber

$$\Omega_{p,q} = \{\gamma \in \Omega : \gamma(0) = p, \ \gamma(1) = q\}$$

is a sub-manifold of codimension $2n$ whose tangent space $T_\gamma\Omega_{p,q}$ is the subspace $H_0^1(\gamma)$ of $H^1(\gamma)$ given by

$$H_0^1(\gamma) = \{\xi \in H^1(\gamma) : \xi(0) = \xi(1) = 0\}.$$

Since π is a submersion, the family of Hilbert spaces $H_0^1(\gamma)$ is a Hilbert bundle $TF(\pi) = \ker T\pi$ over Ω, called the *bundle of tangents along the fibers*. The reference for all the above, and for everything else in this section is [36].

Associated to each pair (g, V) there is an *energy functional* $\bar{E}: \Omega \to \mathbb{R}$ defined by

$$(3.2) \qquad \bar{E}(\gamma) \ = \ \int_0^1 \frac{1}{2}g(\gamma'(x), \gamma'(x))dx \ - \ \int_0^1 V(x, \gamma(x))dx.$$

The critical points of the restriction $E = E_{p,q}$ of $\bar{E}$ to the sub-manifold $\Omega_{p,q}$ are precisely the p-geodesics through p, q with kinetic energy $\frac{1}{2}g(\gamma', \gamma')$ and potential energy V. To be more precise, critical points of the energy functional are weak solutions of (3.1), which turn out to be smooth classical solutions by elliptic regularity.

In [36], we associated two integers to each non degenerate critical point of the energy functional. The first integer is the generalized Morse index $\mu_{\mathrm{spec}}(\gamma)$, defined as the negative of the spectral flow of the Hessians of the energy functionals along the path $\tilde{\gamma}$ canonically induced by γ on Ω. Since several points related to the construction of the generalized Morse index will be used in the proof of our main theorem, we will briefly recall the construction here.

The linearization of the boundary value problem

$$\begin{cases} \frac{D}{dx}\gamma'(x) + \nabla V(x, \gamma(x)) = 0 \\ \gamma(0) = p, \quad \gamma(1) = q \end{cases}$$

at the critical point γ is the equation of Jacobi fields

$$(3.3) \qquad \frac{D^2}{dt^2}\xi(t) + R(\gamma'(t), \xi(t))\gamma'(t) + D_{\xi(t)}\nabla V(t, \gamma(t)) = 0,$$

subjected to Dirichlet boundary conditions $\xi(0) = 0 = \xi(1)$. Here ξ is a vector field along γ, R is the curvature tensor of the connection D and $D_{\xi(x)}\nabla V(x, \gamma(x))$ is the Hessian of $V(x, -)$ with respect to the metric g, viewed as a symmetric operator of $T_{\gamma(x)}M$.

Its weak solutions (H_0^1-*Jacobi fields*) are vector fields $\xi \in H_0^1(\gamma)$ such that for any $\eta \in H_0^1(\gamma)$

$$(3.4) \qquad 0 \ = \ H_\gamma(\xi, \eta) = \int_0^1 g(\tfrac{D}{dx}\xi(x), \tfrac{D}{dx}\eta(x)) \, dx \ +$$

$$- \int_0^1 g(R(\gamma'(x), \xi(x))\gamma'(x) + D_{\xi(x)}\nabla V(x, \gamma(x)), \eta(x)) \, dx.$$

Thus, Jacobi fields in $H_0^1(\gamma)$ are the elements of the kernel of the bilinear form H_γ. The Hessian of $E_{p,q}$ at γ is the quadratic form $h_\gamma : H_0^1(\gamma) \to \mathbb{R}$ associated to H_γ.

Since the embedding of $H_0^1(\gamma)$ into $L^2(\gamma)$ is compact, the form

$$c(\xi) = \int_0^1 g_\lambda(R(\gamma'(x), \xi(x))\gamma'(x) + D_{\xi(x)}\nabla V(x, \gamma(x)), \xi(x))\, dx$$

is weakly semi-continuous. Being a weakly semi-continuous perturbation of a non degenerate form, h_γ is a bounded Fredholm quadratic form. In particular $\ker h_\gamma \equiv \{\xi \,|\, H_\gamma(\xi, \eta) = 0 \text{ for all } \eta\}$ is finite dimensional.

The form h_γ is non degenerate if and only if the instant 1 is not conjugate to 0 along γ. If this holds we will say that the p-geodesic is non degenerate.

Each p-geodesic γ induces in a canonical way a path $\tilde\gamma$ on the manifold Ω of all paths on M. The canonical path $\tilde\gamma \colon frm[o]-- \to \Omega$ associated to γ is defined by $\tilde\gamma(\lambda)(x) = \gamma(\lambda \cdot x)$, $x \in [0, 1]$. Each $\tilde\gamma_\lambda \equiv \tilde\gamma(\lambda)$ is a p-geodesic for (g, V_λ), where $V_\lambda(x, m) = \lambda^2 V(\lambda x, m)$. From the above discussion it follows that the Hessian h_λ of $E_\lambda \equiv E_{p,\gamma(\lambda)}$ at $\tilde\gamma_\lambda$ is degenerate if and only $\gamma(\lambda)$ is a conjugate point to $p = \gamma(0)$ along γ. If γ is non degenerate the path $\tilde\gamma$ has non degenerate end points. The family h_λ, $\lambda \in [0, 1]$ defines a function h on the total space of the pull-back bundle $\tilde\gamma^*(TF)$ over $[0, 1]$ such that its restriction to each fiber is a bounded quadratic form and such that h_0, h_1 are non degenerate. This is what we called in [36] an admissible family of Fredholm quadratic forms. The spectral flow of such a family was defined in [36, Section 2].

Roughly speaking the spectral flow is a measure of how much of a negative subspace for h_λ at $\lambda = 0$ becomes positive at $\lambda = 1$ minus the part of a positive subspace that becomes negative. In general the spectral flow depends on the homotopy class of the path of quadratic forms but in this specific case it depends only on the end points. This allows to define a relative form of Morse index even in the case when the classical Morse indices are infinite. By definition the *generalized Morse index* of a p-geodesic γ is

$$\mu_{\mathrm{spec}}(\gamma) = -\mathrm{sf}(h).$$

The second integer, $\mu_{\mathrm{con}}(\gamma)$, called *conjugate index*, was defined using Brouwer degree in order to count zeroes of a determinant associated to the Jacobi equation by means of trivialization via a parallel orthonormal frame (see [36, Section 4] for a detailed description).

Finally, we proved a generalization of the Morse Index Theorem to perturbed geodesics on semi Riemannian manifolds [36, Theorem 1.1], by establishing the equality

$$(3.5) \qquad\qquad\qquad \mu_{\mathrm{spec}}(\gamma) = \mu_{\mathrm{con}}(\gamma).$$

Taking into account (3.5) we will denote both indices with $\mu(\gamma)$.

4. The main theorem and some consequences

Let $p \colon \mathcal{S}_\nu^2(M) \to M$ be the bundle of all non degenerate symmetric two forms of index ν on the tangent bundle TM. Let $I = [a, b]$ be an interval. A *one parameter family of semi Riemannian metrics on M* is a smooth map $g \colon I \times M \to \mathcal{S}_\nu^2(M)$ such that $pg = \pi$, where π denotes the projection onto the second factor.

Each $g_\lambda = g(\lambda, -)$ is then a semi Riemannian metric on M of index ν. Using the Koszul formula it is easy to see that the family of associated Levi-Civita connections D^λ is smooth with respect to the λ variable in an obvious sense.

If $V \colon I \times [0, 1] \times M \to \mathbb{R}$ is a smooth function, we consider V as a smooth one parameter family of time-dependent *potentials* $V_\lambda \colon [0, 1] \times M \to \mathbb{R}$ defined by $V_\lambda(x, u) = V(\lambda, x, u)$. The data $(M, g_\lambda, V_\lambda)_{\lambda \in I}$ define a one parameter family of time-dependent mechanical systems.

We will also assume the existence of a one parameter family of known p-geodesics. In other words, given a smooth family $(g_\lambda, V_\lambda)_{\lambda \in I}$ as above, we assume that there exists a

smooth map $\sigma \colon [a,b] \times [0,1] \to M$ such that for each $\lambda \in I$, $\sigma_\lambda(x) \equiv \sigma(\lambda, x)$ is a p-geodesic corresponding to the mechanical system (g_λ, V_λ).

We will refer to the family $(\sigma_\lambda)_{\lambda \in [a,b]}$ as *the trivial branch* of perturbed geodesics.

Definition 4.1. A point λ_* is called a *bifurcation point* for p-geodesics from the trivial branch σ if there exists a sequence $(\lambda_n, \gamma_n) \to (\lambda_*, \sigma(\lambda_*))$ in $I \times \Omega$ such that γ_n is a weak solution of equation (3.1) with boundary conditions

$$(4.1) \qquad\qquad \gamma_n(0) = \sigma_{\lambda_n}(0), \quad \gamma_n(1) = \sigma_{\lambda_n}(1)$$

such that γ_n does not belong to $\sigma(I)$.

As stated above the notion of bifurcation point depends on the metric induced on Ω by an arbitrary choice of Riemannian metric on M. As a matter of fact using a slightly weaker topology the concept of bifurcation point can be defined without appealing to the existence of a Riemannian metric on M. There is an embedding of Ω into the space of continuous paths in M endowed with the compact-open topology (see for example [29, Lemma 2.4.6]). Thus, the sequence γ_n in the above definition converges to $\sigma(\lambda_*)$ in $C^0([0,1]; M)$ as well. A metric free definition of bifurcation point would be as follows: it is a point such that any neighborhood of $(\lambda_*, \sigma(\lambda_*))$ in $I \times C^0([0,1]; M)$ contains a solution of the boundary value problem (3.1), (4.1) which is not in the trivial branch. Finally notice that, by bootstrap, γ_n converges to $\sigma(\lambda_*)$ in the Whitney topology of $C^\infty(I; M)$. Therefore, the choice of any of the above approaches gives the same bifurcation points.

For each $\lambda \in I$, let E_λ be the restriction of the energy functional associated to the data (g_λ, V_λ) to the sub-manifold $\Omega_\lambda \equiv \Omega_{\sigma_\lambda(0), \sigma_\lambda(1)}$. The trivial branch σ can be viewed as a smooth path $\sigma \colon I \to \Omega$ of critical points $\sigma(\lambda) = \sigma_\lambda$ of E_λ.

Our main result is the following theorem.

Theorem 4.2. *Let M and $(g_\lambda, V_\lambda)_{\lambda \in I}$ be as above. Let σ be a trivial branch of p-geodesics of $(M, g_\lambda, V_\lambda)_{\lambda \in I}$ such that the end points σ_a and σ_b are non degenerate.*

  (i) *If $\mu(\sigma_a) \neq \mu(\sigma_b)$, then there exists at least one bifurcation point, $\lambda_* \in (a,b)$, from the trivial branch.*
  (ii) *If σ_λ is degenerate only at a finite number of points $\lambda \in I$ then there are at least $|\mu(\sigma_a) - \mu(\sigma_b)|/n$ distinct bifurcation points λ in (a,b).*

Remark 4.3. The hypothesis in part (ii) of Theorem 4.2 is automatically verified in the following two cases:

  (a) The data (M, g, V, σ) are analytic. In this case the singular set $\Sigma = \{\lambda \in I | \ker h_\lambda \neq 0\}$ can be characterized as the set of zeroes of an analytic "determinant" function [21]. Since h_a is non degenerate this function does not vanish on the whole interval I and hence Σ must be discrete.
  (b) The intrinsic derivative (see Section 8) of the generalized family h_λ, $\lambda \in I$ of Hessians at any point $\lambda_* \in \Sigma$ is a non degenerate quadratic form.

Remark 4.4. In the estimate of Theorem 4.2 the dimension n of the manifold arises because it provides a bound for the possible dimensions of the space of H_0^1-Jacobi fields. The estimate can be slightly improved for geodesics ($V \equiv 0$). A H_0^1-Jacobi field along a geodesic γ must be perpendicular to $\gamma'(x)$ at any point $x \in [0,1]$, which implies that the dimension of the space of Jacobi fields vanishing at the end points cannot be greater than $(n-1)$. Therefore in the case of a geodesic we can substitute n with $(n-1)$ in Theorem 4.2.

As already mentioned in the introduction, we will pay special attention to the following particular case: We will take g and the potential V fixed, take a p-geodesic $\gamma \colon [0,1] \to M$ and consider as the trivial branch the path $\tilde{\gamma} \colon [0,1] \to \Omega$ canonically induced by γ.

Definition 4.5. A conjugate point $\gamma(\lambda_*)$, $\lambda_* \in frm[o]--$, will be called a *strictly conjugate* if λ_* is a bifurcation point from the trivial branch $\tilde{\gamma}$ in the sense of Definition 4.1.

Not all conjugate points are strictly conjugate. Here is an example of a p-geodesic on a 3-dimensional semi Riemannian manifold with an isolated conjugate point which is not strictly conjugate.

Example 4.6. Let us consider the manifold $M = \mathbb{R}^3$ endowed with the metric $g_0 = dx^2 - dy^2 + dz^2$ and take as potential the function $V(x, y, z) = 1/2x^2 - 1/2y^2 + 1/3x^3y^3$. Since the gradient of V with respect to g is $\nabla V(z, y, z) = (x + x^2y^3)\frac{\partial}{\partial x} + (y - x^3y^2)\frac{\partial}{\partial y}$, the equations of a p-geodesic are

$$(4.2) \qquad \begin{cases} x'' + x + x^2y^3 &= 0 \\ y'' + y - x^3y^2 &= 0 \\ z'' &= 0 \end{cases}$$

In order to simplify the coefficients we will not normalize the interval: we consider the p-geodesic $\gamma_0(t) = (0, 0, t)$ defined on $[0, 2\pi]$, connecting $(0, 0, 0)$ to $(0, 0, 2\pi)$. We claim that there are no p-geodesics $\gamma \neq \gamma_0$ such that $\gamma(0) = \gamma_0(0)$ and $\gamma(\lambda) = \gamma_0(\lambda)$ for $\lambda \in [0, 2\pi]$.

Indeed, if $\gamma(t) = (x(t), y(t), z(t))$ verifies (4.2) and the above boundary conditions, then multiplying the first equation of (4.2) by $y(t)$, the second by $x(t)$, substracting the second from the first and integrating from 0 to λ we obtain $\int_0^\lambda (x^2y^4 + x^4y^2)\, dt = 0$. Thus both $x, y \equiv 0$ and $\gamma(t) = \gamma_0(t)$.

On the other hand there is a two dimensional subspace of Jacobi fields along γ_0, $\xi(t) = a\,\sin t\,\frac{\partial}{\partial x} + b\,\sin t\,\frac{\partial}{\partial y}$ verifying the Dirichlet boundary conditions $\xi(0) = 0 = \xi(\pi)$. Therefore $\lambda = \pi$ is a conjugate point which is not strictly conjugate.

In order now to construct an example of a geodesic with a conjugate point which is not strictly conjugate we will reduce the problem to the previous one using the well known correspondence between p-geodesics and geodesics of the associated Jacobi metric together with a construction in [26] of a geodesic with assigned curvature operator on the normal space.

Example 4.7. Let M and g_0 be as in the previous example and let us consider the conformally flat metric $g = \exp(2\rho)\, g_0$, where ρ is defined by

$$(4.3) \qquad\qquad \rho(x, y, z) = 1/2y^2 - 1/2x^2 + 1/3x^3y^3.$$

The Levi Civita connexions D of g and D^0 of g_0 are related by

$$D_X Y = D_X^0 Y + d\rho(X)Y + d\rho(Y)X - g_0(X, Y)\nabla^0\rho$$

where ∇^0 is the gradient with respect g_0.

From this it follows easily that $\gamma_0(t) = (0, 0, \sqrt{2}t)$ is a geodesic with energy $\mathcal{E}(\gamma_0) = 1/2g(\gamma'(t), \gamma'(t)) = 1$. Moreover the vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ are parallel along γ_0 and generate the normal space $T^\perp_{\gamma(t)}M$. Calculating the curvature tensor we get $R(\frac{\partial}{\partial z}, \frac{\partial}{\partial x})\frac{\partial}{\partial z} = \frac{\partial}{\partial x}$ and $R(\frac{\partial}{\partial z}, \frac{\partial}{\partial y})\frac{\partial}{\partial z} = \frac{\partial}{\partial y}$. Therefore the Jacobi equations for a normal vector field $\xi(t) = u(t)\frac{\partial}{\partial x} + v(t)\frac{\partial}{\partial y}$ are

$$(4.4) \qquad \begin{cases} u'' + u &= 0 \\ v'' + v &= 0 \end{cases}$$

As before, the Dirichlet problem $\xi(0) = 0 = \xi(\pi)$ for (4.4) has a two dimensional subspace of nontrivial solutions and therefore π is a conjugate instant. However we will show that there cannot be non trivial geodesics bifurcating from the trivial branch γ_0.

Assume that $\gamma(t) = (x(t), y(t), z(t))$ is a geodesic for g with $\gamma(0) = (0,0,0)$ and $\gamma(b) = (0,0,\lambda)$. If γ is close enough to γ_0 then is γ is spacelike and, possibly after changing the interval of definition, we can assume that the energy $\mathcal{E}(\gamma) = 1$. By Jacobi's theorem [1, Theorem 3.7.7] geodesics for g with energy 1 are reparametrizations of the perturbed geodesics for the mechanical system $(M, g_0, -\exp(2\rho))$ with total energy $\mathcal{E} = 0$.

The equations for the corresponding p-geodesic are

$$(4.5) \qquad \begin{cases} x'' + 2\exp 2\rho(x - x^2 y^3) & = & 0 \\ y'' + 2\exp 2\rho(y + x^3 y^2) & = & 0 \\ z'' & = & 0 \end{cases}$$

If $\delta(t) = \gamma(h(t))$ verifies the above equations then multiplying the first equation of (4.5) by $y(t)$, the second by $x(t)$ substracting the second from the first as in the previous example and then integrating from 0 to $t_* = h^{-1}(\lambda)$ we obtain that $x \equiv 0 \equiv y$ and hence $\gamma = \gamma_0$.

We will now discuss some consequences of Theorem 4.2.

Corollary 4.8. *Any regular p-geodesic γ with non-vanishing index has a strictly conjugate point. If the data are analytic there must be at least $|\mu(\gamma)|/n$ of them.*

For example, assume that (M, g) is complete, take a point $p \in M$ and consider the exponential map $exp_p \colon T_p(M) \to M$. It is well known that points conjugate to p along $\gamma(x) = exp_p(x \cdot v)$ are the critical values of the exponential map exp_p lying on γ. Consider

$$B_p = \{m \in M | \text{ for some } v \in \exp^{-1}(m), \exp \text{ is not locally one to one at } v\}.$$

When M is a Riemannian manifold, B_p coincides with the set C_p of all critical values of the exponential map (see [30, Theorem 2.1.12]). On general semi Riemannian manifolds, B_p is only a closed proper subset of C_p. However, by definition of bifurcation point, the exponential $\exp_p$ cannot be a local diffeomorphism at any point of the form $t_* \gamma'(0)$ if t_* is a strictly conjugate instant. Therefore by the previous Corollary any non degenerate geodesic γ starting from p crosses B_p in at least $|\mu(\gamma)|/(n-1)$ points.

In the remaining part of this section we will discuss some criteria for isolated conjugate points to be strictly conjugate.

Let $\mathcal{I}$ be the n-dimensional vector space of all Jacobi fields along γ verifying $\xi(0) = 0$. Consider $\mathcal{I}[x] = \{\xi(x) : \xi \in \mathcal{I}\} \subset T_{\gamma(x)}M$. It is easy to see that $\gamma(\lambda_*)$, $\lambda_* \in [0,1]$, is conjugate to 0 if and only if $\mathcal{I}[\lambda_*] \neq T_{\gamma(\lambda_*)}M$. Let us denote by $\mathcal{I}[x]^\perp$ the g orthogonal of $\mathcal{I}[x]$.

The conjugate instant λ_* is said to be *regular* if the quadratic form $\Gamma(\lambda_*)$ given by the restriction of $g_{\gamma(\lambda_*)}$ to $\mathcal{I}[\lambda_*]^\perp$ is non degenerate. It is shown in [36] that regular conjugate instants are isolated and moreover the variation of the generalized Morse index through λ^* is given by the signature of the quadratic form $\Gamma(\lambda_*)$. More precisely, for ϵ small enough,

$$(4.6) \qquad \mu(\tilde{\gamma}_{\lambda_*+\epsilon}) - \mu(\tilde{\gamma}_{\lambda_*-\epsilon}) = \text{sign}\,\Gamma(\lambda_*).$$

From the main theorem it follows

Corollary 4.9. *Any regular conjugate point such that $\text{sign}(g|_{\mathcal{I}[\lambda_*]^\perp}) \neq 0$ is strictly conjugate.*

Remark 4.10. The conjugate points in the previous examples are both non degenerate but with vanishing signature.

If g is positive definite then any conjugate point is regular and the signature of the form $\Gamma(\lambda_*)$ is simply the dimension of $\mathcal{I}[\lambda_*]^\perp$. Therefore:

Corollary 4.11. *If (M, g, V) is a mechanical system on a Riemannian manifold all conjugate points along a p-geodesic γ are strictly conjugate.*

When λ_* is not regular but yet isolated then its local contribution $\mu_{\mathrm{con}}(\lambda_*)$ to the variation of the Morse index can be still defined as in [36, (4.14)]. Thus λ_* is a bifurcation point whenever $\mu_{\mathrm{con}}(\lambda_*) \neq 0$. The local invariant $\mu_{\mathrm{con}}(\lambda_*)$ can be computed either by the algorithm for computing the degree of an analytic plane vector field as described in [36, Lemma 4.3] or using the calculation of $\mu_{\mathrm{Maslov}}(\lambda_*)$ in the degenerate case via partial signatures as in [25, 43]. However neither the algorithm for the degree nor the formula by partial signatures can be considered effective. This is one of the motivations for the approach adopted in this paper.

5. Computation of the generalized Morse index

In this section we will consider some special cases in which the generalized Morse index can be found explicitly, providing computable criteria for the existence of bifurcation points. As motivation, let us discuss first the Morse index of a geodesic on a Riemannian locally symmetric space.

If $\gamma \colon [0,1] \to M$ is a geodesic, then, by Corollary 4.11, the conjugate instants are isolated and each one of them gives a strictly positive contribution $m(\lambda) = \dim \ker h_\lambda$ to the index $\mu(\gamma)$. The index of a geodesic is not easy to compute in general but when the related second order system has constant coefficients one can easily get explicit formulas.

For example, if M has a constant positive sectional curvature $K > 0$ then the conjugate points correspond to instants $\lambda_k = k\pi/L\sqrt{K}$, where L is the length of the geodesic γ. Moreover the multiplicity of each conjugate point is $(n-1)$. Thus γ is non degenerate if $L\sqrt{K}/\pi$ is not an integer and its index is given by $\mu(\gamma) = (n-1)N_K(\gamma)$, where $N_K(\gamma) = [L\sqrt{K}/\pi]$ is the integral part of $L\sqrt{K}/\pi$. One can use this in Theorem 4.2 in order to estimate the number of geodesics bifurcating from a known trivial branch σ_λ. Assume, for instance, that there exists a smooth family g_λ of metrics on M connecting two metrics of constant positive sectional curvature K, K'. If $\sigma \colon [a,b] \to \Omega(M)$ is a path such that each σ_λ is geodesic for the metric g_λ and σ_a, σ_b are non degenerate, then by Theorem 4.2 there are at least $|N_K(\sigma_a) - N_{K'}(\sigma_b)|$ bifurcation points from the trivial branch σ. In particular, if g is a fixed metric of constant positive curvature then along any branch of geodesics with a large enough variation of length there will be bifurcation points from the branch. All of the above extends immediately to geodesics on locally symmetric spaces by considering the eigenvalues of the matrix of the curvature operator $R(\gamma', v)\gamma'$ in a parallel orthonormal frame.

The generalized Morse index for p-geodesics on a semi Riemannian manifold constructed in [36] is not always positive and hence it is a considerable weaker object than the classical Morse index. Even in the case of locally symmetric spaces ($DR \equiv 0$) formulas for the index are harder to compute than in the Riemannian case. Nevertheless, the above topological picture is still present in this context and leads to estimates for the number of bifurcation points.

Let $\gamma \colon [0,1] \to M$ be a p-geodesic on M for (g, V), with g a semi Riemannian metric, and let $\gamma_\lambda(x) = \gamma(\lambda x)$, $\lambda \in [0,1]$. Choosing a parallel g-orthonormal frame $\mathcal{F} = \{e^i(x) : i = 1, \ldots, n\}$ along γ and trivializing $\gamma_\lambda^*(TM)$ by means of the frame $\mathcal{F}_\lambda = \{e^i(\lambda x) : i = 1, \ldots, n\}$, by the computation performed in [36, Section 4], we have that such a trivialization induces a one to one correspondence between H_0^1-Jacobi fields verifying Dirichlet boundary conditions at 0 and λ and the weak solutions $f = (f_1, \ldots, f_n)$ of the boundary value problem for a second order system of ordinary differential equations

$$(5.1) \qquad \begin{cases} Jf''(x) + S_\lambda(x)f(x) = 0 \\ f(0) = 0 = f(1) \end{cases} \qquad \forall x \in [0,1],$$

where J is the symmetry

$$(5.2) \qquad J = \begin{pmatrix} \mathrm{Id}_{n-\nu} & 0 \\ 0 & -\mathrm{Id}_\nu \end{pmatrix}$$

and $S_\lambda(x) = \lambda^2 S(\lambda x)$, with $S(x) = (S_{ij}(x))$ the $n \times n$ symmetric matrix defined by

$$(5.3) \qquad S_{ij}(x) = g(R(\gamma'(x), e^i(x))\gamma'(x) + D_{e^i(x)}\nabla V(x, \gamma(x)), e^j(x)).$$

Thus weak solutions of the system above are elements in the kernel of the quadratic form $h_\lambda \colon H_0^1([0,1]; \mathbb{R}^n) \to \mathbb{R}$ defined by

$$h_\lambda(f) = \int_0^1 \langle Jf'(x), f'(x) \rangle - \langle S_\lambda f(x), f(x) \rangle \, dx$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^n$. In particular h_λ is a weakly continuous perturbation of a non degenerate form. The quadratic form h_λ is nothing but the expression of h_{γ_λ} in the trivialization of $\gamma_\lambda^*[TF(\pi)]$ induced by the trivialization of $\gamma_\lambda^*(TM)$ given by the parallel frame introduced above (see [36, (3.6)]). It follows from this that

$$(5.4) \qquad \mu(\gamma) = -\mathrm{sf}(h, [0,1]).$$

Now let us compute $\mathrm{sf}(h, [0,1])$ assuming that $S(x) \equiv S$ is a constant matrix. We represent, the associated path of bilinear forms against the scalar product $\langle f, g \rangle_{H_0^1} = \int_0^1 \langle f'(x), g'(x) \rangle dx$ by a path of self-adjoint operators L. Then, each L_λ is defined by

$$\langle L_\lambda f, g \rangle_{H_0^1} = \int_0^1 \langle Jf'(x)g'(x) \rangle - \langle \lambda^2 Sf(x), g(x) \rangle \, dx.$$

Thus, $L_\lambda = J + \lambda^2 K$ where, by abuse of notation, we denote with J the corresponding multiplication operator, while K is the compact operator defined by

$$\langle Kf, g \rangle_{H_0^1} = -\int_0^1 \langle Sf(x), g(x) \rangle \, dx.$$

Any $f \in H_0^1([0,1]; \mathbb{R}^n)$ has a Fourier expansion given by

$$f(x) = \sum_{k=1}^\infty a_k \sin \pi k x$$

where a_k belong to $\mathbb{R}^n$. Hence, $H_0^1([0,1]; \mathbb{R}^n)$ decomposes into an orthogonal direct sum

$$(5.5) \qquad H_0^1([0,1]; \mathbb{R}^n) = \bigoplus_{k=1}^\infty V_k$$

where

$$V_k = \mathrm{span}\{a \sin k\pi x \, : \, a \in \mathbb{R}^n\}.$$

From the relation

$$Kf = g \quad \Longleftrightarrow \quad \begin{cases} g''(x) = Sf(x) \\ g(0) = g(1) = 0 \end{cases}$$

it follows that each V_k is an invariant subspace of L_λ. Since L_λ is self adjoint V_k reduces L_λ and we have an orthogonal decomposition $L_\lambda = \bigoplus_{k=1}^\infty L_\lambda^k$, where L_λ^k is the restriction of L_λ to V_k.

If we choose as basis of V_k the set B_k given by

$$B_k = \{\phi_{i,k}(x) \, : \, 1 \le i \le n\}, \quad \phi_{i,k}(x) = e_i \sin k\pi x,$$

where e_i is the canonical base of $\mathbb{R}^n$, then the $n \times n$ matrix that represents L^k_λ with respect to the base B_k, is given by

$$(5.6) \qquad L^k_\lambda = J - \frac{\lambda^2}{(\pi k)^2} S.$$

By the above formula the restriction of L_λ to $(\bigoplus_{k=1}^m V_k)^\perp$ is an isomorphism for m big enough. Thus by the normalization property (see [36, 19]), the spectral flow of the restriction of the path L to $(\bigoplus_{k=1}^m V_k)^\perp$ is zero. Hence,

$$(5.7) \qquad \mathrm{sf}(L) = \sum_{k=1}^m \mathrm{sf}(L^k, [0,1]) = \sum_{k=1}^m [\nu - \mu_{\mathrm{Morse}}(L^k_1)],$$

since the spectral flow of a path of operators defined on a finite-dimensional space is the difference between the Morse indices of its end points.

Given a constant symmetric matrix S such that (5.1) has only the trivial solution for $\lambda = 1$ we define the index of S with respect to J by

$$(5.8) \qquad \mathrm{ind}_{\mathrm{J}}(S) = \sum_{k=1}^\infty \left[\mu_{\mathrm{Morse}}(L^k_1) - \nu \right].$$

Notice that $\mathrm{ind}_{\mathrm{J}}(S)$ is well defined since by the above discussion, the sum on the right hand side is finite.

Summing up the above discussion, we have proved:

Proposition 5.1. *If the matrix S defined by (5.3) is constant then $\mu(\gamma) = \mathrm{ind}_{\mathrm{J}}(S)$.*

There is a special case, in which the computation of the index turns out to be as simple as in the case of a geodesic on a Riemannian symmetric space.

A matrix S commuting with J will be called *split*. Any split matrix S has the form

$$S = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A and B are two symmetric matrices of dimension respectively $(n-\nu) \times (n-\nu)$ and $\nu \times \nu$. Diagonalizing A and B and we get

$$(5.9) \qquad U^* A U = diag(\lambda_1, \ldots, \lambda_{n-\nu}) \quad \text{and} \quad V^* B V = diag(\mu_1, \ldots, \mu_\nu)$$

and using the invariance of the spectral flow under cogredience we reduce the calculation of $\mathrm{ind}_{\mathrm{J}}(S)$ to that of the diagonalized system

$$(5.10) \qquad \begin{cases} f_1''(x) & + & \lambda^2 \lambda_1 f_1(x) & = & 0 \\ \cdots\cdots & & \cdots\cdots & & \\ f_{n-\nu}''(x) & + & \lambda^2 \lambda_{n-\nu} f_{n-\nu}(x) & = & 0 \\ -f_{n-\nu+1}''(x) & + & \lambda^2 \mu_1 u_{n-\nu+1}(x) & = & 0 \\ \cdots\cdots & & \cdots\cdots & & \\ -f_n''(x) & + & \lambda^2 \mu_\nu f_n(x) & = & 0. \end{cases}$$

Non trivial solutions for $u'' + \lambda^2 a u = 0$ with Dirichlet boundary conditions $u(0) = 0 = u(1)$ arise with multiplicity 1 only for $\lambda^2 \lambda_i = k^2 \pi^2, \ k \in \mathbb{N}$.

Defining, for a real number a,

$$N(a) = \#\{i \in \mathbb{N} \,|\, i^2 \pi^2 < a\},$$

we get:

Proposition 5.2. *If S is constant and split then*

$$(5.11) \qquad \mu(\gamma) = \mathrm{ind}_J(S) = \sum_1^{n-\nu} N(\lambda_i) - \sum_1^{\nu} N(-\mu_i).$$

In general $\mathrm{ind}_J(S)$ can be estimated in terms of the negative real eigenvalues of the J-symmetric matrix JS. For this we will need some well known results about Maslov index. Most of what we will use here can be found in [44, 34].

The Dirichlet boundary value problem for $Ju'' + Su = 0$ is equivalent to the following boundary value problem for a first order Hamiltonian system in $\mathbb{R}^{2n}$

$$(5.12) \qquad \begin{cases} w'(x) = Hw(x) \\ w(0) \in V, w(1) \in V \end{cases}$$

where $w = (u, v) \in \mathbb{R}^n \times \mathbb{R}^n$, $V = \{0\} \times \mathbb{R}^n$ and

$$(5.13) \qquad H = \begin{pmatrix} 0 & J \\ -S & 0 \end{pmatrix}.$$

Since H is a hamiltonian matrix its flow $\Psi(x) = e^{xH}$ is a path in the symplectic group $Sp(2n)$ of all transformations preserving the canonical symplectic form ω on $\mathbb{R}^{2n}$. The symplectic group acts naturally on the manifold $\Lambda(n)$ of all Lagrangian subspaces of $\mathbb{R}^{2n}$, and hence the path Ψ induces a path L on the Lagrangian Grasmannian $\Lambda(n)$ defined by $L(x) = \Psi(x)V$.

According to [36, Proposition 6.1], if $L(1)$ is transverse to V, then $\mathrm{ind}_J(S)$ is defined and, taking $\varepsilon > 0$ small enough,

$$(5.14) \qquad \mathrm{ind}_J S = \mu_V(L, [\varepsilon, 1])$$

where $\mu_V(L)$ is the Maslov index of the path L with respect to V, an integer which counts algebraically the intersections of the path L with the singular cycle $\Sigma(V) = \{V' \in \Lambda(n) \mid \dim V \cap V' > 0\}$ or, what is the same, the number of points $x \in [\varepsilon, 1]$ such that $L(x)$ fails to be transverse to V. On the other hand, the Maslov index $\mu_V(L, [\varepsilon, 1])$ is related with the *Conley Zehnder index* or the *rotation number* of the path Ψ [24, 13]) which can be computed from the spectrum of JS. For this, we need to consider the symplectic space $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ equipped with the symplectic form $\bar{\omega} = -\omega \times \omega$. With this symplectic form the path defined by $M(x) = \mathrm{Graph}\,\Psi(x)$ is a path of Lagrangian subspaces. By [44, Theorem 3.2], if $W = V \times V$ then

$$(5.15) \qquad \mu_V(L, [\varepsilon, 1]) = \mu_W(M, [\varepsilon, 1]).$$

The Maslov index of the path M with respect to the diagonal $\Delta \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ is related to $\mu_W(M, [\varepsilon, 1])$ by

$$(5.16) \qquad \mu_\Delta(M, [\varepsilon, 1]) = \mu_W(M, [\varepsilon, 1]) + 1/2[\mathrm{sign}(W, \Delta, M(1)) - \mathrm{sign}(W, \Delta, M(\varepsilon))]$$

where sign denotes the signature of a triple of Lagrangian subspaces [34].

By our hypothesis $M(1)$ is transverse to W. In what follows, for simplicity, we will assume also that $M(1)$ is transverse to Δ, equivalently, that 1 is not an eigenvalue of e^H. Notice that both conditions are also verified by $M(\varepsilon)$, if we take $\varepsilon > 0$ small enough. Indeed, by [36, Proposition 6.1], $M(x)$ fails to be transverse to W only at a conjugate instant and we showed in [36] that conjugate instants cannot accumulate to 0. On the other hand, 1 is an eigenvalue of $e^{\varepsilon H}$ if and only if $2\frac{\pi}{\varepsilon}i$ is an eigenvalue of H which is impossible for ε small enough.

In this situation, by [34, Proposition 1],

$$(5.17) \qquad |\operatorname{sign}(W, \Delta, M(1))| \leq 1/2 \dim[(\Delta \cap W)^\omega / \Delta \cap W]$$

where $(\Delta \cap W)^\omega$ denotes the symplectic orthogonal of $\Delta \cap W$. Since $\Delta \cap W$ is an n-dimensional isotropic subspace, $|\operatorname{sign}(W, \Delta, M(1))| \leq n$. By the same argument it holds $|\operatorname{sign}(W, \Delta, M(\varepsilon))| \leq n$. Summing up (5.14),(5.15),(5.16) we obtained

$$(5.18) \qquad |\,\mu_\Delta(M, [\varepsilon, 1]) - \operatorname{ind}_J S\,| \leq n.$$

Let us recall that the Conley-Zehnder index of the path $\Psi \colon [0, t] \to Sp(n)$ is the integer $\mu_\Delta(M, [0, t])$. It depends only on the purely imaginary eigenvalues H and can be computed as follows:

Assume first that all purely imaginary eigenvalues of H are simple. If $\lambda = i\alpha$ is an eigenvalue with eigenvector e then $\bar{e}$ is an eigenvector corresponding to $\bar{\lambda}$. It is easy to see that $\frac{1}{2i}\omega(\bar{e}, e)$ is a non vanishing real number. Let $\beta = \pm\alpha$ according to the sign of $\frac{1}{2i}\omega(\bar{e}, e)$. If $\tau = 2\pi/t$ then there exists an $m \in \mathbb{Z}$ such that $m\tau < \beta < (m + 1)\tau$. Set $||\lambda|| = m + 1/2$ and let $j(t, H) = \sum_\lambda ||\lambda||$ then, by [13],

$$(5.19) \qquad \mu_\Delta(M, [0, t]) = j(t, H).$$

When the imaginary eigenvalues are not simple the above formula still holds provided that we count the imaginary part of each eigenvalue λ positively p times and negatively q times according to the number p (resp. q) of positive (resp. negative) eigenvalues of the hermitian form $\frac{1}{2i}\omega(\bar{e}, e)$ defined on the generalized eigenspace of λ.

With this we can estimate $\operatorname{ind}_J(S)$ in terms of the negative eigenvalues of the matrix JS. Indeed, since $H^2 = \operatorname{diag}(JS, SJ)$ the purely imaginary eigenvalues of H are precisely square roots of the negative eigenvalues of JS. Let $j(S) = j(1, H) - j(\varepsilon, H)$.

From (5.18) and the above computation we get $|\operatorname{ind}_J(S) - j(S)| \leq n$ and therefore we obtain:

Proposition 5.3. *Under the above assumptions*

$$j(S)/n - 1 \leq \operatorname{ind}_J(S)/n \leq j(S)/n + 1.$$

Since our main theorem allows us to estimate the number of bifurcation points in terms of $\operatorname{ind}_J(S)/n$, the above result is good enough for our purpose.

6. Applications

A semi Riemannian manifold (M, g) is locally symmetric if for any $p \in M$, there is a local isometry s_p at p, leaving p fixed and such that $T_p s_p = -\operatorname{Id}$. The above condition is equivalent to the vanishing of the covariant derivative of the curvature tensor or, what is the same, that the curvature tensor commutes with parallel translation along paths on M.

The *curvature operator* along a path γ is the g-symmetric endomorphism

$$R_\gamma(x) \colon T_{\gamma(x)}M \to T_{\gamma(x)}M, \quad \text{defined by}$$

$$(6.1) \qquad R_\gamma(x)(v) = R(\gamma'(x), v)\gamma'(x).$$

Notice that here we follow Klingerberg's book [30] in the choice of terminology. Elsewhere this operator is called either "tidal force" or "Jacobi operator" while the term "curvature operator" is reserved to the curvature tensor viewed as a function of the third variable, i.e. to the infinitesimal generators of the restricted holonomy.

By the above discussion the curvature operator along any geodesic path γ on a locally symmetric manifold M commutes with the parallel translation along γ and therefore the matrix

(6.2) $$(R_{i,j}(x)) = g(R(\gamma'(x), e^i(x))\gamma'(x), e_j(x))$$

of $R_{\gamma'(x)}$ in any parallel orthonormal frame $\mathcal{F} = \{e^i(x); i = 1, ..., n\}$ is constant. The index of the matrix $(R_{i,j})$ is clearly independent from the choice of the frame and will be denoted by $\mathrm{ind}_J R$

From Theorem 4.2 and Proposition 5.1 (with $V \equiv 0$) we obtain:

Corollary 6.1. *Let (M, g) be an analytic semi Riemannian locally symmetric space. If γ is a non degenerate geodesic there are at least $|\mathrm{ind}_J(R)|/(n-1)$ strictly conjugate points along γ.*

Let $(M, g) = (M_1, g_1) \times (M_2, -g_2)$ where M_i are Riemannian symmetric spaces of dimension $n - \nu$ and ν respectively. If γ is a non degenerate geodesic on M, the curvature operator along γ splits into a direct sum of the curvature operators along projections of γ on M_1 and M_2. Denoting the corresponding eigenvalues by λ_i and μ_i respectively, from Proposition 5.2 we get

Corollary 6.2. *If M and γ are as above, then along γ there are at least*

$$|\sum_{i=1}^{n-\nu} N(\lambda_i) - \sum_{i=1}^{\nu} N(-\mu_i)|/(n-1)$$

strictly conjugate points.

Now we consider bifurcation of p-geodesics of a family of analytic mechanical systems $(M, g_\lambda, V_\lambda)$ parametrized by $I = [a, b]$.

Assume that both V_a and V_b are time independent. Let q_a and q_b be critical points of V_a and V_b respectively. The constant paths q_a and q_b are p-geodesics for the corresponding mechanical systems. Let $H_i(v) = D_v \nabla V_i(q_i) \colon T_{q_i} M \to T_{q_i} M; i = a, b$ be the Hessians of V_i at q_i. Since the curvature operator along a constant path vanishes, the matrix S of (5.3) reduces in this case to the matrix of the Hessian in an orthonormal basis of $T_{q_i} M$. From the main theorem we obtain:

Corollary 6.3. *Let $(M, g_\lambda, V_\lambda)$ be as above. Assume that the constant p-geodesics q_a and q_b are non degenerate. Then, along any analytic trivial branch σ of p-geodesics such that $\sigma_a = q_a$ and $\sigma_b = q_b$ there must be at least $|\mathrm{ind}_J(H_a) - \mathrm{ind}_J(H_b)|/n$ bifurcation points.*

Our next result deals with bifurcation of nontrivial periodic p-geodesics from a branch of critical points on symmetric spaces.

Let $(M, g_\lambda, V_\lambda)$ be as above. Assume moreover that M is an analytic semi Riemannian symmetric space. Suppose also that each V_λ is time independent and that the trivial branch of p-geodesics is given by an analytic path $q \colon I \to M$ such that each q_λ is a critical point of V_λ. Assume that the potential V_λ is invariant under the symmetry s_λ at q_λ.

Corollary 6.4. *Under the above assumptions, if q_a and q_b are non degenerate p-geodesics then there are at least $|\mathrm{ind}_J(H_a) - \mathrm{ind}_J(H_b)|/n$ points of the branch at which bifurcation of non constant 2-periodic geodesics arises.*

By the above Corollary, there are at least $|\mathrm{ind}_J(H_a) - \mathrm{ind}_J(H_b)|/n$ points $\lambda_* \in I$ such that close to the constant p-geodesic $q_{\lambda*}$ there are p-geodesics with both end points on the trivial branch but not belonging to the trivial branch. Such geodesics cannot be constant.

On the other hand, since s_λ is an isometry and $V_\lambda s_\lambda = V_\lambda$, the energy functional E_λ is invariant under the map induced on Ω_λ by s_λ. Hence, if γ is any p-geodesic for (g_λ, V_λ), so is $s_\lambda \gamma$. By symmetry any non constant p-geodesic γ with end points at q_λ and close to the

trivial branch produces a non constant periodic geodesic $\bar{\gamma}$ of period 2 close to the branch q. For this, it is enough to define

$$\bar{\gamma}(x) = \begin{cases} \gamma(x) & \text{if} \quad 0 \leq x \leq 1 \\ s_\lambda \gamma(2-x) & \text{if} \quad 1 \leq x \leq 2. \end{cases}$$

Observe that Corollary 6.4 it is not applicable to geodesics, since it holds true only if the potential is not trivial.

7. A comparison principle for p-geodesics on semi-Riemannian manifolds

Given a mechanical system (M, g, V) and a path $\gamma \colon [0,1] \to M$ let us consider

$$(7.1) \qquad\qquad S_\gamma(x) \colon T_{\gamma(x)} M \to T_{\gamma(x)} M; \quad S_\gamma(x) = R_\gamma(x) + P_\gamma(x)$$

where $R_\gamma(x)(v) = R(\gamma'(x), v)\gamma'(x)$ and $P_\gamma(x)(v) = D_v \nabla V(x, \gamma(x))$ are the curvature operator and the Hessian of V along γ respectively.

Since both R_γ and P_γ are g-symmetric it follows that S_γ is a symmetric endomorphism of $\gamma^*(TM)$.

Let (M_1, g_1, V_1) and (M_2, g_2, V_2) be two mechanical systems on manifolds of the same dimension n and such that both g_1 and g_2 have the same index ν. Let $p_i \in M_i$; $i = 1, 2$ be two given points. Choose an isometry $\varphi \colon T_{p_1} M_1 \to T_{p_2} M_2$. Given $v_1 \in T_{p_1} M_1$, let γ_1 be the unique p-geodesic corresponding to (g_1, V_1), such that $\gamma_1(0) = p_1$ and $\gamma_1'(0) = v_1$. If $v_2 = \varphi(v_1)$, we denote by γ_2 the unique p-geodesic in M_2 for the system (g_2, V_2), such that $\gamma_2(0) = p_2$ and $\gamma_2'(0) = v_2$. We assume that both γ_1 and γ_2 are defined on the interval $[0,1]$ and are nondegenerate critical points of the corresponding energy functionals.

We will compare the index of γ_1 with the index of γ_2. First we extend the isometry φ to an isometry between $\gamma_1^*(TM)$ and $\gamma_2^*(TM)$. For any $x \in [0,1]$, let $\varphi_x \colon T_{\gamma_1(x)} M_1 \to T_{\gamma_2(x)} M_2$ be defined by

$$(7.2) \qquad\qquad \varphi_x v = \mathcal{P}_x^2 \circ \varphi \circ \mathcal{P}_{-x}^1(v) \quad \forall v \in T_{\gamma_1(x)} M_1,$$

where $\mathcal{P}_x^i \colon T_{p_i} M_i \to T_{\gamma_i(x)} M_i$ is the parallel transport along γ_i.

Theorem 7.1. *Let (M_i, g_i, V_i) and γ_i be as above. Let $S^i(x) = S_{\gamma_i}(x)$. If for each $x \in [0,1]$ and $v \in T_{\gamma_1(x)}$*

$$(7.3) \qquad\qquad g_2\big(S^2(x)\varphi_x v, \varphi_x v\big) \leq g_1\big(S^1(x)v, v\big)$$

then

$$\mu(\gamma_2) \leq \mu(\gamma_1).$$

When both manifolds are Riemannian, the Hessian forms are essentially positive and have a finite Morse index. In this case the theorem is an easy consequence of the fact that Morse index is nonincreasing. The proof in the general case is based on monotonicity properties of the spectral flow and it is given in Section 9. Analogous results can be found in [3] and [4] in a different framework.

Taking $V \equiv 0$ we obtain an extension of the Morse-Schöenberg comparison theorem to general semi Riemannian manifolds.

Corollary 7.2. *Let (M_i, g_i) and γ_i be as above with $V_i \equiv 0$. If for each $x \in [0,1]$,*

$$(7.4) \qquad\qquad g_2\left(R_{\gamma_2}(x)\varphi_x v, \varphi_x v\right) \leq g_1\left(R_{\gamma_1}(x)v, v\right),$$

then $\mu(\gamma_2) \leq \mu(\gamma_1)$.

For lightlike and timelike geodesics on Lorentzian manifolds this was proved by Beem Ehrlich Easley (see for instance [5]).

Comparing an analytic semi Riemannian manifold M_1 with a semi Riemannian locally symmetric space M_2 of the same dimension and index we conclude that

Corollary 7.3. *If (7.4) holds and if* $\mathrm{ind}_J R_2(0)$ *is defined and is positive, then along* γ_1 *there must be at least* $\mathrm{ind}_J R_2(0)/(n-1)$ *strictly conjugate points. The same is true if* $\mathrm{ind}_J R_2(0)$ *is negative and the reverse of inequality (7.4) holds.*

The simplest comparison is with a space form M_2 of index ν and curvature K. The curvature tensor of M_2 is $R_2(X,Y)Z = K[g_2(Z,X)Y - g_2(Z,Y)X]$. Since φ_x is an isometry sending $\gamma_1'(x)$ into $\gamma_2'(x)$ we have

$$g_2\left(R_{\gamma_2}(x)\varphi_x v, \varphi_x v\right) = K[g_1(\gamma_1'(x),\gamma_1'(x))g_1(v,v) - g_1(\gamma_1'(x),v)^2].$$

Identifying $T_{p_1}M_1$ with $T_{p_2}M_2$ via φ, the curvature operator of R_2 restricted to the normal subspace $\gamma_2'(0)^\perp$ coincides with $K[g_1(\gamma_1'(0),\gamma_1'(0))]\mathrm{Id}$ on $\gamma_1'(0)^\perp$. If γ_1 is not a null geodesic, without loss of generality we can assume that $\epsilon(\gamma_1) = g_1(\gamma_1'(0),\gamma_1'(0)) = \pm 1$. By Proposition 5.2 $\mathrm{ind}_J R_2 = N(K\epsilon(\gamma_1))(n-1)$. From the above corollary we conclude that

Corollary 7.4. *If (M,g) is a semi Riemannian manifold such that*

(7.5) $$g(R(X,Y)Y,X) \geq K[g(X,X)g(Y,Y) - g(X,Y)^2]$$

for some real number K*, then any geodesic* γ *on* M *such that* $K\epsilon(\gamma) > 0$ *has at least* $N(K\epsilon(\gamma))$ *strictly conjugate points.*

If the manifold is Riemannian, inequality (7.5) implies that the sectional curvature has to be bounded from below. When the metric is indefinite, if the sectional curvature is bounded from below (or above) then it is automatically bounded. Nevertheless, the hypothesis in Corollary 7.4 does not imply that the sectional curvature of M is constant. Nontrivial examples of semi Riemannian manifolds verifying (7.5) are given in [2]. In this paper and also in [32] several comparison results of Sturm type are proved.

In a similar way we can obtain estimates for the number of strictly conjugate points along a nondegenerate horizontal geodesic in a semi Riemannian warped product with locally symmetric base.

Let B and F be two semi Riemannian manifolds and let ρ be a smooth positive function defined on B. The *warped product* $M_\rho = B \times_\rho F$ is the product manifold $B \times F$ endowed with a twisted product metric. Namely, for $p = (b,f) \in B \times F$ we write the tangent space $T_p B \times F$ as an internal direct sum of $T_p B \equiv T_{(b,f)} B \times \{f\}$ with $T_p F \equiv T_{(b,f)}\{b\} \times F$ and endow it with the metric $g_\rho = g_B + \rho^2 g_F$. Notice that the direct sum decomposition $T_p M_\rho = T_p B \oplus T_p F$ is orthogonal with respect to both g_ρ and the product metric g_1.

We will do the comparison between the manifold M_ρ and the manifold M_1, the semi Riemannian product of B and F. Given a point $p = (b,f)$ we choose an isometry $\varphi\colon T_p M_\rho \to T_{\varphi(p)} M_1$ whose restriction to $T_p B$ is the identity.

Let γ be any horizontal geodesic through p. Then γ is a geodesic under both metrics and decomposing a vector $w \in T_{\gamma(x)}$ as $w = (u,v)$ under the above orthogonal decomposition we have that the curvature operator on M_1 is given by

(7.6) $$R_\gamma(x)(u,v) = R(\gamma'(x),u)\gamma'(x) = R_{\bar\gamma}^B(x)u$$

where $\bar\gamma$ is the projection of γ onto the base B and R^B is the curvature operator on B.

It follows plainly from (7.6) that the family of hessians h_λ of the energy functional on M_1 at γ_λ, $\lambda \in [0,1]$ is a direct sum of the family of hessians $\bar h_\lambda$ of E on B at $\bar\gamma_\lambda$ with a family of nondegenerate quadratic forms defined on $T_{\gamma_\lambda}(F)$. Hence, by normalization and the direct sum property of the spectral flow, $\mu(\gamma) = \bar\mu(\bar\gamma)$, as a geodesic on M_1.

On the other hand by [37, Proposition 42], on M_ρ we have

(7.7) $$R_\gamma(x)(u,v) = R(\gamma'(x),u)\gamma'(x) - \frac{1}{f}\mathrm{Hess}(\rho)(\gamma'(x),\gamma'(x))v.$$

From Corollary 7.2 and the above discussion it follows that if ρ is a convex function then for any horizontal geodesic γ on M_ρ we have $\mu(\bar{\gamma}) \leq \mu(\gamma)$. Hence:

Corollary 7.5. *If B is a locally symmetric space and the warping function ρ is convex then along any nondegenerate horizontal geodesic there are at least $\mathrm{ind}_J R^B_{\bar{\gamma}}(0)/(n-1)$ strictly conjugate points.*

Our final application is an estimate of the number of bifurcation points from the trivial branch for a family $(M, g, V_\lambda, \sigma_\lambda)$ where M is an analytic complete locally symmetric space and V_λ, $\lambda \in [a, b]$, is an analytic family of potentials on M. For $i = a, b$ let $R_i(x) \colon T_{\sigma_i(x)}M \to T_{\sigma_i(x)}M$ and $\mathcal{P}^i_x \colon T_{\sigma_i(0)}M \to T_{\sigma_i(x)}M$ be the curvature operator and the parallel translation along the p-geodesic σ_i respectively.

Theorem 7.6. *Assume that, as symmetric operators,*

$$\mathcal{P}^a_x R_a(0)\mathcal{P}^a_{-x} \geq R_a(x) \ \text{and} \ \mathcal{P}_x R_b(0)\mathcal{P}_{-x} \leq R_b(x).$$

Suppose moreover that the hessians of V_i, $i = a, b$ at each point of σ_i are respectively positive and negative semi-definite. Then, if $\mathrm{ind}_J(R_a)(0) > \mathrm{ind}_J(R_b)(0)$, the number of bifurcation points from the trivial branch along σ is not less than $[\mathrm{ind}_J(R_a(0)) - \mathrm{ind}_J(R_b(0))]/n$.

Proof. We use the comparison principle between (M, g, V_i) and $(M, g, 0)$. Comparing the Hessian of E at the p-geodesics σ_i with the Hessian of the energy at a geodesic having the same origin and initial velocity, we obtain

$$\mu(\sigma_a) \geq \mathrm{ind}_J(R_a)(0) \ \text{and} \ \mu(\sigma_b) \leq \mathrm{ind}_J(R_b)(0).$$

Therefore

$$|\mu(\sigma_a) - \mu(\sigma_b)| \geq \mathrm{ind}_J(R_a)(0) - \mathrm{ind}_J(R_b)(0).$$

The conclusion follows from the main theorem. $\qquad\square$

8. Proof of the main theorem

Before going to the proof of Theorem 4.2 we will prove an abstract bifurcation theorem for critical points of families of smooth functionals. This theorem is an extension of the main theorem in [19] to the geometric framework which naturally arises in dealing with bifurcation of p-geodesics. Moreover, we will improve the result in [19] obtaining estimates from below for the number of bifurcation points.

Definition 8.1. A *smooth family of Hilbert manifolds* $\{X_\lambda\}_{a \leq \lambda \leq b}$ parameterized by a real interval $I = [a, b]$ is a family of manifolds of the form $X_\lambda = p^{-1}(\lambda)$ where $p \colon X \to I$ is a smooth submersion of a Hilbert manifold X onto I.

By the implicit function theorem each fiber X_λ of the submersion is a submanifold of X of codimension one. For each $x \in X_\lambda$ the tangent space $T_x X_\lambda$ coincides with $\ker Tp_x$. Being p a submersion the family of Hilbert spaces $TF(p) = \{\ker Tp_x : x \in X\}$ is a Hilbert sub-bundle of the tangent bundle TX. $TF(p)$ is the bundle of tangents along the fibers or the vertical bundle of the submersion p. A smooth functional $f \colon X \to \mathbb{R}$ defines by restriction to the fibers of p a *smooth family of functionals* $f_\lambda \colon X_\lambda \to \mathbb{R}$

We will assume that there exists a smooth section $\sigma \colon I \to X$ of p such that $\sigma(\lambda)$ is a critical point of the restriction f_λ of the functional f to the fiber X_λ.

We will refer to σ as the *trivial branch* of critical points of the family $\{f_\lambda : \lambda \in I\}$. We will say that $\lambda_* \in I$ is a *point of bifurcation* for critical points of the family $\{f_\lambda : \lambda \in I\}$ from the trivial branch $\sigma(I)$ if there exists a sequence $\lambda_n \to \lambda_*$ and a sequence $x_n \to \sigma(\lambda_*)$ such that $p(x_n) = \lambda_n$ and each x_n is a critical point of f_{λ_n} not belonging to $\sigma(I)$.

In what follows we will denote with h_λ the Hessian of f_λ at the point $\sigma(\lambda)$.

Our next assumption is that, for each $\lambda \in I$, the Hessian h_λ is a Fredholm quadratic form. Moreover we will assume that the end points h_a and h_b are non degenerate.

The family of Hessians h_λ, for $\lambda \in [a, b]$, defines a smooth function h on the total space of the pull-back bundle $\mathcal{H} = \sigma^* TF(p)$ of the vertical bundle $TF(p)$ by the map $\sigma \colon I \to X$ such that the restriction of h to each fiber is a Fredholm quadratic form. The function $h \colon \mathcal{H} \to \mathbb{R}$ is a generalized family of Fredholm quadratic forms. Since the restrictions h_a, h_b to the fibers over a and b are non degenerate, the family is admissible and the spectral flow $\mathrm{sf}(h) = \mathrm{sf}(h, [a, b])$ of the family h is defined by [36, (2.2)].

Theorem 8.2. *Under the above assumptions if* $\mathrm{sf}(h) \neq 0$, *then there exists at least one bifurcation point* $\lambda_* \in (a, b)$ *of critical points of* f *from the trivial branch.*

Moreover, if $\ker h_\lambda \neq 0$ *only for a finite number of points* $\lambda \in I$ *then there are at least* $|\mathrm{sf}(h)|/m$ *distinct bifurcation points in* (a, b), *where* $m = \max\{\dim \ker h_\lambda\}$.

Let us mention that the singular set $\Sigma(h) = \{\lambda \in I \colon \ker h_\lambda \neq 0\}$ is finite if all the data (X, p, f, σ) are analytic. Another situation in which $\Sigma(h)$ is finite is the following one:

If $h \colon \mathcal{H} \to \mathbb{R}$ is a generalized family of quadratic forms, the *intrinsic derivative* $\dot{h}_\lambda \colon \mathcal{H}_\lambda \to \mathbb{R}$ of h at λ is the restriction to $\ker h_\lambda$ of the ordinary derivative of the smooth path of quadratic forms $h_\lambda M_\lambda$ where M is any local trivialization of the bundle $\mathcal{H}$. It is easy to see that the resulting quadratic form $\dot{h}_\lambda$ is independent from the choice of trivialization. A point $\lambda \in \Sigma(h)$ is called *regular* when $\dot{h}_\lambda$ is a nondegenerate quadratic form on $\ker h_\lambda$. By straightforward extension to bundles of the arguments in [20], one shows that each regular point $\lambda \in \Sigma(h)$ is isolated in $\Sigma(h)$. Therefore, if h has only regular singular points they are in finite number.

Remark 8.3. By [20] if all points in $\Sigma(h)$ are regular then

$$\mathrm{sf}(h) = \sum_{\lambda \in \Sigma} \mathrm{sign}\, \dot{h}_\lambda,$$

where sign denotes the signature.

Proof of Theorem 8.2. By the vector bundle neighborhood theorem (Theorem A.1) there exist a trivial Hilbert bundle $\mathcal{H} = I \times H$ over $I = [a, b]$ and a fiber preserving smooth map $\psi \colon \mathcal{H} \to \sigma^* X$ such that $\psi(\lambda, 0) = \sigma(\lambda)$ and such that ψ is a diffeomorphism of $\mathcal{H}$ with an open neighborhood $\mathcal{O}$ of $\sigma(I)$ in X.

Let $\tilde{f} \colon I \times H \to \mathbb{R}$ be defined by $\tilde{f} = f \circ \psi$. Then $\tilde{f}$ is a family of smooth functionals on the Hilbert space H. Since the restriction ψ_λ of ψ to the fiber is a diffeomorphism, we have that $u \in H$ is a critical point of $\tilde{f}_\lambda = f_\lambda \circ \psi_\lambda$ if and only if $x = \psi_\lambda(u)$ is a critical point of f_λ. In particular 0 is a critical point of $\tilde{f}_\lambda$ for each $\lambda \in I$. The Hessian $\tilde{h}_\lambda$ of $\tilde{f}_\lambda$ at 0 is given by $\tilde{h}_\lambda(\xi) = h_\lambda(T_0 \psi_\lambda)([\xi])$. By cogredience and normalization property of the spectral flow (see Propositions 3.1 and 3.2 in [19]), we have that $\mathrm{sf}(h, I) = \mathrm{sf}(\tilde{h}, I) = \mathrm{sf}(L, I)$ where $L \colon I \to \Phi_s(H)$ is the smooth path of self-adjoint Fredholm operators representing the family $\tilde{h}$ with respect to the scalar product, i.e. $\tilde{h}_\lambda(u) = \langle L_\lambda u, u \rangle \quad \forall u \in H$.

In order to prove the first assertion we argue as follows. Our hypothesis implies that $\mathrm{sf}(L, I) \neq 0$, and [19, Theorem 1] ensures the existence of a critical point $u \neq 0$ of some $f_\lambda, \lambda \in I$, in every neighborhood V of 0. Choosing a $u_n \neq 0$ in $V_n = B(0, 1/n)$ we can assume by passing, if necessary, to subsequences that the corresponding sequence λ_n converges to some $\lambda_* \in (a, b)$. Let $x_n = \psi_{\lambda_n}(u_n)$. By construction of ψ each x_n is a critical point of f_{λ_n} not belonging to $\sigma(I)$ and $x_n \to \sigma(\lambda_*)$. Thus λ_* is a bifurcation point.

In order to prove the second assertion let us assume that the set $\Sigma(h)$ is finite. Let $a < \lambda_1 < \lambda_2 < \ldots < \lambda_k < b$ be the points in $\Sigma(h)$. We want to show that at least $|\mathrm{sf}(h)|/m$ of them are bifurcation points.

Taking $I_j = [\lambda_j - \delta, \lambda_j + \delta]$ with $\delta > 0$ sufficiently small so that I_j are disjoint, we have that h at the end points of I_j is non degenerate and hence

$$
(8.1) \qquad\qquad \mathrm{sf}(h, I) = \sum_{j=1}^{k} \mathrm{sf}(h, I_j)
$$

by additivity of the spectral flow. Since 0 is always an isolated point in the spectrum of a self-adjoint Fredholm operator, there exists a positive number η such that the spectrum of L_{λ_j} intersects the interval $[-\eta, \eta]$ only in 0. Then we can choose $\delta > 0$ small enough so that $\eta, -\eta$ do not belong to the spectrum of L_λ, for any $\lambda \in I_j$ and so that (8.1) still holds true. Let P_λ, $\lambda \in [\lambda_j - \delta, \lambda_j + \delta]$, be the orthogonal projection onto the subspace associated with the part of the spectrum of L_λ lying in the interval $[-\eta, \eta]$. Taking if necessary a smaller δ, by [28, Chapter II, Section 6], there exists a continuously differentiable path U on $[\lambda_j - \delta, \lambda_j + \delta]$ of orthogonal operators such that

$$
(8.2) \qquad\qquad U_{\lambda_j} = Id \quad \text{and} \quad U_\lambda P_\lambda U_\lambda^{-1} = P_{\lambda_j} \quad \forall \lambda \in [\lambda_j - \delta, \lambda_j + \delta].
$$

Since the spectral flow is invariant under cogredience, we get

$$
(8.3) \qquad\qquad \mathrm{sf}(L_\lambda, I_j) = \mathrm{sf}(M_\lambda, I_j)
$$

where $M_\lambda = U_\lambda L_\lambda U_\lambda^{-1}$.

From (8.2), it follows that M_λ is reduced by P_{λ_j}, for any $\lambda \in I_j$, being L_λ reduced by P_λ in that interval. Under the splitting $H = \ker L_{\lambda_j} \bigoplus \mathrm{Im}\, L_{\lambda_j}$, M_λ can be written as $M_\lambda = M_\lambda^1 + M_\lambda^2$. But M_λ^2 is an isomorphism and therefore $\mathrm{sf}(M_\lambda^2, I_j) = 0$.

From this and (8.3) we get

$$
\mathrm{sf}(L_\lambda, I_j) = \mathrm{sf}(M_\lambda^1, I_j) = \mu_{\mathrm{Morse}}(M_{\lambda_j - \delta}^1) - \mu_{\mathrm{Morse}}(M_{\lambda_j + \delta}^1)
$$

where the last equality follows from the property that the spectral flow of a path of linear operators defined on a finite dimensional space equals the difference of the Morse Indexes at the end points.

Since $\dim \ker L_{\lambda_j} \leq m$ it follows that

$$
|\mu_{\mathrm{Morse}}(M_{\lambda_j - \delta}^1) - \mu_{\mathrm{Morse}}(M_{\lambda_j + \delta}^1)| \leq m;
$$

Summing up

$$
|\mathrm{sf}(h)| \leq \sum_{j=1}^{k} |\mathrm{sf}(L_{\lambda_j}, I_j)| \leq dm
$$

where d is the number of nonzero addends in the above formula. Since, by the first part, any interval I_j with a nontrivial contribution to the above sum must contain at least one bifurcation point there must be at least $d = |\mathrm{sf}(h)|/m$ distinct bifurcation points in I. $\quad\square$

With this we can finally prove Theorem 4.2.

Let us consider the end-point map $e \colon I \to M \times M$ defined by $e(\lambda) = (\sigma(\lambda)[0], \sigma(\lambda)[1])$. The pullback $e^*(\pi) \colon e^*(\Omega) \to I$ of the submersion $\pi \colon \Omega \to M \times M$ via the map e is defined in the usual way. By standard transversality arguments we have that the total space $e^*(\Omega) = \{(\lambda, \gamma) \in I \times \Omega : e(\lambda) = \pi(\gamma)\}$ is an Hilbert manifold (which inherits the Riemannian metric from Ω) and the map $e^*(\pi) \colon e^*(\Omega) \to I$ given by the projection to the first factor is a submersion (see [33, Chapter 3, Section 3]).

The fiber of $e^*(\pi)$ over $\lambda \in I$ is the submanifold $\Omega_{\sigma(\lambda)[0], \sigma(\lambda)[1]}$. By the commutativity of the following diagram

$$
\begin{array}{ccc}
e^*(\Omega) & \xrightarrow{\quad \iota \quad} & \Omega \\
\downarrow {\scriptstyle e^*(\pi)} & {\scriptstyle \sigma} & \downarrow {\scriptstyle \pi} \\
I & \xrightarrow{\quad e \quad} & M \times M
\end{array}
$$

where ι is the projection to the second factor, we have that $\sigma \colon I \to \Omega$ induces a section $\bar{\sigma} \colon I \to e^*(\Omega)$ of $e^*(\pi)$. By the same argument the vertical bundle $TF(e^*(\pi))$ coincides with $\iota^*(TF(\pi))$ and moreover $\sigma^*(TF(\pi)) = \bar{\sigma}^*[TF(e^*(\pi))]$. The one-parameter family of functionals $\bar{E} = E \circ \iota \colon e^*(\Omega) \to \mathbb{R}$ has the same bifurcation points as the family $\{E_\lambda : \lambda \in I\}$ because ι_λ sends in a one to one fashion critical points of $\bar{E}_\lambda$ into critical points of E_λ. Moreover the isomorphism ι_λ intertwines the Hessian $\bar{h}_\lambda$ of $\bar{E}_\lambda$ at $\bar{\sigma}_\lambda$ with the Hessian h_λ of E_λ at σ_λ. Now Theorem 4.2 follows from Theorem 8.2 and the following proposition

Proposition 8.4. *Under assumptions of Theorem 4.2,*

$$
\mathrm{sf}(h) = \mu(\sigma_a) - \mu(\sigma_b).
$$

Proof. Let $T = [a, b] \times [0, 1]$. Consider the map $\theta \colon T \to \Omega$ defined by

$$
(8.4) \qquad\qquad \theta(\lambda, t)(x) = \sigma_\lambda(tx).
$$

Each $\theta(\lambda, t)$ is a critical point of the energy functional $E_{\lambda, t} \colon \Omega_{e(\lambda)} \to \mathbb{R}$ associated to the data $(M, g_\lambda, t^2 V_\lambda)$. Let $h_{\lambda, t}$ be the Hessian of $E_{\lambda, t}$ at $\theta(\lambda, t)$. As before, $h_{\lambda, t}$ defines a generalized family of Fredholm quadratic forms $h \colon \theta^* TF(\pi) \to \mathbb{R}$. Let $\bar{h}$ be the restriction of this family to the boundary ∂T of T. Trivializing $\theta^* TF(\pi)_{|\partial T}$ and using [19, Poposition 3.6] we conclude that $\mathrm{sf}(\bar{h}) = 0$. On the other hand, the spectral flow is the sum of the spectral flows of each segment of the boundary of T. By definition of $h_{\lambda, t}$ we have that $h_{\lambda, 0}$ is nondegenerate for all λ. Indeed, $\theta(\lambda, 0) \equiv p$ is a constant path. A H^1-vector field ξ along p is simply a path $\xi \in H^1(I; T_p(M))$ and hence $h_0(\xi) = d_p(\xi) = \int_0^1 g(\frac{D}{ds}\xi(x), \frac{D}{ds}\xi(x)) \, dx$ which is clearly nondegenerate.

From the above discussion we have that

$$
\mathrm{sf}(h_{(\cdot, a)}, [0, 1]) - \mathrm{sf}(h_{(1, \cdot)}, [a, b]) = \mathrm{sf}(h_{(\cdot, b)}, [0, 1])
$$

that is $\mathrm{sf}(h) = \mu(\sigma_a) - \mu(\sigma_b)$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

9. Proof of Theorem 7.1

Let γ_1, γ_2 as in (7.1). By (3.4) the Hessian of the energy functional at γ_1 computed on a vector field ξ along γ is given by

$$
(9.1) \qquad h_{\gamma_1}(\xi) = \int_0^1 g_1(\tfrac{D}{dx}\xi(x), \tfrac{D}{dx}\xi(x)) \, dx - \int_0^1 g_1(S^1(x)\xi(x), \xi(x)) \, dx.
$$

Denoting by $\Phi \colon H_0^1(\gamma_1) \to H_0^1(\gamma_2)$ the bounded operator induced by φ we have that

$$
h_{\gamma_2}(\Phi\xi) = \int_0^1 g_2(\tfrac{D}{dx}\varphi_x\xi(x), \tfrac{D}{dx}\varphi_x\xi(x)) \, dx -
$$

$$
(9.2) \qquad\qquad \int_0^1 g_2(S^2(x)\varphi_x\xi(x), \varphi_x\xi(x)) \, dx.
$$

Since parallel translation commutes with $\frac{D}{dx}$ and φ_0 is an isometry it follows that

$$
g(\tfrac{D}{dx}\xi(x), \tfrac{D}{dx}\xi(x)) = g(\tfrac{D}{dx}\varphi_x\xi(x), \tfrac{D}{dx}\varphi_x\xi(x))
$$

pointwise and hence the first two integrals in (9.1), (9.2) coincide. Therefore the inequality

$$g(S^2(x)\varphi_x\xi(x), \varphi_x\xi(x)) \le g(S^1(x)\xi(x), \xi(x))$$

implies that $h_{\gamma_2}(\Phi\xi) \le h_{\gamma_1}(\xi)$, for all $\xi \in H_0^1(\gamma_1)$.

Let $\tilde{\gamma}^i$, $i = 1, 2$ be the canonical paths induced by γ_i on $\Omega(M_i)$ and let h_λ be the Hessian of E_λ at $\tilde{\gamma}_\lambda^i$.

Since the top order terms (those containing $\frac{D}{dx}$) of h_λ^1 and $h_\lambda^2\Phi_\lambda$ coincide, the difference $h_\lambda^1 - h_\lambda^2\Phi_\lambda$ is weakly semi-continuos. By the proof of Proposition 8.4

$$(9.3) \qquad h_0^1(\xi) = \int_0^1 g_1(\tfrac{D}{dx}\xi, \tfrac{D}{dx}\xi)\,dx = \int_0^1 g_2(\tfrac{D}{dx}\varphi_0\xi, \tfrac{D}{dx}\varphi_0\xi)\,dx = h_0^2(\Phi_0\xi)$$

Moreover by hypothesis

$$(9.4) \qquad h_1^2(\Phi_1\xi) = h_{\gamma_2}(\Phi\xi) \le h_{\gamma_1}(\xi) = h_1^1(\xi).$$

On the other hand, $\mu(\gamma_1) = -\mathrm{sf}(h^1)$ while $\mu(\gamma_2) = -\mathrm{sf}(h^2) = -\mathrm{sf}(h^2\Phi)$.

Taking in account (9.3) and (9.4), in order to conclude the proof of the comparison theorem it is enough to show that if h and k are two admissible generalized families of Fredholm quadratic forms on a Hilbert bundle $\mathcal{H}$ over $[a, b]$ whose difference is weakly semi-continuous and such that $h_a = k_a$, $k_b \le h_b$, then $\mathrm{sf}(k) \le \mathrm{sf}(h)$. The rest of the proof is devoted to show this.

Without loss of generality we can assume that $\mathcal{H} = [a, b] \times H$ is trivial. Since $h_\lambda - k_\lambda$ is weakly compact it follows that $q(s, \lambda) = sh_\lambda + (1 - s)k_\lambda$, is a Fredholm quadratic form for all $(s, \lambda) \in [0, 1] \times [a, b]$. The compactness of $T = [0, 1] \times [a, b]$ together with the property that the space of Fredholm quadratic forms is open imply that there exists a small $\eta > 0$ such that $q(s, \lambda) + t\| \cdot \|^2$ is still Fredholm, for any $(s, \lambda) \in T$ and $|t| \le \eta$. Taking possibly a smaller η we can further assume that $h_i + t\| \cdot \|^2$ is non degenerate for $i = a, b$ and $|t| \le \eta$

Define $\tilde{h}_\lambda = h_\lambda + \lambda\eta\| \cdot \|^2$. Then $\tilde{h}$ is a family of Fredholm quadratic forms that is homotopic to h by the admissible homotopy $(t, \lambda) \to h_\lambda + t\lambda\eta\| \cdot \|^2$ and therefore $\mathrm{sf}(\tilde{h}) = \mathrm{sf}(h)$. But now $\tilde{h}_0 = k_0$ and $k_1 < \tilde{h}_1$.

Consider the family of Fredholm quadratic forms defined on T by $\phi(s, \lambda) = s\tilde{h}_\lambda + (1 - s)k_\lambda$. The closed path obtained by restricting ϕ to ∂T has spectral flow zero. In fact, its spectral flow is the sum of the spectral flows of each segment of the boundary of $[0, 1] \times [a, b]$ and by the same arguments as in the proof of Theorem 4.2, we have: $\mathrm{sf}(\tilde{h}) - \mathrm{sf}(k) = \mathrm{sf}(\rho)$, where $\rho(s) = (1 - s)k_1 + s\tilde{h}_1$.

Since $\dot{\rho} = h_1 - k_1$ is positive definite if $\Sigma(\rho)$ is non empty all points $s \in \Sigma(\rho)$ are regular and, by Remark 8.3, each gives a positive contribute $\dim\ker \rho(s)$ to the spectral flow of ρ. Thus $\mathrm{sf}(\rho) \ge 0$ and hence $\mathrm{sf}(k) \le \mathrm{sf}(h)$. $\qquad\square$

In our comparison theorem it is possible to avoid the non degeneracy condition by introducing the concept of extended generalized Morse index analogous to the extended Morse index (index plus nullity) in the classical case (compare with [10, 15]).

Let h be a generalized family of Fredholm quadratic forms on a Hilbert bundle $\mathcal{H}$ over $[a, b]$. Let us consider the self-adjoint Fredholm operators L_λ representing h_λ with respect to the scalar product. Self-adjoint Fredholm operators are characterized by the fact that 0 is an isolated point in the spectrum of finite multiplicity. Thus, there exists a number $\delta_0 > 0$ so that $h_\lambda + \delta\| \cdot \|^2 = 1/2\langle(L + \delta)u, u\rangle$ is not degenerate for $\lambda = a, b$ and for any $0 < \delta \le \delta_0$. Since the set of all Fredholm quadratic forms is open in the set of all quadratic forms, δ_0 can be chosen in such a way that $h_\lambda + \delta\| \cdot \|^2$ is a Fredholm form for $\lambda \in [a, b]$ and $0 < \delta \le \delta_0$.

The *extended spectral flow* of the family h is

$$(9.5) \qquad \overline{\mathrm{sf}}(h) = \mathrm{sf}(h_\lambda + \delta\| \cdot \|^2, \lambda \in [a, b]).$$

for small enough δ.

The normalization property for the spectral flow implies that the right hand side does not depend on the choice of δ. The extended spectral flow is clearly additive under concatenation and direct sum. It is homotopy invariant under homotopies keeping the end points fixed.

The extended (generalized) Morse index is defined for any p-geodesic by

$$(9.6) \qquad\qquad \bar\mu(\gamma) \;=\; -\overline{\mathrm{sf}}(h)$$

where h is the family of hessians at the canonical path $\tilde\gamma$

Arguing as above with the perturbed families we obtain:

Theorem 9.1. *If γ_1, γ_2 are two (possibly degenerate) p-geodesics as in Theorem 7.1, related by (7.3) then*

$$\bar\mu(\gamma_2) \le \bar\mu(\gamma_1).$$

Appendix A. The vector bundle neighborhood Theorem

This section is devoted to a technical result, which is used in the proof of the main theorem.

Theorem A.1. *Let E and B be two C^∞ Banach manifolds such that E admits a C^∞ partition of unity and B is compact. Let $\pi\colon E \to B$ be a C^∞ split submersion, i.e. the differential is surjective and the kernel splits. Let $\tau\colon B \to E$ be a smooth section of π. Then there exist a Banach bundle $\pi'\colon F \to B$, an open neighborhood V of the 0-section in F and a fiber preserving diffeomorphism $j\colon V \hookrightarrow U$, of V with an open neighborhood U of $\tau(B)$ such that $\tau(b) = j(0_b)$ for all $b \in B$*

The proof of this result, follows the construction of tubular neighborhoods of submanifolds based on sprays. However here we need sprays whose exponential map is fiber preserving. The construction is that of Palais in [38], see also [14].

Let $\pi\colon E \to B$ be a split submersion and let $p\colon TF(\pi) \to E$ be the associated bundle of tangents along the fibers. A C^∞-vector field ξ on $TF(\pi)$ is called a *bundle spray* over E if it satisfies the following two conditions:

(1) $Tp(\xi(v)) = v \qquad \forall v \in TF(\pi)$;
(2) $\xi(sv) = T\tilde s(s\xi(v)) \quad \forall s \in \mathbb{R} \;\; \forall v \in TF(\pi)$

where $\tilde s\colon TF(\pi) \to TF(\pi)$ is the multiplication map given by $v \mapsto sv$ for any $v \in TF(\pi)$.

Condition (1) implies that for all $v \in T(E_b)$, also $Tp(\xi(v)) \in T(E_b)$ and then $\xi(v) \in T(T(E_b))$. Therefore, if $u\colon [0,1] \to TF(\pi)$ is an integral curve of ξ with $u(t_0) \in T(E_b)$, then $u([0,1]) \subset T(E_b)$ and so $pu([0,1]) \subset E_b$.

Proof. The existence of a bundle spray ξ over E follows from the implicit function theorem and partition of unity arguments. Indeed, let $e \in E$ be a point. Since $\pi\colon E \to B$ is a split submersion, by the implicit function theorem, there exists a neighborhood U_e of e in E and two open sets $V_b \subset B$ and $W_b \subset \pi^{-1}(b)$ for some $b \in B$ such that $U_e = V_b \times W_b$. Now let us consider the vertical tangent bundle over the open set U_e. By definition, we have

$$TF(U_e) \;=\; TF(V_b \times W_b) \;=\; V_b \times T(W_b).$$

In this way we are able to construct a bundle spray locally on $TF(U_e)$ extending to the product an ordinary spray on $T(W_b)$ in the obvious way. Observing that a convex combination of two bundle sprays is again a bundle spray, we can take a smooth partition of unity on E and glue the local sprays in order to get a bundle spray on $TF(\pi)$.

If ξ is a bundle spray for π, the associated *exponential map*, is defined as follows: for each $v \in TF(E)$ let σ_v be the maximal integral curve of ξ such that $\sigma_v(0) = v$. The second condition for a bundle spray implies that there exists an open neighborhood $\mathcal{D}$ of

the zero section in $TF(E)$ such that for all $v \in \mathcal{D}$, $\sigma_v(1)$ is defined. Now by definition $exp(v) = p(\sigma_v(1)) \quad \forall v \in \mathcal{D}$.

Let $F = \tau^*(TF)$. If $i \colon F \to TF$ is the canonical inclusion and $V = i^{-1}(\mathcal{D})$ we define $j \colon V \to E$ by $j(v) = exp\, i(v)$. By construction of the exponential map j is fiber preserving and clearly $j(0_b) = \tau(b)$. We want to show that j restricted to a (possibly) smaller neighborhood V of the zero section is a diffeomorphism. For this, if $j_b \colon F_b \to E_b$ is the restriction of j to the fiber, identifying F_b with its tangent space in the canonical way and arguing as in [33, Theorem 4.1] we show that $Tj_b(0_b) = id$. Since j is fiber preserving this easily implies that $Tj(0_b)$ is injective and hence invertible since, being B finite dimensional, $Tj(0_b)$ is Fredholm of index 0. Thus j is a local diffeomorphism. Since this holds for every $b \in B$, by a standard compactness argument there must be a (possibly) smaller neighborhood V of the zero section such that the restriction of j to that neighborhood is a (fiber preserving) diffeomorphism with a neighborhood U of $\tau(B)$. This concludes the proof. $\qquad\square$

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References

[1] R. Abraham - J.E. Marsden. Foundations of Mechanics, 2nd edition, *Benjamin/Cummings, Ink. Massachusetts (1978)*.

[2] L. Andersson - R. Howard. *Comparison and rigidity theorems in Semi-Riemannian geometry*, Comm. Anal. Geom. **6** (1998), n.4, 819-877.

[3] S.B. Angenent - R. van der Vorst. *A priori bounds and renormalized Morse indices of solutions of an elliptic system*, Ann. Inst. H. Poincaré Anal. Non Linéaire **17** (2000), n.3, 277-306.

[4] V.I. Arnol'd. *Sturm theorems and symplectic geometry*, Funktsional. Anal. i Prilozhen. **19** (1985), n.4 1-10.

[5] J.K. Beem - P.E. Ehrlich - K.L. Easley. Global Lorentzian Geometry, *Mercel Dekker, Inc. New York and Basel (1996)*.

[6] V. Benci - F. Giannoni - A. Masiello. *Some Properties of the Spectral Flow in Semiriemannian Geometry*, J. Geom. Phys. **27** (1998), 267-280.

[7] A.L. Besse. Manifolds all of whose geodesics are closed. Ergebnisse der Mathematik und ihrer Grenzgebiete 93. Springer-Verlag, (1978).

[8] O. Bolza. Lectures on Calculus of Variation, *Univ. Chicago Press, Chicago (1904)*.

[9] J.W. Bruce - P.J. Giblin. Curves and singularities. A geometrical introduction to singularity theory. Second edition, *Cambridge University Press, Cambridge (1992)*.

[10] S.E. Cappell - R. Lee - E.Y. Miller. *On the Maslov index*, Comm. Pure Appl. Math. **47** (1994), no. 2, 121-186.

[11] I. Chavel. Riemannian geometry: a modern introduction, *Cambridge tracts in Mathematics 108, Cambridge Univerisity Press, (1993)*.

[12] P. Chossat - D. Lewis - J.P. Ortega - T.S. Ratiu. *Bifurcation of relative equilibria in mechanical systems with symmetry.* Adv. in Appl. Math. 31 (2003), no. 1, 10-45.

[13] C. Conley - E. Zehnder. *The Birhoff-Lewis fixed point theorem and a conjecture of V.I Arnold*, Invent. Math. **73** (1983), 33-49.

[14] M. Crabb - I. James. Fibrewise Homotopy Theory, *Springer-Verlag (1998)*.

[15] M. Daniel. *An Extension of a Theorem of Nicolaescu on Spectral Flow and Maslov Index*, Proc. Amer. Math. Soc. **128**, no. 2 (1999), 611-619.

[16] K. Deimling. Nonlinear Functional Analysis, *Springer-Verlag (1985)*.

[17] I. Ekeland. Convexity methods in Hamiltonian systems, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 19, Springer-Verlag, Berlin (1990)*.

[18] Guihua Fei. *Relative Morse index and its application to Hamiltonian systems in the presence of symmetries*, J. Diff. Eq. **122**, (1995), 302-315.

[19] P.M. Fitzpatrick - J. Pejsachowicz - L. Recht. *Spectral flow and bifurcation of critical points of strongly-indefinite functional. Part I. General theory*, J. Functional Analysis **162** (1999), 52-95.

[20] P.M. Fitzpatrick - J. Pejsachowicz - L. Recht. *Spectral flow and bifurcation of critical points of strongly-indefinite functional. Part II. Bifurcation of periodic orbits of Hamiltonian systems*, Journal of Differential Equations **161** (2000), 18-40.

[21] P.M. Fitzpatrick - J. Pejsachowicz. *Parity and generalized multiplicity.*, Trans. Amer. Math. Soc. **326** (1991), 281-305.

[22] A. Floer. *Relative Morse index for the symplectic action*, Comm. Pure Appl. Math., **41** (1989), 335-356.

[23] I.M. Gel'fand - S.V. Fomin. Calculus of Variations, *Prentic-Hall Inc., Englewood Cliffs, New Jersey, USA (1963).*

[24] I.M. Gel'fand - V.B. Lidskii. *On the structure of the regions of stability of linear canonical systems of differential equations with periodic coefficients*, Amer. Math. Soc. Transl. Ser 2 **8** (1958), 143-181.

[25] R. Giambó - P. Piccione - A. Portaluri. *On the Maslov Index of Lagrangian paths that are not transversal to the Maslov cycle. Semi-Riemannian index Theorems in the degenerate case*, preprint 2003.

[26] A.D. Helfer. *Conjugate points on space like geodesics or pseudo self-adjoint Morse-Sturm-Liouville systems*, Pacific J. Math. **164** (1994), 321-340.

[27] J. Jost - X. Li-Jost - X.W. Peng. *Bifurcation of minimal surfaces in Riemannian manifolds.* Trans. Amer. Math. Soc. **347** (1995), no. 1, 51-62.

[28] T. Kato. Perturbation Theory for linear operators, *Grundlehren der Mathematischen Wissenschaften,* **132***, Springer-Verlag (1980).*

[29] W. Klingenberg. Closed geodesics on Riemannian manifolds, *CBMS Regional Conference Series in Mathematics,* **53** *(1983).*

[30] W. Klingenberg. Riemannian Geometry, *de Gruyter, New York, (1995).*

[31] M.A. Krasnoselskii. Topological methods in the theory of nonlinear integral equations, *Pergamon, New York, (1964).*

[32] D.N. Kupeli. *On conjugate and focal points in semi-Riemannian geometry*, Math. Z. **198** (1988), no.4, 569-589.

[33] S. Lang. Differential and Riemannian Manifolds, *Springer-Verlag (1995).*

[34] E. Meinrenken. *Trace formulas and Conley-Zehnder index*, Jour. Geom. and Phys. **13**, (1994), 1-15.

[35] J. Milnor. Morse theory. Based on lecture notes by M. Spivak and R. Wells. *Annals of Mathematics Studies, No. 51 Princeton University Press, Princeton, N.J. 1963.*

[36] M. Musso - J. Pejsachowicz - A. Portaluri. *A Morse Index Theorem and bifurcation for perturbed geodesics on Semi-Riemannian Manifolds*, Topol. Methods Nonlinear Anal. **25** (2005), no. 1, 69–99.

[37] B. O'Neill. Semi-Riemannian geometry with applications to relativity, *Academic Press, New York (1983).*

[38] R.S. Palais. Foundations of global non-linear analysis, *W.A.Benjamin, Inc., New York (1968).*

[39] G. Peano. Lezioni di Analisi infinitesimale Volume I 120-121, Volume II 187-195 *Tipografia editrice G. Candeletti, Torino (1893).*

[40] P. Piccione - A. Portaluri - D.V.Tausk. *Spectral flow, Maslov index and bifurcation of semi- Riemannian geodesics*, Annals of Global Analysis and Geometry **25** (2004), no. 2, 121–149.

[41] A. Portaluri. *A formula for the Maslov index of linear autonomous Hamiltonian systems*, preprint 2004.

[42] A. Portaluri. *Morse Index Theorem and Bifurcation theory on semi-Riemannian manifolds*, PhD. Thesis (2004).

[43] P.J. Rabier. *Generalized Jordan chains and two bifurcation theorems of Krasnosel'skii*, Nonlinear Anal. **13** (1989), 903-934.

[44] J. Robbin - D. Salamon. *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27** (1995), 1-33.

[45] J. Robbin - D. Salamon. *The Maslov index for paths*, Topology **32** (1993), 827-844.

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