

A BIFURCATION RESULT FOR SEMI-RIEMANNIAN TRAJECTORIES OF THE LORENTZ FORCE EQUATION

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ABSTRACT. We obtain a bifurcation result for solutions of the Lorentz equation in a semi-Riemannian manifold; such solutions are critical points of a certain strongly indefinite functionals defined in terms of the semi-Riemannian metric and the electromagnetic field. The flow of the Jacobi equation along each solution preserves the so-called *magnetic symplectic form*, and the corresponding curve in the symplectic group determines an integer valued homology class called the *Maslov index* of the solution.

We study magnetic conjugate instants with symplectic techniques, and we prove at first, an analogous of the semi-Riemannian Morse Index Theorem (see [12]). By using this result, together with recent results on the bifurcation for critical points of strongly indefinite functionals (see [10]), we are able to prove that each non degenerate and non-null magnetic conjugate instant along a given solution of the semi-Riemannian Lorentz force equation is a bifurcation point.

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1. INTRODUCTION

In this paper we will study the occurrence of bifurcation phenomena for solutions of the Lorentz force equation in General Relativity; such solutions represent the trajectories of massive charges moving under the action of gravity and of the electromagnetic field. Conjugate points along a solution z of the Lorentz force equation, called in this paper “magnetic conjugate” points, correspond up to first order infinitesimal to fixed endpoints homotopies of z by a family z_s of solutions of the equation. However, when dealing with phenomena on a very large scale, like for instance when studying trajectories of massive charges in a general relativistic spacetime, such first order approximation is not valid. The aim of this paper is to establish to which extent one has multiplicity of trajectories in the proximities of a magnetic conjugate point of a general relativistic spacetime, i.e., in the language of bifurcation theory, we determine under which circumstances a magnetic conjugate point determines a branch of solutions bifurcating from the given one.

In the classical literature, general relativistic solutions of the Lorentz force equations are studied using the Kaluza-Klein formalism in Lorentzian, or more generally semi-Riemannian geometry. Einstein’s program in general relativity is based on the fact that spacetime is nontrivially curved and that the gravity is the responsible of this curvature. In 1921, Kaluza had postulated that gravitation and electromagnetism could be unified in a five dimensional theory of gravity. The physical interest in the modern Kaluza-Klein theory, which could be considered as the historical precursor of the modern Yang-Mills theory, is due to the fact that some quantities, like for instance the *charge* of a particle, are spacetime related like the *momentum* or the *energy*. The charged particle trajectories correspond to geodesics trajectories in the Kaluza extended manifold.

Although the Kaluza-Klein formalism is very natural, and it provides powerful tools for studying existence and multiplicity results for causal solutions of the Lorentz force equation (see for instance [7, 8]), the theory does not seem to be well suited to study phenomena depending on infinitesimal of second order, like bifurcation theory, and it is practically useless if one wants to relate the Morse theory for solutions of the Lorentz equation with the Morse theory of the corresponding geodesics. This observation is simple consequence of the fact that magnetic conjugate points along solutions of the Lorentz equation do *not* correspond necessarily to conjugate points along the corresponding Kaluza-Klein geodesics, due to the fact that distinct solutions of the Lorentz equation having common endpoints lift to Kaluza-Klein geodesics with possibly distinct endpoints. In particular, the bifurcation result for semi-Riemannian geodesics proven in [19] cannot be applied, and the aim of this paper is to develop a specific theory to study bifurcation of solutions of the Lorentz equation.

In order to state properly our result, let us fix our notations and let us recall a few basic definitions. Let (M, g) be a semi-Riemannian manifold and let $B \in \mathfrak{X}(M)$ be a smooth vector field on M ; the corresponding 1-form $\omega = g(B, \cdot)$ will be called the *magnetic 1-form*. A *trajectory* of a charged massive particle moving under the action of an electromagnetic field B is represented by a curve $z : I \rightarrow M$, where I is an interval of the real line \mathbb{R} , satisfying the Lorentz force equation, given by:

$$\frac{D}{dt} \dot{z} + (\nabla B - \nabla B^*) \dot{z} = 0,$$

where $\frac{D}{dt}$ denotes the covariant derivative along z with respect to the Levi-Civita connection and $(\nabla B)^*$ is the g -adjoint of ∇B . Although the Lorentzian case is of physical interest, we will develop the theory in the more general setting of semi-Riemannian manifold; on the other hand, we will restrict ourselves to the case of *exact* magnetic 2-forms, in

which case the Lorentz equation has a variational structure. Most of the results discussed in this paper hold true also in the more general case of non exact magnetic 2-forms, and very likely the entire theory presented could be extended to such more general case using techniques of non variational bifurcation.

Solutions of the Lorentz equation are critical points of the action functional:

$$F(z) = \int_0^1 \left[\frac{1}{2}g(\dot{z}, \dot{z}) + g(B, \dot{z}) \right] dt = \int_0^1 \left[\frac{1}{2}g(\dot{z}, \dot{z}) + \omega(\dot{z}) \right] dt.$$

where ω is the so-called *magnetic 1-form* on M whose differential is computed as $\text{curl } B$.

Linearization of the Lorentz force equation along a given solution z produces the so-called *electromagnetic Jacobi equation* (see (2.14)), whose solutions are called *electromagnetic Jacobi fields*. The electromagnetic Jacobi equation coincides with the kernel of the so-called *electromagnetic Index form* which is the second variation of the electromagnetic energy functional (see (2.8)). These notions was introduced for the first time in [6].

Now, given a trajectory z , according to [6] we say that an instant $t_0 \in]0, 1]$ is said to be a *magnetic conjugate instant*, if there exists at least one non zero electromagnetic Jacobi field J with $J(0) = 0 = J(t_0)$. The corresponding point $z(t_0)$ is said to be a *magnetic conjugate point* to $z(0)$ along z ; hence, magnetic conjugate points can be interpreted as fixed endpoints of a homotopy of solutions of the magnetic equation, *up to first order approximation*. When dealing with phenomena on a very large scale, such first order approximation is not satisfactory, and the aim of this paper is to establish in which exact terms one has multiplicity of solutions in correspondence to magnetic conjugate points.

It is well known that, in the geodesic case, the flow of the Jacobi deviation equation preserves the symplectic form Ω on TM which is the pull-back via the semi-Riemannian structure g of the standard Liouville form on T^*M ; likewise, the flow associated to the electromagnetic Jacobi equation preserves the so-called *magnetic symplectic form* given by $\mu = \Omega - \pi^*(\text{curl } B)$, where $\pi : TM \rightarrow M$ is the canonical projection. It follows that the fundamental solution of the electromagnetic Jacobi equation, give us a path in the Lie group of all symplectomorphism of the symplectic space \mathbb{R}^{2n} endowed with the magnetic symplectic form, and, in particular, its phase flow induces a path in the Lagrangian Grassmannian manifold of \mathbb{R}^{2n} .

The magnetic conjugate instants along a solution of the Lorentz force equation can be characterized as the intersection of this path with an co-oriented analytic one co-dimensional embedded submanifold of the Lagrangian Grassmannian manifold, also called the Maslov cycle. The relative homology class of this path is defined to be the Maslov index of the solution. One of the main result of this paper is to prove an analogues of the Morse index theorem for trajectories of the Lorentz force equation, relating the Maslov index of a solution with a generalized Morse index of the corresponding path of electromagnetic index forms. Due to the indefiniteness of the semi-Riemannian structure, the Morse Index of the second variation of the electromagnetic energy functional is not well defined and a correct substitute is given by the so-called *spectral flow* associated to a path of symmetric Fredholm forms. The spectral flow is an integer homotopy invariant, originally introduced by Atiyah, Patodi and Singer in [2], which roughly speaking can be thought as the net of change of the eigenvalues which cross the value zero.

Using such an homotopy invariant we will prove the equality between the Maslov index of a trajectory of the Lorentz force equation and the spectral flow associated to the path of Fredholm quadratic forms arising from the Hessians of the electromagnetic energy functionals, up to a sign.

The study of multiplicity results for trajectories of the Lorentz force equation in the Riemannian case or the behavior of these solutions, is well known (see for instance [3, 4, 13]). In the Lorentzian case for spacelike trajectories or more generally for trajectories of any causal character in a semi-Riemannian manifold, a careful analysis of the behavior of such trajectories or a multiplicity result is much more involved. For a better understanding of the behavior of these trajectories in the neighborhood of a magnetic conjugate point, we will introduce the notion of *magnetic bifurcation point* along a solution of the Lorentz equation. A magnetic bifurcation point (or, more precisely, a magnetic bifurcation *instant*) along one such solution z is a point $z(t_0)$ for which there exists a sequence $t_n \rightarrow t_0$ and a sequence of solutions $z_n \neq z$ of the Lorentz equation tending to z as $n \rightarrow \infty$, such that $z_n(t_n) = z(t_n)$ for all n . A natural question to ask is: which magnetic conjugate points along a solution of the Lorentz equation are magnetic bifurcation points?

We will use some recent results on bifurcation theory for strongly indefinite functionals ([10]), we are able to give an answer to the above questions. The main result in [10] is that bifurcation occurs at those singular instants whose contribution to the spectral flow is non null (Proposition 4.3). By a suitable choice of coordinates in the space of paths joining a fixed point p in M and a point variable along a given trajectories of the Lorentz forced equation z starting at p , the magnetic bifurcation problem is reduced to a bifurcation problem for a smooth family of strongly indefinite functionals defined in (an open neighborhood of 0) a fixed Hilbert space (Subsection 5.1). To each magnetic conjugate instant $z(t_0)$ along z we associate a vector space $\mathbb{P}[t_0] \subset T_{z(t_0)}M$, called the *magnetic conjugate plane* (Definition 3.1); when the restriction of the spacetime metric g to $\mathbb{P}[t_0]$ is nondegenerate, then $z(t_0)$ is called nondegenerate, and the signature of such restriction is the *signature* of the magnetic conjugate point. The Maslov index of a solution of the Lorentz equation is computed under generic circumstances as the sum of the signatures of all magnetic conjugate instants (Corollary 3.9); using the index theorem, we get that jumps of the spectral flow occur at those magnetic conjugate points having non null signature. Applying the theory of [10], we get that nondegenerate magnetic conjugate points with non vanishing signature are magnetic bifurcation points; more generally, a magnetic bifurcation point is found in every segment of solution of the Lorentz equation that contains a (possibly non discrete) set of magnetic conjugate points that give a non zero contribution to the Maslov index (Corollary 5.6).

To conclude, we remark that the occurrence of degeneracies of the restriction $g|_{\mathbb{P}[t_0]}$ is yet a rather mysterious phenomenon, that deserves attention. Even more challenging, it is not clear whether non spacelike Lorentzian solutions may admit a non discrete set of magnetic conjugate instants.

2. THE VARIATIONAL PROBLEM

2.1. Geometrical setup and the action functional. We will consider a smooth manifold M endowed with a semi-Riemannian metric tensor g ; by the symbol $\frac{D}{dt}$ we will denote the covariant differentiation of vector fields along a curve with respect to the Levi-Civita connection of g , while R will denote the curvature tensor of ∇ chosen with the sign convention: $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$, where ∇ will denote the covariant derivative of the Levi-Civita connection of g . Let $B \in \mathfrak{X}(M)$ be a smooth vector field on M ; this vector field B defines the so-called *magnetic 1-form* ω on M , defined by:

$$(2.1) \quad \omega = g(B, \cdot);$$

its differential $d\omega$ is easily computed as:

$$d\omega = \text{curl } B,$$

where $\text{curl } B$ is the 2-form:

$$\text{curl } B(X, Y) = g(\nabla_X B, Y) - g(X, \nabla_Y B).$$

Given a smooth n -dimensional manifold, let Ω the set of all paths $z : [0, 1] \rightarrow M$ of Sobolev class H^1 . It is well known that Ω is an infinite dimensional smooth Hilbert manifold modelled on the Hilbert space $H^1([0, 1], \mathbb{R}^n)$. For each $z \in \Omega$, the tangent space $T_z\Omega$ can be identified with the space of vector fields V along z of class H^1 . Now let $\text{ev} : \Omega \rightarrow M \times M$ be the evaluation map given by $\text{ev}(z) = (z(0), z(1))$; this map is a submersion, and therefore for each pair $(p, q) \in M \times M$, the inverse image:

$$\Omega_{p,q} := \text{ev}^{-1}(p, q) = \{z \in \Omega : z(0) = p, z(1) = q\}$$

is a submanifold of codimension $2n$ in Ω , whose tangent space $T_z\Omega_{p,q}$ is identified with the space of vector fields V along z of class H^1 and vanishing at the endpoints. Keeping this identification in mind, we will tacitly apply standard results on Sobolev spaces to tangent spaces $T_z\Omega_{p,q}$.

To each data (g, B) , where g is a semi-Riemannian structure and B is the smooth magnetic vector field, we associate the following *magnetic action functional* $F : \Omega \rightarrow \mathbb{R}$ defined by

$$(2.2) \quad F(z) = \int_0^1 \left[\frac{1}{2}g(\dot{z}, \dot{z}) + g(B, \dot{z}) \right] dt = \int_0^1 \left[\frac{1}{2}g(\dot{z}, \dot{z}) + \omega(\dot{z}) \right] dt.$$

By the smoothness of the data, it follows immediately that F is a smooth function and hence so are the restrictions $F_{p,q}$ of F to $\Omega_{p,q}$. It is not hard to see that, due to the fact that the metric tensor g is indefinite, $F_{p,q}$ is unbounded both from above and from below on $\Omega_{p,q}$, and that the Morse index of its critical points in $\Omega_{p,q}$ is infinite.

2.2. First variation of the action functional. We will now compute the first variation of the functional (2.2); to this aim, let $z \in \Omega_{p,q}$ and $V \in T_z\Omega_{p,q}$ be fixed, and let $\{z_s\}_{s \in]-\varepsilon, \varepsilon[}$ be a variation of z in $\Omega_{p,q}$ with variational vector field V . Recall that this means that $]-\varepsilon, \varepsilon[\ni s \mapsto z_s \in \Omega_{p,q}$ is a C^1 map, with $z_0 = z$ and $\frac{d}{ds}|_{s=0} z_s = V$. Then, $dF_{p,q}(z)V = \frac{d}{ds}|_{s=0} F_{p,q}(z_s)$; this derivative is computed as follows:

$$(2.3) \quad \frac{d}{ds} F_{p,q}(z_s) = \int_0^1 \left[g\left(\frac{D}{ds} \frac{d}{dt} z_s, \frac{d}{dt} z_s\right) + g\left(\frac{D}{ds} B(z_s), \frac{d}{dt} z_s\right) + g\left(B(z_s), \frac{D}{ds} \frac{d}{dt} z_s\right) \right] dt,$$

and evaluating at $s = 0$ we get:

$$(2.4) \quad \begin{aligned} dF_{p,q}(z)V &= \int_0^1 \left[g\left(\frac{D}{dt} V, \dot{z}\right) + g(\nabla_V B, \dot{z}) + g\left(B, \frac{D}{dt} V\right) \right] dt \\ &= \int_0^1 \left[g\left(\frac{D}{dt} V, \dot{z}\right) + d\omega(V, \dot{z}) \right] dt. \end{aligned}$$

Proposition 2.1. *A curve $z \in \Omega_{p,q}$ is a critical point of f if and only if z is smooth and it satisfies the second order equation:*

$$(2.5) \quad \frac{D}{dt} \dot{z} + (\nabla B - \nabla B^*) \dot{z} = \frac{D}{dt} \dot{z} - (\nabla B)^* \dot{z} + \frac{D}{dt} B = 0,$$

where $(\nabla B)^*$ is the g -adjoint¹ of ∇B defined by $g((\nabla B)^* a, b) = g(\nabla_b B, a)$.

¹In order to avoid confusion, in this paper we will denote by \star the adjoint with respect to the bilinear form g , while we will use the customary symbol $*$ to denote the adjoint of linear operators in \mathbb{R}^n with respect to the Euclidean product.

Proof. The regularity of z is obtained by standard boot-strap arguments; equation (2.5) is then obtained immediately performing integration by parts in (2.4) and the Fundamental Lemma of Calculus of Variations. \square

It is also worth recalling that the solutions of (2.5) preserve their causal character:

Lemma 2.2. *If $z : [0, 1] \rightarrow M$ is a solution of (2.5), then the quantity $g(\dot{z}, \dot{z})$ is constant.*

Proof. Contracting the left-hand side of (2.5) with the covector $g(\cdot, \dot{z})$ one gets:

$$0 = g\left(\frac{D}{dt}\dot{z}, \dot{z}\right) - g\left(\frac{D}{dt}B, \dot{z}\right) + g\left(\frac{D}{dt}B, \dot{z}\right) = g\left(\frac{D}{dt}\dot{z}, \dot{z}\right) = \frac{1}{2} \frac{d}{dt} g(\dot{z}, \dot{z}). \quad \square$$

2.3. Second variation of the action functional. Recall that, for a smooth vector field Z on M , the *Hessian* of Z , denoted by $\text{Hess } Z$, is the $(2, 1)$ -tensor field on M given by $\nabla\nabla Z$; more explicitly:

$$\text{Hess } Z(v_1, v_2) = \nabla_{v_1} \nabla_{v_2} Z - \nabla_{\nabla_{v_1} v_2} Z,$$

where V_2 is any local extension of v_2 . Observe that the Hessian $\text{Hess } Z$ is not in general symmetric; its symmetric anti-symmetric parts are computed as:

$$(2.6) \quad \text{Hess}_s Z(v_1, v_2) = \frac{1}{2} R(v_1, v_2)Z, \quad \text{Hess}_a Z(v_1, v_2) = \text{Hess } Z(v_1, v_2) - \frac{1}{2} R(v_1, v_2)Z.$$

Given a tangent vector $v \in T_m M$, we will think of $\text{Hess } Z(v)$ and $\text{Hess}_s Z(v)$ as linear operators on $T_m M$; for the computation of the kernel of the second variation of f we will need the g -adjoint of $\text{Hess}_s Z(v)$, which is computed easily from (2.6) as:

$$(2.7) \quad (\text{Hess}_s Z(v))^*(w) = (\text{Hess } Z(v))^*(w) - \frac{1}{2} R(Z, w)v.$$

Recall that a bounded symmetric bilinear form on a Hilbert space is said to be *Fredholm* if it is realized by a (self-adjoint) Fredholm operator.

Remark 2.3. If $b : H_0^1([0, 1], \mathbb{R}^n) \times H_0^1([0, 1], \mathbb{R}^n) \rightarrow \mathbb{R}$ is a bilinear form such that the map $(V, W) \mapsto b(V, W)$ is continuous in the product topology $C^0 \times H^1$ (or in the topology $H^1 \times C^0$), then b is realized by a compact operator on $H_0^1([0, 1], \mathbb{R}^n)$. This follows easily from the fact that the inclusion of $H^1([0, 1], \mathbb{R}^n)$ into $C^0([0, 1], \mathbb{R}^n)$ is compact, and from the fact that the adjoint of a compact operator is compact.

Proposition 2.4. *Let $z \in \Omega_{p,q}$ be a critical point of $F_{p,q}$; then, the Hessian $\text{Hess } F_{p,q}(z)$ of $F_{p,q}$ at z is the Fredholm form on $T_z \Omega_{p,q}$ given by:*

$$(2.8) \quad \begin{aligned} \text{Hess } F_{p,q}(z)[V, W] = & \int_0^1 \left[g\left(\frac{D}{dt}V, \frac{D}{dt}W\right) + g(R(\dot{z}, V)\dot{z}, W) + g\left(\frac{D}{dt}V, \nabla_W B\right) + g\left(\frac{D}{dt}W, \nabla_V B\right) \right] dt \\ & + \int_0^1 \left[g(\text{Hess}_s B(V, W), \dot{z}) + \frac{1}{2}g(R(V, \dot{z})W, B) + \frac{1}{2}g(R(W, \dot{z})V, B) \right] dt. \end{aligned}$$

Proof. Let $z \in \Omega_{p,q}$ be a critical point of $F_{p,q}$, $V \in T_z \Omega_{p,q}$ and let $\{z_s\}_{s \in]-\varepsilon, \varepsilon[}$ be a variation of z in $\Omega_{p,q}$ with variational vector field V . Then,

$$\text{Hess } F_{p,q}(z)[V, V] = \left. \frac{d^2}{ds^2} \right|_{s=0} F_{p,q}(z_s),$$

which is computed by differentiating (2.3) as follows:

$$\begin{aligned} \frac{d^2}{ds^2} \Big|_{s=0} F_{p,q}(z_s) = & \int_0^1 \left[g\left(\frac{D}{dt}V, \frac{D}{dt}V\right) + g(R(V, \dot{z})V, \dot{z}) + g(R(V, \dot{z})V, B) + 2g(\nabla_V B, \frac{D}{dt}V) \right] dt \\ & + \int_0^1 \left[g\left(\frac{D}{ds} \frac{d}{ds} \Big|_{s=0} B, \dot{z}\right) + g\left(\frac{D}{dt} \frac{D}{ds} \frac{d}{ds} \Big|_{s=0} z_s, B\right) + g\left(\frac{D}{dt} \frac{D}{ds} \frac{d}{ds} \Big|_{s=0} z_s, \dot{z}\right) \right] dt. \end{aligned}$$

Integration by parts in the last two terms of the integral above and the differential equation (2.5) satisfied by z yield:

$$\begin{aligned} \int_0^1 \left[g\left(\frac{D}{dt} \frac{D}{ds} \frac{d}{ds} \Big|_{s=0} z_s, B\right) + g\left(\frac{D}{dt} \frac{D}{ds} \frac{d}{ds} \Big|_{s=0} z_s, \dot{z}\right) \right] dt \\ = - \int_0^1 g((\nabla B)^* \dot{z}, \frac{D}{ds} \frac{d}{ds} \Big|_{s=0} z_s) dt = - \int_0^1 g\left(\nabla \frac{D}{ds} \frac{d}{ds} \Big|_{s=0} z_s, B, \dot{z}\right) dt. \end{aligned}$$

By definition of Hess B we therefore get:

(2.9)

$$\begin{aligned} \text{Hess } F_{p,q}(z)[V, V] = & \int_0^1 \left[g\left(\frac{D}{dt}V, \frac{D}{dt}V\right) + g(R(V, \dot{z})V, \dot{z}) + g(R(V, \dot{z})V, B) \right] dt \\ & + \int_0^1 \left[2g(\nabla_V B, \frac{D}{dt}V) + g(\text{Hess } B(V, V), \dot{z}) \right] dt. \end{aligned}$$

Finally, (2.8) is obtained by polarization of (2.9), using formulas (2.6) and the first Bianchi identity for the curvature tensor R .

The bilinear form B_z defined by $(V, W) \mapsto \int_0^1 g\left(\frac{D}{dt}V, \frac{D}{dt}W\right) dt$ is strongly nondegenerate on $T_z \Omega_{p,q}$, i.e., represented by an isomorphism of $T_z \Omega_{p,q}$. Now, the difference $\text{Hess } f(z) - B_z$ is sum of terms that are continuous with respect to the C^0 -topology in either the first or the second variable. It follows from what observed in Remark 2.3 that $\text{Hess } F_{p,q}(z)$ is Fredholm. \square

In view to future developments, it will be convenient to write the second variation of $F_{p,q}$ in the following form:

$$(2.10) \quad \text{Hess } F_{p,q}(z)[V, W] = \int_0^1 g\left(\frac{D}{dt}V, \frac{D}{dt}W\right) dt + \mathfrak{H}[V, W] + \mathfrak{K}[V, W],$$

where:

(2.11)

$$\begin{aligned} \mathfrak{H}[V, W] = & - \frac{1}{2} \int_0^1 \left[g(V, \nabla \frac{D}{dt}W B) - g(\nabla_V B, \frac{D}{dt}W) + g(W, \nabla \frac{D}{dt}V B) - g(\nabla_W B, \frac{D}{dt}V) \right] dt, \end{aligned}$$

and

(2.12)

$$\begin{aligned} \mathfrak{K}[V, W] &= \int_0^1 \left[g(R(\dot{z}, V) \dot{z}, W) + \frac{1}{2}g(R(V, \dot{z}) W, B) + \frac{1}{2}g(R(W, \dot{z}) V, B) \right] dt \\ &+ \int_0^1 \left[\frac{1}{2}g\left(\frac{D}{dt}V, \nabla_W B\right) + \frac{1}{2}g\left(\nabla_V B, \frac{D}{dt}W\right) + \frac{1}{2}g\left(V, \nabla_{\frac{D}{dt}W} B\right) \right] dt \\ &+ \int_0^1 \left[\frac{1}{2}g\left(\nabla_{\frac{D}{dt}V} B, W\right) + g(\text{Hess}_B(V, W), \dot{z}) \right] dt. \end{aligned}$$

The bilinear forms \mathfrak{H} and \mathfrak{K} are symmetric; formula (2.10) is obtained by a straightforward calculation using (2.11) and (2.12). Moreover, an easy integration by parts yields the following formula:

$$\int_0^1 \left[g\left(\frac{D}{dt}V, \nabla_W B\right) + g\left(V, \nabla_{\frac{D}{dt}W} B\right) \right] dt = - \int_0^1 g(V, \text{Hess } B(\dot{z}, W)) dt,$$

from which it follows that $\mathfrak{K}[V, W]$ can be rewritten as:

(2.13)

$$\begin{aligned} \mathfrak{K}[V, W] &= \int_0^1 \left[g(R(\dot{z}, V) \dot{z}, W) + \frac{1}{2}g(R(V, \dot{z}) W, B) + \frac{1}{2}g(R(W, \dot{z}) V, B) \right] dt + \\ &\int_0^1 \left[g(\text{Hess}_B(V, W), \dot{z}) - \frac{1}{2}g(\text{Hess } B(\dot{z}, V), W) - \frac{1}{2}g(\text{Hess } B(\dot{z}, W), V) \right] dt. \end{aligned}$$

Corollary 2.5. *The kernel of $\text{Hess } F_{p,q}(z)$ in $T_z\Omega_{p,q}$ consists of smooth vector fields $V \in T_z\Omega_{p,q}$ satisfying the following second order linear differential equation:*

$$(2.14) \quad \frac{D^2}{dt^2}V - R(\dot{z}, V)\dot{z} - (\nabla B)^*(\frac{D}{dt}V) + \frac{D}{dt}(\nabla_V B) - \text{Hess } B(V)^*(\dot{z}) - R(\dot{z}, V)B = 0.$$

Proof. The regularity for vector fields in the kernel of $\text{Hess } f(z)$ is obtained by standard bootstrap techniques; equation (2.14) is easily obtained using integration by parts in (2.8), keeping in mind formulas (2.7) and the first Bianchi identity for R . \square

We will denote with the symbol $\mathcal{J}_m(z)$ the magnetic Jacobi differential operator for vector fields along z , i.e., $\mathcal{J}_m(z)V$ is given by the left-hand side of (2.14).

Definition 2.6. The differential equation (2.14) will be called the *magnetic Jacobi equation* along the solution z of the variational problem (2.2).

3. MAGNETIC CONJUGATE POINTS AND THE MASLOV INDEX

In this section we will introduce the notion of *magnetic conjugate points* along a solution of (2.5) and we will define the *Maslov index* of a solution.

3.1. Magnetic conjugate points and magnetic bifurcation points. Let $z : [0, 1] \rightarrow M$ be a solution of (2.5) in (M, g) ; consider the magnetic Jacobi equation along z :

$$(3.1) \quad \frac{D^2}{dt^2}V = \mathfrak{C}(V) + \mathfrak{D}\left(\frac{D}{dt}V\right),$$

where:

$$(3.2) \quad \mathfrak{C}(V) = R(\dot{z}, V)\dot{z} - \text{Hess } B(\dot{z}, V) + \text{Hess } B(V)^*\dot{z} + R(\dot{z}, V)B,$$

and

$$(3.3) \quad \mathfrak{D}(W) = (\nabla B)^*(W) - \nabla_W B.$$

Its solutions are called *magnetic Jacobi fields* along z . Let \mathbb{J}_z denote the n -dimensional space:

$$(3.4) \quad \mathbb{J}_z = \{J \text{ solution of (3.1) such that } J(0) = 0\};$$

for $t_0 \in]0, 1]$, we set:

$$\mathbb{J}_z[t_0] = \{J(t_0) : J \in \mathbb{J}_z\} \subset T_{z(t_0)}M.$$

The evaluation $\mathbb{J}_z \ni J \mapsto J(t_0) \in T_{z(t_0)}$ is a linear map between n -dimensional spaces, hence it is injective if and only if it is surjective. Based on this simple observation, we can now give the following:

Definition 3.1. A point $z(t_0)$, $t_0 \in]0, 1]$ is said to be *magnetic conjugate* to $z(0)$ along z if there exists a non zero magnetic Jacobi field $J \in \mathbb{J}_z$ such that $J(t_0) = 0$, i.e., if $\mathbb{J}_z[t_0] \neq T_{z(t_0)}M$. If $z(t_0)$ is magnetic conjugate to $z(0)$ along z , the *magnetic conjugate space* $\mathbb{P}[t_0]$ is the g -orthogonal complement $\mathbb{J}_z^0[t_0]^\perp$, and its dimension is called the *multiplicity* of the magnetic conjugate point $z(t_0)$, denoted by $\text{mul}(t_0)$. The signature of the restriction of g to $\mathbb{P}[t_0]$ is called the *signature* of $z(t_0)$, and it will be denoted by $\text{sgn}(t_0)$. The magnetic conjugate point $z(t_0)$ is said to be a *nondegenerate* if such restriction is nondegenerate.

Remark 3.2. It is not hard to prove that the nondegenerate magnetic conjugate points are isolated, see Proposition 2.8 in [6]. In particular, $t_0 = 0$ is an isolated magnetic conjugate point along each solution $z : [0, 1] \rightarrow M$ of (2.5), i.e., there exists $\varepsilon > 0$ such that there are no magnetic conjugate instants in $]0, \varepsilon]$ along z .

In analogy with the (geodesic) exponential map of (M, g) , we can define a *magnetic exponential map* $\text{exp}_p^{\text{mag}}$ of (M, g, B) , defined on an open subset of TM containing the zero section and taking values in M , given by:

$$\text{exp}_p^{\text{mag}}(v) = z(1),$$

where z is the unique solution of (2.5) on $[0, 1]$ satisfying $z(0) = p$ and $\dot{z}(1) = v \in T_pM$. The map $\text{exp}_p^{\text{mag}}$ is smooth, and magnetic conjugate instants to a point p correspond to critical values of the map $\text{exp}_p^{\text{mag}}$. We want to investigate the problem of establishing when $\text{exp}_p^{\text{mag}}$ is not injective in neighborhoods of its critical points. More precisely, we give the following definition:

Definition 3.3. Let (M, g) be a semi-Riemannian manifold, $z : [0, 1] \rightarrow M$ be a solution of the Lorentz equation (2.5), and $t_0 \in]0, 1[$. The point $z(t_0)$ is said to be a *magnetic bifurcation point* for z if there exists a sequence $z_n : [0, 1] \rightarrow M$ of solutions of the Lorentz equation and a sequence $(t_n)_{n \in \mathbb{N}} \subset]0, 1[$ satisfying the following properties:

- (1) $z_n(0) = z(0)$ for all n ;
- (2) $z_n(t_n) = z(t_n)$ for all n ;
- (3) $z_n \rightarrow z$ as $n \rightarrow \infty$;
- (4) $t_n \rightarrow t_0$ (and thus $z_n(t_n) \rightarrow z(t_0)$) as $n \rightarrow \infty$.

Remark 3.4. Applying the Inverse Function Theorem to the magnetic exponential map $\text{exp}_p^{\text{mag}}$, it follows that if $z(t_0)$ is a magnetic bifurcation point along z , then necessarily $z(t_0)$ must be magnetic conjugate to $z(0)$ along z .

3.2. The magnetic symplectic structure. We will now describe the symplectic structure of the tangent bundle TM of the semi-Riemannian manifold (M, g) ,² denote by $\pi : TM \rightarrow M$ the canonical projection.

For $m \in M$ and $v \in T_m M$, the tangent space $T_v TM$ can be decomposed as a direct sum

$$T_v TM = \text{Ver}_v \oplus \text{Hor}_v,$$

where Ver_v is the subspace of $T_v TM$ tangent to the fiber $T_p M$, while Hor_v is the *horizontal subspace* determined by the Levi–Civita connection of g . The space Ver_v is naturally identified with $T_p M$, while the differential $d\pi_v : T_v TM \rightarrow T_p M$ restricts to an isomorphism between Hor_v and $T_p M$. We will henceforth identify both spaces Ver_v and Hor_v with $T_m M$ in this fashion; for $\xi \in T_v TM$, we will denote by ξ^{ver} and ξ^{hor} respectively the vertical and the horizontal components of ξ . If $t \mapsto v(t) \in TM$ is a differentiable curve in TM , i.e., $v(t) = (z(t), V(t))$ where z is a differentiable curve in M and V is a vector field along z , then $v'(t)^{\text{ver}} = \dot{z}(t)$ and $v'(t)^{\text{hor}} = \frac{D}{dt}V(t)$.

The *canonical symplectic form* of the semi-Riemannian manifold (M, g) is the closed 2-form Ω on TM defined by:

$$(3.5) \quad \Omega_v(\xi, \eta) = g(\xi^{\text{ver}}, \eta^{\text{hor}}) - g(\xi^{\text{hor}}, \eta^{\text{ver}}), \quad \xi, \eta \in T_v TM.$$

Recalling the definition of the magnetic 1-form ω on M (see (2.1)), we give the following:

Definition 3.5. The *magnetic symplectic form* on TM is the closed 2-form:

$$(3.6) \quad \mu = \Omega - \pi^*(d\omega),$$

where $\pi^*(d\omega)$ is the pull-back of the 2-form $d\omega$ to TM . Explicitly,

$$\mu_v(\xi, \eta) = \Omega_v(\xi, \eta) + \text{curl} B(\xi^{\text{ver}}, \eta^{\text{ver}}), \quad \xi, \eta \in T_v TM.$$

Let now $z : [0, 1] \rightarrow M$ be a solution of (2.5) and let $\mathcal{J}_m(z)$ be the magnetic Jacobi differential operator defined on the space of vector fields along z . By *flow* of $\mathcal{J}_m(z)$ we mean the family of linear maps:

$$\mathcal{F}_z(t) : T_{z(0)}M \oplus T_{z(0)}M \longrightarrow T_{z(t)}M \oplus T_{z(t)}M,$$

defined by:

$$\mathcal{F}_z(t)(v_1, v_2) = (V(t), \frac{D}{dt}V(t)),$$

where V is the unique magnetic Jacobi field along z satisfying the initial conditions:

$$V(0) = v_1, \quad \frac{D}{dt}V(0) = v_2.$$

The following result holds:

Proposition 3.6. *The flow of the magnetic Jacobi equation (2.14) preserves the magnetic symplectic form (3.6).*

Proof. The thesis is equivalent to the fact that, if J_1 and J_2 are magnetic Jacobi fields along a solution $z : [0, 1] \rightarrow M$ of (2.5), then the quantity:

$$h(t) = g(J_1, \frac{D}{dt}J_2) - g(\frac{D}{dt}J_1, J_2) + g(J_1, \nabla_{J_2}B) - g(\nabla_{J_1}B, J_2)$$

²We recall that the *cotangent bundle* TM^* of any differentiable manifold M is naturally a symplectic manifold; however, when M is endowed with a semi-Riemannian metric g , then the symplectic structure of TM^* can be induced on the tangent bundle TM .

is constant on $[0, 1]$. Differentiating the above expression and using the magnetic Jacobi equation (2.14) satisfied by J_1 and J_2 , formulas (2.6) and all the symmetries of the curvature tensor R , we get:

$$\begin{aligned} h'(t) &= g(J_1, \frac{D}{dt^2} J_2) - g(\frac{D}{dt^2} J_1, J_2) + g(J_1, \frac{D}{dt} \nabla_{J_2} B) + g(\frac{D}{dt} J_1, \nabla_{J_2} B) \\ &\quad - g(\frac{D}{dt} \nabla_{J_1} B, J_2) - g(\nabla_{J_1} B, \frac{D}{dt} J_2) \\ &= -2g(\text{Hess}_a B(J_1, J_2), \dot{z}) - g(R(\dot{z}, J_1) B, J_2) + g(R(\dot{z}, J_2) B, J_1) = \\ &= g(-R(J_1, J_2) B + R(B, J_2) J_1 + R(J_1, B) J_2, \dot{z}) = 0. \quad \square \end{aligned}$$

3.3. Flow of the magnetic Jacobi equation. We want to describe the flow of the magnetic Jacobi equation as a curve in the Lie group of symplectomorphisms of a *fixed* symplectic space. Recall that a symplectic space is a real finite dimensional vector space endowed with a nondegenerate anti-symmetric bilinear form. We know from the abstract theory that the only invariant of a symplectic space is its dimension, i.e., given any two symplectic vector spaces there exists an isomorphism between them that preserves their symplectic forms.

Let v_1, \dots, v_n be a g -orthonormal basis of $T_{z(0)}M$ and consider the parallel linear frame V_1, \dots, V_n obtained by parallel transport of the v_i 's along z . This frame gives us isomorphisms $T_{z(0)}M \rightarrow \mathbb{R}^n$ that carry the metric tensor g to a *fixed* symmetric bilinear form on \mathbb{R}^n , still denoted by g . Since each V_i is parallel, the covariant derivative of vector fields along z correspond to the usual differentiation of \mathbb{R}^n -valued maps. For each $t \in [0, 1]$, the map:

$$\mathbb{R}^n \cong T_{z(t)}M \ni v \longmapsto R(\dot{z}(t), v)\dot{z}(t) \in T_{z(t)}M \cong \mathbb{R}^n$$

is a g -symmetric linear operator on \mathbb{R}^n , that will be denoted by $R(t)$; the symbol $R(t)$ will also denote the $n \times n$ matrix that represents $R(t)$ in the canonical basis of \mathbb{R}^n . Moreover, the map:

$$T_{z(t)}M \oplus T_{z(t)}M \cong \mathbb{R}^n \oplus \mathbb{R}^n \ni (v, w) \longmapsto g(v, \nabla_w B) - g(\nabla_v B, w) \in \mathbb{R}$$

is an anti-symmetric bilinear map on \mathbb{R}^n , that will be denoted by $H(t)$; we will also denote with $H(t)$ the $n \times n$ real anti-symmetric matrix that represents the linear operator associated to the bilinear form³ $H(t)$, i.e., such that:

$$(3.7) \quad H(t)(v, w) = H(t)v \cdot w,$$

where \cdot is the Euclidean inner product on \mathbb{R}^n . (More generally, we will always identify bilinear maps on \mathbb{R}^n with linear operators from \mathbb{R}^n to \mathbb{R}^n that realize them with respect to the Euclidean inner product.)

We will need the derivative $H'(t)$ of the curve of operators H , which is computed as follows. Let v, w be fixed, and consider vector fields V, W such that $V(t) = v$ and

³anti-symmetric, so that the matrix $H(t)$ that represent the linear operator corresponding to the bilinear form $H(t)$ is given by $H(t)_{ij} = -H(t)(e_i, e_j)$, where (e_i) is the canonical basis of \mathbb{R}^n . The same observation must be kept in mind also in the sequel, when we will use the matrices associated to the linear operators associated to anti-symmetric bilinear forms.

$W(t) = w$; then:

(3.8)

$$\begin{aligned} H'(t)v \cdot w &= \frac{d}{dt} (H(t)V(t) \cdot W(t)) - H(t)V'(t) \cdot W(t) - H(t)V(t) \cdot W'(t) \\ &= g\left(\frac{D}{dt}V, \nabla_W B\right) + g\left(V, \frac{D}{dt}\nabla_W B\right) - g\left(\frac{D}{dt}\nabla_V B, W\right) - g\left(\nabla_V B, \frac{D}{dt}W\right) \\ &\quad - g\left(\frac{D}{dt}V, \nabla_W B\right) + g\left(\nabla \frac{D}{dt}V, B, W\right) - g\left(V, \nabla \frac{D}{dt}W, B\right) + g\left(\nabla_V B, \frac{D}{dt}W\right) \\ &= g(v, \text{Hess } B(\dot{z}, w)) - g(w, \text{Hess } B(\dot{z}, v)). \end{aligned}$$

Finally, consider the one-parameter family ω_t of symplectic forms on $\mathbb{R}^n \oplus \mathbb{R}^n$ defined by:

$$\omega_t((v_1, w_1), (v_2, w_2)) = g(v_1, w_2) - g(v_2, w_1) + H(t)(v_1, v_2);$$

recall also that the *canonical symplectic form* $\bar{\omega}$ of $\mathbb{R}^n \oplus \mathbb{R}^n$ is defined by:

$$\bar{\omega}((v_1, w_1), (v_2, w_2)) = v_1 \cdot w_2 - v_2 \cdot w_1.$$

In this setup, the linear map $\mathcal{F}_z(t)$ can be seen as a linear automorphism of $\mathbb{R}^n \oplus \mathbb{R}^n$, and Proposition 3.6 tells us that the pull-back of ω_t by $\mathcal{F}_z(t)$ coincides with ω_0 . The matrices representing the linear operator associated to the symplectic forms ω_t and $\bar{\omega}$ in the canonical basis of $\mathbb{R}^n \oplus \mathbb{R}^n$ are given in $n \times n$ blocks by:

$$\omega_t \cong \begin{pmatrix} H(t) & -g \\ g & 0 \end{pmatrix}, \quad \bar{\omega} \cong \begin{pmatrix} 0 & -\text{I} \\ \text{I} & 0 \end{pmatrix},$$

where g denotes the (constant) symmetric matrix representing the bilinear form g on \mathbb{R}^n and I is the identity operator on \mathbb{R}^n . In terms of matrices, we have:

$$(3.9) \quad \mathcal{F}_z(t)^* \omega_t \mathcal{F}_z(t) = \omega_0.$$

where now $*$ denotes the adjoint with respect to the Euclidean product.

For each $t \in [0, 1]$, let us consider the automorphism $L(t) : \mathbb{R}^n \oplus \mathbb{R}^n \rightarrow \mathbb{R}^n \oplus \mathbb{R}^n$ whose matrix in $n \times n$ blocks is:

$$(3.10) \quad L(t) = \begin{pmatrix} \text{I} & 0 \\ -\frac{1}{2}H(t) & g \end{pmatrix};$$

observe that L_0 is invariant by $L(t)$ for all t :

$$(3.11) \quad L(t)(0, w) = (0, gw), \quad \forall w \in \mathbb{R}^n.$$

Let us define the following isomorphisms:

$$(3.12) \quad \Psi(t) = L(t) \circ \mathcal{F}_z(t) \circ L(0)^{-1} : \mathbb{R}^n \oplus \mathbb{R}^n \rightarrow \mathbb{R}^n \oplus \mathbb{R}^n$$

It is easy to see that (3.12) preserves the canonical symplectic form $\bar{\omega}$.

3.4. The Maslov index. We will assume henceforth that z is a solution of (2.5) such that $t = 1$ is not magnetic conjugate along z . Recalling the definition of the Lagrangian plane $L_0 = \{0\} \oplus \mathbb{R}^n$, it follows easily that an instant $t_0 \in]0, 1[$ is magnetic conjugate along z if and only if $(\mathcal{F}_z(t_0)L_0) \cap L_0 \neq \{0\}$. Moreover, since $L(t)$ preserves L_0 , this is also equivalent to the fact that $(\Psi(t_0)L_0) \cap L_0 \neq \{0\}$. Observe that, since Ψ preserves the symplectic form $\bar{\omega}$, then $t \mapsto \Psi(t)L_0$ is a curve of Lagrangian spaces in the symplectic space $(\mathbb{R}^{2n}, \bar{\omega})$.

The geometry of the Grassmannian of all Lagrangian subspaces of a symplectic space is well known (see for instance [12] and the references therein); we recall here briefly

some basic facts. Denote by Λ the set of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \bar{\omega})$ and by $\ell : [0, 1] \rightarrow \Lambda$ the smooth curve in Λ given by:

$$(3.13) \quad \ell(t) = \Psi(t)L_0,$$

and for $k = 1, \dots, n$ we set:

$$\Lambda_k = \{L \in \Lambda : \dim(L \cap L_0) = k\}, \quad \Lambda_{\geq 1} = \bigcup_{k=1}^n \Lambda_k.$$

Each Λ_k is a connected embedded real-analytic submanifold of Λ having codimension $\frac{1}{2}k(k+1)$ in Λ ; the set $\Lambda_{\geq 1} = \bigcup_{k=1}^n \Lambda_k$ is an algebraic variety whose regular part is Λ_1 . Observe that Λ_1 has codimension 1 in Λ , and it has a canonical transverse orientation associated to the symplectic form $\bar{\omega}$. The first relative singular homology group with coefficients in \mathbb{Z} , $H_1(\Lambda, \Lambda \setminus \Lambda_0)$, is infinite cyclic, and can be canonically described in terms of the symplectic form ω .

Definition 3.7. Let z be a solution of (2.5) such that $t = 1$ is not magnetic conjugate. Then, the *Maslov index* of z , denoted by $i_{\text{Maslov}}(z)$, is the integer number:

$$i_{\text{Maslov}}(z) := \mu_{L_0}(\ell|_{[0,1]}),$$

where $\ell(t) = \Psi(t)L_0$ and $\varepsilon > 0$ is chosen in such a way that there are no magnetic conjugate instants along z in $]0, \varepsilon]$ (recall Remark 3.2).

Observe that our definition of $i_{\text{Maslov}}(z)$ does not indeed depend on the choice of a parallel trivialization of TM along z .

3.5. Computation of the Maslov index. We will now develop a technique to compute the value of the Maslov index of a solution z . To this aim, we recall a few results from references [20, 21, 22], where the authors study the Maslov index of a special class of differential systems, called *symplectic differential systems*.

Denote by $\text{Sp}(\mathbb{R}^{2n}, \bar{\omega})$ the Lie group of all automorphisms of \mathbb{R}^{2n} that preserve $\bar{\omega}$, and let $\text{sp}(\mathbb{R}^{2n}, \bar{\omega})$ be its Lie algebra. It is easy to prove that $\text{sp}(\mathbb{R}^{2n}, \bar{\omega})$ consists of all $2n \times 2n$ real matrices X that can be written in $n \times n$ blocks as:

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^* \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are linear operators, with β and γ self-adjoint. In the language of [20], a symplectic differential system is a first order linear system of differential equation in \mathbb{R}^{2n} of the form:

$$(3.14) \quad \frac{d}{dt} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = X(t) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

with $X : [0, 1] \rightarrow \text{sp}(\mathbb{R}^{2n}, \bar{\omega})$ a curve in the Lie algebra $\text{sp}(\mathbb{R}^{2n}, \bar{\omega})$ whose upper right block $\beta(t)$ is invertible for all t .

Given such a system, its fundamental matrix⁴ $\Psi(t)$ is a curve in the Lie group $\text{Sp}(\mathbb{R}^{2n}, \bar{\omega})$, and, provided that the final instant $t = 1$ is not conjugate, a Maslov index of the system (3.14) with initial conditions:

$$(3.15) \quad (w_1(0), w_2(0)) \in L_0$$

is defined in analogy with the theory exposed in Subsection 3.4. An instant $t_0 \in]0, 1]$ is conjugate for the system (3.14) with initial conditions (3.15) if there exists a non zero

⁴i.e., $\Psi(t)$ is defined by $\Psi(t)(w_1(0), w_2(0)) = (w_1(t), w_2(t))$ for all solution (w_1, w_2) of (3.14).

solution (w_1, w_2) of (3.14) and (3.15) such that $w_1(t_0) = 0$. The signature of a conjugate instant t_0 in this context is defined to be the signature of the restriction of the bilinear form $(h, k) \mapsto \beta(t_0)h \cdot k$ to the space

(3.16)

$$\mathbb{A}[t_0] = \{w_2(t_0) : (w_1, w_2) \text{ is a solution of (3.14) and (3.15) satisfying } w_1(t_0) = 0\}.$$

Whenever such restriction is nondegenerate, then the conjugate instant t_0 is said to be nondegenerate, and nondegenerate conjugate instants are isolated. One of the central results for symplectic differential systems ([20, Theorem 2.3.3]) tells us that the Maslov index of (3.14)–(3.15) is given by the sum of the signatures of all its conjugate instants, provided that every conjugate instant is nondegenerate.

Associated to each symplectic differential system (3.14) with coefficient matrix $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ whose upper right $n \times n$ block B is invertible, one associates a bounded symmetric bilinear form I_X (see [20]), the *index form* of the symplectic system, given by:

$$I_X(v, w) = \int_0^1 \left[B^{-1}(v' - Av, w' - Aw) + C(v, w) \right] dt,$$

defined in the space of H^1 vector fields v on $[0, 1]$ satisfying the Lagrangian initial conditions (3.15).

The theory of Maslov index for the solutions of the magnetic equation (2.5) fits into the theory of symplectic differential systems. In order to apply the results of [22] to this case we will show that the curve in the symplectic group $\Psi(t)$ given in (3.12) arises from a symplectic system which is naturally associated to the magnetic Jacobi equation.

Consider the magnetic Jacobi equation (2.14) that, recalling (3.1), (3.2) and (3.3) can be written in the form of system:

$$(3.17) \quad \begin{cases} v_1' = v_2 \\ v_2' = \mathfrak{C}(v_1) + \mathfrak{D}(v_2); \end{cases}$$

(3.17) will be called the *magnetic Jacobi system*. The space \mathbb{J}_z^0 consists of solutions of (3.17) that satisfy the initial conditions:

$$(3.18) \quad (v_1(0), v_2(0)) \in L_0.$$

Again, identifying each tangent space $T_{z(t)}M$ with \mathbb{R}^n by means of a parallel trivialization of TM along z , we will think of (3.17) as a differential system in \mathbb{R}^{2n} , with coefficient matrix given in $n \times n$ blocks by:

$$(3.19) \quad \mathfrak{R} = \begin{pmatrix} 0 & \mathbf{I} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix},$$

where $\mathfrak{C}(t), \mathfrak{D}(t) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are the linear operators corresponding respectively to (3.2) and (3.3). Recalling the definition of the anti-symmetric operator $H(t)$ given in (3.7) and its derivative $H'(t)$ computed in (3.8), from (3.2) and (3.3) we obtain:

$$(3.20) \quad g\mathfrak{C} - \mathfrak{C}^*g = H', \quad \mathfrak{D} = g^{-1}H.$$

The first equality in (3.20) is obtained using (2.6) and the first Bianchi identity as follows:

$$\begin{aligned}
(g\mathfrak{C} - \mathfrak{C}^*g)v \cdot w &= -g(\text{Hess } B(\dot{z}, v), w) + g(\text{Hess } B(\dot{z}, w), v) + 2g(\text{Hess}_s B(v, w), \dot{z}) \\
&\quad + g(R(\dot{z}, v)B, w) - g(R(\dot{z}, w)B, v) \\
&= -g(\text{Hess } B(\dot{z}, v), w) + g(\text{Hess } B(\dot{z}, w), v) \\
&\quad + g(R(v, w)B, \dot{z}) + g(R(w, B)v, \dot{z}) + g(R(B, v)w, \dot{z}) \\
&= -g(\text{Hess } B(\dot{z}, v), w) + g(\text{Hess } B(\dot{z}, w), v) = H'(t)v \cdot w.
\end{aligned}$$

The second equality in (3.20) is immediate.

The following key Proposition 3.8 proved in [6], which gives the link between the theory of symplectic differential systems and the electromagnetic Jacobi equation is as follows:

Proposition 3.8. *Consider the isomorphism $\mathfrak{L} : H^1([0, 1], \mathbb{R}^{2n}) \rightarrow H^1([0, 1], \mathbb{R}^{2n})$ defined by:*

$$(3.21) \quad \mathfrak{L} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} (t) = L(t) \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$$

where $L(t)$ is the $2n \times 2n$ matrix given in (3.10), and set:

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \mathfrak{L} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Then, (v_1, v_2) is a solution of the magnetic Jacobi system (3.17) if and only if (w_1, w_2) is a solution of the symplectic differential system:

$$(3.22) \quad \frac{d}{dt} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = X \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

whose coefficient matrix $X : [0, 1] \rightarrow \text{sp}(\mathbb{R}^{2n}, \bar{\omega})$ is given by:

$$(3.23) \quad X = \begin{pmatrix} \frac{1}{2}g^{-1}H & g^{-1} \\ (g\mathfrak{C})_s + \frac{1}{4}Hg^{-1}H & \frac{1}{2}Hg^{-1} \end{pmatrix},$$

where $(g\mathfrak{C})_s = \frac{1}{2}(g\mathfrak{C} + \mathfrak{C}^*g)$. Moreover,

- (a) the Lagrangian initial conditions (3.18) for (v_1, v_2) correspond to the initial conditions (3.15) for (w_1, w_2) ;
- (b) the conjugate instants of the magnetic Jacobi system (3.17) coincide with those of the symplectic (3.22), and they have the same signatures;
- (c) a conjugate instant $t_0 \in]0, 1]$ is nondegenerate for (3.17) if and only if it is nondegenerate for (3.22);
- (d) the second variation $\text{Hess } F_{p,q}(z)$ (2.8) correspond to the index form I_X of the symplectic differential system (3.22).

Proof. A straightforward computation gives:

$$\frac{d}{dt} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = L' \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + L \frac{d}{dt} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (L' + L\mathfrak{R})L^{-1} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Setting $X = (L' + L\mathfrak{R})L^{-1}$, formula (3.23) is easily obtained from (3.10), (3.19) and (3.20).

The statements (a) and (c) in the thesis are easily proven using the fact that the Lagrangian space L_0 is $L(t)$ -invariant for all t (formula (3.11)), and observing that, if t_0 is a conjugate instant, then $L(t_0)$ carries the magnetic conjugate plane $\mathbb{P}[t_0]$ to $g\mathbb{P}[t_0] = \mathbb{A}[t_0]$

(see (3.16)). As to the equality of the signatures of conjugate instants, observe that the signature of the restriction of g^{-1} to $g\mathbb{P}[t_0]$ equals the signature of the restriction of g to $\mathbb{P}[t_0]$. The equality between the Hessian $\text{Hess } F_{p,q}(z)$ and the index form I_X of the symplectic system (3.22) is obtained by a straightforward direct calculation. \square

We observe that, since H is anti-symmetric, formula (3.23) defines indeed a matrix X in $\text{sp}(\mathbb{R}^{2n}, \bar{\omega})$.

Corollary 3.9. *Let z be a solution of (2.5) such that the instant $t = 1$ is not magnetic conjugate along z . Then, the Maslov index $i_{\text{Maslov}}(z)$ equals the Maslov index of the symplectic system (3.22) with initial conditions (3.15). Moreover, if all the magnetic conjugate instants along z are nondegenerate, then the Maslov index of z equals the sum of the signatures of all the magnetic conjugate instants along z :*

$$i_{\text{Maslov}}(z) = \sum_{t \text{ magnetic conjugate}} \text{sgn}(t).$$

Proof. It follows immediately from [20, Theorem 2.3.3] and Proposition 3.8. \square

4. SPECTRAL FLOW AND BIFURCATION

In this section we will recall (without proofs) a few basic facts on spectral flow, relative dimension and bifurcation for strongly indefinite variational problems. Basic references for a detailed exposition of the material contained in this section are: [1, 10, 18, 19].

4.1. On the relative index of Fredholm forms. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let B a bounded symmetric bilinear form on H ; there exists a unique self-adjoint bounded operator $S : H \rightarrow H$ such that $B = \langle S \cdot, \cdot \rangle$, that will be called the *realization of B* (with respect to $\langle \cdot, \cdot \rangle$). B is nondegenerate if its realization is injective, B is strongly nondegenerate if S is an isomorphism. If B is strongly nondegenerate, or if more generally 0 is not an accumulation point of the spectrum of S , we will call the *negative space* (resp., *the positive space*) of B the closed subspace $V^-(S)$ (resp., $V^+(S)$) of H given by $\chi_{]-\infty, 0[}(S)$ (resp., $\chi_{]0, +\infty[}(S)$), where χ_I denotes the characteristic function of the interval I . We will say that B is *Fredholm* if S is Fredholm, or that B is *RCPPI*, *realized by a compact perturbation of a positive isomorphism*, (resp., *RCPNI*) if S is of the form $S = P + K$ (resp., $S = N + K$) where P is a positive isomorphism of H (N is a negative isomorphism of H) and K is compact. Observe that the properties of being Fredholm, RCPPI or RCPNI do not depend on the inner product, although the realization S and the spaces $V^\pm(S)$ do.

The index (resp., the coindex) of B , denoted by $n_-(B)$ (resp., $n_+(B)$) is the dimension of $V^-(S)$ (resp., of $V^+(S)$); the *nullity* of B , denoted by $n_0(B)$ is the dimension of the kernel of S .

If B is RCPPI (resp., RCPNI), then both its nullity $n_0(B)$ and its index $n_-(B)$ (resp., and its coindex $n_+(B)$) are finite numbers.

Given a closed subspace $W \subset H$, the *B -orthogonal complement of W* , denoted by W^{\perp_B} , is the closed subspace of H :

$$W^{\perp_B} = \{x \in H : B(x, y) = 0 \text{ for all } y \in W\};$$

clearly,

$$W^{\perp_B} = S^{-1}(W^\perp).$$

Let us now recall a few basic things on the notion of commensurability of closed subspaces. Let $V, W \subset H$ be closed subspaces and let P_V and P_W denote the orthogonal

projections respectively onto V and W . We say that V and W are *commensurable* if the restriction to V of the projection P_W is a Fredholm operator from V to W . It is an easy exercise to show that commensurability is an equivalence relation in the Grassmannian of all closed subspaces of H ; observe in particular that, identifying each Hilbert space with its own dual, the adjoint of the operator $P_W|_V : V \rightarrow W$ is precisely $P_V|_W : W \rightarrow V$. If V and W are commensurable the *relative dimension* $\dim_W(V)$ of V with respect to W is defined as the Fredholm index $\text{ind}(P_V|_W : W \rightarrow V)$, which is equal to:

$$\dim_W(V) = \text{ind}(P_W|_V : V \rightarrow W) = \dim(W^\perp \cap V) - \dim(W \cap V^\perp).$$

Clearly, if V and W are commensurable, then V^\perp and W^\perp are commensurable, and:

$$\dim_{W^\perp}(V^\perp) = -\dim_W(V) = \dim_V(W).$$

The commensurability of closed subspaces and the relative dimension do *not* depend on the choice of a Hilbert space inner product on H .

The relative index of a Fredholm bilinear form B can be computed in terms of index and coindex of suitable restrictions of B :

Proposition 4.1. *Let B be a Fredholm symmetric bilinear form on H , S its realization and let $W \subset H$ be a closed subspace which is commensurable with $V^-(S)$. Then the relative index $\text{ind}_W(B)$ is given by:*

$$(4.1) \quad \text{ind}_W(B) = n_-(B|_{W^\perp_B}) - n_+(B|_W).$$

Proof. See [19, Proposition 2.6]. □

4.2. Spectral flow. Let us consider an infinite dimensional separable real Hilbert space H . We will denote by $\mathcal{B}(H)$ and $\mathcal{K}(H)$ respectively the algebra of all bounded linear operators on H and the closed two-sided ideal of $\mathcal{B}(H)$ consisting of all compact operators on H ; the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$ will be denoted by $\mathcal{Q}(H)$, and $\pi : \mathcal{B}(H) \rightarrow \mathcal{Q}(H)$ will denote the quotient map. The *essential spectrum* $\sigma_{\text{ess}}(T)$ of a bounded linear operator $T \in \mathcal{B}(H)$ is the spectrum of $\pi(T)$ in the Calkin algebra $\mathcal{Q}(H)$. Let $\mathcal{F}(H)$ and $\mathcal{F}^{\text{sa}}(H)$ denote respectively the space of all Fredholm (bounded) linear operators on H and the space of all self-adjoint ones. An element $T \in \mathcal{F}^{\text{sa}}(H)$ is said to be *essentially positive* (resp., *essentially negative*) if $\sigma_{\text{ess}}(T) \subset \mathbb{R}^+$ (resp., if $\sigma_{\text{ess}}(T) \subset \mathbb{R}^-$), and *strongly indefinite* if it is neither essentially positive nor essentially negative.

The symbols $\mathcal{F}_+^{\text{sa}}(H)$, $\mathcal{F}_-^{\text{sa}}(H)$ and $\mathcal{F}_*^{\text{sa}}(H)$ will denote the subsets of $\mathcal{F}^{\text{sa}}(H)$ consisting respectively of all essentially positive, essentially negative and strongly indefinite self-adjoint Fredholm operators on H . These sets are precisely the three connected components of $\mathcal{F}^{\text{sa}}(H)$; $\mathcal{F}_+^{\text{sa}}(H)$ and $\mathcal{F}_-^{\text{sa}}(H)$ are contractible, while $\mathcal{F}_*^{\text{sa}}(H)$ is homotopically equivalent to $U(\infty) = \lim_n U(n)$, and it has infinite cyclic fundamental group.

Given a continuous path $S : [0, 1] \rightarrow \mathcal{F}_*^{\text{sa}}(H)$ with $S(0)$ and $S(1)$ invertible, the *spectral flow* of S on the interval $[0, 1]$, denoted by $\text{sf}(S, [0, 1])$, is an integer number which is given, roughly speaking, by the net number of eigenvalues that pass through zero in the positive direction from the start of the path to its end. There exist several equivalent definitions of the spectral flow in the literature; for the purposes of the present paper, we give a description of the spectral flow, which follows the approach in [10]. As we have observed, $\mathcal{F}_*^{\text{sa}}(H)$ is not simply connected, and therefore no non trivial homotopic invariant for curves in $\mathcal{F}_*^{\text{sa}}(H)$ can be defined only in terms of the value at the endpoints. However, in [10] it is shown that the spectral flow can be defined in terms of the endpoints, provided that the path S has the special form $S(t) = \mathfrak{J} + K(t)$, where \mathfrak{J} is a fixed symmetry of H

and $t \mapsto K(t)$ is a path of compact operators. By a *symmetry* of the Hilbert space H it is meant an operator \mathfrak{J} of the form

$$\mathfrak{J} = P_+ - P_-,$$

where P_+ and P_- are the orthogonal projections onto infinite dimensional closed subspaces H_+ and H_- of H such that $H = H_+ \oplus H_-$; assume that such a symmetry \mathfrak{J} has been fixed.

Denote by $\mathcal{B}_o(H)$ the group of all invertible elements of $\mathcal{B}(H)$. There is an action of $\mathcal{B}_o(H)$ on $\mathcal{F}^{sa}(H)$ given by:

$$\mathcal{B}_o(H) \times \mathcal{F}^{sa}(H) \ni (M, S) \longmapsto M^*SM \in \mathcal{F}^{sa}(H);$$

this action preserves the three connected components of $\mathcal{F}^{sa}(H)$. Two elements in the same orbit are said to be *cogredient*; the orbit of each element in $\mathcal{F}^{sa}(H)$ meets the affine space $\mathfrak{J} + \mathcal{K}(H)$, i.e., given any $S \in \mathcal{F}^{sa}(H)$ there exists $M \in \mathcal{B}_o(H)$ such that $M^*SM = \mathfrak{J} + K$, where K is compact. Moreover, using a suitable fiber bundle structure and standard lifting arguments, it is shown in [10] that if $t \mapsto S(t) \in \mathcal{F}^{sa}(H)$ is a path of class C^k , $k = 0, \dots, +\infty$, then one can find a C^k curve $t \mapsto M(t) \in \mathcal{B}_o(H)$ such that $M(t)^*S(t)M(t) = \mathfrak{J} + K(t)$, where $t \mapsto K(t)$ is a C^k curve of compact operators. Among the central results of [10] the authors prove that the spectral flow of a path of strongly indefinite self-adjoint Fredholm operators is invariant by cogredience, and that for paths that are compact perturbation of a fixed symmetry the spectral flow is given as the relative dimension of the negative eigenspaces at the endpoints:

Proposition 4.2. *Let $S : [0, 1] \rightarrow \mathcal{F}_*^{sa}(H)$ be a continuous path such that $S(0)$ and $S(1)$ are invertible, denote by $B(t) = \langle S(t)\cdot, \cdot \rangle$ the corresponding bilinear form on H , and let $M : [0, 1] \rightarrow \mathcal{B}_o(H)$ be a continuous curve with $L(t) := M(t)^*S(t)M(t)$ of the form $\mathfrak{J} + K(t)$, with $K(t)$ compact for all t . Then:*

- (1) $\text{sf}(S, [0, 1]) = \text{sf}(L, [0, 1])$;
- (2) $\text{sf}(L, [0, 1]) = \text{ind}_{V^-(L(1))}(B(0))$
 $= \dim(V^-(L(0)) \cap V^+(L(1))) - \dim(V^+(L(0)) \cap V^-(L(1)))$.

Proof. See [10, Proposition 3.2, Proposition 3.3]. □

Observe that, since $\dim_W(V) = -\dim_V(W)$, the equality in part (2) of Proposition 4.2 can be rewritten as:

$$(4.2) \quad \text{sf}(L, [0, 1]) = -\text{ind}_{V^-(L(0))}(B(1))$$

4.3. Bifurcation for a path of strongly indefinite functionals. Let H be a real separable Hilbert space, $U \subset H$ a neighborhood of 0 and $f_\lambda : U \rightarrow \mathbb{R}$ a family of smooth (i.e., of class C^2) functionals depending smoothly on $\lambda \in [0, 1]$. Assume that 0 is a critical point of f_λ for all $\lambda \in [0, 1]$. An element $\lambda_* \in [0, 1]$ is said to be a *bifurcation value* if there exists a sequence $(\lambda_n)_n$ in $[0, 1]$ and a sequence $(x_n)_n \in U$ such that:

- (1) x_n is a critical point of f_{λ_n} for all n ;
- (2) $x_n \neq 0$ for all n and $\lim_{n \rightarrow \infty} x_n = 0$;
- (3) $\lim_{n \rightarrow \infty} \lambda_n = \lambda_*$.

The main result concerning the existence of a bifurcation value for a path of strongly indefinite functionals is the following:

Proposition 4.3. *Let $S(\lambda) = d^2 f_\lambda(0)$ be the continuous path of self-adjoint Fredholm operators on H given by the second variation of f_λ at 0. Assume that S takes values in $\mathcal{F}_*^{\text{sa}}(H)$ for all $\lambda \in [0, 1]$, and that $S(0)$ and $S(1)$ are invertible. If $\text{sf}(S, [0, 1]) \neq 0$, then there exists a bifurcation value $\lambda_* \in]0, 1[$.*

Proof. See [10, Theorem 1]. □

5. BIFURCATION FOR SOLUTIONS OF THE LORENTZ FORCE EQUATIONS

We will now apply the abstract theory on variational bifurcation to the case of the magnetic action functional (2.2). The first step is to reduce the magnetic variational problem into an abstract analytical setup of a smooth family of functionals on a neighborhood of 0 in a fixed real separable Hilbert space. We will then apply the results of Sections 2 and 4 to obtain the desired bifurcation result for solutions of the Lorentz force equation.

5.1. The analytical setup. Let $z : [0, 1] \rightarrow M$ be a solution of (2.5), with $p = z(0)$ and $q = z(1)$; let us consider again a g -orthonormal basis v_1, \dots, v_n of $T_{\gamma(0)}M$ and assume that the first k vectors v_1, \dots, v_k generate a g -negative space, while the v_{k+1}, \dots, v_n generate a g -positive space. Let us consider again the parallel transport of the v_i 's along γ , that will be denoted by V_1, \dots, V_n . Observe that, since parallel transport is an isometry, then, for all $t \in [0, 1]$, the vectors $V_1(t), \dots, V_k(t)$ generate a g -negative subspace of $T_{\gamma(t)}M$, that will be denoted by D_t^- , and $V_{k+1}(t), \dots, V_n(t)$ generate a g -positive subspace of $T_{\gamma(t)}M$, denoted by D_t^+ .

We fix a positive number $\varepsilon_0 < 1$ such that there are no conjugate points to p along γ in the interval $]0, \varepsilon_0]$. Finally, let us define an auxiliary positive definite inner product on each $T_{\gamma(t)}M$, that will be denoted by g_t , by declaring that the basis $V_1(t), \dots, V_n(t)$ be orthonormal.

For all $s \in [\varepsilon_0, 1]$, let Ω_s denote the manifold of all curves $x : [0, s] \rightarrow M$ of Sobolev class H^1 such that $x(0) = z(0) = p$ and $x(s) = z(s)$. Now let us consider the following energy functional $F_s : \Omega_s \rightarrow \mathbb{R}$, defined by:

$$(5.1) \quad F_s(z) = \int_0^s \left[\frac{1}{2} g(\dot{z}, \dot{z}) + g(B, \dot{z}) \right] dt;$$

it is easy to see that F_s is smooth, and its critical points are precisely the solutions of (2.5) from p to $z(s)$. For each $x \in \Omega_s$, the tangent space $T_x \Omega_s$ is identified with the Hilbertable space:

$$T_x \Omega_s = \{V \text{ vector field along } x \text{ of class } H^1 : V(0) = 0, V(s) = 0\};$$

Let $\rho > 0$ be a positive number, assume for the moment that ρ is less than the injectivity radius of M at $z(s)$ for all $s \in [\varepsilon_0, 1]$; a further restriction for the choice of ρ will be given in what follows. Let \mathcal{W} be the open ball of radius ρ centered at 0 in $H_0^1([0, 1], \mathbb{R}^n) \cong T_z \Omega_1$ and, for all $s \in [\varepsilon_0, 1]$, let \mathcal{W}_s be the neighborhood of 0 in $H_0^1([0, s], \mathbb{R}^n) \cong T_z \Omega_s$ given by the image of \mathcal{W} by the reparameterization map Φ_s defined by:

$$(5.2) \quad H_0^1([0, 1], \mathbb{R}^n) \ni V \longmapsto V(s^{-1} \cdot) \in H_0^1([0, s], \mathbb{R}^n).$$

Finally, for all $s \in [\varepsilon_0, 1]$, let $\widetilde{\mathcal{W}}_s$ be the subset of Ω_s obtained as the image of \mathcal{W}_s by the map:

$$V \longmapsto \text{EXP}(V),$$

where

$$(5.3) \quad \text{EXP}(V)(t) = \exp_{z(t)} V(t).$$

Since $\exp_{z(t)}$ is a local diffeomorphism between a neighborhood of 0 in $T_{z(t)}M$ and a neighborhood of $z(t)$ in M , it is easily seen that the positive number ρ above can be chosen small enough so that, for all $s \in [\varepsilon_0, 1]$, $\widetilde{\mathcal{W}}_s$ is an open subset of Ω_s (containing γ) and EXP is a diffeomorphism between \mathcal{W}_s and $\widetilde{\mathcal{W}}_s$.

In conclusion, we have a family of diffeomorphisms $\Psi_s : \mathcal{W} \rightarrow \widetilde{\mathcal{W}}_s$:

$$\Psi_s = \text{EXP} \circ \Phi_s,$$

and we can define a family $(f_s)_{s \in [\varepsilon_0, 1]}$ of smooth functionals on \mathcal{W} by setting:

$$f_s = F_s \circ \Psi_s;$$

observe that $\Psi_s(0) = z|_{[0, s]}$ for all s .

Proposition 5.1. *$(f_s)_s$ is a smooth family of functionals on \mathcal{W} . For each $s \in [\varepsilon_0, 1]$, a point $x \in \mathcal{W}$ is a critical point of f_s if and only if $\Psi_s(x)$ is a solution of (2.5) in M from p to $z(s)$ in $\widetilde{\mathcal{W}}_s$. In particular, 0 is a critical point of f_s for all s , and every solutions of the Lorentz equation in M from p to $z(s)$ sufficiently close to z in the H^1 -topology is obtained from a critical point of f_s in \mathcal{W} . The second variation of f_s at 0 is given by the bounded symmetric bilinear form I_s on $H_0^1([0, 1], \mathbb{R}^n)$ defined by:*

(5.4)

$$I_s[V, W] = \int_0^1 \left[\frac{1}{s}g(V'(t), W'(t)) + g(R_s^1(t)V(t), W(t)) + sg(R_s^2(t)V(t), W(t)) \right] dt$$

where R^1 is the family of g -symmetric endomorphisms of \mathbb{R}^n corresponding to the third and fourth term in the equation (2.8), and where R^2 is the path of g -symmetric endomorphisms of \mathbb{R}^n corresponding to the remaining terms in (2.8).

Proof. The smoothness of $s \mapsto f_s$ follows immediately from the smoothness of the exponential map and of the reparameterization map $s \mapsto \Phi_s$. Since Ψ_s is a diffeomorphism for all s , the critical points of f_s are precisely the inverse image through Ψ_s of the critical points of F_s , and the second statement of the thesis is clear from our construction. As to the second variation of f_s at 0, formula (5.4) is easily obtained from the classical second variation formula for the action functional F_s at the critical point $z|_{[0, s]}$ with the change of variable $t = \tau s^{-1}$. \square

Now let $z : [0, 1] \rightarrow M$ be a solution of (2.5), and let us consider the second variation formula given in Proposition 2.4. We recall that this is a Fredholm form on $T_z\Omega_{p, q}$ which is realized by a strongly indefinite self-adjoint Fredholm operator.

Set $k = n_-(g)$; a maximal negative distribution along z is a smooth selection $\Delta = (\Delta_t)_{t \in [0, 1]}$ of k -dimensional subspaces of $T_{\gamma(t)}M$ such that $g|_{\Delta_t}$ is negative definite for all t . Given a maximal negative distribution Δ along γ , denote by \mathcal{S}^Δ the closed subspace of $T_z\Omega_{p, q}$ given by:

$$(5.5) \quad \mathcal{S}^\Delta = \left\{ V \in T_z\Omega_{p, q} : V(t) \in \Delta_t, \text{ for all } t \in [0, 1] \right\}.$$

We will now relate the Maslov index of a solution of (2.5) with a difference of index and coindex of restrictions of Fredholm operators arising by the second variation of the magnetic action functional.

Proposition 5.2. *The restriction of $\text{Hess } F_{p, q}(z)$ to \mathcal{S}^Δ is RCPNI and the restriction of $\text{Hess } F_{p, q}(z)$ to $(\mathcal{S}^\Delta)^\perp$ is RCPPI. Moreover, if $z(1)$ is not magnetic conjugate, the index of $\text{Hess } F_{p, q}(z)$ relatively to \mathcal{S}^Δ equals the Maslov index of z :*

$$(5.6) \quad \text{ind}_{\mathcal{S}^\Delta}(\text{Hess } F_{p, q}(z)) = i_{\text{Maslov}}(z).$$

Proof. The first statement in the thesis is proven in [22, Prop. 5.25], the second statement is proven in [20, Lemma 2.6.6]. Equality (5.6) follows from Proposition 4.1 and the semi-Riemannian index theorem for Hamiltonian systems [20], that gives us the equality:

$$i_{\text{Maslov}}(z) = n_- \left(\text{Hess } F_{p,q} \Big|_{(\mathcal{S}^\Delta)^\perp} \right) - n_+ \left(\text{Hess } F_{p,q} \Big|_{\mathcal{S}^\Delta} \right). \quad \square$$

Proposition 5.2 gives us the link between the notion of bifurcation for a smooth family of functionals and the bifurcation problem for the solutions of the Lorentz force equation.

We will now compute the spectral flow of the smooth curve of strongly indefinite self-adjoint Fredholm operators on $H_0^1([0, 1], \mathbb{R}^n)$ associated to the curve of symmetric bilinear forms (5.4).

Lemma 5.3. *For all $s \in [\varepsilon_0, 1]$, the bilinear form I_s of (5.4) is realized by a bounded self-adjoint Fredholm operator S_s on $H_0^1([0, 1], \mathbb{R}^n)$. If $z(1)$ is not conjugate to $z(0)$ along z , then the endpoints of the path*

$$(5.7) \quad]\varepsilon_0, 1] \ni s \longmapsto S_s \in \mathcal{F}_*^{\text{sa}}(H_0^1[0, 1], \mathbb{R}^n)$$

are invertible.

Proof. The bilinear form I_s in (5.4) is symmetric and bounded in the H^1 -topology, hence S_s is a bounded self-adjoint operator. The bilinear form G on $H_0^1([0, 1], \mathbb{R}^n)$ defined by $(V, W) \mapsto \frac{1}{s} \int_0^1 g(V', W') dt$ is realized by an invertible operator, because g is non-degenerate. The difference $I_s - G$ is realized by a self-adjoint compact operator on $H_0^1([0, 1], \mathbb{R}^n)$, because it is clearly continuous in the C^0 -topology, and the inclusion $H_0^1 \hookrightarrow C^0$ is compact. This proves that S_s is Fredholm.

Since S_s is Fredholm of index zero, then S_s is invertible if and only if it is injective, i.e., if and only if I_s has trivial kernel, that is, if and only if $z(s)$ is not conjugate to $z(0)$ along z . Hence, the last statement in the thesis comes from the fact that both $z(\varepsilon_0)$ and $z(1)$ are not conjugate to $z(0)$ along z . \square

Lemma 5.4. *The smooth path \hat{I} of bounded symmetric bilinear forms*

$$]0, 1] \ni s \mapsto \hat{I}_s := s \cdot I_s$$

has a continuous extension to 0 which is obtained by setting:

$$\hat{I}_0(V, W) = \int_0^1 g(V', W') dt.$$

For all $s \in [0, 1]$, let \hat{S}_s be the realization of \hat{I}_s and assume that $z(1)$ is not conjugate to $z(0)$ along z .

The spectral flow of the path $\hat{I} : [0, 1] \rightarrow \mathcal{F}_^{\text{sa}}([0, 1], \mathbb{R}^n)$ is equal to the spectral flow of the path $S :]\varepsilon_0, 1] \rightarrow \mathcal{F}_*^{\text{sa}}([0, 1], \mathbb{R}^n)$ given in (5.7).*

Proof. From (5.4) we get:

$$(5.8) \quad \hat{I}_s[V, W] = \int_0^1 \left[g(V'(t), W'(t)) + sg(R_s^1(t)V(t), W(t)) + s^2g(R_s^2(t)V(t), W(t)) \right]$$

for all $s \in]0, 1]$, and this formula proves immediately the first statement in the thesis.

The cogredience invariance of sf implies that multiplication by a positive map does not change the spectral flow; in particular, the spectral flow of \hat{S} and of S on the interval $[\varepsilon_0, 1]$ coincide. Since \hat{S}_s is invertible for all $s \in [0, \varepsilon_0]$, the spectral flow of S on $[\varepsilon_0, 1]$ coincide with the spectral flow of \hat{S} on $[0, 1]$. \square

5.2. Bifurcation at magnetic conjugate points. We are now ready to compute the spectral flow of the path S in (5.7) using the Morse index theorem [22, 16]:

Proposition 5.5 (Morse Index Theorem for the Magnetic Action Functional). *Assume that $z(1)$ is not magnetic conjugate to $z(0)$ along z . Then the spectral flow of the path S is equal to $-i_{\text{Maslov}}(z)$.*

Proof. It follows from Proposition 3.8, Proposition 4.1, Proposition 4.2, Proposition 5.1, Proposition 5.2 and the Morse Index Theorem for symplectic differential systems in [22, Theorem 6.4]. \square

Corollary 5.6. *Assume that $z(t_0)$ is a nondegenerate magnetic conjugate point along a solution z of (2.5). If $\text{sgn}(t_0) \neq 0$, then $z(t_0)$ is a bifurcation point along z . More generally, if $0 < t_0 < t_1 \leq 1$ are non magnetic conjugate instants along z , if $i_{\text{Maslov}}(z|_{[0,t_0]}) \neq i_{\text{Maslov}}(z|_{[0,t_1]})$ then there exists at least one bifurcation instant $t_* \in]t_0, t_1[$.*

Proof. By the very same argument used in the proof of Proposition 5.5, for all nonconjugate instant $s \in]\varepsilon_0, 1]$ along z , the spectral flow of the path S on the interval $[\varepsilon_0, s]$ equals the Maslov index $i_{\text{Maslov}}(z|_{[0,s]})$. If t_0 is a nondegenerate (hence isolated) conjugate instant, using the additivity by concatenation of sf , for all $\varepsilon > 0$ small enough we then have that the spectral flow of S in the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$ is given by:

$$\begin{aligned} \text{sf}(S, [t_0 - \varepsilon, t_0 + \varepsilon]) &= \text{sf}(S, [\varepsilon_0, t_0 + \varepsilon]) - \text{sf}(S, [\varepsilon_0, t_0 - \varepsilon]) \\ &= -i_{\text{Maslov}}(z|_{[0,t_0+\varepsilon]}) + i_{\text{Maslov}}(z|_{[0,t_0-\varepsilon]}) = -\text{sgn}(t_0). \end{aligned}$$

The conclusion follows from Corollary 3.9, Proposition 4.3 and Proposition 5.1. The proof of the second statement in the thesis is analogous. \square

We have the analogue of a classical result of Morse and Littauer for the exponential map of a Riemannian manifold (see [25]):

Corollary 5.7. *If $z(t_0)$ is a nondegenerate magnetic conjugate point along a solution z of (2.5), with $\text{sgn}(t_0) \neq 0$, then the magnetic exponential map $\exp_{z(0)}^{\text{mag}}$ is not injective on any neighborhood of $t_0 \dot{z}(t_0)$ in $T_{z(0)}M$.* \square

5.3. Final remarks. We have seen that the Maslov index of a solution of the Lorentz force equation in the nondegenerate case is given by the sum of the signatures at each magnetic conjugate instant (Corollary 3.9). This result is analogous to a similar result holding in the case of semi-Riemannian geodesics (see for instance [23]). For causal (i.e., nonspacelike) Lorentzian geodesic, an elementary argument shows that every conjugate point is nondegenerate, and that its signature is positive, and it coincides with its multiplicity. In particular, every conjugate point along a causal Lorentzian geodesic is a bifurcation point (see [19]). In site of this analogy, and also of the fact that, as in the geodesic case, also solutions of the Lorentz equation do preserve their causal character (Lemma 2.2), it is not clear whether conjugate points along timelike or lightlike solutions of (2.5) are nondegenerate, or isolated.

It is an interesting open question to prove or disprove by means of counterexamples that magnetic conjugate points along Lorentzian timelike or lightlike solutions are isolated, and that their signature is positive and equal to the multiplicity.

In the real analytic case, where conjugate instants are necessarily isolated, a more detailed analysis of the degeneracies mentioned can be carried out in terms of higher order Taylor expansion of the symplectic path near an intersection with the Maslov cycle, in the spirit of [24]. The interested reader may find details of this construction for the semi-Riemannian geodesic problem in [12].

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