ON THE MASLOV INDEX OF LAGRANGIAN PATHS THAT ARE NOT TRANSVERSAL TO THE MASLOV CYCLE. SEMI-RIEMANNIAN INDEX THEOREMS IN THE DEGENERATE CASE.

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ABSTRACT. The Maslov index of a Lagrangian path, under a certain transversality assumption, is given by an algebraic count of the intersections of the path with a subvariety of the Lagrangian Grassmannian called the Maslov cycle. In these notes we use the notion of *generalized signatures* at a singularity of a smooth curve of symmetric bilinear forms to determine a formula for the computation of the Maslov index in the case of a real-analytic path having possibly non transversal intersections. Using this formula we give a general definition of Maslov index for continuous curves in the Lagrangian Grassmannian, both in the finite and in the infinite dimensional (Fredholm) case, and having arbitrary endpoints. Other notions of Maslov index are also considered, like the index for pairs of Lagrangian paths, the Kashiwara's triple Maslov index, and Hörmander's four-fold index.

We discuss some applications of the theory, with special emphasis on the study of the Jacobi equation along a semi-Riemannian geodesic. In this context, we prove an extension of several versions of the Morse index theorems for geodesics having possibly conjugate endpoints.

The main results of this paper were announced in [29].

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1. INTRODUCTION

Recent extensions of classical variational theories, like for instance Morse Theory, Ljusternik–Schnirelman theory or bifurcation theory, have shown the increasing relevance of topological techniques in finite or infinite dimensional manifolds. Solutions of different geometrical variational problems, like geodesics, minimal surfaces, harmonic maps, solutions of Hamiltonian systems, etc., are classified by an *index*, typically an integer number, that carries both analytical and geometrical information on the solution. Jumps of the appropriate index function (Morse, Conley, Maslov, spectral, etc.), that can only occur at degeneracy instants of the second variation form, are responsible for topologically non trivial changes in the solution space of the given variational problem. In the non positive definite case, a first order analysis of the degeneracy instant is no longer sufficient to detect a jump of the index and to compute its value, so one needs a more accurate analysis beyond the first derivative test, capable of reckoning fine analytical details of the degeneracy.

In the symplectic world, a recurrent notion is that of Maslov index, that appears naturally in many different contexts, especially in relation with solutions of Hamiltonian systems. The natural environment for the notion of Maslov index is the Lagrangian Grassmannian Λ of a finite or infinite dimensional symplectic space; the Maslov index is a $\mathbb{Z}$-valued homotopical invariant for continuous curves in Λ that gives an algebraic measure of the intersections with the Maslov cycle, which is an algebraic subvariety of Λ. Under generic circumstances, the intersections of a curve with the Maslov cycle occur at its regular part, and they are transversal (hence isolated). In this case the computation of the Maslov index is done via well established results obtained by differential topological methods, and its relations with the geometrical and analytical invariants of the variational problem are clear. Typically, the non transversal case is studied by perturbative techniques, which allow to extend to this case the results involving quantities that are stable by uniformly small perturbations.

There are several reasons to develop a non perturbative analysis of the non transversal intersections, and that motivated the research exposed in this paper. In first place, perturbation arguments do not work properly when non transversal intersections occur at the endpoints; namely, in this case arbitrarily small perturbations may destroy the intersection. Observe that the arguments needed to prove the genericity of the transversality property in specific contexts become drastically more involved (if not impossible at all) if one restricts to fixed endpoints homotopies. For instance, in the case of periodic solutions of Hamiltonian systems, in order to obtain multiplicity results one needs to consider iteration formulas for the Maslov index, and it is not evident whether transversality is generic in spaces of inextensible paths.

Second, and more important, perturbative methods preserve global quantities, but destroy the information concerning each single intersection, which of course may be relevant in the problem under consideration. This is the case, for instance, in bifurcation theory, if one wants to know whether a given instant bifurcates or not, and not just whether bifurcation occurs *somewhere*. Also, in semi-Riemannian geometry the presence of conjugate points, that correspond to intersections with the Maslov cycle obtained from the flow of the Jacobi deviation equation, have global geometrical implications on their own, and perturbing the data (i.e., the metric) would not be a very meaningful procedure. Along a semi-Riemannian geodesic, non transversal intersections with the Maslov cycle (that may occur only if the metric is not positive definite) correspond to degenerate conjugate points; the presence of this kind of conjugate points is responsible for a series of new and interesting phenomena in the semi-Riemannian vs. the Riemannian world, hence it deserves a specific analysis. The occurrence of degenerate intersections arising from semi-Riemannian geodesics has been somewhat overlooked in the classical literature, and only recently the relevance of such occurrences has been recognized; for a more detailed discussion on this topic, see for instance [38, Subsection 5.4] and [48]. The result of these notes contributes understanding such degeneracy phenomena in the real-analytic case; for instance, we give conditions under which the exponential map is not one-to-one on any neighborhood of a possibly degenerate semi-Riemannian conjugate point (Corollary 4.13), extending a classical Riemannian result of Morse and Littauer (see [54]).

In this paper we will be essentially interested in the notion of Maslov index associated to geodesics and, more generally, to solutions of Hamiltonian systems. The Hamiltonian flow on a symplectic manifold preserves the symplectic form, and by a suitable trivialization of the tangent bundle of the manifold along a given solution we get a smooth curve in the symplectic group of a fixed symplectic space. Likewise, the evolution of Lagrangian initial conditions by the flow of the Hamilton equation produces a curve in the Grassmannian of all Lagrangian subspaces of a fixed symplectic space. The local geometry of the Lagrangian Grassmannian of a symplectic space, both in finite and the infinite dimensional case, is the geometry of symmetric bilinear forms, and intersections with the Maslov cycle correspond to degeneracy instants of the forms; non transversality with the Maslov cycle corresponds, in local charts, to the fact that the corresponding curve $B(t)$ of symmetric bilinear forms has derivative $B'(t_0)$ which is degenerate when restricted to $\mathrm{Ker}\big(B(t_0)\big)$. The Maslov index of a curve measures at each intersection the change of value of the index of the symmetric form, and a computation of this jump is studied using higher order derivatives and the introduction of the notions of "generalized Jordan chains" and of "partial signatures" at an isolated degeneracy instant. Given a smooth curve in the Lagrangian Grassmannian having only isolated intersections with the Maslov cycle, using local charts we associate a sequence of integers to each such intersection, and we give a formula to compute the Maslov index of the curve in terms of these integers. The theory works well under the assumption that the curve be real-analytic, or, more generally, when it is possible to find a smoothly varying Hilbert basis of the space that diagonalizes the curve of symmetric forms. Important examples for the theory of Maslov index of real-analytic curves are encountered in the study of eigenvalue problems for some differential operators, whose solutions depend analytically on the eigenvalue; in this paper we will discuss the theory in the case of the Morse–Sturm–Liouville equation.

The partial signatures at a degeneracy instant have appeared in several different contexts in the literature, associated to "jumps" of integer valued invariants, like the spectral flow, or the so-called eta-invariant associated to an elliptic self-adjoint operator. It is not an easy task to establish where exactly the notion of partial signature at an isolated degeneracy instant of a path of symmetric (or hermitian) forms has first appeared. To the authors' knowledge, one of the first references where such notion appears is [31], where the partial signatures are introduced as generalized multiplicities of characteristic values of a meromorphic operator valued function.

The partial signatures appear in a paper of Rabier (ref. [49]) as an evolution of Magnus' generalized algebraic multiplicity for nonlinear eigenvalues of operators between Banach spaces, aiming at results in bifurcation theory. Our approach follows more closely the paper by Farber and Levine [23], where the partial signatures have been used to determine a formula for the jumps of the so-called *eta-invariant*. The link between the Maslov index and the theory of partial signatures has been suggested to the authors by some recent results of Fitzpatrick, Pejsachowicz and Recht on the bifurcation theory for strongly indefinite functionals (see [28]). The main result of Rabier in [49] relates bifurcation phenomena to generalized Jordan chains for a sequence of symmetric linear operators on finite dimensional vector spaces. The central result of [28] is, roughly speaking, that bifurcation for a smooth family of (strongly indefinite) functionals having a trivial branch of critical point at which the Hessians are Fredholm, occurs at those instants when the *spectral flow* of the path of second variations jumps. Finally, a geometrical bifurcation problem in the context of semi-Riemannian geodesics has been recently studied in [44], where it is proven that the Maslov index of a geodesic equals the spectral flow of the corresponding curve of index forms. An alternative approach that leads naturally to discovering a link between the notions of spectral flow, of Maslov index and of partial signatures is given in [40]. In this paper, the authors introduce a new topological invariant for smooth paths of self-adjoint Fredholm operators using a certain line integral, as in [31], that can be computed in terms of partial signatures. In view of these results, it now appears to be the case that these seemingly different results are indeed different aspects of the same mathematical theory.

As an application of our theory, we will discuss a very general version of the semi-Riemannian Morse index theorem (Theorem 4.9), relating the spectral data of the Jacobi differential operator with the spectral flow of the index form along the geodesic, and with the conjugate points along the geodesic. The theorem gives an equality of three different indexes, called the *spectral index*, the *generalized Morse index* and the *Maslov index* of the geodesic, which is shown to hold without any assumption on the final endpoint of the geodesic. The notion of spectral index, which plays a central role in our theory, is defined as the spectral flow of the curve of index form for the eigenvalue Jacobi differential problem as the eigenvalue runs from $-\infty$ to 0. The dependence of this path of Fredholm forms on the eigenvalue λ is analytic (namely, affine), and the method of generalized Jordan chains can be applied directly. An explicit formula for the spectral index of a semi-Riemannian geodesic with possibly conjugate endpoint is computed in Proposition 4.8.

Motivated by infinite dimensional Morse theory, and by the analysis of first order elliptic operators on closed manifolds, the study of relations between the Maslov index and the spectral flow is a quite active research field, and a rather extensive literature on this topic is available nowadays. The equality between the Maslov (or the Morse) index for paths of self-adjoint Fredholm operators has been proven in several contexts. Starting from the celebrated Morse index theorem in Riemannian geometry, the Maslov index has been used by Duistermaat [21] to prove an index theorem for convex Hamiltonian systems. Several versions of the Morse index theorem were proven in the last decade in the case of geodesics in manifolds endowed with non positive definite metrics and for non convex Hamiltonian systems (see for instance [4, 30, 38, 45, 46]). Basic references on the notion of Maslov index, from which the authors of the present paper have taken inspiration, are the articles by Robbin and Salamon [50], by de Gosson [19], and that of Cappell, Lee and Miller [9]; analogies and differences between the results of [9], [50] and those of the present paper will be discussed at the beginning of Section 3. As to the notion of spectral flow for a path of self-adjoint Fredholm operators, the literature available is enormous, starting from the pioneering work of Atiyah, Patodi and Singer [3]. We found particularly simple and elegant the approach in a paper by Phillips [43], where functional calculus is used. Phillips' definition of Maslov index has inspired the proof of our Proposition 3.2 that gives an abstract method for constructing arbitrary group valued index functions for continuous paths in topological spaces. However, for the purposes of the present paper, the most appropriate definition of spectral flow seems to be the one given in [28], that uses the theory of relative dimension in Hilbert spaces and the cogredient action of the general linear group; this approach is particularly useful when dealing with index and coindex of symmetric bilinear forms (see Proposition A.11). The notion of Maslov index has been related to the conjugate points along a Riemannian geodesic by Morvan in [39], while in the semi-Riemannian (i.e., non positive definite) case, Helfer in [32] was the first to introduce the notion of Maslov index of a geodesic. The equality of the spectral index of a path $D(t)$ of Dirac operators on a vector bundle over an odd dimensional Riemannian manifold M split along an hypersurface Σ and the Maslov index of the pair of curves of Lagrangian spaces $(L_1(t), L_2(t))$ obtained as the Cauchy data spaces of $D(t)$ has been proven in [42] by Nicolaescu in the case of nondegenerate endpoints, extending a previous result of Yoshida [53] in dimension 3; Yoshida–Nicolaescu's theorem has been extended by Daniel in [14] to the possibly degenerate case. A similar result is proven by Cappell, Lee and Miller in [11], where the authors obtain several formulas expressing the spectral flow of a one-parameter family of self-adjoint elliptic operators on a closed manifold as a sum of terms computed from a decomposition of the manifold into two submanifolds. In the infinite dimensional case, the relationship between spectral flow and Maslov index for curves in the Fredholm Lagrangian Grassmannian has been studied by Booss–Bavnbek, Furutani and Otsuki in [6, 7, 8, 27], where index theorems are proven under very general circumstances.

Our main interest in studying connections between the Maslov index and the spectral flow comes from recent developments of infinite dimensional Morse theory (see [2]), which follows the lines of the celebrated works of Floer [24, 25, 26]. The index theorem proven by Robbin and Salamon in [51] gives an equality between the spectral flow of a one-parameter family $A(t)$ of unbounded self-adjoint operators on a Hilbert space H with the Fredholm index of the densely defined operator $\frac{\mathrm{d}}{\mathrm{d}t} + A(t) : L^2(I\!R, H) \to L^2(I\!R, H)$. Such result aims, as in our case, to Morse homology, where $A(t)$ is the Hessian of a Morse function along an orbit of the gradient flow between two critical points. In this case, the spectral flow of $A(t)$ gives the dimension of the intersection between the stable and the unstable manifold at the endpoints; the finiteness of such dimension is a central point in the construction of the Morse–Witten complex. A crucial assumption in Robbin and Salamon's proof is that the limits $\lim\limits_{t\to\pm\infty} A(t) = A^{\pm}$ be hyperbolic, hence invertible. Assuming invertibility at the endpoints is an obstruction for developing Morse homology in case of degenerate critical points, like for instance in the case of closed (semi-)Riemannian geodesics, in which case all critical points are degenerate due to the equivariant action of the group $\mathrm{O}(2)$. Our degenerate index theorem aims at providing the tools for treating infinite dimensional Morse theory in presence of equivariant, or otherwise well-behaved, degenerate critical points.

It is very likely that one can obtain a somewhat simpler proof of many of the index theorems mentioned above using the theory of partial signature discussed in this paper along the lines of the proof of Theorem 4.9:

- a direct (finite dimensional) homotopy argument proves the equality between the Maslov index of the path of Lagrangians associated to the path of self-adjoint operators and the Maslov index of a real-analytic curve obtained from the corresponding eigenvalue problem;
- an infinite dimensional homotopy argument proves the equality between the spectral flow and the "spectral index" of the data, which is the spectral flow of a real-analytic curve depending on the spectral parameter;
- the equality between the index of the two real-analytic curves is obtained by proving the equality of the partial signatures at each degeneracy instant.

Typically, the spectral index is the spectral flow of a path depending affinely on the parameter, in which case the computation of the partial signatures can be done explicitly (see Subsection 3.9).

The paper is ideally split into two distinct parts. Sections 2 and 3 deal with the theory of partial signatures and its applications to the abstract index theory. Several notions of Maslov index appearing in the literature are discussed in our framework in the spirit of the article [9], including the Maslov index of pairs of Lagrangian paths and for symplectic paths (Conley–Zehnder index), the triple index (Kashiwara's index) and the four-fold index (Hörmander index). The notions of triple and four-fold index are given in terms of the Maslov index for paths; it is an interesting observation that, conversely, the Maslov index can be constructed using only the axioms of Kashiwara's index (formulas (3.16) and (3.17)). The second part of the paper, contained in Section 4, deals with the geometric applications of the theory in the context of semi-Riemannian geodesics. The central results are the Index Theorem 4.9 and its geometrical version Theorem 4.10, that generalize [46, Theorem 5.2] and [13, Theorem 3.3] to the case of possibly conjugate endpoints.

Two appendices have been added at the end of the paper. Appendix A contains the basic definitions and some technical results concerning functional analytical aspects of the index theory, with special emphasis on the notion of commensurability of closed subspaces of Hilbert spaces, relative dimension and relative index of Fredholm bilinear forms. Appendix B contains the proof of the connectedness of the Grassmannian of maximal negative subspaces of a finite dimensional vector space endowed with a nondegenerate symmetric bilinear form; such result is used in a homotopy argument employed in the proof of a geometrical version of the index theorem (Subsection 4.8).

In order to facilitate the reading, each section of the paper has been divided into small subsections that should help the reader to keep track of the several notions introduced and to localize cross references.

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2. GENERALIZED SIGNATURES AT AN ISOLATED SINGULARITY

Let V be a real vector space; we will denote by $\mathrm{B}_{\mathrm{sym}}(V)$ be the vector space of all symmetric bilinear forms $B : V \times V \to I\!R$ on V. When V is endowed with a positive definite inner product $\langle\cdot,\cdot\rangle$, we will denote by $\mathcal{L}_{\mathrm{sym}}(V)$ the vector space of all linear maps $T : V \to V$ that are symmetric relatively to $\langle\cdot,\cdot\rangle$. There is an identification $\mathrm{B}_{\mathrm{sym}}(V) \cong \mathcal{L}_{\mathrm{sym}}(V)$ via the map $T \mapsto B = \langle T\cdot,\cdot\rangle$, and such identification will be made implicitly in many parts of the paper, although in some occasion (especially when the choice of a fixed inner product is not done explicitly) it will be convenient to maintain a conceptual distinction between linear operators and bilinear forms. For $B \in \mathrm{B}_{\mathrm{sym}}(V)$, denote by $n^-(B)$, $n_0(B)$ and $n^+(B)$ respectively the *index*, the *degeneracy* (or *nullity*) and the *coindex* of B, that are respectively the number of -1's, the number of 1's and the number of 0's in the canonical form of B as given in Sylvester's Inertia Theorem. For the purposes of the paper, it will be interesting to introduce the notations

$$\mathring{n}^+(B) = n^+(B) + n_0(B), \quad \text{and} \quad \mathring{n}^-(B) = n^-(B) + n_0(B)$$

respectively for the *extended coindex* and the *extended index* of B. The *signature* $\sigma(B)$ of a bilinear form[1] B is the difference $n^+(B) - n^-(B) = \mathring{n}^+(B) - \mathring{n}^-(B)$; B is said to be *nondegenerate* if $n_0(B) = 0$. Clearly, $n^-(B)$, and $n^+(B)$ are respectively the number of negative and of positive eigenvalues of the symmetric linear map T corresponding to B, the degeneracy $n_0(B)$ is the multiplicity of 0 as an eigenvalue of T, and B is nondegenerate exactly when T is an isomorphism. With a slight abuse of terminology, we will use the notation $n^\pm$ (and $\mathring{n}^\pm$) for symmetric linear maps, meaning the (extended) index and the

[1] The use of the symbol σ for the signature of a bilinear form will be used quite frequently throughout the paper, and for this reason the customary notation $\sigma(T)$ for the *spectrum* of a linear operator T will be replaced with $\mathfrak{s}(T)$.

coindex of the corresponding element in $\mathrm{B}_{\mathrm{sym}}(V)$. Finally, by $\mathrm{Ker}(B)$ we will mean the kernel of the corresponding linear map T, which can be described also as

$$\mathrm{Ker}(B) = \big\{w \in V : B(v,w) = 0 \text{ for all } v \in V\big\}.$$

In next two subsections we will assume that V is a fixed finite dimensional real vector space, and for simplicity we will assume that $\langle\cdot,\cdot\rangle$ is a fixed positive definite inner product on V. Also in the infinite dimensional case, we will deal with spaces endowed with a fixed Hilbert space inner product. The choice of a specific inner product is by no means essential, and we will prove at the end that all the notions introduced are independent of such choice (Remark 2.17).

2.1. **Root functions and partial signatures at a degeneracy instant.**

Let $\mathbf{L} : [t_0 - \varepsilon, t_0 + \varepsilon] \to \mathcal{L}_{\mathrm{sym}}(V)$ be a curve such that $t = t_0$ is an isolated singularity for $\mathbf{L}(t)$. We are interested in determining the jump of the functions $n^+\big(\mathbf{L}(t)\big)$ and $n^-\big(\mathbf{L}(t)\big)$ as t passes through t_0; the following elementary result is well known:

Proposition 2.1. *Assume that the restriction B_1 of $\langle \mathbf{L}'(t_0)\cdot,\cdot\rangle$ to $\mathrm{Ker}\big(\mathbf{L}(t_0)\big)$ is nondegenerate. Then:*

$$n^+\big(\mathbf{L}(t_0+\varepsilon)\big) - n^+\big(\mathbf{L}(t_0)\big) = n^+(B_1), \quad n^+\big(L(t_0)\big) - n^+\big(\mathbf{L}(t_0-\varepsilon)\big) = -n^-(B_1),$$
$$n^+\big(\mathbf{L}(t_0+\varepsilon)\big) - n^+\big(\mathbf{L}(t_0-\varepsilon)\big) = \sigma(B_1).$$

Proof. See for instance [30, Proposition 2.5]. □

Attempts to generalize the result of Proposition 2.1 by replacing the nondegeneracy assumption with an assumption concerning higher order derivatives of $\mathbf{L}(t)$ at $t = t_0$ in $\mathrm{Ker}\big(\mathbf{L}(t_0)\big)$ fail, as the following example shows:

Example 2.2. Consider the curves $B, \widetilde{B}\colon I\!R \to \mathrm{B}_{\mathrm{sym}}(I\!R^2)$ given by:

$$B(t) = \begin{pmatrix} 1 & t \\ t & t^3 \end{pmatrix}, \quad \widetilde{B}(t) = \begin{pmatrix} 1 & t^2 \\ t^2 & t^3 \end{pmatrix};$$

we have $B(0) = \widetilde{B}(0)$ and $N = \mathrm{Ker}\big(B(0)\big) = \mathrm{Ker}\big(\widetilde{B}(0)\big) = \{0\} \oplus I\!R$. Observe that $B(t)|_{N\times N} = \widetilde{B}(t)|_{N\times N}$ for all $t \in I\!R$, so that the Taylor expansion of B coincides with that of $\widetilde{B}$ in N; on the other hand, for $\varepsilon > 0$ sufficiently small, we have:

$$n^+\big(B(\varepsilon)\big) - n^+\big(B(-\varepsilon)\big) = 1 - 1 = 0,$$
$$n^+\big(\widetilde{B}(\varepsilon)\big) - n^+\big(\widetilde{B}(-\varepsilon)\big) = 2 - 1 = 1.$$

We will now discuss a method for computing the jump of the coindex involving higher order derivatives; let us recall that, given $k \ge 1$, a smooth map $v : \,]t_0 - \varepsilon, t_0 + \varepsilon[\, \to V$ is said to have a *zero of order* k at $t = t_0$ if $v(t_0) = v'(t_0) = \ldots = v^{(k-1)}(t_0) = 0$ and $v^{(k)}(t_0) \neq 0$. In order to set up properly our framework we give the following:

Definition 2.3. A *root function* for $\mathbf{L}(t)$ at $t = t_0$ is a smooth map $u : [t_0 - \varepsilon, t_0 + \varepsilon] \to V$, $\varepsilon > 0$, such that $u(t_0) \in \mathrm{Ker}\big(\mathbf{L}(t_0)\big)$. The *order* $\mathrm{ord}(u)$ of the root function u is the (possibly infinite) order of zero at $t = t_0$ of the map $t \mapsto \mathbf{L}(t)u(t)$.

Associated to the degeneracy instant t_0 for the curve $\mathbf{L}$, we will now define a filtration of vector spaces $W_k \subset V$ and a sequence of bilinear forms $B_k : W_k \times W_k \to I\!R$, $k \ge 1$, as follows. Set:

$$W_k := \big\{u_0 \in V : \exists \text{ a root function } u \text{ with } \mathrm{ord}(u) \ge k \text{ and } u(t_0) = u_0\big\}, \tag{2.1}$$

and

$$B_k(u_0, v_0) := \frac{1}{k!}\Big\langle \tfrac{\mathrm{d}^k}{\mathrm{d}t^k}\Big|_{t=t_0}\big[\mathbf{L}(t)u(t)\big], v_0\Big\rangle, \quad u_0, v_0 \in W_k, \tag{2.2}$$

where u is any root function with $\mathrm{ord}(u) \geq k$ and $u(t_0) = u_0$. We will prove next that the right hand side of the equality in (2.2) does not indeed depend on the choice of the root function u.

Proposition 2.4. *W_k is a subspace of $\mathrm{Ker}\big(\mathbf{L}(t_0)\big)$, and $W_{k+1} \subseteq W_k$ for all $k \geq 1$. The bilinear forms B_k are well defined and symmetric for all $k \geq 1$.*

Proof. The first statement follows trivially from the definition of W_k.

For the second statement, observe that for $u_0, v_0 \in W_k$, if $u(t)$ and $v(t)$ are root functions with $\mathrm{ord}(u), \mathrm{ord}(v) \geq k$, $u(t_0) = u_0$, $v(t_0) = v_0$, keeping in mind that $\mathbf{L}(t)u(t)$ and $\mathbf{L}(t)v(t)$ have vanishing derivatives of order less than or equal to $k-1$ at $t = t_0$, then:

$$\begin{aligned} \big\langle \tfrac{\mathrm{d}^k}{\mathrm{d}t^k}\big|_{t=t_0} \big[\mathbf{L}(t)u(t)\big], v_0 \big\rangle &= \tfrac{\mathrm{d}^k}{\mathrm{d}t^k}\big|_{t=t_0} \langle \mathbf{L}(t)u(t), v(t) \rangle \\ &= \tfrac{\mathrm{d}^k}{\mathrm{d}t^k}\big|_{t=t_0} \langle u(t), \mathbf{L}(t)v(t) \rangle = \big\langle u_0, \tfrac{\mathrm{d}^k}{\mathrm{d}t^k}\big|_{t=t_0} \big[\mathbf{L}(t)v(t)\big] \big\rangle. \end{aligned} \tag{2.3}$$

The first term in (2.3) does not depend on the choice of the root function $v(t)$ and the last term does not depend on the choice of a root function $u(t)$. This proves at the same time that B_k is well defined and that it is symmetric. □

Remark 2.5. Here is an alternative definition of the spaces W_k and of the bilinear forms B_k. Consider the Taylor expansion of $\mathbf{L}(t)$ centered at $t = t_0$:

$$\mathbf{L}_0 + (t-t_0)\mathbf{L}_1 + (t-t_0)^2\mathbf{L}_2 + \ldots + (t-t_0)^k\mathbf{L}_k + \ldots,$$

where $\mathbf{L}_k = \frac{1}{k!}\mathbf{L}^{(k)}(t_0) \in \mathcal{L}_{\mathrm{sym}}(V)$ for all k. Then, the space W_{k+1} can be described as the set of those $u_0 \in V$ such that there exist $u_1, \ldots, u_k \in V$ for which the following system of linear equations is satisfied:

$$\begin{cases} \mathbf{L}_0 u_0 = 0, \\ \mathbf{L}_1 u_0 + \mathbf{L}_0 u_1 = 0, \\ \mathbf{L}_2 u_0 + \mathbf{L}_1 u_1 + \mathbf{L}_0 u_2 = 0, \\ \quad \vdots \\ \sum_{j=0}^k \mathbf{L}_{k-j} u_j = 0. \end{cases} \tag{2.4}$$

A root function $u(t)$ with $\mathrm{ord}(u) \geq k+1$ and $u(t_0) = u_0$ would be given in this case by $u(t) = \sum_{j=0}^k (t-t_0)^j u_j$; in the language of [49], a sequence $(u_0, u_1, \ldots, u_k)$ satisfying (2.4) is called a *generalized Jordan chain* for $\mathbf{L}(t)$ at $t = t_0$ starting at u_0. The *length* of a generalized Jordan chain is defined to be the number of its elements; a generalized Jordan chain $(u_0, \ldots, u_k)$ is said to be *extendible* if there exists $u_{k+1} \in V$ such that $(u_0, \ldots, u_k, u_{k+1})$ is a generalized Jordan chain. Thus, W_{k+1} is the space of those u_0 for which there exists a generalized Jordan chain of length $k+1$ starting at u_0.

Non extendible Jordan chains are said to be *maximal*; observe that maximal generalized Jordan chains starting at the same element u_0 may have different lengths.[2]

The first equation of (2.4) tells us that $u_0 \in \mathrm{Ker}(\mathbf{L}_0)$; it is immediate to observe that, if $(u_0, \ldots, u_k)$ is a generalized Jordan chain for $\mathbf{L}$ starting at u_0, then it is extendible if and only if:

$$\sum_{j=0}^k \mathbf{L}_{k+1-j} u_j \in \mathrm{Im}\big(\mathbf{L}_0\big) = \mathrm{Ker}(\mathbf{L}_0)^\perp.$$

Moreover, the bilinear form B_{k+1} can be defined in terms of generalized Jordan chains by:

$$B_{k+1}(u_0, v_0) = \sum_{j=0}^k \big\langle \mathbf{L}_{k+1-j} u_j, v_0 \big\rangle, \tag{2.5}$$

[2]For instance, suppose that (u_0, u_1, u_2) is a generalized Jordan chain starting at u_0 and that $\xi \in \mathrm{Ker}(\mathbf{L}_0)$ is such that $\mathbf{L}_1\xi \notin \mathrm{Im}(\mathbf{L}_0)$. Then $(u_0, u_1 + \xi)$ is maximal.

where $(u_0, u_1, \ldots, u_k)$ is *any* generalized Jordan chain for $\mathbf{L}$ of length $k+1$ starting at u_0. System (2.4) and formula (2.5) appear in reference [23] (see [23, (23i)] and [23, Corollary 3.13]).

From the very definition of B_k, one sees immediately that we have an inclusion $W_{k+1} \subset \mathrm{Ker}(B_k)$; observe in particular that if B_k is nondegenerate for some k, then $W_j = (0)$ for all $j > k$. We will show below that, in fact, $W_{k+1} = \mathrm{Ker}(B_k)$ for all $k \geq 1$ (Corollary 2.10).

Definition 2.6. For all $k \geq 1$, the integer numbers

$$n_k^-(\mathbf{L}, t_0) := n^-(B_k), \quad n_k^+(\mathbf{L}, t_0) := n^+(B_k), \quad \sigma_k(\mathbf{L}, t_0) := \sigma(B_k) \tag{2.6}$$

are called respectively the *k-th partial index*, the *k-th partial coindex*, and the *k-th partial signature* of $\mathbf{L}(t)$ at $t = t_0$. The *k-th partial extended index* and *coindex* $\mathring{n}^{\pm}{}_k(\mathbf{L}, t_0)$ are defined similarly.

The integers $n_k^{\pm}(\mathbf{L}, t_0)$, $\mathring{n}^{\pm}{}_k(\mathbf{L}, t_0)$ and $\sigma_k(\mathbf{L}, t_0)$ will be referred to collectively as the "partial signatures" of the curve of bilinear forms $\mathbf{L}$ at the degeneracy instant t_0.

Remark 2.7. It is immediate from the definition that $W_1 = \mathrm{Ker}(\mathbf{L}_0)$ and that B_1 coincides with the restriction of $\langle \mathbf{L}_1 \cdot, \cdot \rangle$ (recall Proposition 2.1).

Example 2.8. Let us compute explicitly the spaces W_k, the bilinear forms B_k and the partial signatures σ_k for the curves in Example 2.2 at the instant $t = 0$.

For $\mathbf{L}(t) = \begin{pmatrix} 1 & t \\ t & t^3 \end{pmatrix}$, one computes easily:

$$W_1 = \{0\} \oplus I\!\!R, \quad \mathbf{L}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{L}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{L}_2 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

For all $u_0 = (0, a)$, the system $\mathbf{L}_1 u_0 + \mathbf{L}_0 u_1$ has the unique solution $u_1 = (-a, 0)$, i.e., $W_2 = W_1 = \{0\} \oplus I\!\!R$; moreover, it is easily seen that the generalized Jordan chain $\big((0,a), (-a,0)\big)$ is maximal, i.e., the system $\mathbf{L}_2 u_0 + \mathbf{L}_1 u_1 + \mathbf{L}_0 u_2$ has no solution $u_2 \in I\!\!R^2$ and therefore $W_3 = (0)$.

Using (2.5) we compute:

$$B_1\big((0,a),(0,b)\big) = \big\langle \mathbf{L}_1 u_0, v_0 \big\rangle = 0, \quad \forall\, a, b \in I\!\!R, \quad n_1^{\pm} = 0,\ \sigma_1 = 0$$
$$B_2\big((0,a),(0,b)\big) = \big\langle \mathbf{L}_2 u_0 + \mathbf{L}_1 u_1, v_0 \big\rangle = -ab, \quad n_2^+ = 0,\ n_2^- = 1,\ \sigma_2 = -1.$$

Clearly, $W_k = (0)$, $B_k = 0$ and $\sigma_k = 0$ for all $k \geq 3$.

A similar computation for $\mathbf{L}(t) = \begin{pmatrix} 1 & t^2 \\ t^2 & t^3 \end{pmatrix}$ gives:

$$W_1 = \{0\} \oplus I\!\!R, \quad \mathbf{L}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{L}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{L}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{L}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Set $u_0 = (0, a) \in W_1$; the system $\mathbf{L}_1 u_0 + \mathbf{L}_0 u_1$ has solutions $u_1 = (0, \beta)$, with $\beta \in I\!\!R$; hence, $W_2 = W_1$. The system $\mathbf{L}_2 u_0 + \mathbf{L}_1 u_1 + \mathbf{L}_0 u_2$ has the unique solution $u_2 = (-a, 0)$, and so $W_3 = W_2$. Finally, the generalized Jordan chain (u_0, u_1, u_2) is maximal, because the system $\mathbf{L}_3 u_0 + \mathbf{L}_2 u_1 + \mathbf{L}_1 u_2 + \mathbf{L}_0 u_3 = 0$ has no solution, i.e., $W_k = (0)$ for all $k \geq 4$.

Using (2.5) we compute:

$$B_1\big((0,a),(0,b)\big) = \big\langle \mathbf{L}_1 u_0, v_0 \big\rangle = 0, \quad n_1^{\pm} = 0,\ \sigma_1 = 0,$$
$$B_2\big((0,a),(0,b)\big) = \big\langle \mathbf{L}_2 u_0 + \mathbf{L}_1 u_1, v_0 \big\rangle = \langle (a,0), (0,b) \rangle = 0, \quad n_2^{\pm} = 0,\ \sigma_2 = 0,$$
$$B_3\big((0,a),(0,b)\big) = \big\langle \mathbf{L}_3 u_0 + \mathbf{L}_2 u_1 + \mathbf{L}_1 u_2, v_0 \big\rangle = \big\langle (\beta, a), (0,b) \big\rangle = ab,$$
$$n_3^+ = 1,\ n_3^- = 0,\ \sigma_3 = 1.$$

2.2. **Computation of the partial signatures and the spectral flow.**

When V has a smoothly varying basis of eigenvectors of $\mathbf{L}(t)$, then the computation of the spaces W_k and of the bilinear forms B_k can be simplified as explained in the following:

Proposition 2.9. *Let* $\mathbf{L}:[t_0-\varepsilon,t_0+\varepsilon]\to\mathcal{L}_{\mathrm{sym}}(V)$ *be a smooth curve having a unique degeneracy instant at* $t=t_0$. *Assume that the following regularity condition for the eigensystem of* $\mathbf{L}(t)$ *is satisfied: denoting by* $\lambda_1(t),\ldots,\lambda_n(t)$ *the smooth functions of eigenvalues of* $\mathbf{L}(t)$,

(a) *each non constant* λ_i *has a zero of finite order at* $t=t_0$;
(b) *there exist* smooth *functions* $\mathfrak{v}_i:[t_0-\varepsilon,t_0+\varepsilon]\to V$, $i=1,\ldots,n$, *of pairwise orthogonal unit eigenvectors of the* $\lambda_i(t)$*'s.*

Then, the following statements hold:

(1) $W_k=\mathit{span}\Big\{\mathfrak{v}_i(t_0):i\in\{1,\ldots,n\}$ *is such that* $\lambda_i^{(j)}(t_0)=0$ *for all* $j<k\Big\}$;
(2) *if* $v\in W_k$ *is an eigenvector of* $\lambda_i(t_0)$, *where* $\lambda_i(t_0)=\lambda_i'(t_0)=\ldots=\lambda_i^{(k-1)}(t_0)=0$, *then* $B_k(v,w)=\frac{1}{k!}\lambda_i^{(k)}(t_0)\langle v,w\rangle$, *for all* $w\in W_k$;
(3) $n^+\big(\mathbf{L}(t_0+\varepsilon)\big)-n^+\big(\mathbf{L}(t_0)\big)=\sum\limits_{k\ge1}n_k^+(\mathbf{L},t_0)$,
$n^+\big(\mathbf{L}(t_0)\big)-n^+\big(\mathbf{L}(t_0-\varepsilon)\big)=-\sum\limits_{k\ge1}\big(n_{2k-1}^-(\mathbf{L},t_0)+n_{2k}^+(\mathbf{L},t_0)\big)$, *and*
$n^+\big(\mathbf{L}(t_0+\varepsilon)\big)-n^+\big(\mathbf{L}(t_0-\varepsilon)\big)=\sum\limits_{k\ge1}\sigma_{2k-1}(\mathbf{L},t_0)$,
where all the sums appearing in these formulas have at most a finite number of non zero terms. Similarly:
$\mathring{n}^+\big(\mathbf{L}(t_0+\varepsilon)\big)-\mathring{n}^+\big(\mathbf{L}(t_0)\big)=\sum\limits_{k\ge1}n_k^+(\mathbf{L},t_0)-\dim\big(\mathrm{Ker}(\mathbf{L}(t_0))\big)$,
$\mathring{n}^+\big(\mathbf{L}(t_0)\big)-\mathring{n}^+\big(\mathbf{L}(t_0-\varepsilon)\big)=-\sum\limits_{k\ge1}\big(n_{2k-1}^-(\mathbf{L},t_0)+n_{2k}^+(\mathbf{L},t_0)\big)+\dim\big(\mathrm{Ker}(\mathbf{L}(t_0))\big)$.

Proof. To prove (1) argue as follows. If $i\in\{1,\ldots,n\}$ is such that $\lambda_i(t_0)=\lambda_i'(t_0)=\ldots=\lambda_i^{(k-1)}(t_0)=0$, then setting $u_0=\mathfrak{v}_i(t_0)$ and $u(t)=\mathfrak{v}_i(t)$ it follows immediately that u is a root function of order greater than or equal to k at $t=t_0$ with $u(t_0)=u_0$. This proves that $\mathfrak{v}_i(t_0)\in W_k$, i.e., that the span of such $\mathfrak{v}_i(t_0)$'s is contained in W_k. On the other hand, assume that $u_0\in W_k$; let $u(t)$ be a root function with $\mathrm{ord}(u)\ge k$, $u(t_0)=u_0$, and set $u(t)=\sum_i\mu_i(t)\mathfrak{v}_i(t)$. Then:

$$\mathbf{L}(t)u(t)=\sum_i\mu_i(t)\lambda_i(t)\mathfrak{v}_i(t);$$

from the above equality it follows easily that $\mathbf{L}(t)u(t)$ has a zero of order greater than or equal to k at $t=t_0$ if and only if the function $(\lambda_i\mu_i)$ has a zero of order greater than or equal to k at $t=t_0$ for all i. In particular, this implies that $\mu_i(t_0)=0$ unless $\lambda_i(t_0)=\lambda_i'(t_0)=\ldots=\lambda_i^{(k-1)}(t_0)=0$, which proves (1).

The proof of part (2) is immediate using the definition of B_k, taking a proper multiple of $\mathfrak{v}_i$ as a root function for v.

In view of assumption (a) in the hypotheses, part (1) and (2) of the thesis, for each $k\ge1$, the index $n^-(B_k)$ (resp., the coindex $n^+(B_k)$) is given by the number of i's in $\{1,\ldots,n\}$ such that $\lambda_i(t)$ has a zero of order k at $t=t_0$ and whose k-th derivative is negative (resp., positive) at $t=t_0$. The five formulas in part (3) of the thesis follow easily from this observation; note in particular that the third formula is obtained by addition of the first two. □

We observe that the assumptions of Proposition 2.9 are satisfied when $t\mapsto\mathbf{L}(t)$ is a real-analytic map with an isolated degeneracy at $t=t_0$. Namely, in this case both the eigenvalues and the eigenvectors are real-analytic functions (see [34, Chapter 2, § 1]), which implies immediately assumptions (a) and (b) in the statement. Part (1) and (2) of

Proposition 2.9 are used as definition of the spaces W_k and the bilinear forms B_k in [35]. Another situation in which Proposition 2.9 can be applied is when $\dim\big(\mathrm{Ker}(L_0)\big) = 1$, in which case obviously the entire statement and the proof can be simplified; the case of *simple* eigenvalues can be treated in a much easier way by means of the Implicit Function theorem (see for instance [5, 37]), and the occurrence of such circumstance will not be further commented in this paper.

Observe also that part (3) of the thesis of Proposition 2.9 is a generalization of the result of Proposition 2.1.

Corollary 2.10. *Let* $\mathbf{L} : [t_0 - \varepsilon, t_0 + \varepsilon] \to \mathcal{L}_{\mathrm{sym}}(V)$ *be any smooth curve having a unique degeneracy instant at* $t = t_0$*. Then* $W_{k+1} = \mathrm{Ker}(B_k)$ *for all* $k \geq 1$*.*

Proof. If $\mathbf{L}$ is real analytic, the conclusion follows immediately from part (1) and part (2) of Proposition 2.9.

Assume that $\mathbf{L}$ is smooth; recall from Remark 2.5 that, for each $k \geq 1$, the definition of the space W_k and of the bilinear form B_k depends only on $\mathbf{L}_0$ and the first k derivatives $\mathbf{L}_1, \ldots, \mathbf{L}_k$ of $\mathbf{L}(t)$ at $t = t_0$. In particular, the object W_j and B_j, $j = 1, \ldots, k$ do not change if we replace $\mathbf{L}$ by its k-th order Taylor polynomial $\widetilde{\mathbf{L}}(t) = \sum_{j=0}^{k} \mathbf{L}_j (t - t_0)^j$, which is a real-analytic map and the first part of the proof applies. □

Corollary 2.11. *Under the assumptions of Proposition 2.9,*

$$\sum_{k \geq 1} \Big[n_k^+\big(\mathbf{L}; t_0\big) + n_k^-\big(\mathbf{L}; t_0\big) \Big] = \dim\big(\mathrm{Ker}(\mathbf{L}(t_0))\big). \tag{2.7}$$

Proof. By Corollary 2.10, for all $k \geq 1$:

$$n_k^+\big(\mathbf{L}; t_0\big) + n_k^-\big(\mathbf{L}; t_0\big) = \dim(W_k) - \dim(W_{k+1}).$$

The conclusion follows from an easy induction argument, keeping in mind that, under the assumption of Proposition 2.9, $W_{k+1} = \{0\}$ for k sufficiently large. □

Example 2.12. For $\mathbf{L}(t) = \begin{pmatrix} 1 & t \\ t & t^3 \end{pmatrix}$ one computes easily:

$$\lambda_1(t) = \frac{1}{2}\big(1 + t^3 - \sqrt{1 + 4t^2 - 2t^3 + t^6}\big), \quad \lambda_2(t) = \frac{1}{2}\big(1 + t^3 + \sqrt{1 + 4t^2 - 2t^3 + t^6}\big),$$

and:

$$\lambda_1(0) = \lambda_1'(0) = 0,\ \lambda_1''(0) = -2,\ \mathfrak{v}_1(0) = (0, 1),\ \lambda_2(0) = 1.$$

Using Proposition 2.9 one computes immediately:

$$W_1 = \mathrm{Ker}\big(\mathbf{L}(0)\big) = \{0\} \oplus I\!R,\ B_1 = 0,\ n_1^{\pm} = 0,\ \sigma_1 = 0$$
$$W_2 = \{0\} \oplus I\!R,\ B_2\big((0, a), (0, b)\big) = -ab,\ n_2^+ = 0,\ n_2^- = 1,\ \sigma_2 = -1,$$
$$W_k = (0) \text{ and } n_k^{\pm} = \sigma_k = 0 \text{ for all } k \geq 3.$$

Taking now $\mathbf{L}(t) = \begin{pmatrix} 1 & t^2 \\ t^2 & t^3 \end{pmatrix}$, one has:

$$\lambda_1(0) = 0,\ \lambda_1'(0) = 0,\ \lambda_1''(0) = 0,\ \lambda_1'''(0) = 6,\ \lambda_2(0) = 1, \mathfrak{v}_1(0) = (0, 1).$$

Hence:

$$W_1 = \{0\} \oplus I\!R,\ B_1 = 0,\ n_1^{\pm} = 0,\ \sigma_1 = 0,$$
$$W_2 = \{0\} \oplus I\!R,\ B_2 = 0,\ n_2^{\pm} = 0,\ \sigma_2 = 0,$$
$$W_3 = \{0\} \oplus I\!R,\ B_3 = \big((0, a), (0, b)\big) = ab,\ n_3^+ = 1,\ n_3^- = 0,\ \sigma_3 = 1,$$
$$W_k = (0) \text{ and } n_k^{\pm} = \sigma_k = 0 \text{ for all } k \geq 4.$$

Given a curve $[a,b] \ni t \mapsto B(t)$ of symmetric bilinear forms on a finite dimensional vector space, the difference

$$\mathring{n}^+\big(B(b)\big) - \mathring{n}^+\big(B(a)\big) = n^-\big(B(a)\big) - n^-\big(B(b)\big)$$

is called the *spectral flow of the curve B on* $[a,b]$, and it will be denoted by $\mathrm{sf}\big(B,[a,b]\big)$. Such definition will be extended to the infinite dimensional case in next subsection.

2.3. **Spectral flow and relative dimension in Hilbert spaces.** The result of Proposition 2.9 can be extended easily to analytic paths of self-adjoint Fredholm operators, after proper rephrasing of the statement.

Let $\mathcal{H}$ denote a separable real Hilbert space, by $\mathcal{L}^{\mathrm{sa}}$ the space of self-adjoint bounded linear operators on $\mathcal{H}$, and by $\mathcal{F}^{\mathrm{sa}}(\mathcal{H})$ the open subset of $\mathcal{L}^{\mathrm{sa}}(\mathcal{H})$ of Fredholm operators. If $\mathbf{L} : [a,b] \to \mathcal{F}^{\mathrm{sa}}(\mathcal{H})$ is a continuous curve, then an integer number is naturally associated to $\mathbf{L}$, called the *spectral flow* of $\mathbf{L}$ over $[a,b]$. Such number, denoted by:

$$\mathrm{sf}\big(\mathbf{L},[a,b]\big)$$

is roughly speaking the integer given by the number of negative eigenvalues of $\mathbf{L}(a)$ that become nonnegative as the parameter t goes from a to b minus the number of nonnegative eigenvalues of $\mathbf{L}(a)$ that become negative. Observe that for paths of strongly indefinite self-adjoint operators[3] both the index and the coindex functions are infinite; we refer to [43] for a concise introduction to the spectral flow for a continuous path of self-adjoint Fredholm operators, although the reader will find that in the literature it is most frequently treated only the case of paths having invertible endpoints. We observe here that for paths with degenerate endpoints there are several options for the choice of a definition of spectral flow, depending on how one wants to consider the contribution given by the kernel at the endpoints. For the sake of consistency with our definition of Maslov index in the finite dimensional case (Corollary 3.5), we will use the infinite dimensional analogue of the variation of the *extended coindex* $\mathring{n}^+$. Let us recall a formula proven in [28] that gives the spectral flow in terms of relative dimension of closed subspaces[4] of a Hilbert space. If $\mathbf{L}(t)$ is of the form $\mathfrak{J} + K(t)$, where $\mathfrak{J}$ is a self-adjoint *symmetry* of $\mathcal{H}$ (i.e., $\mathfrak{J}^2 = \mathrm{Id}$) and $K(t)$ is a compact self-adjoint operator on $\mathcal{H}$, then

(2.8)

$$\begin{aligned}\mathrm{sf}\big(\mathbf{L},[a,b]\big) &= \\ &\dim_{V^-(\mathfrak{J}+K(b))}\big(V^-(\mathfrak{J}+K(a))\big) = \\ &\dim_{V^+(\mathfrak{J}+K(a))}\big(V^+(\mathfrak{J}+K(b))\big) + \dim\big(\mathrm{Ker}(\mathfrak{J}+K(b))\big) - \dim\big(\mathrm{Ker}(\mathfrak{J}+K(a))\big),\end{aligned}$$

where $V^-(S)$ and $V^+(S)$ denote respectively the *negative* and the *positive eigenspace*[5] of the operator S. The second equality in (2.8) is proven easily using the result of Proposition A.1. The computation of the spectral flow of an arbitrary continuous path of self-adjoint Fredholm operators is then reduced to the above case using the cogredient action of the general linear group of $\mathcal{H}$ (see Subsection 2.4).

Remark 2.13. Observe that if $\mathbf{L}(a)$ (hence $\mathbf{L}(t)$ for all t) is an essentially positive operator, i.e., its essential spectrum is contained in $[0,+\infty[$, then

$$\mathrm{sf}\big(\mathbf{L},[a,b]\big) = n^-\big(\mathbf{L}(a)\big) - n^-\big(\mathbf{L}(b)\big).$$

---

[3] i.e., whose essential spectrum is contained neither in $\mathbb{R}^+$ nor in $\mathbb{R}^-$.

[4] Two closed subspace $V, W \subset \mathcal{H}$ are *commensurable* if $P_V|_W : W \to V$ is a Fredholm operator; if V and W are commensurable the *relative dimension* $\dim_V(W)$ of W with respect to V is defined as:

$$\dim_V(W) = \dim(W \cap V^\perp) - \dim(W^\perp \cap V).$$

Some basic facts of the theory of relative dimension of closed subspaces of Hilbert spaces will be recalled in Appendix A.

[5] The negative (resp., the positive) eigenspace of a self-adjoint operator S can be defined, for instance, using functional calculus as $\chi_{]-\infty,0[}(S)$ (resp., $\chi_{]0,+\infty[}(S)$), where χ_I is the characteristic function.

Likewise, if $\mathbf{L}(a)$ is essentially negative, then

$$\mathrm{sf}\big(\mathbf{L},[a,b]\big)=\mathring{n}^+\big(\mathbf{L}(b)\big)-\mathring{n}^+\big(\mathbf{L}(a)\big).$$

The jumps of the spectral flow of a path occur precisely at the degeneracy instants, and, using a Galerkin approximation, the computation of the jump is reduced to a dimension counting of finite rank projections. Observe also that, by the finite dimensionality of the kernel, one can define the partial signatures $n_k^{\pm}(\mathbf{L},t_0)$, $\mathring{n}^{\pm}{}_k(\mathbf{L},t_0)$ and $\sigma_k(\mathbf{L},t_0)$ at an isolated singularity $t=t_0$ of a path $\mathbf{L}$ in $\mathcal{F}^{sa}(\mathcal{H})$ exactly as in Definition 2.6. Given a real-analytic path $\mathbf{L}:[t_0-\varepsilon,t_0+\varepsilon]\to\mathcal{F}^{sa}(\mathcal{H})$ having a unique degeneracy at $t=t_0$, then, for t sufficiently close to t_0 and for $a>0$ small enough, the intersection of $[-a,a]$ with the spectrum of $\mathbf{L}(t)$ consists of a finite number of eigenvalues $\lambda_1(t),\ldots,\lambda_N(t)$ having bounded multiplicity, that are real-analytic functions of t and that vanish (possibly) only at $t=t_0$. In this situation, the spectral flow of $\mathbf{L}$ over $[t_0-\varepsilon,t_0+\varepsilon]$ is computed, as in the finite dimensional case, by looking at the change of sign of the λ_i's through t_0. Using the same arguments in proof of Proposition 2.9 one obtains the following:

Corollary 2.14. *Let $\mathbf{L}:[t_0-\varepsilon,t_0+\varepsilon]\to\mathcal{F}^{sa}(\mathcal{H})$ be a real-analytic curve of self-adjoint Fredholm operators on the real separable Hilbert space $\mathcal{H}$ having a unique degeneracy instant at $t=t_0$. Then, the spectral flow of $\mathbf{L}$ is computed as:*

$$\begin{aligned}\mathrm{sf}\big(\mathbf{L},[t_0-\varepsilon,t_0]\big)&=-\sum_{k\ge1}\big(n^-_{2k-1}(\mathbf{L},t_0)+n^+_{2k}(\mathbf{L},t_0)\big)+\dim\big(\mathrm{Ker}(\mathbf{L}(t_0))\big),\\ \mathrm{sf}\big(\mathbf{L},[t_0,t_0+\varepsilon]\big)&=\sum_{k\ge1}n^+_k(\mathbf{L},t_0)-\dim\big(\mathrm{Ker}(\mathbf{L}(t_0))\big),\\ \mathrm{sf}\big(\mathbf{L},[t_0-\varepsilon,t_0+\varepsilon]\big)&=\sum_{k\ge1}\sigma_{2k-1}(\mathbf{L},t_0).\quad\square\end{aligned}$$

Explicit computations of the partial signatures of paths of self-adjoint Fredholm operators will be done in Section 3 (see Proposition 3.29 and Corollary 3.30).

We conclude this subsection with an elementary result concerning the spectral flow of paths of self-adjoint Fredholm operators having a common degeneracy instant at the right end point, and at which the partial signatures differ by the sign.

Proposition 2.15. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real separable Hilbert spaces; let be given two real analytic paths $[t_0-\varepsilon,t_0]\ni t\mapsto\mathbf{L}^{(1)}(t)\in\mathcal{F}^{sa}\big(\mathcal{H}_1\big)$ and $[t_0-\varepsilon,t_0]\ni t\mapsto\mathbf{L}^{(2)}(t)\in\mathcal{F}^{sa}\big(\mathcal{H}_2\big)$ having a unique degeneracy instant at $t=t_0$. Assume that there exists an isomorphism $Z:\mathrm{Ker}\big(\mathbf{L}^{(1)}(t_0)\big)\to\mathrm{Ker}\big(\mathbf{L}^{(2)}(t_0)\big)$ such that:*

(a) $Z\big(W_k(\mathbf{L}^{(1)},t_0)\big)=W_k\big(\mathbf{L}^{(2)},t_0\big)$;
(b) *the pull-back* $Z^*\big(B_k(\mathbf{L}^{(2)},t_0)\big)$ *coincides with* $-B_k\big(\mathbf{L}^{(1)},t_0\big)$,

for all $k\ge1$. Then,

$$\mathrm{sf}\big(\mathbf{L}^{(1)},[t_0-\varepsilon,t_0]\big)=-\mathrm{sf}\big(\mathbf{L}^{(2)},[t_0-\varepsilon,t_0]\big)+\dim\big(\mathrm{Ker}(\mathbf{L}^{(2)}(t_0))\big).\tag{2.9}$$

Proof. Using Corollary 2.14, the equalities $\dim\big(\mathrm{Ker}(\mathbf{L}^{(1)}(t_0))\big)=\dim\big(\mathrm{Ker}(\mathbf{L}^{(2)}(t_0))\big)$,

$$n^-\big(B_{2k-1}(\mathbf{L}^{(1)},t_0)\big)=n^+\big(B_{2k-1}(\mathbf{L}^{(2)},t_0)\big),\ n^+\big(B_{2k}(\mathbf{L}^{(1)},t_0)\big)=n^-\big(B_{2k}(\mathbf{L}^{(2)},t_0)\big)$$

and Corollary 2.10, we compute easily:

$$\begin{aligned}
&\mathrm{sf}\big(\mathbf{L}^{(1)},[t_0-\varepsilon,t_0]\big)+\mathrm{sf}\big(\mathbf{L}^{(2)},[t_0-\varepsilon,t_0]\big)=\\
&\quad=2\dim\big(\mathrm{Ker}(\mathbf{L}^{(2)}(t_0)\big)-\sum_{k\ge1}\big[n^+\big(B_k(\mathbf{L}^{(2)},t_0)\big)+n^-\big(B_k(\mathbf{L}^{(2)},t_0)\big)\big]\\
&\quad=2\dim\big(\mathrm{Ker}(\mathbf{L}^{(2)}(t_0)\big)-\sum_{k\ge1}\dim\big(W_k(\mathbf{L}^{(2)},t_0)\big)+\sum_{k\ge1}\dim\big(\mathrm{Ker}(B_k(\mathbf{L}^{(2)},t_0))\big)\\
&\quad=2\dim\big(\mathrm{Ker}(\mathbf{L}^{(2)}(t_0)\big)-\sum_{k\ge1}\dim\big(W_k(\mathbf{L}^{(2)},t_0)\big)+\sum_{k\ge1}\dim\big(W_{k+1}(\mathbf{L}^{(2)},t_0)\big)\\
&\quad=2\dim\big(\mathrm{Ker}(\mathbf{L}^{(2)}(t_0)\big)-\dim\big(W_1(\mathbf{L}^{(2)},t_0)\big)=\dim\big(\mathrm{Ker}(\mathbf{L}^{(2)}(t_0)\big).
\end{aligned}$$

□

Clearly, a similar statement holds for the spectral flow on an interval of the form $[t_0, t_0+\varepsilon]$.

2.4. **Invariance properties of the partial signatures.** For the computation of the spectral flow of a curve of self-adjoint Fredholm operators on a real separable Hilbert space $\mathcal{H}$, it will be useful to consider the *cogredient action* of the general linear group $\mathrm{GL}(\mathcal{H})$ of $\mathcal{H}$ on $\mathcal{F}^{\mathrm{sa}}(\mathcal{H})$. Recall that the cogredient action is the map:

$$\mathrm{GL}(\mathcal{H})\times\mathcal{F}^{\mathrm{sa}}(\mathcal{H})\ni(S,T)\longmapsto S^*TS\in\mathcal{F}^{\mathrm{sa}}(\mathcal{H});$$

for instance, it is proven in [28] that, given any curve $\mathbf{L}:[a,b]\to\mathcal{F}^{\mathrm{sa}}(\mathcal{H})$ of class C^k and any symmetry $\mathfrak{J}$ of $\mathcal{H}$, there exists a curve $M:[a,b]\to\mathrm{GL}(\mathcal{H})$ of class C^k such that $M(t)^*\mathbf{L}(t)M(t)-\mathfrak{J}$ is compact for all t. Recall also that the spectral flow is invariant by the cogredience (see Corollary 2.18 below).

In view of these observations and of the result of Corollary 2.14, we are naturally led to the following:

Proposition 2.16. *Let $\mathbf{L},\widetilde{\mathbf{L}}:[a,b]\to\mathcal{F}^{sa}(\mathcal{H})$ be cogredient smooth curves. If $t_0\in\,]a,b[$ is an isolated degeneracy instant for the two curves, then $\mathbf{L}$ and $\widetilde{\mathbf{L}}$ have the same partial signatures at $t=t_0$.*

Proof. Assume that $\mathbf{L}(t)=M(t)^*\widetilde{\mathbf{L}}(t)M(t)$, where $M:[a,b]\to\mathrm{GL}(\mathcal{H})$ is a smooth curve; denote respectively by W_k, B_k and $\widetilde{W}_k,\widetilde{B}_k$ the objects (vector space space and symmetric bilinear form) defined in Subsection 2.1 relatively to the curves $\mathbf{L}$ and $\widetilde{\mathbf{L}}$ at the instant $t=t_0$. Clearly, $\widetilde{W}_1=\mathrm{Ker}\big(\widetilde{\mathbf{L}}(t_0)\big)=M(t_0)\big(W_1\big)$; we will prove that $\widetilde{W}_k=M(t_0)\big(W_k\big)$ for all k, and that B_k is the pull-back of $\widetilde{B}_k$ by the isomorphism $M(t_0)$.

To this aim, fix $k\ge1$, choose $u_0,v_0\in W_k$ and let $u(t)$ and $v(t)$ be root functions for $\mathbf{L}(t)$ at $t=t_0$ with $\mathrm{ord}(u),\mathrm{ord}(v)\ge k$, $u(t_0)=u_0$, $v(t_0)=v_0$. Set $\tilde u_0=M(t_0)u_0$, $\tilde v_0=M(t_0)v_0$, $\tilde u(t)=M(t)u(t)$ and $\tilde v(t)=M(t)v(t)$; in first place observe that:

$$\widetilde{\mathbf{L}}(t)\tilde u(t)=M(t)^{*-1}\mathbf{L}(t)u(t),$$

which implies that $\tilde u$ is a root function for $\widetilde{\mathbf{L}}(t)$ at $t=t_0$ with $\mathrm{ord}(\tilde u)\ge k$ and $\tilde u(t_0)=\tilde u_0$. This says that $\widetilde{W}_k=M(t_0)\big(W_k\big)$ for all $k\ge1$.

Moreover, using (2.3), we compute easily:

$$\begin{aligned}
B_k(u_0,v_0)&=\frac{1}{k!}\,\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big|_{t=0}\big\langle\mathbf{L}(t)u(t),v(t)\big\rangle=\frac{1}{k!}\,\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big|_{t=0}\big\langle M(t)^*\widetilde{\mathbf{L}}(t)M(t)u(t),v(t)\big\rangle\\
&=\frac{1}{k!}\,\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big|_{t=0}\big\langle\widetilde{\mathbf{L}}(t)\tilde u(t),\tilde v(t)\big\rangle=\widetilde{B}_k(\tilde u_0,\tilde v_0),
\end{aligned}$$

which concludes the proof. □

Remark 2.17. The notion of partial signatures and of spectral flow can be given for curves of Fredholm symmetric bilinear forms[6] on a Hilbert space, by considering the associated curve of Fredholm self-adjoint operators that realize the forms relatively to the inner product of $\mathcal{H}$. If $\mathbf{L}$ and $\widetilde{\mathbf{L}}$ are smooth curves in $\mathcal{F}^{\mathrm{sa}}(\mathcal{H})$ associated to the curve of Fredholm symmetric bilinear forms $\mathbf{B}$ on $\mathcal{H}$ using two different Hilbert space inner products on $\mathcal{H}$, then $\mathbf{L}$ and $\widetilde{\mathbf{L}}$ are cogredient. It follows that the notion of partial signatures at an isolated degeneracy instant of a curve of Fredholm symmetric bilinear forms do not depend on the choice of a Hilbert space inner product on $\mathcal{H}$.

Observe that, using Corollary 2.14, Proposition 2.16, the density of the set of real analytic curves in $C^0\big([a,b],\mathcal{F}^{\mathrm{sa}}(\mathcal{H})\big)$ and the homotopy invariance of the spectral flow, one gets an alternative and simple proof of the following (see [28, Proposition 3.2]):

Corollary 2.18. *The spectral flow is invariant by cogredience.* □

We will also be interested in a different invariance property of the partial signatures (see Proposition 3.4):

Proposition 2.19. *Let $\mathbf{L}:[a,b]\to\mathcal{F}^{sa}(\mathcal{H})$ be a smooth curve having an isolated degeneracy instant at $t=t_0\in\,]a,b[$. Suppose that $h:[a,b]\to\mathrm{GL}(\mathcal{H})$ is a smooth curve of isomorphisms of $\mathcal{H}$ such that:*

(1) *$h(t)^*\mathbf{L}(t)=\mathbf{L}(t)h(t)$ for all $t\in[a,b]$;*
(2) *$h(t_0)$ is the identity on $\mathrm{Ker}\big(\mathbf{L}(t_0)\big)$.*

Then, t_0 is an isolated degeneracy instant of $\widetilde{\mathbf{L}}$, and the curves $\mathbf{L}$ and $\widetilde{\mathbf{L}}:=\mathbf{L}\circ h$ have the same partial signatures at $t=t_0$.

Proof. Condition (1) guarantees that $\widetilde{\mathbf{L}}$ is a smooth curve in $\mathcal{F}^{\mathrm{sa}}(\mathcal{H})$. Condition (2) tells us that t_0 is an isolated degeneracy instant for $\widetilde{\mathbf{L}}$, and that

$$W_1=\mathrm{Ker}\big(\mathbf{L}(t_0)\big)=\mathrm{Ker}\big(\widetilde{\mathbf{L}}(t_0)\big)=\widetilde{W}_1.$$

If $u_0\in W_1$ and $u(t)$ is a root function for $\mathbf{L}$ starting at u_0, then $\tilde{u}(t)=h(t)^{-1}\circ u(t)$ is a root function for $\widetilde{\mathbf{L}}$ starting at $h(t_0)^{-1}u_0=u_0$, and $\mathrm{ord}(u)=\mathrm{ord}(\tilde{u})$ because $\mathbf{L}(t)u(t)=\widetilde{\mathbf{L}}(t)\tilde{u}(t)$. This proves that $W_k=\widetilde{W}_k$ for all k. Finally, using (2.2), the equality $B_k=\widetilde{B}_k$ is easily checked by:

$$B_k(u_0,v_0)=\big\langle\tfrac{\mathrm{d}^k}{\mathrm{d}t^k}\big|_{t=0}\mathbf{L}u,v_0\big\rangle=\big\langle\tfrac{\mathrm{d}^k}{\mathrm{d}t^k}\big|_{t=0}\widetilde{\mathbf{L}}\tilde{u},v_0\big\rangle=\widetilde{B}_k(u_0,v_0).$$

This concludes the proof. □

Arguing as above, we get the following invariance property for the spectral flow:

Corollary 2.20. *Let $\mathbf{L}$, h and $\widetilde{\mathbf{L}}$ be as in Proposition 2.19, assuming that assumption* (2) *holds for all $t_0\in[a,b]$. Then, $\mathrm{sf}(\mathbf{L},[a,b])=\mathrm{sf}(\widetilde{\mathbf{L}},[a,b])$.* □

2.5. **Spectral flow of restrictions.** Let us conclude this chapter with a few simple observations on the computation of the partial signatures of restrictions of Fredholm bilinear forms. Let us assume that $\mathbf{L}:[-\varepsilon,\varepsilon]\to\mathcal{F}^{\mathrm{sa}}(\mathcal{H})$ is a real-analytic map and let us denote by $S(t)=\langle\mathbf{L}(t)\cdot,\cdot\rangle$ the corresponding map of symmetric bilinear forms. Let $V_1,V_2\subset\mathcal{H}$ be closed subspaces such that the restrictions $S_1(t)=S(t)|_{V_1}$ and $S_2(t)=S(t)|_{V_2}$ are Fredholm, and let us assume that $t=0$ is a degeneracy instant for both S_1 and S_2. Obviously, in general it will be $\mathrm{sf}\big(S_1,[-\varepsilon,\varepsilon]\big)\neq\mathrm{sf}\big(S_2,[-\varepsilon,\varepsilon]\big)$, even in the case that $\mathrm{Ker}\big(S_1(0)\big)=\mathrm{Ker}\big(S_2(0)\big)\subset V_1\cap V_2$. On the other hand, if $W_k(S_1,0)=W_k(S_2,0)$ for all $k\geq1$, then obviously it must be $n_k^{\pm}(S_1,0)=n_k^{\pm}(S_2,0)$ for all $k\geq1$, hence $\mathrm{sf}\big(S_1,[-\varepsilon,\varepsilon]\big)=\mathrm{sf}\big(S_2,[-\varepsilon,\varepsilon]\big)$ for $\varepsilon>0$ small enough. This follows easily from the

[6]A bilinear form B on a Hilbert space $\mathcal{H}$ is said to be Fredholm if it is realized by a Fredholm operator on $\mathcal{H}$. Such notion does not indeed depend on the choice of a Hilbert space inner product on $\mathcal{H}$.

observation that the derivatives of S_i coincide with the restriction to V_i of the derivatives of S, $i=1,2$.

Assuming that $S(t)$ has a degeneracy instant at $t=0$ on $\mathcal{H}$, it is an interesting question to ask how to find a "minimal" closed subspace V of $\mathcal{H}$ containing $\mathrm{Ker}\big(S(0)\big)$ and with the property that the restriction of $S(t)$ to V has the same partial signatures at $t=0$. Denote by $\mathfrak{v}_i(t)$, $i=1,\ldots,N=\dim\big(\mathrm{Ker}(S(0))\big)$, a smooth orthonormal family of unit eigenvectors corresponding to the eigenvalues $\lambda_i(t)$ of $S_i(t)$ such that $\lambda_i(0)=0$. Examining the proof of Proposition 2.9 suggests that the desired subspace V can be obtained by considering the span of all the vectors $\mathfrak{v}_i(0)$, together with their derivatives $\mathfrak{v}_i'(0)$, $\mathfrak{v}_i''(0)$, …, $\mathfrak{v}_i^{(r_i)}(0)$, where $i=1,\ldots,N$ and $r_i>0$ is such that $\lambda_i^{(r_i-1)}(0)$ and $\lambda_i^{(r_i)}(0)\neq 0$. Such a space V is finite dimensional, and repeating the argument at each degeneracy instant of a real-analytic path in $\mathcal{F}^{\mathrm{sa}}(\mathcal{H})$ we have obtained the following:

Proposition 2.21. *Given any real-analytic path $\mathbf{L}:[a,b]\to\mathcal{L}^{sa}(\mathcal{H})$ there exists a finite dimensional subspace V of $\mathcal{H}$ such that, denoting by $S(t)$ the bilinear form $\langle\mathbf{L}(t)\cdot,\cdot\rangle$ on $\mathcal{H}$ and by $\widetilde{S}(t)$ its restriction to V, then S and $\widetilde{S}$ have precisely the same degeneracy instants in $[a,b]$, and $n_k^{\pm}(S,t_0)=n_k^{\pm}(\widetilde{S},t_0)$ for all degeneracy instant t_0. In particular,* $\mathrm{sf}\big(S,[a,b]\big)=\mathrm{sf}\big(\widetilde{S},[a,b]\big)$. ☐

Example 2.22. Let us consider a real separable Hilbert space $\mathcal{H}$, a self-adjoint isomorphism $g:\mathcal{H}\to\mathcal{H}$ and a g-symmetric Fredholm operator $T:\mathcal{H}\to\mathcal{H}$, i.e., T is such that $\langle gT\cdot,\cdot\rangle$ is a symmetric bilinear form on $\mathcal{H}$; assume that 0 is (an isolated point) in the spectrum of T. Consider the following real-analytic path of Fredholm self-adjoint operators on $\mathcal{H}$: $\mathbf{L}(t)=gT-tg$, $t\in[-\varepsilon,\varepsilon]$; then, $t=0$ is an isolated degeneracy instant for $\mathbf{L}$. It is not hard to see that T has Fredholm index 0, that there exists $n_0>0$ such that $\mathrm{Ker}(T^n)=\mathrm{Ker}(T^{n_0})$ for all $n\geq n_0$, with $\mathrm{Ker}(T^{n_0})$ a finite dimensional subspace of $\mathcal{H}$ (see Proposition 3.29). If $S(t)$ denotes the symmetric bilinear form $\langle\mathbf{L}(t)\cdot,\cdot\rangle$ on $\mathcal{H}$ and $\widetilde{S}(t)$ its restriction to $\mathrm{Ker}(T^{n_0})$, then $n_k^{\pm}(S,0)=n_k^{\pm}(\widetilde{S},0)$ for all $k\geq 1$. To see this observe that a sequence $(u_0,u_1,\ldots,u_k)$ of vectors in $\mathcal{H}$, with $u_0\in\mathrm{Ker}(T)$, is a generalized Jordan chain for S at $t=0$ iff $Tu_r=u_{r-1}$ for all $r=1,\ldots,k$. This in particular implies $T^ru_r\in\mathrm{Ker}(T)$, i.e., $u_r\in\mathrm{Ker}(T^{r+1})$ for all r, which proves our assertion.

3. On the Maslov index of Lagrangian paths

We will apply the result of Section 2 to the study of the Maslov index of a path in the Lagrangian Grassmannian of a symplectic space. Our first goal is to give a general definition of Maslov index for arbitrary continuous curves which is invariant by homotopies with endpoints varying in a stratum of the Maslov cycle and additive by concatenation. Such number will depend on the choice of a Lagrangian L_0 that is used to define the Maslov cycle, except for the case of closed paths (Corollary 3.20). There are several definitions of Maslov index available in the literature, not always equivalent. Duistermaat's definition of Maslov index in [21] does not depend on the choice of L_0, but it is not additive by concatenation. A semi-integer valued Maslov index, which has the two properties above has been introduced by Robbin and Salamon in [50] by considering first regular curves having transversal intersections with the Maslov cycle and then extending by homotopy invariance. The definition of Maslov index given by de Gosson (see [16, 17, 18, 19, 20]) is based on the notion of *Leray's index* for pairs in the universal covering of the Lagrangian Grassmannian Λ. Booss–Bavnbek and Furutani have given in [6] a functional analytical definition of Maslov index, both in the finite and infinite dimensional case, of a Fredholm Lagrangian paths. This is obtained by defining a one-parameter operator family of operators associated to the path, whose spectrum oscillates on the unit circle around $e^{i\pi}$ in the complex plane, and giving an appropriate algebraic count of the passages through a fixed gauge. The construction is done locally, and then patched together following Phillips' definition of spectral flow in [43]. A simple relation holds between the indices of Robbin/Salamon and Booss–Bavnbek/Furutani for smooth transversal paths (see [6, Section 2]). We simplify greatly the approach of [6] using a van Kampen type theorem for the fundamental groupoid of a topological space (Proposition 3.2), that reduces the proof of the well-definiteness of the Maslov index to a simple local compatibility condition in the space of symmetric bilinear forms (Proposition 3.4). In turn, the latter compatibility condition is proven as an immediate application of our partial signature theory; it should also be emphasized that our construction allows to define index functions taking values in arbitrary groups. We start with an abstract result on how to construct group-valued homomorphisms on the fundamental groupoid of a topological space, and then we will study the case of Lagrangian paths. Several other notions of Maslov index will be discussed in the last part of the Section.

3.1. **An abstract index theory for continuous paths.** Let X be a topological space. We denote by $\pi(X)$ the *fundamental homotopy groupoid* of X, i.e., the set of all fixed-endpoint homotopy classes $[\gamma]$ of continuous paths $\gamma:[0,1]\to X$, endowed with the partial binary operation of concatenation $\diamond$, i.e., $[\gamma]\cdot[\mu]=[\gamma\diamond\mu]$, where $\gamma\diamond\mu$ is defined by $\gamma\diamond\mu(t)=\gamma(2t)$, for $t\in\left[0,\frac12\right]$ and $\gamma\diamond\mu(t)=\mu(2t-1)$, for $t\in\left[\frac12,1\right]$; obviously the concatenation $[\gamma]\cdot[\mu]$ is defined whenever $\gamma(1)=\mu(0)$. If $\gamma:[a,b]\to X$ is a continuous path defined on an arbitrary compact interval $[a,b]$, we will identify γ with the curve $[0,1]\ni t\mapsto\gamma\big(t(b-a)+a\big)$, and we will write $[\gamma]\in\pi(X)$. Given a group G then by a *G-valued homomorphism* ϕ on $\pi(X)$ we mean a map $\phi:\pi(X)\to G$ such that $\phi\big([\gamma]\cdot[\mu]\big)=\phi\big([\gamma]\big)\phi\big([\mu]\big)$, for all $[\gamma],[\mu]\in\pi(X)$ with $\gamma(1)=\mu(0)$.

Example 3.1. Given an arbitrary map $\tau:X\to G$ then one can define a G-valued homomorphism on $\pi(X)$ by setting $\phi\big([\gamma]\big)=\tau\big(\gamma(0)\big)^{-1}\tau\big(\gamma(1)\big)$, for all $[\gamma]\in\pi(X)$.

We will now discuss the existence (and the uniqueness) of a group valued homomorphism defined globally on the fundamental groupoid of a topological space, once that the values of such homomorphism are given on “short” curves. The following proposition is a van Kampen type result for the fundamental groupoid of a topological space; the proof presented takes inspiration on the construction of the spectral flow given in [43, Proposition 3]. Recall that the classical statement of van Kampen’s theorem in elementary algebraic topology gives the fundamental group $\pi_1(X)$ of a topological space X as the quotient of a free product of fundamentals groups $\pi_1(U_i)$, where $\{U_i\}_{i\in I}$ is an open cover of X. Such quotient is characterized by a universal property concerning the existence and uniqueness of an extension $\phi:\pi_1(X)\to G$ of a family of group homomorphisms $\phi_i:\pi_1(U_i)\to G$, provided that some compatibility assumption on the ϕ_i is satisfied. This is precisely the spirit of our fundamental groupoid version of this result:

Proposition 3.2. *Let X be a topological space, G a group and $X=\bigcup_{i\in I}U_i$ an open cover of X. Assume that it is given a G-valued homomorphism ϕ_i on $\pi(U_i)$ for each $i\in I$ such that $\phi_i\big([\gamma]\big)=\phi_j\big([\gamma]\big)$ for all $[\gamma]\in\pi(U_i\cap U_j)$ and all $i,j\in I$. Then, there exists a unique G-valued homomorphism ϕ on $\pi(X)$ such that $\phi\big([\gamma]\big)=\phi_i\big([\gamma]\big)$ for all $\gamma\in\pi(U_i)$ and all $i\in I$.*

Proof. Let $\gamma:[0,1]\to X$ be a continuous curve; by compactness there exists a partition $0=t_0<t_1<\ldots<t_{N+1}=1$ of the interval $[0,1]$ and a map $r:\{0,1,\ldots,N\}\to I$ such that $\gamma\big([t_k,t_{k+1}]\big)\subset U_{r(k)}$ for all $k=0,\ldots,N$ (the choice of both the partition and the map r is highly non unique). Define:

$$\widetilde{\phi}(\gamma):=\phi_{r(0)}\big([\gamma|_{[t_0,t_1]}]\big)\cdot\phi_{r(1)}\big([\gamma|_{[t_1,t_2]}]\big)\cdot\ldots\cdot\phi_{r(N)}\big([\gamma|_{[t_N,t_{N+1}]}]\big);\tag{3.1}$$

in order to have a well-defined map on $\pi(X)$ we need to show that the value $\widetilde{\phi}(\gamma)$ does not depend on the choice of the partition $(t_k)_k$ and of the function r as above. In first place, one observes that passing to a finer partition does not change the value of $\widetilde{\phi}$, by the concatenation multiplicativity of the maps ϕ_i. Secondly, if for some k it happens that

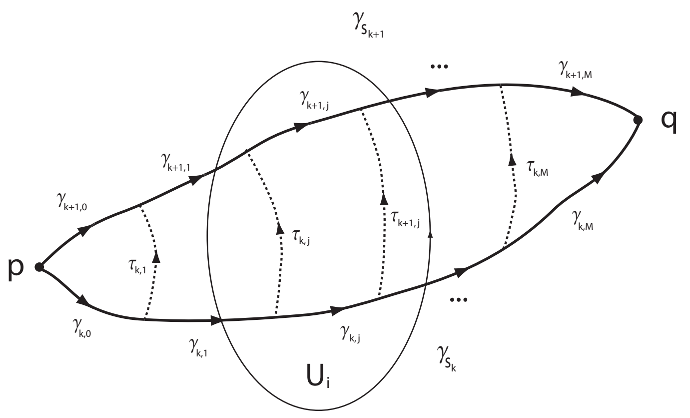

FIGURE 1. The curves defined in the proof of Proposition 3.2

$\gamma\big([t_k,t_{k+1}]\big)\subset U_i\cap U_j$, then $\phi_i\big([\gamma|_{[t_k,t_{k+1}]}]\big)=\phi_j\big([\gamma|_{[t_k,t_{k+1}]}]\big)$, so that the value of $\widetilde{\phi}$ does not depend on the choice of the function r. Finally, given two different partitions of $[0,1]$, one can find a third partition which is finer than both of them, and by what has been observed, this implies that $\widetilde{\phi}$ is well defined.

Uniqueness is also clear, since the concatenation multiplicativity implies that (3.1) must hold.

As to the homotopy invariance, observe first that the multiplicativity by concatenation and the homotopy invariance of the ϕ_i's imply that:

- if γ is a constant curve, then $\widetilde{\phi}(\gamma)=0$;
- if γ^- denotes the backwards reparameterization of γ, then $\widetilde{\phi}(\gamma^-)=\widetilde{\phi}(\gamma)^{-1}$.

Assume that p, q are points of X and $H:[0,1]\times[0,1]\to X$ is a continuous map with $H(s,0)=p$ and $H(s,1)=q$ for all $s\in[0,1]$. Set $\gamma_0=H(0,\cdot)$ and $\gamma_1=H(1,\cdot)$; we want to prove that $\widetilde{\phi}(\gamma_0)=\widetilde{\phi}(\gamma_1)$. Choose partitions $0=s_0<s_1<\ldots<s_{N+1}=1$ and $0=t_0<t_1<\ldots<t_{M+1}=1$ of the interval $[0,1]$ such that, for all $k=0,\ldots,N$ and all $j=0,\ldots,M$, the image $H\big([s_k,s_{k+1}]\times[t_j,t_{j+1}]\big)$ is contained in some U_i. For all $k=0,\ldots,N+1$, set $\gamma_{s_k}=H(s_k,\cdot)$; we will prove that $\widetilde{\phi}(\gamma_{s_k})=\widetilde{\phi}\big(\gamma_{s_{k+1}}\big)$ for all k, which will conclude the argument. Define the following curves (see Figure 1):

$$\begin{aligned}\tau_{k,j}&=H(\cdot,t_j)\Big|_{[s_k,s_{k+1}]},\\ \gamma_{k,j}&=\gamma_{s_k}\Big|_{[t_j,t_{j+1}]}, \qquad\qquad k=0,\ldots,N,\quad j=0,\ldots,M,\end{aligned}$$

and

$$\widetilde{\gamma}_{s_k}=\gamma_{k,0}\diamond\tau_{k,1}\diamond\tau_{k,1}^-\diamond\gamma_{k,1}\diamond\tau_{k,2}\diamond\tau_{k,2}^-\diamond\ldots\diamond\tau_{k,M}\diamond\tau_{k,M}^-\diamond\gamma_{k,M}.$$

Clearly, $\widetilde{\phi}(\gamma_{s_k})=\widetilde{\phi}(\widetilde{\gamma}_{s_k})$, because $\widetilde{\phi}(\tau_{k,j}\diamond\tau_{k,j}^-)=0$ for all j. Moreover, $\gamma_{k,0}\diamond\tau_{k,1}$ is homotopic with fixed endpoints to $\gamma_{k+1,0}$ in some U_i, $\tau_{k,1}^-\diamond\gamma_{k,1}\diamond\tau_{k,2}$ is homotopic with fixed endpoints to $\gamma_{k+1,1}$ in some other U_i, etc., and $\tau_{k,M}^-\diamond\gamma_{k,M}$ is homotopic with fixed endpoints to $\gamma_{k+1,M}$ in still some other U_i. Since the ϕ_i's are invariant by fixed endpoint homotopies, the multiplicativity by concatenation once more implies that $\widetilde{\phi}(\widetilde{\gamma}_{s_k})=\widetilde{\phi}(\gamma_{s_{k+1}})$. The desired map ϕ is thus obtained by setting $\phi\big([\gamma]\big)=\widetilde{\phi}(\gamma)$. □

Example 3.3. Assume that $X=\bigcup_{i\in I}U_i$ is an open cover of X and that for each $i\in I$ one is given a map $\tau_i:U_i\to G$ such that the map $U_i\cap U_j\ni g\mapsto\tau_i(g)^{-1}\tau_j(g)\in G$ is constant on each arc-connected component of $U_i\cap U_j$, for all $i,j\in I$. By Example 3.1

and Proposition 3.2, there exists a unique G-valued homomorphism ϕ on $\pi(X)$ such that $\phi([\gamma]) = \tau_i\big(\gamma(0)\big)^{-1}\tau_i\big(\gamma(1)\big)$ for all $\gamma \in \pi(U_i)$ and all $i \in I$.

3.2. **Charts on the Lagrangian Grassmannian of a symplectic space.** Throughout this subsection, (V,ω) will denote a finite dimensional symplectic space, i.e., V is a real vector space, and ω is a nondegenerate skew-symmetric bilinear form on V; set $\dim(V) = 2n$. The *symplectic group* $\mathrm{Sp}(V,\omega)$ is the closed subgroup of $\mathrm{GL}(V)$ consisting of those linear maps on V that preserve ω.

Recall that a *Lagrangian subspace* of V is an n-dimensional subspace $L \subset V$ on which ω vanishes. The set of all Lagrangian subspaces of V, denoted by Λ, has the structure of a compact, real-analytic submanifold of the Grassmannian of all n-dimensional subspaces of V. The dimension of Λ equals $\frac{1}{2}n(n+1)$, and a real-analytic atlas on Λ is given as follows.

For all $L \in \Lambda$ and $k \in \{0,\ldots,n\}$, set:

$$\Lambda_k(L) = \big\{L' \in \Lambda : \dim(L \cap L') = k\big\};$$

in particular, $\Lambda_0(L)$ is the set of all Lagrangian subspaces that are transversal to L, and it is a dense open subset of Λ. Given a pair $L_0, L_1 \in \Lambda$ of complementary Lagrangians, i.e., $L_0 \cap L_1 = \{0\}$, then one defines a map:

$$\varphi_{L_0,L_1} : \Lambda_0(L_1) \longrightarrow \mathrm{B}_{\mathrm{sym}}(L_0)$$

as follows. Any Lagrangian $L \in \Lambda_0(L_1)$ is the graph of a unique linear map $T : L_0 \to L_1$; then, φ_{L_0,L_1} is defined to be the restriction of the bilinear map $\omega(T\cdot,\cdot)$ to $L_0 \times L_0$. It is easy to see that, due to the fact that L is Lagrangian, such bilinear map is symmetric. Observe that:

$$\mathrm{Ker}\big(\varphi_{L_0,L_1}(L)\big) = L \cap L_0, \quad \forall\, L \in \Lambda_0(L_1). \tag{3.2}$$

The collection of all φ_{L_0,L_1}, when (L_0,L_1) runs in the set of all pairs of complementary Lagrangians, is a real-analytic atlas on Λ. If L_1 and L_1' are complementary to a given L_0, then the transition function:

$$\varphi_{L_0,L_1'} \circ \varphi_{L_0,L_1}^{-1} : \varphi_{L_0,L_1}\big(\Lambda_0(L_1')\big) \subset \mathrm{B}_{\mathrm{sym}}(L_0) \longrightarrow \mathrm{B}_{\mathrm{sym}}(L_0)$$

is given by:

$$\varphi_{L_0,L_1'} \circ \varphi_{L_0,L_1}^{-1}(B) = B \circ \big(\mathrm{Id} + (\pi_0'|_{L_1}) \circ \rho_{L_0,L_1}^{-1} \circ B\big)^{-1}, \tag{3.3}$$

where $\pi_0' : L_0 \oplus L_1' \to L_0$ is the projection onto the first summand, and $\rho_{L_0,L_1} : L_1 \to L_0^*$ is the map $v \mapsto \omega(v,\cdot)|_{L_0}$, which is an isomorphism. Observe that in formula (3.3) the bilinear form B is seen as a map $L_0 \to L_0^*$. The map $\mathrm{Id} + (\pi_0'|_{L_1}) \circ \rho_{L_0,L_1}^{-1} \circ B$ is an automorphism of L_0 whose inverse, denoted by $\mathfrak{h}$, satisfies $B \circ \mathfrak{h} = \mathfrak{h}^* \circ B$; observe that $\mathfrak{h}$ is the identity on $\mathrm{Ker}(B)$.

Recall that every symplectic space is isomorphic $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ endowed with the *standard symplectic form*:

$$\omega_0\big((v,\alpha),(w,\beta)\big) = \beta(v) - \alpha(w).$$

More generally, given Lagrangians $L_0, L_1 \subset V$ with $L_0 \cap L_1 = \{0\}$, then there exists a symplectic isomorphism $\phi : V \to \mathbb{R}^n \oplus \mathbb{R}^{n*}$ (i.e., the pull-back $\phi_*\omega_0$ coincides with ω) such that $\phi(L_0) = \{0\} \oplus \mathbb{R}^{n*}$ and $\phi(L_1) = \mathbb{R}^n \oplus \{0\}$.

3.3. **Maslov index of a symplectic path.** Let now L_0 be a fixed Lagrangian in Λ.

The L_0-Maslov index of a continuous curve γ with endpoints in $\Lambda_0(L_0)$ can be defined in terms of the first relative homology group of the pair $\big(\Lambda, \Lambda_0(L_0)\big)$ (see [38, 46]). We will give an alternative and more general definition of Maslov index in the case of continuous curves in Λ with arbitrary endpoints.

Proposition 3.4. *Given Lagrangians L_0, L_1, L_1' in V with $L_1, L_1' \in \Lambda_0(L_0)$, then the map:*

$$\Lambda_0(L_1) \cap \Lambda_0(L_1') \ni L \longmapsto \mathring{n}^+\big(\varphi_{L_0,L_1'}(L)\big) - \mathring{n}^+\big(\varphi_{L_0,L_1}(L)\big) \in \mathbb{Z}$$

is constant on each connected component of the open set $\Lambda_0(L_1) \cap \Lambda_0(L_1') \subset \Lambda$.

Proof. In first place we observe that $\mathrm{Ker}\big(\varphi_{L_0,L_1'}(L)\big) = L \cap L_0 = \mathrm{Ker}\big(\varphi_{L_0,L_1}(L)\big)$, hence

$$\mathring{n}^+\big(\varphi_{L_0,L_1'}(L)\big) - \mathring{n}^+\big(\varphi_{L_0,L_1}(L)\big) = n^+\big(\varphi_{L_0,L_1'}(L)\big) - n^+\big(\varphi_{L_0,L_1}(L)\big).$$

As we have observed, it is not restrictive to assume that $V = I\!R^n \oplus I\!R^{n*}$ is the standard symplectic space, $L_0 = \{0\} \oplus I\!R^{n*}$ and $L_1 = I\!R^n \oplus \{0\}$. Then L_1' is of the form:

$$L_1' = \big\{(v, Zv) : v \in I\!R^n\big\},$$

where $Z : I\!R^n \to I\!R^{n*}$ is some symmetric linear map. Given $L \in \Lambda_0(L_1)$, we set $B = \varphi_{L_0,L_1}(L) \in \mathrm{B_{sym}}(I\!R^{n*})$; it is easily checked that $L \in \Lambda_0(L_1')$ if and only if $\mathrm{Id} - ZB$ is invertible and that $\widetilde{B} = \varphi_{L_0,L_1'}(L)$ is given by:

$$\widetilde{B} = B(\mathrm{Id} - ZB)^{-1}. \tag{3.4}$$

The proof will be completed once we show that $n^+(\widetilde{B}) - n^+(B)$ is constant on each connected component of the open set $\mathcal{B} \subset \mathrm{B_{sym}}(I\!R^{n*})$ consisting of those B with $\mathrm{Id} - ZB$ invertible, where $\widetilde{B}$ is defined by (3.4). Given B_0 and B_1 in some connected component of $\mathcal{B}$ then we can find a real-analytic curve $B : [0,1] \to \mathcal{B}$, with $B(0) = B_0$, $B(1) = B_1$. Setting $\widetilde{B}(t) = B(t) \circ h(t)$, with $h(t) := \big(\mathrm{Id} - ZB(t)\big)^{-1}$, then $\widetilde{B} : [0,1] \to \mathrm{B_{sym}}(I\!R^{n*})$ is a real-analytic curve of symmetric bilinear forms; observe that $h(t)$ is the identity on $\mathrm{Ker}(B(t))$ for all $t \in [0,1]$. So, by Proposition 2.19, $n_k^{\pm}(B, t_0) = n_k^{\pm}(\widetilde{B}, t_0)$, $\sigma_k(B, t_0) = \sigma_k(\widetilde{B}, t_0)$ for all $t_0 \in [0,1]$ and for all $k \in I\!N$. From Proposition 2.9, part (3) it follows that:

$$n^+\big(B(1)\big) - n^+\big(B(0)\big) = n^+\big(\widetilde{B}(1)\big) - n^+\big(\widetilde{B}(0)\big).$$

Hence $n^+\big(\widetilde{B}(1)\big) - n^+\big(B(1)\big) = n^+\big(\widetilde{B}(0)\big) - n^+\big(B(0)\big)$, which concludes the proof. □

From Proposition 3.2, Example 3.3 and Proposition 3.4 we obtain immediately:

Corollary 3.5. *For all $L_0 \in \Lambda$, there exists a unique $\mathbb{Z}$-valued groupoid homomorphism μ_{L_0} on $\pi(\Lambda)$ such that:*

$$\mu_{L_0}([\gamma]) = \mathring{n}^+\big(\varphi_{L_0,L_1}(\gamma(1))\big) - \mathring{n}^+\big(\varphi_{L_0,L_1}(\gamma(0))\big) \tag{3.5}$$

for all continuous curve $\gamma : [0,1] \to \Lambda_0(L_1)$ and for all $L_1 \in \Lambda_0(L_0)$. □

Remark 3.6. From what has been observed in the proof of Proposition 3.4, it is clear that the result of Corollary 3.5 holds also if one replaces the extended coindex $\mathring{n}^+$ with the coindex n^+ in (3.5). When the endpoints of γ are transversal to L_0, then such distinction does not affect the value of μ_{L_0}, while in the case of degenerate endpoints the function μ_{L_0} obtained using the coindex would give a different value. The choice of one or another definition is merely a matter of personal taste; on the other hand, such choice should be made consistently with the choice of a notion of spectral flow in the case of degenerate endpoints. In our case, the Maslov index defined in Corollary 3.5 is such that, when applied to the case of Riemannian geodesics (see Subsection 4.2), it gives the total number of conjugate points along the geodesic, including that possibly occurring at the final instant.

Definition 3.7. Given any continuous curve γ in Λ, the integer $\mu_{L_0}(\gamma) := \mu_{L_0}([\gamma])$ will be called the L_0-*Maslov index* (or simply, the *Maslov index* when the choice of L_0 is clear from the context) of the curve γ.

Observe that the homotopy invariance implies, in particular, that $\mu_{L_0}(\gamma)$ is independent on the parameterization of γ. We collect in the following statement the main properties of the integer valued map μ_{L_0}, whose proof follows almost immediately from the definition:

Lemma 3.8. *The Maslov index μ_{L_0} satisfies the following properties:*

(1) *replacing the symplectic form ω by $-\omega$ produces a change in the sign of μ_{L_0};*

(2) *given a continuos curve $\gamma : [a,b] \to \Lambda$, if $\dim\big(\gamma(t) \cap L_0\big)$ is constant on $[a,b]$, then $\mu_{L_0}(\gamma) = 0$;*

(3) *if $H : [a,b] \times [c,d] \to \Lambda$ is a continuous map, then:*

$$\mu_{L_0}\big(t \mapsto H(t,c)\big) + \mu_{L_0}\big(s \mapsto H(b,s)\big) = \mu_{L_0}\big(t \mapsto H(t,d)\big) + \mu_{L_0}\big(s \mapsto H(a,s)\big);$$

(4) (symplectic invariance) *if $\phi : (V,\omega) \to (V',\omega')$ is a symplectomorphism and $\gamma : [a,b] \to \Lambda(V,\omega)$ is continuous, then:*

$$\mu_{L_0}(\gamma) = \mu_{\phi(L_0)}\big(\phi \circ \gamma\big);$$

(5) (symplectic additivity) *if $\gamma : [a,b] \to (V,\omega)$ and $\widetilde{\gamma} : [a,b] \to (\widetilde{V},\widetilde{\omega})$ are continuous, $L_0 \in \Lambda(V,\omega)$ and $\widetilde{L}_0 \in \Lambda(\widetilde{V},\widetilde{\omega})$, then*

$$\mu_{L_0 \oplus \widetilde{L}_0}\big(\gamma \oplus \widetilde{\gamma}\big) = \mu_{L_0}(\gamma) + \mu_{\widetilde{L}_0}(\widetilde{\gamma}). \quad \square$$

It will be useful to single out the following additional properties of the Maslov index:

Corollary 3.9. *The Maslov index μ_{L_0} satisfies also the following:*

(a) *If $\gamma_1, \gamma_2 : [a,b] \to \Lambda$ are continuous curves that are homotopic by a homotopy with free endpoints in some $\Lambda_k(L_0)$, i.e., there exists a continuous map $H : [0,1] \times [a,b] \to \Lambda$ such that $H(0,\cdot) = \gamma_1$, $H(1,\cdot) = \gamma_2$, and with $\dim\big(H(s,a) \cap L_0\big)$ and $\dim\big(H(s,b) \cap L_0\big)$ constant on $[0,1]$, then $\mu_{L_0}(\gamma_1) = \mu_{L_0}(\gamma_2)$;*

(b) *if γ_1 and γ_2 are continuous loops in Λ that are* freely *homotopic, then $\mu_{L_0}(\gamma_1) = \mu_{L_0}(\gamma_2)$;*

(c) *if $[a,b] \ni t \mapsto \phi(t)$ is a continuous curve in the symplectic group $\mathrm{Sp}(V,\omega)$ such that $\phi(t)\big(L_0\big) = L_0$ for all t, and $\gamma : [a,b] \to \Lambda$ is continuous, then:*

$$\mu_{L_0}(\gamma) = \mu_{L_0}\big(t \mapsto \phi(t)(\gamma(t))\big).$$

Proof. (a) and (b) follow easily from part (3) of Lemma 3.8. Observe indeed that, if H denotes the given homotopy between γ_1 and γ_2, then the curves $s \mapsto H(a,s)$ and $s \mapsto H(b,s)$ coincide in the case (b) of freely homotopic loops; they are curves in $\Lambda_k(L_0)$ for some fixed k in the case (a), so their Maslov index vanishes by Lemma 3.8 part (2).

To prove (c), observe that the curve $t \mapsto \phi(t)(\gamma(t))$ is homotopic to the curve $t \mapsto \phi(a)(\gamma(t))$ by the homotopy:

$$H(s,t) = \phi\big((1-s)t\big)\big(\gamma(t)\big).$$

Since $\phi(t)(L_0) = L_0$ for all t, by part (4) of Lemma 3.8 it follows that

$$\mu_{L_0}\big(t \mapsto \phi(a)(\gamma(t))\big) = \mu_{L_0}\big(t \mapsto \phi(t)(\gamma(t))\big).$$

Moreover, the above homotopy is by curves with endpoints varying in some fixed $\Lambda_k(L_0)$, and the conclusion follows from part (a). $\square$

The above definition of Maslov index generalizes that in [38, 46], where μ_{L_0} had been defined in terms of the first relative homology group $H_1(\Lambda, \Lambda_0(L_0))$. Some properties of μ_{L_0} that were discussed in [38, 46] using homological techniques and functoriality properties will have to be reproven in this more general context; to this aim we will now discuss a method for computing $\mu_{L_0}(\gamma)$ when γ is a real-analytic curve.

3.4. **Partial signatures and Maslov index.** We have seen that the computation of the Maslov index of a curve in Λ is reduced using local charts to the study of the jumps of the extended coindex of curves of symmetric bilinear forms, and this is where the result of Proposition 2.9 comes about.

Assume that $\gamma : [a,b] \to \Lambda$ is a smooth curve. From (3.2) it is clear that the jumps of the coindex of $\varphi_{L_0,L_1}(\gamma)$ occur precisely at those instants $t \in \,]a,b[$ when $\gamma(t)$ intersects the set:

$$\Sigma_{L_0} := \bigcup_{k\geq 1} \Lambda_k(L_0).$$

We will call Σ_{L_0} the *Maslov cycle with vertex at* L_0. We remark here that each $\Lambda_k(L_0)$ is an embedded submanifold of Λ, but that Σ_{L_0} is *not* a submanifold of Λ. Assume that $t_0 \in \,]a,b[$ is an isolated intersection of γ with Σ_{L_0} and choose any Lagrangian $L_1 \in \Lambda_0(L_0)$ which is transversal also to $\gamma(t_0)$. By continuity, L_1 is transversal to $\gamma(t)$ for t near t_0, and we can define a smooth curve $[t_0 - \varepsilon, t_0 + \varepsilon] \ni t \mapsto \varphi_{L_0,L_1}\big(\gamma(t)\big) \in \mathrm{B}_{\mathrm{sym}}(L_0)$, for $\varepsilon > 0$ small enough. As we have observed, such a curve has an isolated degeneracy instant at $t = t_0$, and we can define the sequences:

$$n_k^-(\gamma, t_0; L_0, L_1), \mathring{n}_k^-(\gamma, t_0; L_0, L_1),\; n_k^+(\gamma, t_0; L_0, L_1),$$
$$\mathring{n}_k^+(\gamma, t_0; L_0, L_1),\; \sigma_k(\gamma, t_0; L_0, L_1), \quad k \geq 1,$$

respectively as the partial (extended) indexes, partial (extended) coindexes and partial signatures of the curve of symmetric bilinear forms $\varphi_{L_0,L_1} \circ \gamma$ at $t = t_0$.

Lemma 3.10. *The integers* $n_k^\pm(\gamma, t_0; L_0, L_1)$*,* $\mathring{n}_k^\pm(\gamma, t_0; L_0, L_1)$ *and* $\sigma_k(\gamma, t_0; L_0, L_1)$ *do not depend on the choice of the Lagrangian* L_1*.*

Proof. Choose two Lagrangian spaces $L_1, L_1' \in \Lambda_0\big(\gamma(t_0)\big) \cap \Lambda_0(L_0)$ and set $L(t) = \varphi_{L_0,L_1}\big(\gamma(t)\big)$, $\widetilde{L}(t) = \varphi_{L_0,L_1'}\big(\gamma(t)\big)$ for t sufficiently near t_0.

Using formula (3.3) we get that $\widetilde{L}(t) = L(t)h(t)$, where $h(t)$ is a curve of automorphisms of L_0 such that $h(t_0)$ is the identity on $\mathrm{Ker}\big(L(t_0)\big) = \gamma(t_0) \cap L_0$. The conclusion follows immediately from Proposition 2.19. □

We are now entitled to talk about the L_0-partial signatures $n_k^\pm(\gamma, t_0; L_0)$, $\mathring{n}^\pm{}_k(\gamma, t_0; L_0)$ and $\sigma_k(\gamma, t_0; L_0)$ at an isolated intersection of γ with Σ_{L_0}, without specifying the choice of a Lagrangian L_1.

Proposition 3.11. *Let* $\gamma : [a,b] \to \Lambda$ *be a real-analytic curve which is not entirely contained in the Maslov cycle* Σ_{L_0}*. Then, the* L_0*-Maslov index of* γ *is given by:*

(3.6)

$$\mu_{L_0}(\gamma) = \sum_{\substack{t_0 \in \gamma^{-1}(\Sigma_{L_0}) \\ t_0 \in]a,b[}} \left[\sum_{k\geq 1} \sigma_{2k-1}\big(\gamma, t_0; L_0\big) \right] + \dim\big(\gamma(b) \cap L_0\big) - \dim\big(\gamma(a) \cap L_0\big)$$
$$+ \sum_{k\geq 1} n_k^+(\gamma, a; L_0) - \sum_{k\geq 1} \big(n_{2k-1}^-(\gamma, b; L_0) + n_{2k}^+(\gamma, b; L_0)\big),$$

where all the sums on the right hand side of (3.6) *have a finite number of non zero terms.*

Proof. In first place we observe that γ has at most a finite number of intersections with Σ_{L_0}; namely, in local coordinates φ_{L_0,L_1}, such intersections correspond to zeroes of the real-analytic function $t \mapsto \det\big(\varphi_{L_0,L_1}(\gamma(t)\big)$. Such function is not identically zero because γ is not entirely contained in the Maslov cycle.

From the definition, given such a curve γ, its Maslov index $\mu_{L_0}(\gamma)$ is given by the sum of the jumps of the coindex function of $\varphi_{L_0,L_1} \circ \gamma$ at the instants in $\gamma^{-1}\big(\Sigma_{L_0}\big)$, and the conclusion follows readily from part (3) of Proposition 2.9, observing that

$$\mathrm{Ker}\big(\varphi_{L_0,L_1}\circ\gamma(t)\big)=\gamma(t)\cap L_0. \qquad\Box$$

We have proven in Corollary 3.9 that, given a continuous curve of Lagrangians $\gamma:[a,b]\to\Lambda$ and a continuous curve of symplectomorphisms $\phi:[a,b]\to\mathrm{Sp}(V,\omega)$ such that $\phi(t)(L_0)=L_0$ for all t, the L_0-Maslov index of the curves $t\mapsto\gamma(t)$ and $t\mapsto\phi(t)(\gamma(t))$ coincide. We will conclude this subsection with the proof that, in the smooth case, also all the partial signatures of the curves $t\mapsto\gamma(t)$ and $t\mapsto\phi(t)(\gamma(t))$ at each intersection with the Maslov cycle coincide:

Lemma 3.12. *Let $\gamma:[a,b]\to\Lambda$ a continuous curve and $t_0\in[a,b]$ such that $\gamma(t_0)\in\Sigma_{L_0}$, and let $\phi:[a,b]\to\mathrm{Sp}(V,\omega)$ be a smooth curve of symplectomorphisms such that $\phi(t)(L_0)=L_0$ for all t. Denote by $\widetilde\gamma$ the curve in Λ given by $\widetilde\gamma(t)=\phi(t)\big(\gamma(t)\big)$; then $\widetilde\gamma(t_0)\in\Sigma_{L_0}$ and all the partial signatures of $\widetilde\gamma$ at t_0 coincide with those of γ.*

Proof. Since $\phi(t_0)$ is an isomorphism and $\phi(t_0)(L_0)=L_0$, then $\phi(t_0)\big(\Lambda_k(L_0)\big)=\Lambda_k(L_0)$ for all k, hence $\widetilde\gamma(t_0)\in\Sigma_{L_0}$. To prove the invariance of the partial signatures, let us choose a Lagrangian $L_1\in\Lambda_0(L_0)\cap\Lambda_0\big(\gamma(t_0)\big)$; observe that then $L_1'=\phi(t_0)(L_1)\in\Lambda_0(L_0)\cap\Lambda_0\big(\phi(t_0)(\gamma(t_0))\big)$. The partial signatures of γ at t_0 are computed using the curve of symmetric bilinear forms $B(t)=\varphi_{L_0,L_1}\circ\gamma(t)$ on L_0, while the partial signatures of $\widetilde\gamma$ at t_0 are computed using the curve $\widetilde B(t)=\varphi_{L_0,L_1'}\circ\widetilde\gamma(t)$, for t near t_0.

Set $C(t)=\varphi_{L_0,\phi(t)(L_1)}\big(\phi(t)(\gamma(t))\big)\in\mathrm{B}_{\mathrm{sym}}(L_0)$; an immediate calculation using the very definition of the charts φ_{L_0,L_1} shows that $C(t)$ is the pull-back of the bilinear form $\varphi_{L_0,L_1}\circ\gamma(t)$ by the isomorphism $\phi(t)^{-1}:L_0\to L_0$. By Proposition 2.16, the curves $B(t)$ and $C(t)$ have the same partial signatures at the degeneracy instant t_0.

Recalling formula (3.3), we compute:

$$\widetilde B(t)=\varphi_{L_0,L_1'}\circ\widetilde\gamma(t)=\varphi_{L_0,L_1'}\circ\varphi^{-1}_{L_0,\phi(t)(L_1)}\big(C(t)\big)=C(t)\big(\mathrm{Id}+Z(t)C(t)\big)^{-1},$$

where $Z(t):L_0^*\to L_0$ is a homomorphism depending smoothly on t. Observing that $h(t)=\big(\mathrm{Id}+Z(t)C(t)\big)^{-1}$ is the identity on $\mathrm{Ker}\big(C(t)\big)$, the conclusion follows from Proposition 2.19. $\Box$

3.5. **Infinite dimensional Lagrangian Grassmannians.** Let us now consider an infinite dimensional, separable real Hilbert space $\mathcal H$ with inner product $\langle\cdot,\cdot\rangle$ endowed with a symplectic form ω; let $J:\mathcal H\to\mathcal H$ be the corresponding almost complex structure, i.e., $\omega=\langle J\cdot,\cdot\rangle$. Then, J is an anti-symmetric bounded operator on $\mathcal H$ such that $J^2=-1$. A (necessarily closed) subspace L of $\mathcal H$ will be called *Lagrangian* if $L=L^{\perp_\omega}=(JL)^\perp$.

Recall that a pair (V_1,V_2) of closed subspaces of $\mathcal H$ is called a *Fredholm pair* if $V_1\cap V_2$ is finite dimensional and if the sum V_1+V_2 is closed and it has finite codimension in $\mathcal H$.

Given a pair (L_0,L_1) of complementary Lagrangian subspaces of $\mathcal H$ (i.e., $L_0\cap L_1=(0)$ and $L_0+L_1=\mathcal H$), then one can define a real-analytic chart φ_{L_0,L_1} on $\Lambda_0(L_1)$ and taking values in $\mathcal L^{\mathrm{sa}}(L_0)$ by setting $\varphi_{L_0,L_1}(L)=P_0JS$, where $S:L_0\to L_1$ is the unique bounded operator whose graph in $L_0\oplus L_1$ is L, and $P_0:\mathcal H\to L_0$ is the orthogonal projection.

Observe that $\mathrm{Ker}\big(\varphi_{L_0,L_1}(L)\big)=L\cap L_0$; more generally, it is not hard to prove that (L_0,L) is a Fredholm pair if and only if $\varphi_{L_0,L_1}(L)\in\mathcal F^{\mathrm{sa}}(L_0)$ for some, hence for all, $L_1\in\Lambda_0(L_0)\cap\Lambda_0(L)$. In particular, the set $\mathcal F\Lambda$ of all Lagrangian subspaces L of $\mathcal H$ such that (L_0,L) is Fredholm, is an open submanifold of the manifold Λ of all Lagrangian subspaces of $\mathcal H$.[7] The entire theory described in subsection 3.2 concerning the Lagrangian Grassmannian of a symplectic space extends from the finite dimensional case to the case of the Fredholm Lagrangian Grassmannian of an infinite dimensional symplectic space; in particular, formulas of the type (3.3) hold for the transition maps $\varphi_{L_0,L_1'}\circ\varphi^{-1}_{L_0,L_1}$.

[7] In the infinite dimensional case, the full Lagrangian Grassmannian Λ is contractible (see for instance [42]).

In order to define the notion of Maslov index for continuous paths in $\mathcal{F}\Lambda$, we prove the analogue of Corollary 3.5 for the infinite dimensional case:

Proposition 3.13. *There exists a unique $\mathbb{Z}$-valued groupoid homomorphism μ_{L_0} on $\pi(\mathcal{F}\Lambda)$ such that:*

$$\mu_{L_0}\big([\gamma]\big) = \mathrm{sf}\big(\varphi_{L_0,L_1}\circ\gamma,[a,b]\big), \tag{3.7}$$

for all Lagrangian L_1 complementary to L_0 and for all continuous curve $\gamma:[a,b]\to\mathcal{F}\Lambda$ having image contained in $\Lambda_0(L_1)$.

Proof. Using the result of Proposition 3.2, it suffices to show that the right hand side in formula (3.7) gives a well defined $\mathbb{Z}$-valued groupoid homomorphism on $\pi\big(\Lambda_0(L_1)\big)$, and that

$$\mathrm{sf}\big(\varphi_{L_0,L_1}\circ\gamma,[a,b]\big) = \mathrm{sf}\big(\varphi_{L_0,L_1'}\circ\gamma,[a,b]\big)$$

if γ has image in $\Lambda_0(L_1)\cap\Lambda_0(L_1')$.

This last equality follows easily from the formula of the transition maps (3.3) and the corresponding invariance property for the spectral flow stated in Corollary 2.20.

As to the first part of the proof, the claim is equivalent to the fact that the spectral flow is invariant by fixed-endpoints homotopies and additive by concatenation. These facts are proven in [28] in the case that γ has endpoints in $\Lambda_0(L_0)$, i.e., that $\varphi_{L_0,L_1}\circ\gamma$ has invertible endpoints. The very same proof in [28], which uses formula (2.8) and it is based on an argument of homotopy lifting in fiber bundles, holds for the general case of possibly degenerate endpoints. □

Following closely the theory in subsection 3.4 one defines partial indexes, coindexes and signatures at an isolated intersection of a smooth curve $\gamma:[a,b]\to\mathcal{F}\Lambda$ with the Maslov cycle Σ_{L_0}, and as in the finite dimensional case one obtains the following:

Proposition 3.14. *Let $\gamma:[a,b]\to\mathcal{F}\Lambda$ be a real-analytic curve whose image is not entirely contained in the Maslov cycle. Then, the Maslov index $\mu_{L_0}(\gamma)$ is given by* (3.6). □

3.6. **On the notion of Maslov index for pairs of Lagrangian paths.** There exists in the literature a slightly different notion of Maslov index for *pairs* (γ_1,γ_2) of continuous curves $\gamma_1,\gamma_2:[a,b]\to\Lambda$ (see for instance [9]), as an integer valued measure of the set of instants $t\in[a,b]$ at which the Lagrangians $\gamma_1(t)$ and $\gamma_2(t)$ are not transversal. Using our partial signatures theory, we will discuss below the definition of Maslov index for such pairs; we will consider, as in [9], the case of arbitrary pairs (γ_1,γ_2) without any transversality assumption at the endpoints.

For all $L_0\in\Lambda$, consider the real-analytic fibration $\beta_{L_0}:\mathrm{Sp}(2n,\mathbb{R})\longrightarrow\Lambda$:

$$\beta_{L_0}(\phi) = \phi(L_0); \tag{3.8}$$

given any curve $\gamma:[a,b]\to\Lambda$ of class C^k, $k=1,\dots,\infty,\omega$, it can be lifted to a curve $\psi:[a,b]\to\mathrm{Sp}(2n,\mathbb{R})$ of class C^k, i.e., $\gamma(t)=\psi(t)(L_0)$ for all $t\in[a,b]$. We will call such a curve ψ a L_0*-lifting* of γ. Observe that, given two curves $\gamma_1,\gamma_2:[a,b]\to\Lambda$ and any L_0-lifting $\psi:[a,b]\to\mathrm{Sp}(V,\omega)$ of γ_2, the non transversality instants for the curves γ_1 and γ_2 correspond to the intersections of the curve $t\mapsto\psi(t)^{-1}(\gamma_1(t))$ with the Maslov cycle Σ_{L_0}.

In order to define the Maslov index of a pair (γ_1,γ_2) one needs the following:

Lemma 3.15. *Let $L_0\in\Lambda$ be fixed, let $(\gamma_1,\gamma_2):[a,b]\to\Lambda\times\Lambda$ be a pair of continuous curves in Λ, and let $\psi:[a,b]\to\mathrm{Sp}(2n,\mathbb{R})$ be any L_0-lifting of γ_2. Then, the Maslov index $\mu_{L_0}\big(t\mapsto\psi(t)^{-1}(\gamma_1(t))\big)$ does not depend on the choice of ψ. Moreover, if $\widetilde{L}_0$ is another fixed Lagrangian and $\widetilde{\psi}:[a,b]\to\mathrm{Sp}(2n,\mathbb{R})$ is a continuous curve such that $\gamma_2(t)=\widetilde{\psi}(t)(\widetilde{L}_0)$, then*

$$\mu_{L_0}\big(t\mapsto\psi(t)^{-1}\gamma_1(t)\big) = \mu_{\widetilde{L}_0}\big(t\mapsto\widetilde{\psi}(t)^{-1}\gamma_1(t)\big).$$

Likewise, if γ_1 and γ_2 are smooth curves, and t_0 is an isolated non transversality instant for γ_1 and γ_2, then the L_0-partial signatures of the curve $t \mapsto \psi(t)^{-1}\gamma_1(t)$ at $t = t_0$ do not depend on the choice of the Lagrangian L_0 and of the L_0-lifting ψ of γ_2.

Proof. If $\psi_1, \psi_2 : [a,b] \to \mathrm{Sp}(2n, I\!R)$ are any two continuous L_0-liftings of γ_2, then clearly $\phi(t) = \psi_2(t)^{-1}\psi_1(t)$ is a continuous curve in $\mathrm{Sp}(2n, I\!R)$ such that $\phi(t)(L_0) = L_0$ for all t. Using part (c) of Lemma 3.9 we obtain:

$$\mu_{L_0}\big(t \mapsto \psi_2(t)^{-1}(\gamma_1(t))\big) = \mu_{L_0}\big(t \mapsto \phi(t)\psi_1(t)^{-1}(\gamma_1(t))\big) = \mu_{L_0}\big(t \mapsto \psi_1(t)^{-1}(\gamma_1(t))\big),$$

which proves the independence on the lifting.

The proof of the second part of the statement follows from the symplectic invariance of the Maslov index (part (4) of Lemma 3.8). Namely, choose a symplectomorphism $h \in \mathrm{Sp}(2n, I\!R)$ such that $L_0 = h(\widetilde{L}_0)$, let $\widetilde{\psi} : [a,b] \to \mathrm{Sp}(2n, I\!R)$ be a continuous $\widetilde{L}_0$-lifting of $\gamma_2(t)$, and set $\psi(t) = \widetilde{\psi}(t)h^{-1}$, so that ψ is an L_0-lifting of $\gamma_2(t)$. Then:

$$\begin{aligned}\mu_{L_0}\big(t \mapsto \psi(t)^{-1}(\gamma_1(t))\big) &= \mu_{L_0}\big(t \mapsto h\widetilde{\psi}(t)^{-1}(\gamma_1(t))\big)\\ &= \mu_{h^{-1}(L_0)}\big(t \mapsto \widetilde{\psi}(t)^{-1}(\gamma_1(t))\big) = \mu_{\widetilde{L}_0}\big(t \mapsto \widetilde{\psi}(t)^{-1}(\gamma_1(t))\big),\end{aligned}$$

which concludes the proof of the first part of the statement. The proof of the last statement is totally analogous, and it uses the invariance property of the partial signatures discussed in Lemma 3.12. ☐

We can now give the following:

Definition 3.16. The Maslov index $\mu(\gamma_1, \gamma_2)$ of a pair $(\gamma_1, \gamma_2) : [a,b] \to \Lambda$ of continuous curves in Λ is the L_0-Maslov index of the curve $t \mapsto \psi(t)^{-1}\gamma_2(t)$, where $L_0 \in \Lambda$ is any fixed Lagrangian and $\psi : [a,b] \to \mathrm{Sp}(V, \omega)$ is any L_0-lifting of γ_2. If γ_1 and γ_2 are smooth and $t_0 \in [a,b]$ is an isolated non transversality instant for γ_1 and γ_2, then the partial signatures of the pair (γ_1, γ_2) at $t = t_0$, denoted by $n_k^{\pm}(\gamma_1, \gamma_2, t_0)$, $\mathring{n}_k^{\pm}(\gamma_1, \gamma_2, t_0)$ and $\sigma_k(\gamma_1, \gamma_2, t_0)$, are defined as the corresponding L_0-partial signatures of the symplectic path $t \mapsto \psi(t)^{-1}(\gamma_2(t))$ at $t = t_0$, where L_0 is any Lagrangian and $\psi : [a,b] \to \mathrm{Sp}(V, \omega)$ any smooth L_0-lifting of γ_2.

The Maslov index of a pair (γ_1, γ_2) of continuous curves in the Lagrangian Grassmannian of a symplectic space (V, ω) is better described in terms of the Maslov index of a single curve in a suitably "doubled" symplectic space V^2. Given (V, ω), denote by (V^2, ω^2) the symplectic space $V \oplus V$ endowed with the symplectic form $\omega^2 = \omega \oplus (-\omega)$:

$$\omega^2((v_1 \oplus v_2), (v_3 \oplus v_4)) = \omega(v_1, v_3) - \omega(v_2, v_4).$$

Clearly, if L_1, L_2 are two Lagrangians in (V, ω), then $L_1 \oplus L_2$ is Lagrangian in (V^2, ω^2), i.e., there is an injection $\Lambda(V, \omega) \times \Lambda(V, \omega)$ into $\Lambda(V^2, \omega^2)$; moreover, the diagonal $\Delta \subset V \oplus V$ is Lagrangian in (V^2, ω^2).

We will look at the Maslov index of curves in $\Lambda(V^2, \omega^2)$ computed relatively to the Lagrangian Δ. An easy calculations shows that, if $L_0 \in \Lambda(V, \omega)$ and $\gamma : [a,b] \to \Lambda(V, \omega)$ is continuous, then,

$$\mu_\Delta\big(t \mapsto \gamma(t) \oplus L_0\big) = \mu_{L_0}\big(t \mapsto \gamma(t)\big). \tag{3.9}$$

Proposition 3.17. *Let $\gamma_1, \gamma_2 : [a,b] \to \Lambda(V, \omega)$ be a pair of continuous curves; then,*

$$\mu(\gamma_1, \gamma_2) = \mu_\Delta(\gamma_1 \oplus \gamma_2). \tag{3.10}$$

Moreover, if γ_1 and γ_2 are smooth and if $t_0 \in [a,b]$ is an isolated non transversality instant for γ_1 and γ_2, then the partial signatures of the pair (γ_1, γ_2) at $t = t_0$ coincide with the corresponding Δ-partial signatures of the curve $t \mapsto \gamma_1(t) \oplus \gamma_2(t) \in \Lambda(V^2, \omega^2)$.

Proof. Choose a Lagrangian $L_0 \in \Lambda(V,\omega)$ and a continuous L_0-lifting $\phi : [a,b] \to \mathrm{Sp}(V,\omega)$ of $\gamma_2(t)$. Clearly, $\phi(t)^{-1} \oplus \phi(t)^{-1} \in \mathrm{Sp}(V^2,\omega^2)$ and $\phi(t)^{-1} \oplus \phi(t)^{-1}(\Delta) = \Delta$ for all t. Hence, using part (c) of Corollary 3.9 and formula (3.9), we get:

$$\begin{aligned}\mu_\Delta(\gamma_1 \oplus \gamma_2) &= \mu_\Delta\big(t \mapsto \phi(t)^{-1}(\gamma_1(t)) \oplus \phi(t)^{-1}(\gamma_2(t))\big) \\ &= \mu_\Delta\big(t \mapsto \phi(t)^{-1}(\gamma_1(t)) \oplus L_0\big) = \mu_{L_0}\big(t \mapsto \phi(t)^{-1}(\gamma_1(t))\big) = \mu(\gamma_1,\gamma_2).\end{aligned}$$

This proves the first statement of the thesis; the last statement is proven similarly, using Lemma 3.12. □

We will collect below a few properties of the map $(\gamma_1,\gamma_2) \mapsto \mu(\gamma_1,\gamma_2)$:

Proposition 3.18. *The Maslov index for pairs of curves in Λ satisfies the following:*

(1) *the induced map $\mu : \pi(\Lambda) \oplus \pi(\Lambda) \to \mathbb{Z}$ is a groupoid homomorphism;*
(2) *μ is anti-symmetric: $\mu(\gamma_1,\gamma_2) = -\mu(\gamma_2,\gamma_1)$;*
(3) *if $\gamma : [a,b] \to \Lambda$ is a continuous curve and $L_0 \in \Lambda$ is fixed, then $\mu(\gamma, L_0) = \mu_{L_0}(\gamma)$.*

Proof. (1) is proven observing that $\mu = \mu_\Delta \circ \mathfrak{i}$, where $\mathfrak{i} : \pi(\Lambda) \oplus \pi(\Lambda) \to \pi(\Lambda \times \Lambda)$ is the groupoid homomorphism induced by the immersion $\Lambda(V,\omega) \times \Lambda(V,\omega) \hookrightarrow \Lambda(V^2,\omega^2)$.

(2) follows from (1) and (4) of Lemma 3.8, observing that the map $V^2 \ni x \oplus y \mapsto y \oplus x \in V^2$ is an anti-symplectomorphism that preserves Δ.

(3) follows immediately from Proposition 3.17 and formula (3.9). □

The homotopy invariance of the Maslov index for pairs of Lagrangian paths can be used to obtain a series of interesting facts, otherwise not so evident, about the Maslov index for single curves in Λ. Here is an example:

Lemma 3.19. *Let $\gamma_1,\gamma_2 : [a,b] \to \Lambda$ be a pair of continuous curves. The following equality holds:*

$$\mu_{\gamma_1(a)}(\gamma_2) - \mu_{\gamma_1(b)}(\gamma_2) = \mu_{\gamma_2(b)}(\gamma_1) - \mu_{\gamma_2(a)}(\gamma_1). \tag{3.11}$$

Proof. Consider the continuous map

$$[a,b] \times [a,b] \ni (s,t) \longmapsto H(s,t) = \gamma_1(s) \oplus \gamma_2(t) \in \Lambda(V^2,\omega^2)$$

and apply part (3) of Lemma 3.8. The conclusion follows easily from Proposition 3.18. □

Corollary 3.20. *If $\gamma : [a,b] \to \Lambda$ is a continuous loop, then the value of $\mu_{L_0}(\gamma)$ does not depend on the choice of L_0.*

Proof. Choose any two Lagrangians $L_0, L_1 \in \Lambda$, and any continuous curve $\gamma_2 : [a,b] \to \Lambda$ with $\gamma_2(a) = L_0$ and $\gamma_2(b) = L_1$. Set $\gamma_1 = \gamma$ and apply Lemma 3.19 to γ_1 and γ_2. Since γ_1 is a loop, the left hand side of (3.11) vanishes, yielding:

$$\mu_{L_0}(\gamma) = \mu_{L_1}(\gamma),$$

which was to be proven. □

3.7. **On the Maslov triple and four-fold indexes.**

Let us now discuss a different notion of Maslov index, originally due to Kashiwara (see [36]), and further investigated by Cappell, Lee and Miller in [9, Section 8].

Assume that (V,ω) is a fixed (finite dimensional) symplectic space; given three Lagrangians $L_1, L_2, L_3 \in \Lambda(V,\omega)$, the *Maslov triple index* $\tau_V(L_1,L_2,L_3)$ is defined as the signature of the (symmetric bilinear form associated to the) quadratic form $Q : L_1 \oplus L_2 \oplus L_3 \to I\!R$ given by:

$$Q(x_1,x_2,x_3) = \omega(x_1,x_2) + \omega(x_2,x_3) + \omega(x_3,x_1).$$

It is proven in [9, Section 8] that τ_V is the unique integer valued map on $\Lambda \times \Lambda \times \Lambda$ satisfying the following properties:

[P1] (skew symmetry) If σ is a permutation of the set $\{1,2,3\}$,

$$\tau_V(L_{\sigma(1)}, L_{\sigma(2)}, L_{\sigma(3)}) = \text{sign}(\sigma)\,\tau_V(L_1, L_2, L_3);$$

[P2] (symplectic additivity) given symplectic spaces (V,ω), $(\widetilde{V},\widetilde{\omega})$, and Lagrangians $L_1, L_2, L_3 \in \Lambda(V,\omega)$, $\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_3 \in \Lambda(\widetilde{V},\widetilde{\omega})$, then:

$$\tau_{V\oplus\widetilde{V}}(L_1 \oplus \widetilde{L}_1, L_2 \oplus \widetilde{L}_2, L_3 \oplus \widetilde{L}_3) = \tau_V(L_1, L_2, L_3) + \tau_{\widetilde{V}}(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_3);$$

[P3] (symplectic invariance) if $\phi : (V,\omega) \to (\widetilde{V},\widetilde{\omega})$ is a symplectomorphism, then:

$$\tau_V(L_1, L_2, L_3) = \tau_{\widetilde{V}}(\phi(L_1), \phi(L_2), \phi(L_3));$$

[P4] (normalization) if $V = \mathbb{R}^2$ is endowed with the canonical symplectic form[8], and $L_1 = \mathbb{R}(1,0)$, $L_2 = \mathbb{R}(1,1)$, $L_3 = \mathbb{R}(0,1)$, then

$$\tau_V(L_1, L_2, L_3) = 1.$$

We will now proceed to a geometrical description of the triple index τ_V using the notion of Maslov index for paths; we will introduce to this aim a four-fold index, i.e. a map

$$\mathfrak{q} : \Lambda \times \Lambda \times \Lambda \times \Lambda \longrightarrow \mathbb{Z}.$$

Lemma 3.21. *Given four Lagrangians $L_0, L_1, L_0', L_1' \in \Lambda$ and any continuous curve $\gamma : [a,b] \to \Lambda$ such that $\gamma(a) = L_0'$ and $\gamma(b) = L_1'$, then the value of the quantity $\mu_{L_1}(\gamma) - \mu_{L_0}(\gamma)$ does* not *depend on the choice of γ.*

Proof. An easy application of Corollary 3.20. □

An analogous result has been proven by Robbin and Salamon for their half-integer valued Maslov index (see [50, Theorem 3.5]). We are now entitled to give the following:

Definition 3.22. Given four Lagrangians $L_0, L_1, L_0', L_1' \in \Lambda$, the *four-fold Maslov index* $\mathfrak{q}(L_0, L_1; L_0', L_1')$ is the integer number $\mu_{L_1}(\gamma) - \mu_{L_0}(\gamma)$, where $\gamma : [a,b] \to \Lambda$ is any continuous curve with $\gamma(a) = L_0'$ and $\gamma(b) = L_1'$.

The four-fold Maslov index $\mathfrak{q}$, also known in the literature as the *Hörmander's index*, satisfies some symmetries that resemble those satisfied by the curvature tensor of a symmetric connection:

Proposition 3.23. *Let $L_0, L_1, L_0', L_1', L \in \Lambda$ be five Lagrangians. The following identities hold:*

(a) $\mathfrak{q}(L_0, L_1; L_0', L_1') = -\mathfrak{q}(L_1, L_0; L_0', L_1')$;
(b) $\mathfrak{q}(L_0, L_1; L_0', L_1') = -\mathfrak{q}(L_0, L_1; L_1', L_0')$;
(c) $\mathfrak{q}(L_0, L_1; L_0', L_1') = -\mathfrak{q}(L_0', L_1'; L_0, L_1)$;[9]
(d) $\mathfrak{q}(L_0, L_1; L_0', L) + \mathfrak{q}(L_0, L_1; L, L_1') = \mathfrak{q}(L_0, L_1; L_0', L_1')$.

Proof. (a) and (b) are obvious by the definition of $\mathfrak{q}$, while (d) is simply the additivity by concatenation of the Maslov index. Part (c) follows easily from Lemma 3.19. □

A whole series of different identities satisfied by the four-fold index $\mathfrak{q}$ are easily obtained by combining the equalities above; for instance:

$$\mathfrak{q}(L_0, L_1; L_0', L_1') = -\mathfrak{q}(L_1', L_0'; L_1, L_0). \tag{3.12}$$

[8] i.e., $\omega((x_1,y_1),(x_2,y_2)) = x_1y_2 - x_2y_1$

[9] the curvature tensor R of a symmetric connection satisfies the identity

$$R(x_0, x_1; x_0', x_1') = R(x_0', x_1'; x_0, x_1),$$

and in turn, such symmetry leads to the Bianchi identity for R. In the case of the Maslov four-fold index, the anti-symmetry (c) leads to the *cocycle identity* (3.15).

We can now establish the relation between the four-fold Maslov index $\mathfrak{q}$ and the Maslov triple index τ_V. Let us define $\overline{\mathfrak{q}} : \Lambda \times \Lambda \times \Lambda \to \mathbb{Z}$ by:

$$\overline{\mathfrak{q}}(L_0, L_1, L_2) := \mathfrak{q}(L_0, L_1; L_2, L_0). \tag{3.13}$$

Observe that the function $\mathfrak{q}$ is completely determined by $\overline{\mathfrak{q}}$, because of the following identity:

$$\begin{aligned} \mathfrak{q}(L_0, L_1; L_0', L_1') &= \mathfrak{q}(L_0, L_1; L_0', L_0) + \mathfrak{q}(L_0, L_1; L_0, L_1') \\ &= \overline{\mathfrak{q}}(L_0, L_1, L_0') - \overline{\mathfrak{q}}(L_0, L_1, L_1'), \end{aligned} \tag{3.14}$$

however, the symmetries of the four-fold Maslov index are easier to detect thanks to the clear geometrical meaning of $\mathfrak{q}$.

Proposition 3.24. *The map $\overline{\mathfrak{q}}$ coincides with the Maslov triple index τ_V.*

Proof. By uniqueness, it suffices to prove that $\overline{\mathfrak{q}}$ satisfies the properties [P1], [P2], [P3] and [P4] above. [P2] and [P3] are easily checked using respectively the symplectic additivity and the symplectic invariance of the Maslov index of paths. [P4] is also easily checked by an explicit calculation, whose details are omitted. Property [P1], the skew-symmetry, is the non obvious part of the statement; it suffices to prove the two equalities $\overline{\mathfrak{q}}(L_0, L_1, L_2) = -\overline{\mathfrak{q}}(L_0, L_2, L_1)$ and $\overline{\mathfrak{q}}(L_0, L_1, L_2) = -\overline{\mathfrak{q}}(L_1, L_0, L_2)$. The first of the two equalities is obtained using (3.12), while the second is obtained as follows:

$$\begin{aligned} &\overline{\mathfrak{q}}(L_1, L_0, L_2) = \mathfrak{q}(L_1, L_0; L_2, L_1) \\ &\quad = \mathfrak{q}(L_1, L_0; L_2, L_0) + \mathfrak{q}(L_1, L_0; L_0, L_1) = \mathfrak{q}(L_1, L_0; L_2, L_0) = -\overline{\mathfrak{q}}(L_0, L_1, L_2). \end{aligned}$$

This concludes the proof. □

Using Proposition 3.23, it is easy to check that $\overline{\mathfrak{q}}$ satisfies the following cocycle identity (see [9, p. 163]):

$$\overline{\mathfrak{q}}(L_1, L_2, L_3) = \overline{\mathfrak{q}}(L_1, L_2, L_4) + \overline{\mathfrak{q}}(L_2, L_3, L_4) + \overline{\mathfrak{q}}(L_3, L_1, L_4). \tag{3.15}$$

Let us conclude our discussion on the triple and the four-fold index with the observation that it is possible to give an alternative construction of the Maslov index for Lagrangian paths using only the function $\mathfrak{q}$ (or $\overline{\mathfrak{q}}$). Namely, assume that one is given two fixed Lagrangians $L_0, L_1 \in \Lambda$ and a continuous curve $\gamma : [a, b] \to \Lambda$ whose image is contained in $\Lambda_0(L_1)$, i.e., $\gamma(t)$ is transversal to L_1 for all $t \in [a, b]$. In this case $\mu_{L_1}(\gamma) = 0$, and thus the quadruple index $\mathfrak{q}(L_0, L_1; \gamma(a), \gamma(b))$ coincides with the negative Maslov index $-\mu_{L_0}(\gamma)$. In the general case, the interval $[a, b]$ admits a finite partition $t_0 = a < t_1 < \ldots < t_M = b$ such that $\gamma\big([t_{i-1}, t_i]\big)$ is contained in $\Lambda_0(L_i)$, for some $L_i \in \Lambda$ and $i \in \{1, \ldots, M\}$, and by the concatenation additivity:

$$\mu_{L_0}(\gamma) = -\sum_{i=1}^{M} \mathfrak{q}\big(L_0, L_i; \gamma(t_{i-1}), \gamma(t_i)\big). \tag{3.16}$$

Clearly, the choice of the partition $(t_i)_{i=1}^{M-1}$ and of the Lagrangians $(L_i)_{i=1}^{M}$ is not unique. Using (3.14) we get:

$$\mu_{L_0}(\gamma) = \sum_{i=1}^{M} \Big[\overline{\mathfrak{q}}\big(L_0, L_i, \gamma(t_i)\big) - \overline{\mathfrak{q}}\big(L_0, L_i, \gamma(t_{i-1})\big)\Big]. \tag{3.17}$$

3.8. **Maslov index of symplectic paths.** Let $\Phi : [a, b] \to \mathrm{Sp}(V, \omega)$ be a continuous curve; it is easy to see that, for each $t \in [a, b]$, the graph $\mathrm{Gr}\big(\Phi(t)\big)$ is a Lagrangian subspace of the symplectic space (V^2, ω^2) defined in Subsection 3.6. We can therefore give the following:

Definition 3.25. Given a continuous curve Φ in the symplectic group $\mathrm{Sp}(V,\omega)$, the Maslov index $\mathrm{i}_{\text{Maslov}}(\Phi)$ of Φ is the Δ-Maslov index of the curve $t \mapsto \mathrm{Gr}\big(\Phi(t)\big) \in \Lambda(V^2,\omega^2)$:

$$\mathrm{i}_{\text{Maslov}}(\Phi) := \mu_\Delta\big(t \mapsto \mathrm{Gr}(\Phi(t))\big).$$

The Δ-Maslov index of a symplectic path Φ is also known in the literature as the *Conley–Zehnder index* of Φ (see [12, 50, 52]).

Recall that each $L_0 \in \Lambda(V,\omega)$ gives a smooth map $\beta_{L_0} : \mathrm{Sp}(V,\omega) \to \Lambda(V,\omega)$ (defined in (3.8)), and, with the help of the four-fold Maslov index $\mathfrak{q}$, we can compare the Maslov index of the curve Φ with the L_0-Maslov index of the curve $\beta_{L_0} \circ \Phi : [a,b] \to \Lambda(V,\omega)$. To this aim, we first give the following:

Lemma 3.26. *Let $\Phi : [a,b] \to \mathrm{Sp}(V,\omega)$ be a continuous curve and let $L_0, L_1, L_1' \in \Lambda(V,\omega)$ be fixed. Then:*

$$\mu_{L_0}\big(\beta_{L_0} \circ \Phi\big) - \mu_{L_0}\big(\beta_{L_1'} \circ \Phi\big) = \mathfrak{q}\big(L_1, L_1'; \Phi(a)^{-1}(L_0), \Phi(b)^{-1}(L_0)\big).$$

Proof. Using the Maslov index for pairs and the symplectic invariance, we compute as follows:

$$\mu_{L_0}\big(\beta_{L_1} \circ \Phi\big) = \mu\big(\beta_{L_1} \circ \Phi, L_0\big) = \mu\big(L_1, t \mapsto \Phi(t)^{-1}(L_0)\big) = -\mu_{L_1}\big(t \mapsto \Phi(t)^{-1}(L_0)\big).$$

Similarly,

$$\mu_{L_0}\big(\beta_{L_1'} \circ \Phi\big) = -\mu_{L_1'}\big(t \mapsto \Phi(t)^{-1}(L_0)\big).$$

The conclusion follows easily from the definition of $\mathfrak{q}$. □

Proposition 3.27. *Let $\Phi : [a,b] \to \mathrm{Sp}(V,\omega)$ be a continuous curve and $L_0, \ell_0 \in \Lambda(V,\omega)$ be fixed. Then:*

$$\mathrm{i}_{\text{Maslov}}(\Phi) + \mu_{L_0}\big(\beta_{\ell_0} \circ \Phi\big) = \mathfrak{q}\big(\Delta, L_0 \oplus \ell_0; \mathrm{Gr}\big(\Phi(a)^{-1}\big), \mathrm{Gr}\big(\Phi(b)^{-1}\big)\big).$$

In particular, if Φ is a loop, then $\mathrm{i}_{\text{Maslov}}(\Phi) = -\mu_{L_0}(\beta_{\ell_0} \circ \Phi)$.

Proof. We compute:

$$\mathrm{i}_{\text{Maslov}}(\Phi) = \mu_\Delta\big(t \mapsto (\mathrm{Id} \oplus \Phi(t)(\Delta)\big)$$

and, using the properties of the Maslov index for pairs of curves,

$$\mu_{L_0}\big(\beta_{\ell_0} \circ \Phi\big) = -\mu_\Delta\big(t \mapsto L_0 \oplus \beta_{\ell_0} \circ \Phi(t)\big) = -\mu_\Delta\big(t \mapsto (\mathrm{Id} \oplus \Phi(t))(L_0 \oplus \ell_0)\big).$$

The result follows now easily applying Lemma 3.26 to the curve $t \mapsto \mathrm{Id} \oplus \Phi(t) \in \mathrm{Sp}(V^2,\omega^2)$ and to the Lagrangians $\Delta, L_0 \oplus \ell_0 \in \Lambda(V^2,\omega^2)$. □

3.9. **Spectral flow of affine paths.** Let us now discuss an application of Proposition 3.11 that will be used to compute the spectral flow of affine paths (Proposition 3.29, Corollary 3.30), and in the spectral index theorem (Subsection 4.5):

Example 3.28. Let V be a real, finite dimensional vector space, $g : V \times V \to \mathbb{R}$ a nondegenerate symmetric bilinear form, and $T : V \to V$ a nilpotent ($T^n = 0$) linear g-symmetric endomorphism (i.e., such that the bilinear form $g^T := g(T\cdot,\cdot)$ is symmetric) of V. For a subspace $W \subset V$, denote by $W^{\perp_g}$ the g-orthogonal space to W, defined by

$$W^{\perp_g} = \big\{v \in V : g(v,w) = 0 \text{ for all } w \in W\big\}.$$

For all $k = 1,\ldots,n$, define a vector space $W_k = T^{k-1}\big(\mathrm{Ker}(T^k)\big)$ and a bilinear form $B_k : W_k \times W_k \to \mathbb{R}$ by:

$$B_k(a,b) = g(c,b),$$

where $a = T^{k-1}(c)$. Using the identities:[10]

(3.18)
$$\mathrm{Ker}(T^\alpha)^{\perp_g} = \mathrm{Im}(T^\alpha), \quad \mathrm{Im}(T^\alpha) \cap \mathrm{Ker}(T^\beta) = T^\alpha\big(\mathrm{Ker}(T^{\alpha+\beta})\big), \quad \forall\, \alpha, \beta = 1, \ldots, n,$$

it is not hard to see directly that $W_k \supseteq W_{k+1}$, that B_k is a well defined symmetric bilinear form on W_k, and that $W_{k+1} = \mathrm{Ker}(B_k)$ for all k; in particular, B_n is nondegenerate. We get to the same conclusions indirectly observing that the spaces W_k and the bilinear forms B_k can be obtained using the construction of Section 2 (Remark 2.5) applied to the real-analytic path of symmetric bilinear forms

$$[-\varepsilon, \varepsilon] \ni \lambda \longmapsto B(\lambda) = g^T - \lambda g \in \mathrm{B_{sym}}(V)$$

at the isolated singularity $\lambda = 0$.

Using the homotopy invariance of the Maslov index, we will now show that the following identities hold:

$$\begin{aligned} &\sum_{k\ge 1} \big(n^-(B_{2k-1}) + n^+(B_{2k})\big) = \mathring{n}^-(g^T) - n^-(g) = n^+(g) - n^+\big(g^T\big), \\ &\sum_{k\ge 1} n^+(B_k) = \mathring{n}^-(g^T) - n^+(g) = n^-(g) - n^+\big(g^T\big), \\ &\sum_{k\ge 1} \sigma(B_{2k-1}) = -\sigma(g). \end{aligned} \tag{3.19}$$

By Corollary 2.14,

$$\begin{aligned} &\sum_{k\ge 1} \big(n^-(B_{2k-1}) + n^+(B_{2k})\big) = \dim(\mathrm{Ker}T) - \mathrm{sf}\big(B, [-\varepsilon, 0]\big), \\ &\sum_{k\ge 1} n^+(B_k) = \dim(\mathrm{Ker}T) + \mathrm{sf}\big(B, [0, \varepsilon]\big), \\ &\sum_{k\ge 1} \sigma(B_{2k-1}) = \mathrm{sf}\big(B, [-\varepsilon, \varepsilon]\big). \end{aligned}$$

To prove the equalities (3.19), consider the two-parameter map

$$[0, 1] \times [-\varepsilon, \varepsilon] \ni (r, \lambda) \longmapsto \ell(r, \lambda) = rg^T - \lambda g \in \mathrm{B_{sym}}(V);$$

observe that $B(\lambda) = \ell(1, \lambda)$, and that the bilinear forms $\ell(r, -\varepsilon), \ell(r, \varepsilon)$ are nondegenerate for all $r \in [0, 1]$, because 0 is the unique eigenvalue of T. It follows (see Figure 2))

$$\begin{aligned} \mathrm{sf}\big(B, [-\varepsilon, 0]\big) &= \mathrm{sf}\big(\ell(0, \cdot), [-\varepsilon, 0]\big) + \mathrm{sf}\big(\ell(\cdot, 0), [0, 1]\big), \\ \mathrm{sf}\big(B, [0, \varepsilon]\big) &= -\mathrm{sf}\big(\ell(\cdot, 0), [0, 1]\big) + \mathrm{sf}\big(\ell(0, \cdot), [0, \varepsilon]\big). \end{aligned}$$

An immediate computation gives:

$$\begin{aligned} \mathrm{sf}\big(\ell(0, \cdot), [-\varepsilon, 0]\big) &= \mathring{n}^+(0) - \mathring{n}^+(\varepsilon g) = \dim(V) - n^+(g) = n^-(g), \\ \mathrm{sf}\big(\ell(\cdot, 0), [0, 1]\big) &= \mathring{n}^+(g^T) - \mathring{n}^+(0) = \mathring{n}^+(g^T) - \dim(V) = -n^-(g^T), \\ \mathrm{sf}\big(\ell(0, \cdot), [0, \varepsilon]\big) &= \mathring{n}^+(-\varepsilon g) - \mathring{n}^+(0) = n^-(g) - \dim(V) = -n^+(g), \end{aligned}$$

from which equalities (3.19) follow easily.

Proposition 3.29. *Let $\mathcal{H}$ be a real, separable Hilbert space, $g : \mathcal{H} \to \mathcal{H}$ an invertible self-adjoint linear operator and $T : \mathcal{H} \to \mathcal{H}$ a linear Fredholm operator such that $gT = T^*g$. Then:*

[10] The second identity in (3.18) holds for any linear operator T on any vector space V. The first one needs the assumption that T is g-symmetric and, in addition, that $\dim(V) < +\infty$, or, if V is an infinite dimensional Hilbert space, that g is *strongly nondegenerate* on V (i.e., realized by a self-adjoint isomorphism of V) and that T is a Fredholm operator of index 0. Identities (3.18) will be used also in the proof of Propositions 3.29 and 3.31 under these more general assumptions.

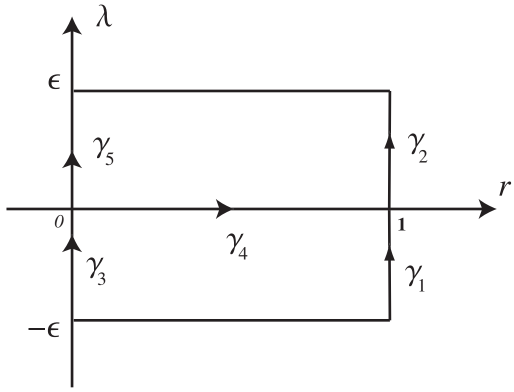

FIGURE 2. The curves of symmetric bilinear forms used in Example 3.28. In the picture, γ_1 is the curve $[-\varepsilon, 0] \ni \lambda \mapsto B(\lambda)$, γ_2 is the curve $[0, \varepsilon] \ni \lambda \mapsto B(\lambda)$, γ_3 is the curve $[-\varepsilon, 0] \ni \lambda \mapsto \ell(0, \lambda)$, γ_4 is the curve $[0, 1] \ni r \mapsto \ell(r, 0)$ and γ_5 is the curve $[0, \varepsilon] \ni \lambda \mapsto \ell(0, \lambda)$

(1) $\dim\big(\bigcup_{n\ge1}\mathrm{Ker}(T^n)\big) < +\infty$, *i.e., there exists* $n_0 \ge 0$ *such that* $\mathrm{Ker}(T^n) = \mathrm{Ker}(T^{n_0})$ *for all* $n \ge n_0$ *and* $\dim\big(\mathrm{Ker}(T^{n_0})\big) < +\infty$*;*
(2) *the bilinear form* $\langle g\cdot, \cdot\rangle$ *is nondegenerate on* $\mathrm{Ker}\big(T^{n_0}\big)$*;*
(3) *there are no degeneracy instants* $t \neq 0$ *near* 0 *for the affine path* $t \mapsto \mathbf{L}(t) = gT - tg$ *of self-adjoint Fredholm operators, and for* $\varepsilon > 0$ *small enough:*

$$\begin{aligned} &\mathrm{sf}\big(\mathbf{L}, [-\varepsilon, 0]\big) = n^+\big(\mathcal{B}^T\big) - n^+(\mathcal{B}) + \dim(\mathrm{Ker}T) = \mathring{n}^+\big(\mathcal{B}^T\big) - n^+(\mathcal{B}),\\ &\mathrm{sf}\big(\mathbf{L}, [0, \varepsilon]\big) = n^-(\mathcal{B}) - n^+\big(\mathcal{B}^T\big) - \dim(\mathrm{Ker}T) = n^-(\mathcal{B}) - \mathring{n}^+\big(\mathcal{B}^T\big),\\ &\mathrm{sf}\big(\mathbf{L}, [-\varepsilon, \varepsilon]\big) = -\sigma(\mathcal{B}), \end{aligned} \tag{3.20}$$

where $\mathcal{B} = \langle g\cdot, \cdot\rangle|_{\mathrm{Ker}(T^{n_0})}$ *and* $\mathcal{B}^T = \langle gT\cdot, \cdot\rangle|_{\mathrm{Ker}(T^{n_0})}$.

Proof. We start observing that if $\mathrm{Ker}(T) = \{0\}$ then the entire statement is trivial; we observe also that, since g is an isomorphism, the equality $gT = T^*g$ implies that T is a Fredholm operator of index 0.[11] If $\mathrm{Ker}(T) \neq \{0\}$, $t = 0$ is an isolated degeneracy instant of $\mathbf{L}(t)$, because 0 must be isolated in the spectrum of the Fredholm operator T. As in Example 3.28, the spaces W_k and the bilinear forms B_k can be computed explicitly as:

$$W_k = T^{k-1}\big(\mathrm{Ker}(T^k)\big), \quad B_k(x, y) = g(z, y),$$

where $x, y \in W_k$ and $T^k(z) = x$. Since the path $\mathbf{L}$ is real-analytic, then there exists $n_0 \in I\!N$ such that $W_{n_0} \neq \{0\}$, B_{n_0} is nondegenerate on W_{n_0}, and $W_n = \{0\}$ for $n > n_0$. The equality $W_n = \{0\}$ clearly implies $\mathrm{Ker}(T^n) = \mathrm{Ker}(T^{n-1})$, hence $\mathrm{Ker}(T^n) = \mathrm{Ker}(T^{n_0})$ for all $n \ge n_0$, which proves part (1). Consider now $u \in W_{n_0}$, $u \neq 0$. If $x = T^{n_0-1}(u) \neq 0$, then, since B_{n_0} is nondegenerate, there exists $y \in W_{n_0}$ such that $g(u, y) = B_{n_0}(x, y) \neq 0$. On the other hand, if $u \in \mathrm{Ker}(T^{n_0-1})$ and $g(u, v) = 0$ for all $v \in \mathrm{Ker}(T^{n_0})$, then, recalling the identities (3.18):

$$\begin{aligned} u \in \mathrm{Ker}(T^{n_0-1}) \cap \mathrm{Ker}(T^{n_0})^{\perp_g} &= \\ \mathrm{Ker}(T^{n_0-1}) \cap \mathrm{Im}(T^{n_0}) &= T^{n_0}\big(\mathrm{Ker}(T^{2n_0-1})\big) = T^{n_0}\big(\mathrm{Ker}(T^{n_0})\big) = \{0\}, \end{aligned}$$

[11] Recall that the *index* $\mathrm{ind}(T)$ of a Fredholm operator T is the integer $\dim\big(\mathrm{Ker}(T)\big) - \mathrm{codim}\big(\mathrm{Im}(T)\big)$; in particular, the index of an isomorphism is zero. Composition of Fredholm operators is Fredholm, and the Fredholm index is additive by composition; moreover, $\mathrm{ind}(T^*) = -\mathrm{ind}(T)$. The equality $gT = T^*g$ implies $\mathrm{ind}(T) = \mathrm{ind}(g) + \mathrm{ind}(T) = \mathrm{ind}(gT) = \mathrm{ind}(T^*g) = \mathrm{ind}(g) + \mathrm{ind}(T^*) = -\mathrm{ind}(T)$, hence $\mathrm{ind}(T) = 0$.

which proves part (2). Finally, part (3) is obtained immediately from Example 3.28 (formulas (3.19)) and from Corollary 2.14 by considering the restriction of g and T to the space $V = \mathrm{Ker}(T^{n_0})$ (which is clearly invariant by T), and observing that the spectral flow of the path $t \mapsto gT - tg$ on $\mathcal{H}$ coincides with the spectral flow of its restriction[12] to $\mathrm{Ker}(T^{n_0})$ (see Example 2.22, Subsection 2.5). □

We can now compute the spectral flow of arbitrary affine paths of self-adjoint Fredholm operators:

Corollary 3.30. *Let A and K be self-adjoint operators on $\mathcal{H}$, with A invertible and K compact. Assume that $\lambda_0 \in I\!R \setminus \{0\}$ and that λ_0^{-1} is in the spectrum of $-A^{-1}K$, so that λ_0 is an isolated singularity of the affine path $\lambda \mapsto \mathbf{S}(\lambda) = A + \lambda K$ in $\mathcal{F}^{sa}(\mathcal{H})$. Then, for $\varepsilon > 0$ small enough:*

(3.21)

$$\mathrm{sf}\big(\mathbf{S}, [\lambda_0 - \varepsilon, \lambda_0]\big) = n^+(\mathcal{B}_2) - n^+(\mathcal{B}_1) + \dim\big(\mathrm{Ker}(A + \lambda_0 K)\big) = \mathring{n}^+(\mathcal{B}_2) - n^+(\mathcal{B}_1),$$
$$\mathrm{sf}\big(\mathbf{S}, [\lambda_0, \lambda_0 + \varepsilon]\big) = n^-(\mathcal{B}_1) - n^+(\mathcal{B}_2) - \dim\big(\mathrm{Ker}(A + \lambda_0 K)\big) = n^-(\mathcal{B}_1) - \mathring{n}^+(\mathcal{B}_2),$$
$$\mathrm{sf}\big(\mathbf{S}, [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]\big) = -\sigma(\mathcal{B}_1),$$

where $\mathcal{B}_1 = \langle A\cdot, \cdot\rangle|_{\mathcal{H}_{\lambda_0}}$, $\mathcal{B}_2 = \langle (A + \lambda_0 K)\cdot, \cdot\rangle|_{\mathcal{H}_{\lambda_0}}$, and $\mathcal{H}_{\lambda_0}$ is the finite dimensional subspace of $\mathcal{H}$ given by

$$\mathcal{H}_{\lambda_0} = \bigcup_{n \geq 1} \mathrm{Ker}\big(A^{-1}K + \tfrac{1}{\lambda_0}\mathrm{Id}\big)^n.$$

Proof. Assume first $\lambda_0 > 0$; by the cogredient invariance, for λ near λ_0 the spectral flow of $\mathbf{S}(\lambda)$ equals the spectral flow of

$$\tfrac{1}{\lambda}\mathbf{S}(\lambda) = (\tfrac{1}{\lambda} - \tfrac{1}{\lambda_0})A + A\big(A^{-1}K + \tfrac{1}{\lambda_0}\mathrm{Id}\big).$$

To obtain (3.21), set $t = \frac{1}{\lambda_0} - \frac{1}{\lambda}$, $g = A$, $T = A^{-1}K + \frac{1}{\lambda_0}\mathrm{Id}$ and apply Proposition 3.29 to this setup. Observe that the identities (3.18) can now be used because T is Fredholm of index 0 (it is a compact perturbation of the isomorphism $\frac{1}{\lambda_0}\mathrm{Id}$).

The proof in the case $\lambda_0 < 0$ is obtained from the previous case, replacing λ_0 with $-\lambda_0$ and K with $-K$, observing that the definition of the objects $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{H}_{\lambda_0}$ are unchanged when both λ_0 and K are taken with the opposite sign. □

Using similar arguments, one proves the following version of Proposition 3.29, which is better suited to study the case of Fredholm bilinear forms obtained from unbounded operators paired with compact bilinear forms. Recall that a densely defined linear operator T is said to be *discrete* if for some (hence for all) λ not in the spectrum $\mathfrak{s}(T)$ of T, the resolvent $(T - \lambda)^{-1}$ is a compact operator; the spectrum of a discrete operator T is a discrete subset of $\mathbb{C}$, and every element in the spectrum is an eigenvalue of T (see [22, Chapter 19]).

Proposition 3.31. *Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{D}_2 \subset \mathcal{D}_1 \subset \mathcal{H}$ be dense algebraic subspaces, each of which is endowed with a Hilbert space structure that makes each inclusion a bounded operator, and the inclusion of $\mathcal{D}_2$ into $\mathcal{H}$ compact. Let $G : \mathcal{H} \to \mathcal{H}$ be a self-adjoint isomorphism and $T : \mathcal{D}_2 \to \mathcal{H}$ a G-symmetric Fredholm operator of index 0. Assume that the symmetric bilinear form $\langle GT\cdot, \cdot\rangle$ defined on $\mathcal{D}_2$ admits a (bounded) Fredholm extension $\widehat{g}^T$ to $\mathcal{D}_1$. Then:*

1. *T is a discrete operator (and therefore its spectrum $\mathfrak{s}(T)$ consists of eigenvalues of T).*

Assume further that the following regularity property holds: if $x \in \mathcal{D}_1$ and $z \in \mathcal{H}$ are such that $\widehat{g}^T(x, y) = \langle Gz, y\rangle$ for all $y \in \mathcal{D}_2$, then $x \in \mathcal{D}_2$ (and necessarily $Tx = z$). Then:

[12] here, by restriction of a linear operator L we mean the restriction of the corresponding bilinear form $\langle L\cdot, \cdot\rangle$.

(2) *for all* $\lambda \in \mathfrak{s}(T) \cap I\!R$*, the generalized eigenspace* $\mathcal{H}_\lambda = \bigcup_{n\geq 1} \mathrm{Ker}\big((T-\lambda)^n\big)$ *is finite dimensional, and the restriction of* $\widehat{g} = \langle G\cdot,\cdot\rangle$ *to* $\mathcal{H}_\lambda$ *is nondegenerate;*

(3) *for any compact interval* $[a,b]$*, the spectral flow of the path of self-adjoint Fredholm operators* $[a,b] \ni \lambda \mapsto \mathbf{S}(\lambda) = \widehat{g}^T - \lambda\widehat{g} \in \mathcal{F}^{sa}(\mathcal{D}_1)$ *is given by:*

(3.22)

$$\mathrm{sf}\big([\mathbf{S},[a,b]]\big) = n^-\big(\widehat{g}|_{\mathcal{H}_a}\big) - \mathring{n}^+\big(\widehat{g}^T_a|_{\mathcal{H}_a}\big) - \sum_{\lambda_0\in\mathfrak{s}(T)\cap]a,b[} \sigma\big(\widehat{g}|_{\mathcal{H}_{\lambda_0}}\big) + \mathring{n}^+\big(\widehat{g}^T_b|_{\mathcal{H}_b}\big) - n^+\big(\widehat{g}|_{\mathcal{H}_b}\big)$$

where $\widehat{g}^T_\lambda = \big\langle G(T-\lambda)\cdot,\cdot\big\rangle$.

Proof. Let λ be an element[13] in $\mathfrak{s}(T)$ and consider the Fredholm self-adjoint operator $S_\lambda = G(T-\lambda) : \mathcal{D}_2 \to \mathcal{H}$; to prove that T is discrete, observe that $(T-\lambda)^{-1} = S_\lambda^{-1}G$ and that G is a compact operator on $\mathcal{D}_2$, due to the fact that the inclusion of $\mathcal{D}_2$ into $\mathcal{H}$ is compact. This proves (1).

Parts (2) and (3) are now proven repeating *verbatim* the proof of Proposition 3.29, keeping in mind the following:

(a) the regularity property implies that $\mathrm{Ker}(\mathbf{S}(\lambda)) = \mathrm{Ker}(T-\lambda)$ for all $\lambda \in [a,b]$, and that, for all $\lambda_0 \in \mathfrak{s}(T) \cap [a,b]$, the spaces W_k obtained from the partial signatures construction for the curve $\mathbf{S}(\lambda)$ at the degeneracy instant λ_0 are given by $(T-\lambda_0)^{k-1}\big(\mathrm{Ker}(T-\lambda_0)^k\big)$ for all $k \geq 1$;

(b) the identities (3.18) can be used in this context thanks to the assumption that T is Fredholm of index 0 and that G is a self-adjoint isomorphism.

Formula (3.22) is obtained easily from (3.20) using the additivity by concatenation of the spectral flow. □

4. SEMI-RIEMANNIAN GEODESICS

As an application of the theory discussed, we will now describe how the partial signature method can be applied to the study of the Maslov index of a semi-Riemannian geodesic.

The semi-Riemannian geodesic problem is a central example of strongly indefinite variational problem to which some recent extension of the classical Morse theory (see [2]), as well as of the bifurcation theory (see [28, 44]), can be applied to obtain global geometrical results. In these theories, the Maslov index of a geodesic plays the role of a generalized Morse index, and it is an essential point to understand how to compute it in terms of the conjugate points along the geodesic. In the Riemannian case, the well-known Morse index theorem gives the equality between the number of conjugate points along the geodesic and the Morse index of the geodesic action functional, which is also equal to the number of negative eigenvalues of the Jacobi differential operator. We will establish a similar result for the semi-Riemannian case, where suitable definitions of "Morse index", "number of conjugate points" and "number of negative eigenvalues" have to be introduced. More precisely, we will define the following notions associated to a semi-Riemannian geodesic:

- *Maslov index*, as an appropriate count of the conjugate points (Subsection 4.1);
- *generalized Morse index*, as the spectral flow of the path of index forms along the geodesic (Subsection 4.4);
- *spectral index*, as an appropriate count of the nonpositive eigenvalues of the Jacobi differential operator (Subsection 4.5).

We will prove the equality of the three integer numbers in the general case (Theorem 4.9), extending the results of [13, 30, 32, 38, 46] to the case of degenerate endpoints.

Two different notions of index are used in the classical literature to count conjugate points. The *geometric index*, i.e., the sum of the multiplicities of the conjugate points,

[13]for the sake of precision, the argument presented here works for only for *real* λ's. For the complex case, one needs to consider the complexification of $\mathcal{H}$ endowed with the inner product given by the sesquilinear extension of $\langle\cdot,\cdot\rangle$, and the complex linear extensions of T and of G.

which is not a very meaningful notion outside the Riemannian or the causal Lorentzian context, and the *conjugate index* (as in [32], or *focal index* in [38]), that coincides with the Maslov index in the case of nondegenerate (see Subsection 4.1) conjugate points. In spite of the fact that such nondegeneracy holds generically, the conjugate index is not a good measure of the conjugate points (see [38, §5.4] and Subsection 4.7 below).

4.1. **Conjugate points and Maslov index.** Let us recall briefly the definition of Maslov index for a semi-Riemannian geodesic; the background material for this section can be found in references [32, 38, 46].

Let (M, g) be an n-dimensional semi-Riemannian manifold, ∇ the covariant derivative of the Levi–Civita connection of g and R its curvature tensor, chosen with the sign convention: $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

Given a geodesic $\theta : [0, 1] \to M$, the Jacobi equation along θ is the second order linear equation $V'' = R(\dot\theta, V)\dot\theta$ for vector fields V along θ; here prime means covariant differentiation along θ. Solutions of the Jacobi equation are called Jacobi fields. Let us recall that $t_0 \in \,]0, 1]$ is sad to be a *conjugate instant along* θ if there exists a non zero Jacobi field V such that $V(0) = V(t_0) = 0$. The *multiplicity* $\mathrm{mul}(t_0)$ of a conjugate instant t_0 is defined to be the dimension of the vector space of all Jacobi fields V satisfying $V(0) = V(t_0) = 0$; for all conjugate instant t_0, $\mathrm{mul}(t_0) \le n - 1$.

By a parallel trivialization of the tangent bundle TM along θ (or of the normal bundle $\dot\theta^\perp$ in the non lightlike case), then the metric g can be seen as a constant nondegenerate bilinear form on $I\!R^n$, and the Jacobi equation becomes simply $V'' = RV$, where now the prime symbol denotes the standard derivative of $I\!R^n$-valued maps, and $R(t)$ is a smooth curve of g-symmetric endomorphisms of $I\!R^n$. We will implicitly identify vector fields along θ with $I\!R^n$-valued maps via such trivialization. Let us consider the flow of the Jacobi equation, which is the smooth curve of isomorphisms

$$\Phi_t : I\!R^{2n} \to I\!R^{2n}$$

defined by:

$$\Phi_t\big(V(0), V'(0)\big) = \big(V(t), V'(t)\big),$$

for all solutions V of $V''(t) = R(t)V(t)$. An immediate calculation shows that Φ_t preserves the symplectic form $\omega_g\big((v_1, v_2), (w_1, w_2)\big) = g(v_1, w_2) - g(v_2, w_1)$, hence we get a smooth curve in the Lie group $\mathrm{Sp}(I\!R^{2n}, \omega_g)$.

Setting $L_0 = \{0\} \oplus I\!R^n$, which is Lagrangian relatively to ω_g, we get a smooth curve $\gamma(t) = \Phi_t(L_0)$ in the Lagrangian Grassmannian of $(I\!R^{2n}, \omega_g)$. Conjugate points along θ correspond to instants t_0 at which $\gamma(t_0)$ is not transversal to L_0, and the Maslov index of θ, denoted by $\mathrm{i}_{\mathrm{Maslov}}(\theta)$, is defined to be the sum:

$$\mathrm{i}_{\mathrm{Maslov}}(\theta) = \mu_{L_0}(\gamma) + n^-(g), \tag{4.1}$$

where $\mu_{L_0}(\gamma)$ is the L_0-Maslov index μ_{L_0} of the curve γ, as defined in subsection 3.3. Such definition does not depend on the choice of a parallel trivialization of TM along θ; a proof of this fact in the case that the final endpoint $\theta(1)$ is not conjugate is proven in [46], while for the general case the proof will be done in Lemma 4.2 below.

Remark 4.1. It should be observed that the curve γ obtained by the above construction from the Jacobi equation along a semi-Riemannian geodesic $\theta : [0, 1] \to M$ is not entirely contained in the Maslov cycle. This is due to the well known fact that, although γ always intersects the Maslov cycle at the initial instant $t = 0$ (i.e., $t = 0$ is always a conjugate instant in a trivial sense), there are no conjugate instants in $]0, \varepsilon]$ for $\varepsilon > 0$ small enough (see for instance [38, Proposition 2.7]). Based on this observation, the Maslov index of a semi-Riemannian geodesic θ whose final endpoint $\theta(1)$ is not conjugate had been defined in references [32, 38, 46, 48] as the Maslov index of the restriction $\gamma|_{[\varepsilon,1]}$, in order to exclude the contribution of the initial conjugate instant. Such contribution can be computed easily using the theory below, and it is equal to $-n^-(g)$ (see formula 4.4), i.e., for $\varepsilon > 0$ small enough:

$$\mu_{L_0}(\gamma) = \mu_{L_0}\big(\gamma|_{[\varepsilon,1]}\big) - n^-(g), \tag{4.2}$$

which shows that the definition of $\mathrm{i}_{\text{Maslov}}(\theta)$ given in (4.1) is consistent with that of references [32, 38, 46, 48]. Observe that in the Riemannian case $\mathrm{i}_{\text{Maslov}}(\theta) = \mu_{L_0}(\gamma)$

It is well known that conjugate points along θ can accumulate away from the instant $t = 0$ (see [48]) unless g is positive (or negative!) definite. Nevertheless, when (M, g) is real-analytic, then so is also every solution V of the Jacobi equation along θ and also the curve γ above, and this implies that there is only a finite number of conjugate points.

Let $\theta(t_0)$, $t_0 \in \,]0, 1]$, be a conjugate point along θ and define:

$$\mathbb{J}[t_0] = \big\{J(t_0) : J \in \mathbb{J}\big\},$$

where:

$$\mathbb{J} = \big\{J : J \text{ is a Jacobi field along } \theta\text{, with } J(0) = 0\big\}; \tag{4.3}$$

observe that $\mathbb{J}$ is an n-dimensional vector space. Then, since $\gamma(t_0)$ is conjugate, $\mathbb{J}[t_0] \neq I\!R^n$, and the *signature* $\sigma(t_0)$ of the conjugate point $\theta(t_0)$ is defined to be the signature of the restriction of g to $\mathbb{J}[t_0]^\perp$, where now $\perp$ denotes orthogonality relative to g; an easy argument shows that the following equality holds:

$$\mathbb{J}[t_0]^\perp = \big\{J'(t_0) : J \in \mathbb{J},\ J(t_0) = 0\big\}.$$

It is also easy to see that, for all conjugate instant t_0 along θ:

$$\mathrm{mul}(t_0) = \dim\big(\mathbb{J}[t_0]^\perp\big) = \mathrm{codim}\big(\mathbb{J}[t_0]\big).$$

When the restriction of g to $\mathbb{J}[t_0]^\perp$ is nondegenerate, then $\theta(t_0)$ is said to be a *nondegenerate conjugate point*. Nondegenerate conjugate points correspond to transversal intersections of the curve γ with the Maslov cycle Σ_{L_0}; if all the conjugate points along θ are nondegenerate, and if the final instant $t = 1$ is not conjugate, then the Maslov index $\mathrm{i}_{\text{Maslov}}(\theta)$ equals the sum of the signatures of all conjugate instants in $]0, 1[$ along θ. Using the theory developed in the present paper we are now able to compute the Maslov index of any geodesic without any nondegeneracy assumption and any assumption on the final instant $t = 1$.

4.2. **Partial signatures at a conjugate instant and Maslov index.** It is natural to define $n_k^-(\theta, t_0)$ ($\mathring{n}_k^-(\theta, t_0)$), $n_k^+(\theta, t_0)$ ($\mathring{n}_k^+(\theta, t_0)$) and $\sigma_k(\theta, t_0)$ respectively as the k-th partial (extended) index, (extended) coindex and signature of the curve γ at t_0. For the sake of precision, we must show that these quantities do not depend on the choice of a trivialization of TM:

Lemma 4.2. *Let $\gamma, \widetilde{\gamma} : [0, 1] \to \Lambda$ be curves associated to the semi-Riemannian geodesic θ by two different trivializations of the tangent bundle TM along θ. Then, for all conjugate instant t_0, the partial signatures of γ and $\widetilde{\gamma}$ at t_0 coincide.*

Proof. The conclusion follows easily from Lemma 3.12, observing that the curves γ and $\widetilde{\gamma}$ are related by the formula $\widetilde{\gamma}(t) = \varsigma\big(\gamma(t)\big)$, where ς is a fixed symplectomorphism of $(I\!R^{2n}, \omega_g)$. More precisely, if $T : I\!R^n \to I\!R^n$ is the isomorphism relating the two different trivializations of TM along θ, then ς is given by:

$$\varsigma(x, y) = (Tx, g^{-1}T^{*-1}y). \qquad \square$$

A method for computing the partial signatures of a geodesic at a conjugate instant will be given in Subsection 4.3. Using Proposition 3.11 and formulas (4.1), (4.2) we obtain immediately:

Proposition 4.3. *If (M, g) is a real-analytic semi-Riemannian manifold and $\theta : [0, 1] \to M$ is a geodesic in M, then the Maslov index of θ is given by:*

$$\mathrm{i}_{\textit{Maslov}}(\theta) =$$

$$\sum_{t_0 \text{ conjugate instant in }]0,1[} \left[\sum_{k\geq 1} \sigma_{2k-1}(\theta, t_0)\right] - \sum_{k\geq 1} \left[n^-_{2k-1}(\theta, 1) + n^+_{2k}(\theta, 1)\right] + \mathrm{mul}(1).$$

In particular, the definition of Maslov index of θ does not depend on the choice of the trivialization of TM along θ. □

Observe that if g is Riemannian, i.e., positive definite, then for all conjugate instant t_0 along θ, $B_1(\theta, t_0)$ is positive definite, hence its signature coincides with the multiplicity of t_0 as a conjugate instant. It follows that $B_k(\theta, t_0) = 0$ for all $k \geq 2$, and the Maslov index of θ is equal to the sum of the multiplicities of all the conjugate instants along θ in $]0, 1]$. Likewise, for any semi-Riemannian metric g, $B_1(\theta, 0)$ coincides with the metric g on $T_{\theta(0)}M$, which is nondegenerate hence $B_k(\theta, 0) = 0$ for all $k > 1$, and the contribution to the Maslov index $\mu_{L_0}(\gamma)$ given by the initial instant 0 can be computed from formula (3.6):

$$n^+\big(B_1(\theta, 0)\big) - \dim\big(T_{\theta(0)}M\big) = n^+(g) - n = -n^-(g). \tag{4.4}$$

4.3. **Computation of the partial signatures at a conjugate instant.** We will now give an operational method for computing the partial signatures of a geodesic $\theta : [0, 1] \to M$ at a conjugate instant $t_0 \in]0, 1]$:

Proposition 4.4. *Assume that the map $\mathbb{J} \ni J \mapsto J'(t_0) \in I\!R^n$ is injective (hence an isomorphism); then the partial signatures of θ at t_0 coincide with the partial signatures of the curve $\mathcal{B}$ of symmetric bilinear forms on $\mathbb{J}$ given by:*

$$\mathcal{B}_t(J_1, J_2) = g\big(J_1(t), J_2'(t)\big). \tag{4.5}$$

Similarly, if $T : I\!R^n \to I\!R^n$ is a g-symmetric isomorphism of $I\!R^n$ such that the map $\mathbb{J} \ni J \mapsto J(t_0) - TJ'(t_0) \in I\!R^n$ is injective, then the partial signatures of θ at t_0 coincide with the partial signatures of the curve $\mathcal{B}$ of symmetric bilinear forms on $\mathbb{J}$ given by:

$$\mathcal{B}_t(J_1, J_2) = g\big(J_1(t), J_2'(t) - T^{-1}J_2(t)\big). \tag{4.6}$$

Proof. The condition that the map $S : \mathbb{J} \to I\!R^n$, $S(J) = J'(t_0)$, be an isomorphism is equivalent to the transversality of the Lagrangian

$$\gamma(t_0) = \Phi_{t_0}(L_0) = \big\{\big(J(t_0), J'(t_0)\big) : J \in \mathbb{J}\big\}$$

(recall that Φ has been defined in Subsection 4.1 as the flow of the Morse–Sturm equation $V'' = RV$) with the Lagrangian $L_1 = I\!R^n \oplus \{0\}$. Let us use the chart φ_{L_0,L_1} around $\gamma(t_0)$ (Subsection 3.2), and let us identify the space $\mathbb{J}$ with L_0 via the map:

$$L_0 \ni v \longmapsto J_v \in \mathbb{J} \tag{4.7}$$

where J_v is the unique Jacobi field in $\mathbb{J}$ determined by the initial condition $J'(0) = v$. A straightforward calculation shows that $\varphi_{L_0,L_1} \circ \gamma$ is given by:

$$\varphi_{L_0,L_1} \circ \gamma(t)(v_1, v_2) = g\big(J_{S^{-1}(v_1)}(t_0), v_2\big), \quad t \sim t_0. \tag{4.8}$$

The conclusion follows from Proposition 2.16, observing that (4.5) and (4.8) are cogredient:

$$\mathcal{B}_t(v_1, v_2) = \varphi_{L_0,L_1} \circ \gamma(t)\big(Sv_1, Sv_2\big).$$

Similarly, the proof of the second part of the statement reduces to a straightforward direct calculation of $\varphi_{L_0,\widetilde{L}_1} \circ \gamma$, where $\widetilde{L}_1 = \big\{(Tw, w) : w \in I\!R^n\big\} = \mathrm{Gr}(T^{-1})$. Note that $\widetilde{L}_1$ is Lagrangian because T is g-symmetric, transversal to L_0 because T is invertible, and transversal to $\gamma(t_0)$ due to the assumption that $\mathbb{J} \ni J \mapsto J(t_0) - TJ'(t_0) \in I\!R^n$ is injective. □

Clearly, different choices of the isomorphism T as in the assumptions of Proposition 4.4 produce cogredient curves of bilinear forms as in (4.6), so that the partial signatures of (4.6) do not depend on the choice of the isomorphism T. Observe that the set of isomorphisms T as in the assumptions of Proposition 4.4 is diffeomorphic to $\Lambda_0\big(\gamma(t_0)\big)\cap\Lambda_0(L_0)\cap\Lambda_0(L_1)$, which is a dense open subset of Λ by Baire's theorem.

Using Proposition 4.4, the computation of the Maslov index of a geodesic reduces to simple computations involving the curvature tensor and its derivatives. For instance, in the case of a simple conjugate instant t_0 (i.e., a conjugate instant of multiplicity one), if $J \in \mathbb{J}$ is a nontrivial Jacobi field vanishing at t_0, then the derivatives of the map $h(t) = g\big(J(t), J'(t)\big)$ at $t = t_0$ are given by:

$$h'(t_0) = g\big(J'(t_0), J'(t_0)\big),\ h''(t_0) = g\big(J'(t_0), R(t_0)J'(t_0)\big),$$
$$h^{(3)}(t_0) = g\big(J'(t_0), R'(t_0)J'(t_0)\big),\ \ldots,\ h^{(k)}(t_0) = g\big(J'(t_0), R^{(k-2)}(t_0)J'(t_0)\big), \ldots$$

From Proposition 4.4 we also get the following result:

Corollary 4.5. *Let $t_0 \in \,]0,1]$ be a conjugate instant along θ and let $J_0 \in \mathbb{J}$ be such that $J_0(t_0) = 0$; set $v_0 = J_0'(0)$. Then, $v_0 \in B_k$ if and only if there exists a smooth curve $v : \,]t_0-\varepsilon, t_0+\varepsilon[\, \to T_{\theta(0)}M$ with $v(t_0) = v_0$ such that the map $]t_0-\varepsilon, t_0+\varepsilon[\, \ni t \mapsto J_{v(t)}(t)$ has a zero of order greater than or equal to k at $t = t_0$.*

Proof. Choose a g-symmetric linear operator $T : I\!R^n \to I\!R^n$ as in the second part of Proposition 4.4 and consider the corresponding path of bilinear forms $\mathcal{B}_t$ as in (4.6). Observe that when J runs in $\mathbb{J}$, $t \mapsto J'(t) - T^{-1}J(t)$ is a smooth map that takes arbitrary values in $I\!R^n$ at $t = t_0$. Using this fact, an easy induction argument on k shows that a curve $t \mapsto J_t$ in $\mathbb{J}$ is a root function of order greater than or equal to k for $\mathcal{B}_t$ at $t = t_0$ if and only if the map $t \mapsto J_t(t)$ has a zero of order greater than or equal to k at $t = t_0$. This concludes the proof. □

4.4. **A generalized Morse index.** For all $t \in \,]0,1]$, define H_t as the Sobolev space $H_0^1\big([0,t], I\!R^n\big)$; let S_t^θ be the bounded symmetric bilinear form on H_t given by:

$$S_t^\theta(V,W) = \int_0^t \Big[g\big(V'(s), W'(s)\big) + g\big(R(s)V(s), W(s)\big)\Big]\,\mathrm{d}s, \tag{4.9}$$

where g is a nondegenerate symmetric bilinear form on $I\!R^n$ and $t \mapsto R(t)$ is a smooth curve of g-symmetric endomorphisms of $I\!R^n$. The objects g and R are obtained respectively from the metric g and the curvature tensor R of g via a parallel trivialization of TM along θ. By identifying the spaces H_t with H_1 via the map $h_t : H_t \to H_1$, $(h_tV)(s) = V(ts)$, we get a smooth curve of bounded symmetric bilinear forms $\bar{S}_t^\theta$ on H_1 obtained by the push-forward of S_t^θ by h_t. More explicitly:

$$\bar{S}_t^\theta(V,W) = \int_0^1 \Big[\tfrac{1}{t}g\big(V'(s), W'(s)\big) + tg\big(R(ts)V(s), W(s)\big)\Big]\,\mathrm{d}s, \quad V, W \in H_1.$$

The kernel $\mathcal{N}_t$ of $\bar{S}_t^\theta$, which is clearly given by the image of $\mathrm{Ker}(S_t^\theta)$ by h_t, consists of smooth vector fields V on $[0,1]$ such that $V(0) = V(1) = 0$ and satisfying the linear equation:

$$V''(s) = t^2R(ts)V(s), \quad s \in [0,1].$$

The map $]0,1] \ni t \mapsto \bar{S}_t^\theta$ is a smooth map of Fredholm bounded symmetric bilinear forms on H_1, and the map: $\mathcal{S}_t^\theta := t\bar{S}_t^\theta$:

$$\mathcal{S}_t^\theta(V,W) = \int_0^1 \Big[g\big(V'(s), W'(s)\big) + t^2g\big(R(ts)V(s), W(s)\big)\Big]\,\mathrm{d}s, \tag{4.10}$$

admits a real-analytic extension to $t = 0$ obtained by setting:

$$\mathcal{S}_0^\theta(V,W) = \int_0^1 g\big(V'(s), W'(s)\big)\,\mathrm{d}s.$$

It is easy to prove that, for $\varepsilon > 0$ small enough, $\mathcal{S}^\theta_t$ is nondegenerate for all $t \in [0, \varepsilon]$, hence:

$$\mathrm{Ker}\big(\mathcal{S}^\theta_0\big) = \{0\} \quad \text{and} \quad \mathrm{sf}\big(\mathcal{S}^\theta, [0,1]\big) = \mathrm{sf}\big(\mathcal{S}^\theta, [\varepsilon,1]\big) = \mathrm{sf}\big(\bar{S}^\theta, [\varepsilon,1]\big).$$

Definition 4.6. The *generalized Morse index* of θ is defined as:

$$\mathrm{i}_{\text{Morse}}(\theta) := \dim\big(\mathrm{Ker}(\mathcal{S}^\theta_1)\big) - \mathrm{sf}\big(\mathcal{S}^\theta, [0,1]\big).$$

Recalling Remark 2.13, if (M, g) is Riemannian, then the generalized Morse index of θ coincides with the *extended Morse index* of the geodesic action functional at the critical point θ.

We observe here that, for all $t \in \,]0,1]$, $\mathcal{S}^\theta_t$ is realized by a compact perturbation of $\mathcal{S}^\theta_0$, which in turn is realized by a self-adjoint symmetry of the Hilbert space $\mathcal{H}^\theta$. Hence, formula (2.8) can be used to compute the spectral flow of $\mathcal{S}^\theta$, and we get:

$$\mathrm{i}_{\text{Morse}}(\theta) = \dim\big(\mathrm{Ker}(\mathcal{S}^\theta_1)\big) - \dim_{V^-(\mathcal{S}^\theta_1)}\big(V^-(\mathcal{S}^\theta_0)\big) = \dim_{V^+(\mathcal{S}^\theta_0)}\big(V^+(\mathcal{S}^\theta_1)\big). \tag{4.11}$$

4.5. **The spectral index.** Important classes of examples where one can apply the Maslov index theory for real-analytic curves arises naturally when one studies certain eigenvalue problems for ODE's, whose solutions depend analytically on the eigenvalue by standard regularity results.

We will consider in what follows the case of Morse–Sturm–Liouville equation in $I\!R^n$, whose spectral index is given as the Maslov index of a certain curve parameterized by the spectral parameter λ. In this case, each negative (real) eigenvalue of the equation gives a contribution to the Maslov index, and it is possible to compute explicitly the bilinear forms B_k at each eigenvalue.

Let us consider a nondegenerate symmetric bilinear form $g : I\!R^n \times I\!R^n \to I\!R$ and a continuous map $[0,1] \ni t \mapsto R(t)$ of g-symmetric endomorphisms of $I\!R^n$. The *Morse–Sturm–Liouville equation* with data (g and) R is given by:

$$-v'' + (R - \lambda)v = 0, \tag{4.12}$$

where $v : [0,1] \to I\!R^n$ and $\lambda \in I\!R$. The corresponding differential operator, denoted by $\mathfrak{J}_\lambda$:

$$\mathfrak{J}_\lambda = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (R - \lambda),$$

defined on the domain $\mathcal{D} = H^1_0\big([0,1], I\!R^n\big) \cap H^2\big([0,1], I\!R^n\big)$, is an unbounded and, unless g is positive or negative definite, non normal linear operator. We will consider the following symmetric bounded and nondegenerate bilinear form $\widehat{g}$ on $L^2\big([0,1], I\!R^n\big)$:

$$\widehat{g}(v, w) = \int_0^1 g\big(v(t), w(t)\big)\,\mathrm{d}t.$$

It is easy to see that $\mathfrak{J}_\lambda$ is $\widehat{g}$-symmetric:

$$I^\theta_\lambda(v,w) := \widehat{g}\big(\mathfrak{J}_\lambda v, w\big) = \int_0^1 \Big[g\big(v'(t), w'(t)\big) + g\big((R(t)-\lambda)v(t), w(t)\big)\Big]\mathrm{d}t = \widehat{g}\big(v, \mathfrak{J}_\lambda w\big),$$

for all $v, w \in \mathcal{D}$; in particular, $I^\theta_0 = \widehat{g}(\mathfrak{J}_0 \cdot, \cdot)$ coincides with the index form S^θ_1 as defined in 4.9. Moreover, it is easy to see that I^θ_λ is a Fredholm bilinear form for all λ (it is a compact perturbation of I^θ_0).

The spectral properties of $\mathfrak{J}_0$ have been studied in [13], we will recall here some facts: $\mathfrak{J}_0$ is discrete (i.e., it has compact resolvent), its spectrum $\mathfrak{s}(\mathfrak{J}_0)$ is a discrete subset of the strip:

$$\big\{z \in \mathbb{C} : \Re(z) \geq -\|R\|_\infty,\ |\Im(z)| \leq \|R\|_\infty\big\},$$

where $\|R\|_\infty$ denotes the supremum norm of R.

Definition 4.7. The *spectral index* $\mathrm{i}_{\text{spectral}}(\theta)$ of the geodesic θ is defined to be integer:

$$\mathrm{i}_{\text{spectral}}(\theta) = \dim\big(\mathrm{Ker}(I_0^\theta)\big) - \mathrm{sf}\big(I_\lambda^\theta, [-M_0, 0]\big), \tag{4.13}$$

where $M_0 > \|R\|_\infty$.

Both the generalized Morse index $\mathrm{i}_{\text{Morse}}(\theta)$ and the spectral index $\mathrm{i}_{\text{spectral}}(\theta)$ do not depend on the choice of the parallel trivialization of TM along θ; this fact can be proven directly, or obtained as a consequence of Theorem 4.9.

As an easy application of Proposition 3.31, we obtain the following:

Proposition 4.8. *The* generalized eigenspace $\mathcal{H}_\lambda = \bigcup_{n\ge1}\mathrm{Ker}\big((\lambda - \mathfrak{J}_0)^n\big)$ *is finite-dimensional for all* $\lambda \in \mathfrak{s}(\mathfrak{J}_0)$*, and the restriction of* $\widehat{g}$ *to* $\mathcal{H}_\lambda$ *is nondegenerate. The spectral index of* θ *is given by:*

$$\mathrm{i}_{\textit{spectral}}(\theta) = n^+\big(\widehat{g}|_{\mathcal{H}_0}\big) - n^+\big(I_0^\theta|_{\mathcal{H}_0}\big) + \sum_{\substack{\lambda\in\mathfrak{s}(\mathfrak{J}_0)\\ \lambda\in]-\infty,0[}} \sigma\big(\widehat{g}|_{\mathcal{H}_\lambda}\big). \tag{4.14}$$

Proof. Apply Proposition 3.31 to the Hilbert spaces

$$\mathcal{H} = L^2\big([0,1], I\!R^n\big), \quad \mathcal{D}_2 = H^2\big([0,1], I\!R^n\big)\cap H_0^1\big([0,1], I\!R^n\big), \quad \mathcal{D}_1 = H_0^1\big([0,1], I\!R^n\big),$$

the operator $T = \mathfrak{J}_0$, and the self-adjoint isomorphism $G : \mathcal{H} \to \mathcal{H}$ given by pointwise composition with the (constant) symmetric endomorphism g of $I\!R^n$.

Observe that $-\frac{\mathrm{d}^2}{\mathrm{d}t^2} : \mathcal{D}_2 \to \mathcal{H}$ is and isomorphism, and T is a compact perturbation of such isomorphism, hence a Fredholm operator of index 0. As to the "regularity" condition assumed in the hypotheses of Proposition 3.31, in our case it follows easily from the fact that, using standard bootstrap arguments, if $v \in H_0^1\big([0,1], I\!R^n\big)$ is such that there exists $z \in H_0^1\big([0,1], I\!R^n\big)$ with

$$\int_0^1 g(v', w')\,\mathrm{d}t = \int_0^1 g(z, w)\,\mathrm{d}t$$

for all $w \in H_0^1\big([0,1], I\!R^n\big)$, then $v \in H^2\big([0,1], I\!R^n\big)$.

In order to obtain (4.14) from (3.22) keep in mind that the boundary term corresponding to $\lambda = -M_0$ is null, because $-M_0 \notin \mathfrak{s}(\mathfrak{J}_0)$. □

Observe that when g is Riemannian, then $\mathfrak{J}_0$ is indeed self-adjoint, $\mathcal{H}_\lambda = \mathrm{Ker}(\lambda - \mathfrak{J}_0)$, in particular $S_1^\theta|_{\mathcal{H}_0} = 0$ and thus $n^+\big(\mathcal{S}_1^\theta\big|_{\mathcal{H}_0}\big) = 0$. Moreover, since $\widehat{g}$ is positive definite, $n^+\big(\widehat{g}|_{\mathcal{H}_\lambda}\big) = \sigma\big(\widehat{g}|_{\mathcal{H}_\lambda}\big) = \dim\big(\mathrm{Ker}(\lambda - \mathfrak{J}_0)\big)$, and $n^-(\widehat{g}) = 0$; this shows that in the Riemannian case the spectral index of θ coincides with the *extended Morse index* of the index form $\mathcal{S}_1^\theta = I_0^\theta$.

4.6. **The index theorem.** Before we get into the aimed index theorem, we will need to introduce the following notation. For all $\lambda \in I\!R$, let $\Phi_\lambda : [0,1] \to \mathrm{Sp}(I\!R^{2n}, \omega_g)$ denote the flow of the Morse–Sturm–Liouville equation (4.12), i.e., $\Phi_\lambda(t) : I\!R^{2n} \to I\!R^{2n}$ is the linear isomorphism defined by:

$$\Phi_\lambda(t)\big(v(0), v'(0)\big) = \big(v(t), v'(t)\big),$$

for all solutions v of (4.12).

We choose $M_0 > \|R\|_\infty$ and we consider the curve $\ell : [-M_0, 0] \to \Lambda$:

$$\ell(\lambda) = \Phi_\lambda(1)\big(L_0\big). \tag{4.15}$$

Observe that λ is real-analytic; moreover, the intersections of ℓ with the Maslov cycle occur precisely at each real nonpositive eigenvalue of the Jacobi differential operators $\mathfrak{J}_0$.

Everything is now ready to state and prove the following:

Theorem 4.9 (Index Theorem in the Degenerate Case)**.** *Let (M,g) be a semi-Riemannian manifold and $\theta:[0,1]\to M$ a geodesic. Then:*

$$\mathrm{i}_{\mathit{Maslov}}(\theta)=\mathrm{i}_{\mathit{spectral}}(\theta)=\mathrm{i}_{\mathit{Morse}}(\theta).$$

Proof. The equality $\mathrm{i}_{\mathrm{spectral}}(\theta)=\mathrm{i}_{\mathrm{Morse}}(\theta)$ is proven by an infinite dimensional homotopy argument. Namely, consider the two-parameter smooth map C of symmetric bilinear forms in $H^1_0\big([0,1],I\!R^n\big)$:

$$C(t,\lambda)(V,W)=\int_0^1\Big[g\big(V'(s),W'(s)\big)+t^2g\big((R(ts)-\lambda)V(s),W(s)\big)\Big]\,\mathrm{d}s, \tag{4.16}$$

$(t,\lambda)\in[0,1]\times[-M_0,0]$.

Observe that, by definition, $-\mathrm{i}_{\mathrm{spectral}}(\theta)+\dim\big(\mathrm{Ker}(C(1,0))\big)$ equals the spectral flow of the curve $[-M_0,0]\ni\lambda\mapsto C(1,\lambda)$, while $-\mathrm{i}_{\mathrm{Morse}}(\theta)+\dim\big(\mathrm{Ker}(C(1,0))\big)$ equals the spectral flow of $[0,1]\ni t\mapsto C(t,0)$. The equality $\mathrm{i}_{\mathrm{spectral}}(\theta)=\mathrm{i}_{\mathrm{Morse}}(\theta)$ follows from the fixed-endpoints homotopy invariance and the additivity by concatenation of the spectral flow, observing that the maps $[-M_0,0]\ni\lambda\mapsto C(0,\lambda)$ and $[0,1]\ni t\mapsto C(t,-M_0)$ have null spectral flow, due to the fact that $C(0,\lambda)$ and $C(t,-M_0)$ are always nondegenerate.

By a similar homotopy argument in Λ, one proves that $\mathrm{i}_{\mathrm{Maslov}}(\theta)$ equals the Maslov index of the curve ℓ defined in (4.15). Namely, $\mathrm{i}_{\mathrm{Maslov}}(\theta)$ is by definition the Maslov index of the curve γ given by $[\varepsilon,1]\ni t\mapsto\Phi_0(t)(L_0)$, and the two-parameter map $[\varepsilon,1]\times[-M_0,0]\ni(t,\lambda)\mapsto\Phi_\lambda(t)(L_0)\in\Lambda$ gives a continuous homotopy between γ and ℓ. In this case, observe that the curve $[\varepsilon,1]\ni t\mapsto\Phi_{-M_0}(t)(L_0)$ does not intersect the Maslov cycle for all $\varepsilon\ge0$, while the curve $[-M_0,0]\ni\lambda\mapsto\Phi_\lambda(\varepsilon)(L_0)$ does not intersect the Maslov cycle provided that $\varepsilon>0$ is chosen sufficiently small.

Finally, the crucial part of the proof consists in showing that $\mathrm{i}_{\mathrm{spectral}}(\theta)$ equals $\mathrm{i}_{\mathrm{Maslov}}(\ell)$; in this case a direct homotopy argument cannot be used, because $\mathrm{i}_{\mathrm{spectral}}(\theta)$ is the negative spectral flow of the path of Fredholm bilinear forms $\lambda\mapsto C(1,\lambda)$ on $H^1_0([0,1],I\!R^n)$, while $\mathrm{i}_{\mathrm{Maslov}}(\ell)$ is the Maslov index of a curve in Λ. However, as we have observed both curves are real-analytic, and they have precisely the same degeneracy instants.

Observe that for this equality one can use the partial signatures theory, since both integers are Maslov indexes of real-analytic paths: for each eigenvalue $\lambda_0\in[-M_0,0]$, we prove that the spaces W_{k+1} and the bilinear forms B_{k+1} obtained from the two constructions coincide, up to the sign.

Let $\lambda_0\in[-M_0,0]$ be a degenerate value for $\lambda\mapsto C(1,\lambda)$ and let $T:I\!R^n\to I\!R^n$ be a g-symmetric linear endomorphism such that the Lagrangian $L_1'=\mathrm{Gr}(T^{-1})$ is transversal to $\Phi(1,\lambda_0)(L_0)$. Let us consider the following real analytic path $\lambda\mapsto\mathfrak{B}_\lambda$ of Fredholm symmetric bilinear forms:

$$\mathfrak{B}_\lambda(V,W)=\int_0^1\big[g\big(V',W'\big)+g\big((R-\lambda)V,W\big)\big]\mathrm{d}s-g\big(V(1),T^{-1}W(1)\big)$$

defined on the Hilbert space:

$$\mathcal{H}=\big\{V\in H^1([0,1],I\!R^n):V(0)=0\big\}.$$

Observe that the restriction of $\mathfrak{B}_\lambda$ to $H^1_0\big([0,1],I\!R^n\big)$ coincides with $C(1,\lambda)$, while the restriction of $\mathfrak{B}_\lambda$ to the finite dimensional space:

$$\mathbb{J}_\lambda:=\big\{J\in C^2\big([0,1],I\!R^n\big):J''=(R-\lambda)J,\ J(0)=0\big\}\subset\mathcal{H}$$

gives the symmetric bilinear form $D_\lambda:\mathbb{J}_\lambda\times\mathbb{J}_\lambda\to I\!R$:

$$D_\lambda(J_1,J_2)=g\big(J_1(1),J_2'(1)-T^{-1}J_2(1)\big).$$

Thus, recalling Proposition 4.4, if $\kappa_\lambda:L_0\to\mathbb{J}_\lambda$ is the isomorphism $v\mapsto J_v$, where J_v is the unique element of $\mathbb{J}_\lambda$ satisfying $J_v'(0)=v$, we consider the pull-back $(\kappa_\lambda)^*(D_\lambda)$, which is the symmetric bilinear form $\widetilde{D}_\lambda : L_0 \times L_0 \to I\!R$ given by:

$$\widetilde{D}_\lambda(v,w) := B_\lambda\big(\kappa_\lambda(v), \kappa_\lambda(w)\big) = \varphi_{L_0,L_1'}\big(\ell(\lambda)\big)(v,w),$$

for all $v, w \in L_0$. We observe that:

$$\mathrm{Ker}\big(C(1,\lambda_0)\big) = \mathbb{J}_{\lambda_0} \cap H_0^1\big([0,1], I\!R^n\big) = \mathrm{Ker}(D_{\lambda_0}) = \kappa_{\lambda_0}\big(\mathrm{Ker}(\widetilde{D}_{\lambda_0})\big).$$

Our aim is to show that, for all $k \geq 0$,

$$W_k\big(C(1,\lambda);\lambda_0\big) = \kappa_{\lambda_0}\big(W_k\big(\widetilde{D}_\lambda;\lambda_0\big)\big), \tag{4.17}$$

and that:

$$B_k\big(C(1,\lambda);\lambda_0\big) = -(\kappa_{\lambda_0})^*\big(B_k\big(\widetilde{D}_\lambda;\lambda_0\big)\big), \tag{4.18}$$

for all eigenvalue $\lambda_0 \in [-M_0, 0]$. Recalling Proposition 2.15, the equality $\mathrm{i}_{\mathrm{spectral}}(\theta) = \mathrm{i}_{\mathrm{Maslov}}(\ell)$ will follow at once from (4.17) and (4.18).

Let $k \geq 1$ be fixed, and choose $v_0 \in L_0$; if $\lambda \mapsto v_\lambda$ is a root function of order greater than or equal to k for $\widetilde{D}_\lambda$ at $\lambda = \lambda_0$, and such that $v_{\lambda_0} = v_0$, then by Corollary 4.5 the map $\lambda \mapsto \kappa_\lambda(v_\lambda)(1)$ has a zero of order greater than or equal to k at $\lambda = \lambda_0$. Thus, the map $\lambda \mapsto \widetilde{J}_\lambda \in H_0^1\big([0,1], I\!R^n\big)$ defined by:

$$\widetilde{J}_\lambda(t) = J_\lambda(t) - tJ_\lambda(1),$$

where

$$J_\lambda = \kappa_\lambda(v_\lambda),$$

is a root function of order greater than or equal to k for $C(1,\lambda)$ at $\lambda = \lambda_0$, with $\widetilde{J}_{\lambda_0} = \kappa_{\lambda_0}(v_0)$. For, an easy computation shows that for all $W \in H_0^1\big([0,1], I\!R^n\big)$:

$$C(1,\lambda)\big(\widetilde{J}_\lambda, W\big) = -\int_0^1 tg\big((R(t)-\lambda)J_\lambda(1), W(t)\big)\,\mathrm{d}t.$$

This shows that we have an inclusion that $W_k\big(C(1,\lambda);\lambda_0\big) \supset \kappa_{\lambda_0}\big(W_k\big(\widetilde{D}_\lambda;\lambda_0\big)\big)$.

To prove the opposite inclusion, we use the result of part (1) in Proposition 2.9 arguing as follows. Choose a positive definite inner product g^+ in $I\!R^n$ and let $A : I\!R^n \to I\!R^n$ the g-symmetric automorphism such that $g^+ = g(A\cdot,\cdot)$; define the following Hilbert space inner product in $H_0^1\big([0,1], I\!R^n\big)$:

$$\langle\!\langle V, W\rangle\!\rangle := \int_0^1 g^+\big(V'(t), W'(t)\big)\,\mathrm{d}t = \int_0^1 g\big(AV'(t), W'(t)\big)\,\mathrm{d}t.$$

Let $\lambda \mapsto \mathbf{L}_\lambda \in \mathcal{F}^{\mathrm{sa}}\big(H_0^1\big([0,1], I\!R^n\big)\big)$ be the real-analytic path of self-adjoint operators that realize $C(1,\lambda)$ with respect to the above inner product, and let $\lambda \mapsto \sigma(\lambda) \in I\!R$, $\lambda \mapsto V_\lambda \in H_0^1\big([0,1], I\!R^n\big)$ be real-analytic maps such that:

- $\mathbf{L}_\lambda(V_\lambda) = \sigma(\lambda)V_\lambda$ for all λ;
- σ has a zero of order greater than or equal to k at $\lambda = \lambda_0$.

Then, for all $W \in H_0^1\big([0,1], I\!R^n\big)$ the following equality holds:

$$\int_0^1 \Big[g\big((\mathrm{Id} - \sigma(\lambda)A)V_\lambda'(t), W'(t)\big) + g\big(R(t)-\lambda)V_\lambda(t), W(t)\big)\Big]\,\mathrm{d}t = 0,$$

from which it follows that V_λ is a map of class C^2 that satisfies the "perturbed" Jacobi equation:

$$V_\lambda'' = \big(\mathrm{Id} - \sigma(\lambda)A\big)^{-1}(R-\lambda)V_\lambda.$$

Observe that for λ near λ_0, the operator $\mathrm{Id} - \sigma(\lambda)A$ is invertible. Observe that also V_{λ_0} is in $\mathrm{Ker}\big(C(1,\lambda_0)\big)$, and by Proposition 2.9, the space $W_{k+1}\big(C(1,\lambda);\lambda_0\big)$ is generated by such functions V_{λ_0}. The map $\lambda \mapsto v_\lambda = V_\lambda'(0) \in L_0$ is real-analytic, and we claim that it is a root function for $\widetilde{D}_\lambda$ at $\lambda = \lambda_0$ of order greater than or equal to k, with $\kappa_{\lambda_0}(v_{\lambda_0}) = V_{\lambda_0}$.

By Corollary 4.5, to prove our assertion it suffices to show that the map $\lambda \mapsto \kappa_\lambda(v_\lambda)(1) \in \mathbb{R}^n$ has a zero of order greater than or equal to k at $\lambda = \lambda_0$. To prove this fact, observe that $J_\lambda = \kappa_\lambda(v_\lambda)$ is the solution of the equation $J_\lambda'' = (R - \lambda)J_\lambda$ satisfying $J_\lambda(0) = 0$ and $J_\lambda'(0) = v_\lambda$. For λ near λ_0, say $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$, $\delta > 0$, we set:

$$B(\lambda) = \sum_{n=1}^{\infty} \sigma(\lambda)^{n-1} A^n,$$

so that $(\mathrm{Id} - \sigma(\lambda)A)^{-1} = \mathrm{Id} + \sigma(\lambda)B(\lambda)$; then, for $s \in [0, 1]$:

$$\begin{aligned}
\big|V_\lambda(s) - J_\lambda(s)\big| &\le \int_0^s \big|V_\lambda'(\tau) - J_\lambda'(\tau)\big|\,\mathrm{d}\tau \le \int_0^s \mathrm{d}\tau \int_0^\tau \big|V_\lambda''(r) - J_\lambda''(r)\big|\,\mathrm{d}r \\
&= \int_0^s \mathrm{d}\tau \int_0^\tau \big|(R(r) - \lambda)(V_\lambda(r) - J_\lambda(r)) + \sigma(\lambda)B(\lambda)(R(r) - \lambda)V_\lambda\big|\,\mathrm{d}r \\
&= \int_0^s \mathrm{d}r \int_r^s \big|(R(r) - \lambda)(V_\lambda(r) - J_\lambda(r)) + \sigma(\lambda)B(\lambda)(R(r) - \lambda)V_\lambda\big|\,\mathrm{d}\tau \\
&= \int_0^s (s - r)\big|(R(r) - \lambda)(V_\lambda(r) - J_\lambda(r)) + \sigma(\lambda)B(\lambda)(R(r) - \lambda)V_\lambda\big|\,\mathrm{d}r \\
&\le \int_0^s \big|(R(r) - \lambda)(V_\lambda(r) - J_\lambda(r))\big| + |\sigma(\lambda)|\big|B(\lambda)(R(r) - \lambda))V_\lambda(r)\big|\,\mathrm{d}r \\
&\le d_0 \int_0^s \big|(V_\lambda(r) - J_\lambda(r))\big|\,\mathrm{d}r + d_1|\sigma(\lambda)|,
\end{aligned}$$

where:

$$d_0 = \max_{\substack{r\in[0,1]\\ \lambda\in[\lambda_0-\delta,\lambda_0+\delta]}} \|R(r) - \lambda\,\mathrm{Id}\|, \quad d_1 = \max_{\substack{r\in[0,1]\\ \lambda\in[\lambda_0-\delta,\lambda_0+\delta]}} \|B(\lambda)(R(r) - \lambda\,\mathrm{Id})\| \cdot |V_\lambda(r)|.$$

From Gronwall's Lemma, we then obtain:

$$\|V_\lambda - J_\lambda\|_\infty \le d_1 e^{d_0} |\sigma(\lambda)|,$$

and since $V_\lambda(1) = 0$, it follows that $J_\lambda(1)$ has a zero of order greater than or equal to k. This argument shows that we have an inclusion $W_k\big(C(1,\lambda);\lambda_0\big) \subset \kappa_{\lambda_0}\big(W_k\big(\widetilde{D}_\lambda;\lambda_0\big)\big)$ and (4.17) is proven.

Finally, we will now prove equality (4.18); it will suffice to show that, given root functions $\lambda \mapsto u_\lambda, v_\lambda \in L_0$ of order greater than or equal to k for $\widetilde{D}_\lambda$ at $\lambda = \lambda_0$, then:

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0} \widetilde{D}_\lambda\big(u_\lambda, v_\lambda\big) = -\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0} C(1,\lambda)\big(\widetilde{J}_\lambda^{(1)}, \widetilde{J}_\lambda^{(2)}\big), \tag{4.19}$$

where:

$$\widetilde{J}_\lambda^{(1)}(t) = J_\lambda^{(1)}(t) - tJ_\lambda^{(1)}(1), \quad \widetilde{J}_\lambda^{(2)}(t) = J_\lambda^{(2)}(t) - tJ_\lambda^{(2)}(1),$$

and

$$J_\lambda^{(1)} = \kappa_\lambda(u_\lambda), \quad J_\lambda^{(2)} = \kappa_\lambda(v_\lambda).$$

A direct computation gives:

$$\widetilde{D}_\lambda\big(u_\lambda, v_\lambda\big) = g\big(J_\lambda^{(1)}(1), (J_\lambda^{(2)})'(1) - T^{-1}J_\lambda^{(2)}(1)\big),$$

and since $g\big(J_\lambda^{(1)}(1), T^{-1}J_\lambda^{(2)}(1)\big)$ has a zero of order greater than or equal to $2k$ at $\lambda = \lambda_0$, it follows:

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0} \widetilde{D}_\lambda\big(u_\lambda, v_\lambda\big) = \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0} g\big(J_\lambda^{(1)}(1), (J_\lambda^{(2)})'(1)\big). \tag{4.20}$$

On the other hand, integration by parts yields the following:

$$C(1,\lambda)\big(\widetilde{J}_\lambda^{(1)}, \widetilde{J}_\lambda^{(2)}\big) = -\int_0^1 t\, g\big((R(t) - \lambda)J_\lambda^{(1)}(1), \widetilde{J}_\lambda^{(2)}(t)\big)\,\mathrm{d}t;$$

and, again because of the fact that $g\big(J_\lambda^{(1)}(1), J_\lambda^{(2)}(1)\big)$ has a zero of order greater than or equal to $2k$ at $\lambda = \lambda_0$, we get:

$$\begin{aligned}
&\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0} C(1,\lambda)\big(\widetilde{J}_\lambda^{(1)}, \widetilde{J}_\lambda^{(2)}\big)\\
&\quad= -\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0} \int_0^1 tg\big((R(t)-\lambda)J_\lambda^{(1)}(1), J_\lambda^{(2)}(t)\big)\,\mathrm{d}t\\
&\quad= -\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0} \int_0^1 tg\big(J_\lambda^{(1)}(1), (R(t)-\lambda)J_\lambda^{(2)}(t)\big)\,\mathrm{d}t\\
&\quad= -\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0} \int_0^1 tg\big(J_\lambda^{(1)}(1), (J_\lambda^{(2)})''(t)\big)\,\mathrm{d}t\\
&\quad= \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0} \Big[\int_0^1 g\big(J_\lambda^{(1)}(1), (J_\lambda^{(2)})'(t)\big)\,\mathrm{d}t - g\big(J_\lambda^{(1)}(1), (J_\lambda^{(2)})'(1)\big)\Big]\\
&\quad= \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0} \Big[g\big(J_\lambda^{(1)}(1), J_\lambda^{(2)}(1)\big) - g\big(J_\lambda^{(1)}(1), (J_\lambda^{(2)})'(1)\big)\Big]\\
&\quad= -\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0} g\big(J_\lambda^{(1)}(1), (J_\lambda^{(2)})'(1)\big).
\end{aligned} \tag{4.21}$$

Comparison of (4.20) and (4.21) give (4.19), and the proof is concluded. □

4.7. **On a counterexample for the equality of the conjugate and the Maslov index.** The sum of the signatures of the conjugate points along a geodesic in a real-analytic semi-Riemannian manifold is called the *conjugate index* in reference [32] (or *focal index* in reference [38]). It was erroneously stated in [32] that the conjugate index is equal to the Maslov index of a semi-Riemannian geodesic, and in [38, Subsection 5.4] the authors have given a counterexample to such equality, occurring in the case of a degenerate conjugate point along a Lorentzian spacelike geodesic. Using the results of the present paper we are now able to have a better view of the phenomenon.

Recall that the counterexample mentioned consists in a spacelike geodesic $\theta : [-\varepsilon, \varepsilon] \to M$, where M is a real-analytic three-dimensional Lorentzian manifold, $\varepsilon > 0$, having a unique conjugate point at $t = 0$. By a parallel transport of the normal bundle $\dot\theta^\perp$ along θ, and using suitable coordinate systems in the Lagrangian Grassmannian as explained above, the Maslov index of θ is computed as the spectral flow through $t = 0$ of the curve of symmetric bilinear forms on $I\!R^2$ given by:

$$[-\varepsilon, \varepsilon] \ni t \longmapsto \mathbf{L}(t) = \begin{pmatrix} x(t) & z(t) \\ z(t) & y(t) \end{pmatrix},$$

where:

$$x(t) = -2t^3 - \frac{54}{5}t^5, \quad y(t) = -1 - 6t + 18t^2 - 54t^3, \quad z(t) = -3t^2.$$

One computes easily:

$$\mathbf{L}_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{L}_1 = \begin{pmatrix} 0 & 0 \\ 0 & -6 \end{pmatrix}, \quad \mathbf{L}_2 = \begin{pmatrix} 0 & 3 \\ 3 & 18 \end{pmatrix}, \quad \mathbf{L}_0 = \begin{pmatrix} -2 & 0 \\ 0 & -54 \end{pmatrix},$$

hence $W_1 = \mathrm{Ker}(\mathbf{L}_0) = I\!R \oplus \{0\}$ and $B_1 = \langle L_1 \cdot, \cdot\rangle|_{W_1} = 0$, and $t = 0$ is a degenerate conjugate instant having signature equal to 0. Nevertheless, the Maslov index of θ, which is equal to the spectral flow of $\mathbf{L}$, is easily computed as:

$$n^+\big(\mathbf{L}(\varepsilon)\big) - n^+\big(\mathbf{L}(-\varepsilon)\big) = -1,$$

providing a counterexample for the equality between the Maslov and the conjugate index in the case of degenerate conjugate points. An elementary direct computation gives:

$$W_2 = I\!R \oplus \{0\},\ B_2 = 0, \quad W_3 = I\!R \oplus \{0\},\ B_3\big((\alpha, 0), (\beta, 0)\big) = -2\alpha\beta,$$

hence $\sigma_1 = \sigma_2 = 0$, $\sigma_3 = -1$ and $\sigma_k = 0$ for all $k > 3$. Obviously, $\mathrm{i}_{\text{Maslov}}(\theta)$ equals the sum $\sum_{k\geq 1} \sigma_k(\theta, 0)$, as it must be in compliance with Proposition 4.3.

4.8. **A geometrical version of the semi-Riemannian index theorem.** Using an abstract result on the computation of the relative index of Fredholm bilinear forms, we will now give a geometrical version of the index theorem, in the spirit of the semi-Riemannian index theorem in [46]. The Maslov index of a semi-Riemannian geodesic can be computed as the difference between the index and the coindex of suitable restrictions of the index form S_1^θ (4.9); this version of the index theorem gives a link with variational problems and aims at developments of Morse homology in the infinite dimensional Hilbert manifold of all paths in M joining two fixed points, as in [30, 47].

Let (M, g) be a semi-Riemannian manifold, set $k = n^-(g)$. Let $\theta : [0, 1] \to M$ be a geodesic; a *maximal negative distribution* along θ is a smooth family $\mathcal{D}_t \subset T_{\theta(t)}M$ of k-dimensional subspaces, $t \in [a, b]$, such that $g|_{\mathcal{D}_t}$ is negative definite for all t. By "smooth", we mean that $\mathcal{D}_t$ is the span of $Y_1(t), \ldots, Y_k(t)$ for all $t \in [0, 1]$, where $Y_1, \ldots, Y_k$ is a family of smooth vector fields along θ; such a family $Y_1, \ldots, Y_k$ will be called a *frame* for $\mathcal{D}$. Associated to each choice of a maximal negative distribution $\mathcal{D}$ along θ one can define two closed spaces of variational vector fields along θ: the space of vector fields along θ taking values in $\mathcal{D}$, denoted by $\mathcal{Q}$, and the space of vector fields along θ that are "Jacobi in the directions of $\mathcal{D}$", denoted by $\mathcal{K}$. More precisely, denote by $\mathcal{H}^\theta$ the space of all vector fields of Sobolev class H^1 along θ vanishing at the endpoints; fix a frame $Y_1, \ldots, Y_k$ for $\mathcal{D}$ and define:

$$\mathcal{Q} = \big\{ v \in \mathcal{H}^\theta : v(t) \in \mathcal{D}_t, \text{ for all } t \in [0, 1] \big\},$$

$$\mathcal{K} = \big\{ v \in \mathcal{H}^\theta : g(v', Y_i) \text{ is of class } H^1, \, g(v', Y_i)' = g(v', Y_i') + g\big(R(\dot\theta, v)\,\dot\theta, Y_i\big)\ \forall i \big\}.$$

Observe that a vector field v of class C^2 along θ belongs to $\mathcal{K}$ if and only if $v'' - R(\dot\theta, v)\dot\theta$ is pointwise orthogonal to $\mathcal{D}$, i.e., fields in $\mathcal{K}$ are interpreted as "Jacobi fields in the directions of $\mathcal{D}$"; geometrical and analytical descriptions of the spaces $\mathcal{Q}$ and $\mathcal{K}$ can be found in [46].

Proposition 4.10. *The restriction S_1^θ to $\mathcal{K}$ has finite index, and the restriction of S_1^θ to $\mathcal{Q}$ has finite coindex; moreover, the following equality holds*

$$\mathrm{i}_{\text{Morse}}(\theta) = n^-\big(S_1^\theta|_{\mathcal{K}}\big) - n^+\big(S_1^\theta|_{\mathcal{Q}}\big) + \dim\big(\mathrm{Ker}(S_1^\theta)\big). \tag{4.22}$$

Proof. The restriction of S_1^θ to $\mathcal{Q}$ is realized by a compact perturbation of a negative isomorphism of $\mathcal{Q}$ ([46, Corollary 5.25]), hence it has finite coindex. The restriction of S_1^θ to $\mathcal{K}$ is realized by a compact perturbation of a positive isomorphism of $\mathcal{K}$ ([45, Lemma 2.6.6]), hence it has finite coindex. Moreover, an immediate calculation shows that $\mathcal{K} = \mathcal{Q}^{\perp_{S_1^\theta}}$.

By Lemma A.8, $\mathcal{Q}$ is commensurable with $V^-(\mathcal{S}_1^\theta)$, and using the abstract result of Proposition A.11, the relative dimension $\dim_{\mathcal{Q}}\big(V^-(\mathcal{S}_1^\theta)\big)$ can be computed as:

$$\dim_{\mathcal{Q}}\big(V^-(\mathcal{S}_1^\theta)\big) = n^-\big(S_1^\theta|_{\mathcal{K}}\big) - n^+\big(S_1^\theta|_{\mathcal{Q}}\big), \tag{4.23}$$

where $V^-(\mathcal{S}_1^\theta)$ is the negative eigenspace of the realization of $\mathcal{S}_1^\theta$ relatively to any Hilbert structure on $\mathcal{H}^\theta$. In order to compute the left hand side in equality (4.23), we will first show that its value does not depend on the choice of a maximal negative distribution along θ.

The idea to prove the independence of the relative index from the choice of a maximal negative distribution consists in showing that any two maximal negative distributions can be joined by a "continuous" selection of maximal negative distributions, and that the relative index depends "continuously" on such selection. Let us make the argument formal, as follows. In first place, using a parallel trivialization of the tangent bundle TM along θ, the problem is reduced to studying the equality of the relative dimensions:

$$\dim_{\mathcal{Q}^0}\big(\mathcal{H}_0\big) \quad \text{and} \quad \dim_{\mathcal{Q}^1}\big(\mathcal{H}_0\big),$$

where

- $\mathcal{H}_0$ is a closed subspace ($\cong V^-(\mathcal{S}_1^\theta)$) of the Sobolev space $\mathcal{H} = H_0^1\big([a,b], I\!R^n\big)$ ($\cong \mathcal{H}^\theta$)
- g is a fixed nondegenerate symmetric bilinear form on $I\!R^n$ of index k;
- denoting by $G_k^-(n)$ the Grassmannian of all k-dimensional subspaces of $I\!R^n$ on which g is negative definite, $\mathcal{D}^0, \mathcal{D}^1 : [a,b] \to G_k^-(n)$ are continuous curves;
- $\mathcal{Q}^i = \big\{v \in \mathcal{H} : v(t) \in \mathcal{D}^i(t) \text{ for all } t \in [a,b]\big\}$, $i = 0,1$, are closed subspaces of $\mathcal{H}$ that are commensurable to $\mathcal{Q}_0$ and $\mathcal{Q}_1$.

The set $G_k^-(n)$ is open in the Grassmannian $G_k(n)$, and it is an arc-connected set (see Appendix B); hence, the curves $\mathcal{D}^0$ and $\mathcal{D}^1$ are homotopic. Choose a continuous map $[a,b]\times[0,1] \ni (t,s) \mapsto \mathcal{D}_{t,s} \in G_k^-(n)$ such that $\mathcal{D}_{t,i} = \mathcal{D}_t^i$ for $i = 0,1$ and for all $t \in [a,b]$, and for all $s \in [0,1]$ set:

$$\mathcal{Q}^s = \big\{v \in \mathcal{H} : v(t) \in \mathcal{D}_{s,t} \text{ for all } t \in [a,b]\big\};$$

using the argument above, $\mathcal{Q}^s$ is commensurable with $V^-(\mathcal{S}_1^\theta)$ for all s.

Let us prove that the map $s \mapsto \mathcal{Q}^s$ is a *continuous family of closed subspaces* of $\mathcal{H}$ (see Appendix A), i.e., that $\mathcal{Q}^s$ is the image of a fixed closed subspace $\mathcal{Q}^*$ of $\mathcal{H}$ via a continuous family of isomorphisms $\phi_s : \mathcal{H} \to \mathcal{H}$. For all fixed $S^* \in G_k^-(n)$, the map $\mathrm{GL}(n, I\!R) \ni U \mapsto U(S^*) \in G_k(n)$ is a smooth fibration; choose a continuous lifting $[a,b]\times[0,1] \ni (t,s) \mapsto U_{t,s} \in \mathrm{GL}(n,I\!R)$ of the map $(t,s) \mapsto \mathcal{D}_{t,s}$, i.e.,

$$U_{t,s}(S^*) = \mathcal{D}_{t,s}, \qquad \forall\,(t,s) \in [a,b]\times[0,1].$$

Finally, define

$$\mathcal{Q}^* = \big\{v \in \mathcal{H} : v(t) \in S^* \text{ for all } t \in [a,b]\big\},$$

and for all $s \in [0,1]$ let $\phi_s : \mathcal{H} \to \mathcal{H}$ be the isomorphism given by:

$$\phi_s(x)(t) = U_{t,s}\big(x(t)\big), \quad x \in \mathcal{H};$$

clearly, $\phi_s(\mathcal{Q}^*) = \mathcal{Q}^s$ for all s, and $s \mapsto \phi_s \in \mathrm{GL}(\mathcal{H})$ is continuous. This proves that $s \mapsto \mathcal{Q}^s$ is a continuous family of closed subspaces of $\mathcal{H}$, and thus, by Corollary A.5, $\dim_{\mathcal{Q}^s}(\mathcal{H}_0)$ is constant.

Once the independence on the choice of the maximal negative distribution has been established, to prove equality (4.23) we will now choose a maximal negative distribution $\mathcal{D}^- = \{D_t^-\}_{t\in[0,1]}$ which is obtained by the parallel transport along θ of a maximal negative subspace of $T_{\theta(0)}M$; let us also denote by $\mathcal{D}^+ = \{\mathcal{D}_t^+\}_{t\in[0,1]}$ a maximal positive distribution along θ which is obtained by the parallel transport along θ of a maximal positive subspace of $T_{\theta(0)}M$. In order to compute the left hand side of (4.23) we will also have to choose a Hilbert space inner product in $\mathcal{H}^\theta$; to this aim, a convenient choice is to set:

$$\langle V, W\rangle = \int_0^1 g_t^+\big(V'(t), W'(t)\big)\,\mathrm{d}t, \qquad V, W \in \mathcal{H}^\theta,$$

where g_t^+ is the unique positive definite inner product on $T_{\theta(t)}M$ for which $\mathcal{D}_t^+$ and $\mathcal{D}_t^-$ are orthogonal spaces, that coincides with g on $\mathcal{D}_t^+$ and with $-g$ on $\mathcal{D}_t^-$. It is easy to see that, with such choice, the space $\mathcal{Q}$ is precisely the negative eigenspace of $\mathcal{S}_0^\theta$, and recalling (4.11), we compute easily:

$$\begin{aligned}\dim_{\mathcal{Q}}\big(V^-(\mathcal{S}_1^\theta)\big) &= \dim_{V^-(\mathcal{S}_0^\theta)}\big(V^-(\mathcal{S}_1^\theta)\big) = -\dim_{V^-(\mathcal{S}_1^\theta)}\big(V^-(\mathcal{S}_0^\theta)\big)\\ &= \mathrm{i}_{\text{Morse}}(\theta) - \dim\big(\mathrm{Ker}(\mathcal{S}_1^\theta)\big).\end{aligned} \tag{4.24}$$

Equality (4.22) follows immediately from (4.23) and (4.24). □

4.9. **Bifurcation of geodesics at a conjugate instant.** As a further application of our theory, we will discuss briefly a simple consequence of Proposition 4.3 obtained using recent results in bifurcation theory for strongly indefinite variational problems (see [28]).

Let us recall the definition of bifurcation for a smooth family of functionals at a common critical point. Given a family $\{f_r\}_{r\in[c,d]}$ of smooth functionals on some Hilbert space $\mathcal{H}$ depending smoothly on the parameter r, assume that $x = 0$ is a critical point for f_r for all $r \in [c,d]$. Consider the set $C = \big\{(x,r) : \mathrm{d}f_r(x) = 0\big\}$ endowed with the relative topology of $\mathcal{H} \times [c,d]$ and assume that the segment $D = \{0\} \times [c,d]$ is entirely contained in C, i.e., that $x = 0$ is a critical point for f_r for all $r \in [c,d]$. An instant $r_0 \in [c,d]$ is said to be a *bifurcation point* for the family $\{f_r\}$ if $(0, r_0)$ is an accumulation point for $C \setminus D$, i.e., if there exists a sequence $(r_n)_n$ in $[c,d]$ tending to r_0 and a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{H} \setminus \{0\}$ tending to 0 such that $\mathrm{d}f_{r_n}(x_n) = 0$ for all n. Assume that for some (hence for all) $r \in [c,d]$, the Hessian $\mathrm{d}^2 f_r(0)$ is a (self-adjoint) Fredholm operator on $\mathcal{H}$. A sufficient condition for the existence of a bifurcation instant in the strongly indefinite case has been proven recently in [28]: if $\mathrm{sf}\big(\mathrm{d}^2 f_r(0), [c,d]\big) \neq 0$, then there exists a bifurcation instant for $\{f_r\}_r$ in $[c,d]$. This result can be applied to the case of bifurcation of semi-Riemannian geodesics; let us recall from [44] the definition of bifurcation point along a semi-Riemannian geodesic.

Definition 4.11. Let (M,g) be a semi-Riemannian manifold, $\theta : [0,1] \to M$ be a geodesic in M and $t_0 \in \,]0,1[$. The point $\theta(t_0)$ is said to be a *bifurcation point along* θ if there exists a sequence $\theta_n : [0,1] \to M$ of geodesics in M and a sequence $(t_n)_{n\in\mathbb{N}} \subset \,]0,1[$ satisfying the following properties:

(1) $\theta_n(0) = \theta(0)$ for all n;
(2) $\theta_n(t_n) = \theta(t_n)$ for all n;
(3) $\theta_n \to \theta$ as $n \to \infty$;
(4) $t_n \to t_0$ (and thus $\theta_n(t_n) \to \theta(t_0)$) as $n \to \infty$.

An immediate application of the inverse function theorem tells us that bifurcation points occur necessarily at conjugate instants, however, in the non Riemannian (or causal Lorentzian) case it is not clear which conjugate points determine bifurcation. Note that if $\theta(t_0)$ is a bifurcation point along θ, then the exponential map $\exp_{\theta(0)}$ in not one-to-one on *any* neighborhood of $t_0\theta'(0)$ in $T_{\theta(0)}M$.

Conditions for the bifurcation at a nondegenerate conjugate instant have been discussed in [44]; using the theory of partial signatures it is now an easy game to extend the result to the possibly degenerate real-analytic case:

Proposition 4.12. *Let (M,g) be a real-analytic semi-Riemannian manifold, let $\theta : [0,1] \to M$ be a geodesic and let $t_0 \in \,]0,1[$ be a conjugate instant along θ. If the sum of the odd partial signatures $\sum_{k\ge1} \sigma_{2k-1}(\theta,t_0)$ is different from 0, then $\theta(t_0)$ is a bifurcation point along θ.*

Proof. By a standard local construction (see [44, Section 5.1] for details), the geodesic bifurcation problem is cast into a bifurcation problem for a smooth family of functionals defined in a neighborhood of 0 in a *fixed* Hilbert space $\mathcal{H}$. The spectral flow of the corresponding path of second variations is precisely generalized Morse index, whose jumps, by Theorem 4.9, occur at those conjugate instants giving a non zero contribution to the Maslov index, i.e., those conjugate instants $t_0 \in \,]0,1[$ along θ such that, for $\varepsilon > 0$ small enough, $\mathrm{i}_{\mathrm{Maslov}}\big(\theta|_{[0,t_0-\varepsilon]}\big) \neq \mathrm{i}_{\mathrm{Maslov}}\big(\theta|_{[0,t_0+\varepsilon]}\big)$. The conclusion follows easily from Proposition 4.3. □

The result of Proposition 4.12 gives an important link between the theory of bifurcation, for which the method of partial signatures was originally conceived, and the theory of Maslov index in the context of semi-Riemannian and symplectic geometry. Moreover, as an easy application of Proposition 4.12, we get an extension of a classical result of

Morse and Littauer (see Warner's proof in [54]) that the exponential map of a Riemannian manifold is never one-to-one on any neighborhood of a conjugate point.

Corollary 4.13. *Let (M,g) be a real-analytic semi-Riemannian manifold, let $\theta:[0,1]\to M$ be a geodesic having a conjugate instant $t_0\in\,]0,1[$ such that $\sum_{k\ge1}\sigma_{2k-1}(\theta,t_0)\ne0$. Then, the exponential map $\exp_{\theta(0)}$ is not injective on any neighborhood of $t_0\dot\theta(0)$.*

In particular, the result holds true if (M,g) is Riemannian, or if (M,g) is Lorentzian and θ is nonspacelike. □

APPENDIX A. RELATIVE INDEX OF FREDHOLM BILINEAR FORMS ON HILBERT SPACES

The goal of this appendix is to provide the reader with a formal proof of a result (Proposition A.11) that gives the relative index of a form as the difference between the index and the coindex of suitable restrictions of the form. A large portion of the material presented is borrowed from [44, Section 2].

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$; we will denote by $\mathrm{Lin}(\mathcal{H})$ the space of all bounded linear operators on $\mathcal{H}$ and by $\mathrm{GL}(\mathcal{H})$ the group of invertible operators in $\mathrm{Lin}(\mathcal{H})$. Let B a bounded symmetric bilinear form on $\mathcal{H}$; there exists a unique self-adjoint bounded operator $S:\mathcal{H}\to\mathcal{H}$ such that $B=\langle S\cdot,\cdot\rangle$, that will be called the *realization of* B (with respect to $\langle\cdot,\cdot\rangle$). B is nondegenerate if its realization is injective, B is strongly nondegenerate if S is an isomorphism. If B is strongly nondegenerate, or if more generally 0 is not an accumulation point of the spectrum of S (for instance, if S is Fredholm), we will call the *negative eigenspace* (resp., *the positive eigenspace*) of B the closed subspace $V^-(S)$ (resp., $V^+(S)$) of $\mathcal{H}$ given by $\chi_{]-\infty,0[}(S)$ (resp., $\chi_{]0,+\infty[}(S)$), where χ_I denotes the characteristic function of the interval I. The spaces $V^-(S)$ and $V^+(S)$ are S-invariant, and they are both orthogonal and B-orthogonal. We will say that B is *Fredholm* if S is Fredholm, or that B is ***RCPPI***, *realized by a compact perturbation of a positive isomorphism*, (resp., ***RCPNI***) if S is of the form $S=P+K$ (resp., $S=N+K$) where P is a positive isomorphism of $\mathcal{H}$ (N is a negative isomorphism of $\mathcal{H}$) and K is compact. The properties of being Fredholm, RCPPI or RCPNI do not depend on the inner product, although the realization S and the spaces $V^\pm(S)$ do.

If B is RCPPI (resp., RCPNI), then both its nullity $n_0(B)$ and its index $n^-(B)$ (resp., and its coindex $n^+(B)$) are finite numbers. Given a closed subspace $W\subset\mathcal{H}$, the *B-orthogonal complement of* W, denoted by $W^{\perp_B}$, is the closed subspace of $\mathcal{H}$:

$$W^{\perp_B}=\big\{x\in\mathcal{H}:B(x,y)=0\text{ for all }y\in W\big\}=S^{-1}(W^\perp).$$

If B is Fredholm, and let S be its realization and $W\subset\mathcal{H}$ is any subspace, then the following properties hold:

- B is nondegenerate iff it is strongly nondegenerate;
- $n_0(B)<+\infty$;
- $(W^{\perp_B})^{\perp_B}=\overline{W}+\mathrm{Ker}(S)$;
- if W is closed and $B|_W$ (i.e., the restriction of B to $W\times W$) in nondegenerate, then also $B|_{W^{\perp_B}}$ is nondegenerate and $\mathcal{H}=W\oplus W^{\perp_B}$.

Let us now recall a few basic things on the notion of commensurability of closed subspaces (see reference [1] for more details). Let $V,W\subset\mathcal{H}$ be closed subspaces and let P_V and P_W denote the orthogonal projections respectively onto V and W. We say that V and W are *commensurable* if the restriction to V of the projection P_W is a Fredholm operator from V to W.

It is an easy exercise to show that commensurability is an equivalence relation in the Grassmannian of all closed subspaces of $\mathcal{H}$; observe in particular that, identifying each Hilbert space with its own dual, the adjoint of the operator $P_W|_V:V\to W$ is precisely $P_V|_W:W\to V$. If V and W are commensurable the *relative dimension* $\dim_W(V)$ of V

with respect to W is defined as the Fredholm index $\mathrm{ind}\big(P_V|_W : W \to V\big)$, which is equal to:

$$\dim_W(V) = \mathrm{ind}\big(P_W|_V : V \to W\big) = \dim(W^\perp \cap V) - \dim(W \cap V^\perp).$$

Clearly, if V and W are commensurable, then $V^\perp$ and $W^\perp$ are commensurable, and:

$$\dim_{W^\perp}(V^\perp) = -\dim_W(V) = \dim_V(W).$$

The commensurability of closed subspaces and the relative dimension do *not* depend on the choice of a Hilbert space inner product on $\mathcal{H}$.

Using the basic properties of Fredholm index, it is easy to prove the following:

Proposition A.1. *Let $V, W \subset \mathcal{H}$ be closed commensurable subspaces, and let V', W' be finite dimensional subspaces such that $V' \subset V^\perp$ and $W' \subset W^\perp$. Then, $V + V'$ and $W + W'$ are commensurable, and*

$$\dim_{W+W'}(V + V') = \dim_W(V) + \dim(V') - \dim(W'). \quad \square$$

Let us recall that a family $\{S_t\}_{t\in[a,b]}$ of closed subspaces of a Hilbert space is said to be *continuous* if for all $t_0 \in [a,b]$ there exists $\varepsilon > 0$, a closed subspace $S \subseteq H$ and a continuous map $]t_0 - \varepsilon, t_0 + \varepsilon[\; \ni t \mapsto \phi_t \in \mathrm{GL}(\mathcal{H})$ such that $\phi_t(S) = S_t$ for all t. Such a map $t \mapsto \phi_t$ will be called a *local trivialization* of the family S_t around t_0. The relative dimension is continuous with respect to this notion of continuity for paths of closed subspaces. In order to prove this, it is useful to characterize continuous families of closed subspaces in terms of graphs of continuous families of linear operators:

Lemma A.2. *Let $[a,b] \ni t \mapsto S_t$ be a continuous family of closed subspaces of $\mathcal{H}$. Then, for all $t_0 \in [a,b]$ there exists an orthogonal decomposition $\mathcal{H} = W_0 \oplus W_1$ of $\mathcal{H}$ into closed subspaces, and a continuous map defined around t_0, $t \mapsto L_t \in \mathrm{Lin}(W_0, W_1)$ of bounded linear operators from W_0 to W_1 such that $S_t = \mathrm{Gr}(L_t)$ for all t.*

Proof. Choose a closed subspace $S \subset \mathcal{H}$, a local trivialization $t \mapsto \phi_t$ of S_t around t_0, with $\phi_t(S) = S_t$; set $W_0 = S_{t_0}$, $W_1 = W_0^\perp$. Denote by $\pi_i : \mathcal{H} \to W_i$ the orthogonal projection, $i = 0, 1$, and define:

$$L_t = \big(\pi_1 \circ \phi_t|_S\big) \circ \big(\pi_0 \circ \phi_t|_S\big)^{-1}.$$

Observe that $\pi_0 \circ \phi_{t_0}|_S : S \to W_0$ is an isomorphism, and by continuity, $\pi_0 \circ \phi_t|_S : S \to W_0$ is an isomorphism for t near t_0. Clearly, $t \mapsto L_t$ is continuous, and an easy computation shows that $\mathrm{Gr}(L_t) = S_t$ for all t. $\square$

Lemma A.3. *Let $\mathcal{H} = W_0 \oplus W_1$ be an orthogonal direct sum. The map $\mathrm{Lin}(W_0, W_1) \to \mathrm{Lin}(\mathcal{H})$ given by $L \mapsto P_{\mathrm{Gr}(L)}$ is continuous.*

Proof. A straightforward computation gives:

$$P_{\mathrm{Gr}(L)}(x_0 + x_1) = \big((\mathrm{I} + L^*L)^{-1}(x_0 + L^*x_1), L(\mathrm{I} + L^*L)^{-1}(x_0 + L^*x_1)\big),$$

for $x_0 \in W_0$, $x_1 \in W_1$. The conclusion follows easily. $\square$

Corollary A.4. *If $t \mapsto S_t \subset \mathcal{H}$ is a continuous family of closed subspaces, then the map $t \mapsto P_{S_t} \in \mathrm{Lin}(\mathcal{H})$ is continuous.*

Proof. It follows immediately from Lemmas A.2 and A.3. $\square$

Corollary A.5. *Let $[a,b] \ni t \mapsto V_t \subset \mathcal{H}$ and $[c,d] \ni s \mapsto W_s \subset \mathcal{H}$ be continuous families of closed subspaces, and let $(t_0, s_0) \in [a,b] \times [c,d]$ be such that V_{t_0} is commensurable with W_{s_0}. Then, V_t is commensurable with W_s and the relative dimension $\dim_{W_s}(V_t)$ is constant for (s,t) near (s_0, t_0); if V_t is commensurable with W_s for all s and t, then $\dim_{W_s}(V_t)$ is constant on $[a,b] \times [c,d]$.*

Proof. Choose closed subspaces $V, W \subset \mathcal{H}$ and local trivializations ϕ_t and ψ_s such that $\phi_t(V) = V_t$ and $\psi_s(W) = W_s$ for (t,s) near (t_0, s_0). Since the Fredholm index is additive by composition, and the Fredholm index of an isomorphism is 0, then:

$$\dim_{W_s}(V_t) = \mathrm{ind}\big(P_{W_s}|_{V_t} : V_t \to W_s\big) = \mathrm{ind}\big(\psi_s^{-1} \circ P_{W_s} \circ \phi_t|_V : V \to W\big).$$

The conclusion follows observing that the set of Fredholm operators is open, the map $(s,t) \mapsto \psi_s^{-1} \circ P_{W_s} \circ \phi_t$ is continuous (Corollary A.4), and the Fredholm index is locally constant. □

Compact perturbations preserve the commensurability class of positive and negative eigenspaces of Fredholm operators:

Proposition A.6. *Let S, T be linear bounded self-adjoint operators on $\mathcal{H}$ whose difference $K = S - T$ is compact. Then $V^-(S)$ (resp., $V^+(S)$) is commensurable with $V^-(T)$ (resp., with $V^+(T)$). Conversely, assume that S is a bounded self-adjoint Fredholm operator on $\mathcal{H}$, and let $\mathcal{H} = W^- \oplus W^+$ be an orthogonal decomposition of $\mathcal{H}$ such that W^- is commensurable with $V^-(S)$ and W^+ is commensurable with $V^+(S)$. Then there exists an invertible self-adjoint operator T on $\mathcal{H}$ such that $V^-(T) = W^-$, $V^+(T) = W^+$ and such that $S - T$ is compact.*

Proof. See [1, Proposition 2.3.2 and Proposition 2.3.5]. □

Let us now study under which conditions a closed subspace $W \subset \mathcal{H}$ is commensurable with the negative eigenspace of a Fredholm symmetric bilinear form B on $\mathcal{H}$.

Lemma A.7. *Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, let $S : \mathcal{H} \to \mathcal{H}$ be a self-adjoint Fredholm operator and set $B = \langle S\cdot,\cdot\rangle$. Let $W \subset \mathcal{H}$ be a closed subspace with the following properties:*

(a) *$B|_W$ is* strongly negative definite*, i.e., there exists $k > 0$ such that $-B(x,x) \geq k\|x\|^2$ for all $x \in W$;*

(b) *$B|_{W^{\perp_B}}$ is positive semi-definite, i.e., $B(x,x) \geq 0$ for all $x \in W^{\perp_B}$.*

Then W is commensurable with $V^-(S)$, and $\dim_W\big(V^-(S)\big) = 0$.

Proof. Set $H_- = V^-(S)$ and $H_+ = \mathrm{Ker}(S) \oplus V^+(S)$, so that $\mathcal{H} = H_- \oplus H_+$, the direct sum being orthogonal and also B-orthogonal; let $P_- : \mathcal{H} \to H_-$ be the orthogonal projection. Observe that H_- and H_+ are S-invariant; assumption (a) means that the restriction $S_-|_{H_-} : H_- \to H_-$ is a negative isomorphism, i.e., its spectrum is contained in $]-\infty, 0[$; moreover, by (a), $W \cap H_+ = \{0\}$, and, by (b), $W^{\perp_B} \cap H_- = \{0\}$. The thesis is equivalent to the condition that the restriction $P_-|_W : W \to H_-$ be an isomorphism, which is what we will prove now.

In first place, observe that $P_-|_W$ is injective, because $\mathrm{Ker}\big(P_-|_W\big) = H_+ \cap W = \{0\}$. Next, $P_-|_W$ has dense image, because its adjoint $P_W|_{H_-}$ is injective: $\mathrm{Ker}\big(P_W|_{H_-}\big) = W^\perp \cap H_- = \{0\}$. For, if $x \in W^\perp \cap H_-$, since the restriction $S|_{H_-} : H_- \to H_-$ is an isomorphism, then $x = Sy$ for some $y \in H_-$, and for all $w \in W$ it is $B(y,w) = \langle Sy, w\rangle = \langle x, w\rangle = 0$, so that $y \in W^{\perp_B} \cap H_- = \{0\}$, i.e., $x = 0$.

Finally, we must prove that $P_-|_W$ has closed image; it suffices to show that there exists $\beta > 0$ such that $\|P_-(x)\|^2 \geq \beta\|x\|^2$ for all $x \in W$. To this aim, define:

$$k = \inf_{y \in W, \|y\|=1} -B(y,y) > 0,$$

and let $x \in W$ be fixed with $\|x\| = 1$, $x = x_- + x_+$, $x_- \in H_-$, $x_+ \in H_+$. Then,

$$B(x_-, x_-) \leq B(x_-, x_-) + B(x_+, x_+) = B(x,x) \leq -k;$$

moreover, since B is strongly negative definite on H_-, then $-B$ is a Hilbert space inner product on H_- equivalent to $\langle\cdot,\cdot\rangle$, hence there exists $\alpha > 0$ such that:

$$-B(y,y) \leq \alpha\|y\|^2, \quad \forall\, y \in H_-.$$

From the last two inequalities we obtain:

$$\|P_-(x)\|^2 = \|x_-\|^2 \geq \frac{k}{\alpha} > 0,$$

which concludes the proof. □

Lemma A.8. *Let B be a Fredholm symmetric bilinear form on the Hilbert space $\mathcal{H}$ and let $W \subset \mathcal{H}$ be a closed subspace. Then, the following are equivalent:*

(a) *$B|_W$ is RCPNI and $B|_{W^{\perp_B}}$ is RCPPI;*
(b) *W is commensurable with $V^-(S)$, where S is the realization of B.*

In particular, condition (b) is independent on the choice of an inner product on $\mathcal{H}$.

Proof. Assume that W is commensurable with $V^-(S)$; then $W^\perp$ is commensurable with $V^-(S)^\perp = V^+(S) \oplus \mathrm{Ker}(B)$. Moreover, since $\mathrm{Ker}(B)$ is finite dimensional, then $W^\perp$ is also commensurable with $V^+(S)$. By Proposition A.6, there exists an invertible self-adjoint operator $T : \mathcal{H} \to \mathcal{H}$ such that $V^-(T) = W$, $V^+(T) = W^\perp$, and with $S = T + K$, with K compact. It follows easily that $B|_W$ is RCPNI (namely, if P denotes the orthogonal projection onto W, the realization of $B|_W$ is $PS|_W = (PT + PK)|_W = (T + PK)|_W$), and $B|_{W^\perp}$ is RCPPI. Observe in particular that $W \cap W^{\perp_B} = \mathrm{Ker}(B|_W)$ is finite dimensional. To prove that $B|_{W^{\perp_B}}$ is RCPPI we argue as follows; denote by P the orthogonal projection onto W and by $P^\perp = 1 - P$ the orthogonal projection onto $W^\perp$. As we have observed, $W^{\perp_B} = S^{-1}(W^\perp)$; hence, for all $x, y \in W^{\perp_B}$ we have:

$$\begin{aligned} B(x,y) &= \langle Sx, y\rangle = \langle Sx, P^\perp y\rangle = \langle SPx, P^\perp y\rangle + \langle SP^\perp x, P^\perp y\rangle = \\ &= \langle P^\perp KPx, y\rangle + \langle P^\perp TP^\perp x, y\rangle + \langle P^\perp KP^\perp x, y\rangle. \end{aligned} \tag{A.1}$$

In the above equality we have used the fact that W and $W^\perp$ are T-invariant. From (A.1) we deduce that $B|_{W^{\perp_B}}$ is represented by a compact perturbation of the operator $\widetilde{T} : W^{\perp_B} \to W^{\perp_B}$ given by $\widetilde{T} = P^{\perp_B} P^\perp T P^\perp|_{W^{\perp_B}}$ (where $P^{\perp_B}$ is the orthogonal projection onto $W^{\perp_B}$) which is positive semi-definite. The kernel of $\widetilde{T}$ is easily computed as the finite dimensional space $W^{\perp_B} \cap T^{-1}\big(W \cap W^{\perp_B}\big)$; it follows that $\widetilde{T}$ is a compact perturbation of a positive isomorphism of $W^{\perp_B}$, which proves that (b) implies (a).

Conversely, assume that $B|_W$ is RCPNI and $B|_{W^{\perp_B}}$ is RCPPI. Using functional calculus, write $W = W_0 \oplus W_1$, with W_0 and W_1 orthogonal and B-orthogonal, $B|_{W_0}$ strongly negative definite, $B|_{W_1}$ positive semi-definite, and $\dim(W_1) < +\infty$; then,

$$\mathcal{H} = W_0 \oplus W_0^{\perp_B}.$$

Clearly, $W^{\perp_B} \subset W_0^{\perp_B}$; we claim that $W^{\perp_B}$ has finite codimension in $W_0^{\perp_B}$. Namely,

$$W_0^{\perp_B} \cap W_1^{\perp_B} \subset W^{\perp_B} \subset W_0^{\perp_B},$$

and since $W_1^{\perp_B}$ has finite codimension in $\mathcal{H}$, $W_0^{\perp_B} \cap W_1^{\perp_B}$ (and *a fortiori* $W^{\perp_B}$) has finite codimension[14] in $W_0^{\perp_B}$. Since $B|_{W^{\perp_B}}$ is RCPP, it follows that also $B|_{W_0^{\perp_B}}$ is RCPPI, and again we can write $W_0^{\perp_B}$ as a B-orthogonal direct sum $W_2 \oplus W_3$ with $B|_{W_2}$ negative definite, $B|_{W_3}$ positive semi-definite, and $\dim(W_2) < +\infty$.

Set $Z := W_0 \oplus W_2$; then $B|_Z$ is strongly negative definite; moreover, $Z^{\perp_B} = W_3$, hence $B|_{Z^{\perp_B}}$ is positive semi-definite; by Lemma A.7, Z is commensurable with $V^-(S)$ Since W_0 has finite codimension in Z, then W_0 and Z are commensurable, and since W_0 has finite codimension in W, then W_0 and W are commensurable. By transitivity, W is commensurable with $V^-(S)$, and the proof is concluded. □

[14]If X is a vector space, and $S, Y \subset X$ subspaces. If S has finite codimension in X, then $S \cap Y$ has finite codimension in Y. Namely, $\mathrm{codim}_Y(S \cap Y)$ equals the dimension of the image of Y by the projection $X \to X/S$ onto the finite dimensional space X/S.

Assume now that B is a symmetric bilinear form, S is its realization; if W is closed subspace of $\mathcal{H}$ which is commensurable with $V^-(S)$, the one defines the *relative index* of B with respect to W, denoted by $\mathrm{ind}_W(B)$, the integer number:

$$\mathrm{ind}_W(B) = \dim_W\big(V^-(S)\big).$$

A subspace Z of $\mathcal{H}$ is said to be *isotropic* for the symmetric bilinear form B if $B|_Z \equiv 0$.

Lemma A.9. *Let B be a RCPPI nondegenerate symmetric bilinear form on $\mathcal{H}$, and let $Z \subset \mathcal{H}$ be an isotropic subspace of B. Then:*

$$n^-(B) = n^-\big(B|_{Z^{\perp_B}}\big) + \dim(Z).$$

Proof. Since B is RCPPI, then the index $n^-(B)$ is finite, and so $n^-\big(B|_{Z^{\perp_B}}\big)$ and $\dim(Z)$ are finite. Clearly, $Z \subset Z^{\perp_B}$; let $U \subset Z^{\perp_B}$ be a closed subspace such that $Z^{\perp_B} = Z \oplus U$, so that $B|_U$ is nondegenerate and $\mathcal{H} = U \oplus U^{\perp_B}$. Moreover:

$$n^-(B) = n^-\big(B|_U\big) + n^-\big(B|_{U^{\perp_B}}\big).$$

Since Z is isotropic, then $n^-\big(B|_U\big) = n^-\big(B|_{Z^{\perp_B}}\big)$; to conclude the proof we need to show that $n^-\big(B|_{U^{\perp_B}}\big) = \dim(Z)$. To this aim, observe first that $\dim(U^{\perp_B}) = 2\dim(Z)$. Namely, $\dim(U^{\perp_B}) = \mathrm{codim}(U)$; moreover, $\mathrm{codim}_{Z^{\perp_B}}(U) = \dim(Z)$, and $\mathrm{codim}(Z^{\perp_B}) = \dim(Z)$. Thus, keeping in mind that the dimension of an isotropic subspace of a nondegenerate symmetric bilinear form is less than or equal to the index and the coindex, we have:

$$n^-\big(B|_{U^{\perp_B}}\big) + n^+\big(B|_{U^{\perp_B}}\big) = \dim(U^{\perp_B}) = 2\dim(Z) \le n^-\big(B|_{U^{\perp_B}}\big) + n^+\big(B|_{U^{\perp_B}}\big),$$

which proves that $n^-\big(B|_{U^{\perp_B}}\big) = n^+\big(B|_{U^{\perp_B}}\big) = \dim(Z)$ and concludes the proof. □

Lemma A.10. *Let B be a nondegenerate Fredholm symmetric bilinear form on $\mathcal{H}$ and $W \subset \mathcal{H}$ be a closed subspace such that $B|_{W^{\perp_B}}$ is RCPPI. Let $\widetilde{W}$ be any closed complement of $W \cap W^{\perp_B}$ in W. Then the following identity holds:*

$$n^-\big(B|_{\widetilde{W}^{\perp_B}}\big) = n^-\big(B|_{W^{\perp_B}}\big) + \dim\big(W \cap W^{\perp_B}\big).$$

Proof. We start with the observation that $\mathrm{Ker}\big(B|_W\big) = \mathrm{Ker}\big(B|_{W^{\perp_B}}\big) = W \cap W^{\perp_B}$; this implies in particular that $B|_{\widetilde{W}}$ and $B|_{\widetilde{W}^{\perp_B}}$ are nondegenerate. Since $B|_{W^{\perp_B}}$ is RCPPI, then $n^-\big(B|_{W^{\perp_B}}\big)$ and $\dim(W \cap W^{\perp_B}) = n$ are finite numbers.

Since $\mathrm{codim}_{\widetilde{W}^{\perp_B}}\big(W^{\perp_B}\big) = n$, then:

$$n^-\big(B|_{\widetilde{W}^{\perp_B}}\big) \le n^-\big(B|_{W^{\perp_B}}\big) + n,$$

from which it follows that $n^-\big(B|_{\widetilde{W}^{\perp_B}}\big)$ is finite; moreover, $B|_{\widetilde{W}^{\perp_B}}$ is RCPPI. The conclusion now follows easily from Lemma A.9, applied to the nondegenerate bilinear form $B|_{\widetilde{W}^{\perp_B}}$ and the isotropic space $Z = W \cap W^{\perp_B}$. □

Proposition A.11. *Let B be a Fredholm symmetric bilinear form on $\mathcal{H}$, S its realization and let $W \subset \mathcal{H}$ be a closed subspace which is commensurable with $V^-(S)$. Then the relative index $\mathrm{ind}_W(B)$ is given by:*

$$\mathrm{ind}_W(B) = n^-\big(B|_{W^{\perp_B}}\big) - n^+\big(B|_W\big). \tag{A.2}$$

Proof. Assume first that B is nondegenerate on W; then we have a direct sum decomposition $\mathcal{H} = W \oplus W^{\perp_B}$. The relative $\mathrm{ind}_W(B)$ does not change if we change the inner product of $\mathcal{H}$; we can therefore assume that W and $W^{\perp_B}$ are orthogonal subspaces of $\mathcal{H}$. Then, $S = S^- \oplus S^+$, where $S^- : W \to W$ is the realization of $B|_W$ and $S^+ : W^{\perp_B} \to W^{\perp_B}$

is the realization of $B|_{W^{\perp_B}}$. Moreover, $V^-(S) = V^-(S^-) \oplus V^-(S^+)$. An immediate calculation yields:

$$\begin{aligned}\mathrm{ind}_W(B) &= \dim\big(V^-(S)\cap W^{\perp_B}\big) - \dim\big(V^-(S)^\perp \cap W\big)\\ &= \dim\big(V^-(S)\cap W^{\perp_B}\big) - \mathrm{codim}_W\big(V^-(S^-)\big)\\ &= \dim\big(V^-(S^+)\big) - \mathrm{codim}_W\big(V^-(S^-)\big)\\ &= n^-\big(B|_{W^{\perp_B}}\big) - n^+\big(B|_W\big).\end{aligned}$$

Let us consider now the case that $B|_W$ is degenerate; by Lemma A.8, $B|_W$ is RCPNI, and so $\dim\big(W\cap W^{\perp_B}\big) = n < +\infty$. Set $\widetilde W = \big(W\cap W^{\perp_B}\big)^\perp \cap W$, so that $B|_{\widetilde W}$ is nondegenerate; moreover, $V^-(S)$ is commensurable with $\widetilde W$, because it has finite codimension in W. We can then apply the first part of the proof, and we obtain:

$$\mathrm{ind}_{\widetilde W}(B) = n^-\big(B|_{\widetilde W^{\perp_B}}\big) - n^+\big(B|_{\widetilde W}\big). \tag{A.3}$$

Clearly,

$$n^+\big(B|_{\widetilde W}\big) = n^+\big(B|_W\big); \tag{A.4}$$

moreover, by definition of relative index:

$$\mathrm{ind}_{\widetilde W}(B) = \mathrm{ind}_W(B) + n. \tag{A.5}$$

Finally, by Lemma A.8, $B|_{W^{\perp_B}}$ is RCPPI, and by Lemma A.10:

$$n^-\big(B|_{\widetilde W^{\perp_B}}\big) = n^-\big(B|_{W^{\perp_B}}\big) + n. \tag{A.6}$$

Formulas (A.3), (A.4), (A.5) and (A.6) yield (A.2) and conclude the proof. □

APPENDIX B. CONNECTEDNESS OF THE GRASSMANNIAN OF g-NEGATIVE SUBSPACES.

We will prove briefly in this appendix that, given a nondegenerate bilinear form g on $\mathbb{R}^n$ having index k and denoting by $G_k^-(n)$ the open subset of the Grassmannian $G_k(n)$ consisting of those k-dimensional planes in $\mathbb{R}^n$ on which g is negative definite, then $G_k^-(n)$ is (arc) connected. The non trivial case is when $0 < k < n$. Clearly, it is not restrictive to assume, as we will, that g is the bilinear form whose matrix representation in the canonical basis of $\mathbb{R}^n$ is given by:

$$g = \begin{pmatrix} -\mathrm{I}_k & 0\\ 0 & \mathrm{I}_{n-k}\end{pmatrix};$$

the group of isometries of g will be denoted by $\mathrm{O}(n,k)$:

$$\mathrm{O}(n,k) = \big\{A\in \mathrm{GL}(n,\mathbb{R}) : A^*gA = g\big\} = \big\{A\in\mathrm{GL}(n,\mathbb{R}) : gA^*g = A^{-1}\big\},$$

and its Lie algebra by $\mathrm{so}(n,k)$:

$$\mathrm{so}(n,k) = \big\{H\in\mathrm{gl}(n,\mathbb{R}) : gH + H^*g = 0\big\}.$$

To prove our assertion we will show that the connected component of the identity of $\mathrm{O}(n,k)$ acts transitively on $G_k^-(n)$. Given $A\in\mathrm{O}(k,n)$, denote by A_{up} and A_{low} respectively the upper left $k\times k$ block of A and the lower right $(n-k)\times(n-k)$ block of A; it is easy to see that both A_{up} and A_{low} are invertible, and we will show that the sign of the determinant of the two blocks distinguish the connected components of $\mathrm{O}(k,n)$:

Lemma B.1. *If $0 < k < n$, $\mathrm{O}(k,n)$ has four connected components. The connected component of the identity consists of those $A\in\mathrm{O}(k,n)$ such that $\det(A_{up})$ and $\det(A_{low})$ are positive.*

Proof. The four components are determined by the choice of the signs of $\det(A_{\rm up})$ and $\det(A_{\rm low})$. These sets are clearly open; let us prove that they are connected. Denote by by $\mathrm{Sym}(n)$ the vector space of $n \times n$ symmetric matrices, and by $\mathrm{Sym}_+(n)$ the subset of the positive definite ones; recall that the exponential map gives a diffeomorphism $\exp : \mathrm{Sym}(n) \to \mathrm{Sym}_+(n)$. Consider the diffeomorphism:

$$\mathrm{GL}(n, I\!R) \xrightarrow{\cong} \mathrm{O}(n) \times \mathrm{Sym}_+(n) \tag{B.1}$$

given by the polar decomposition $A \mapsto (U, P)$, $U = A|A|^{-1}$, $P = |A|$. It is easy to see that, given $A \in \mathrm{GL}(n, I\!R)$, A belongs to $\mathrm{O}(k, n)$ if and only if $U(A)$ and $P(A)$ do. Namely, using the fact that $g^2 = \mathrm{I}$, if $gA^*g = A^{-1}$, then $gPU^*g = (gPg)(gU^*g) = P^{-1}U^{-1}$ and so $(gUg)(gP^{-1}g) = UP = A$. By the uniqueness of the polar decomposition, $gUg = U$ and $gP^{-1}g = P$, i.e., $U, P \in \mathrm{O}(n, k)$.

Observe the following:

1. $\mathrm{O}(n, k) \cap \mathrm{O}(n) = \mathrm{O}(k) \times \mathrm{O}(n - k)$;
2. from the uniqueness of the square root, it follows easily that if $A \in \mathrm{O}(n, k) \cap \mathrm{Sym}_+(n)$, then $A^{\frac{1}{2}} \in \mathrm{O}(n, k)$;
3. if $H \in \mathrm{Sym}(n)$, then $H \in \mathrm{so}(n, k)$ if and only if $\exp(H) \in \mathrm{O}(n, k)$.[15]

Thus, the restriction of the diffeomorphism (B.1) is a diffeomorphism:

$$\big(\mathrm{O}(k) \times \mathrm{O}(n - k)\big) \times \big(\mathrm{so}(n, k) \cap \mathrm{Sym}(n)\big) \xrightarrow{\cong} \mathrm{O}(n, k),$$

given by $(U, Z) \mapsto U \exp(Z)$. The conclusion follows easily from the observation that $\mathrm{so}(n, k) \cap \mathrm{Sym}(n)$ is a vector space, hence contractible, and from the fact that, for $r > 0$, $\mathrm{O}(r)$ has exactly two connected components determined by the sign of the determinant. □

Corollary B.2. *The connected component of the identity of* $\mathrm{O}(n, k)$ *acts transitively on* $G_k^-(n)$*; in particular,* $G_k^-(n)$ *is arc-connected.*

Proof. Let $S \in G_k^-(n)$ be fixed arbitrarily; let us show that there exists T in the connected component of the identity of $\mathrm{O}(n, k)$ such that $T\big(I\!R^k \oplus \{0\}\big) = S$. Set $S' = S_g^\perp$, choose a g-orthonormal basis $b_1, \ldots, b_k$ of S, a g-orthonormal basis $b_{k+1}, \ldots, b_n$ if S' and, denoting by $e_1, \ldots, e_n$ the canonical basis of $I\!R^n$, let $\widetilde{T} \in \mathrm{GL}(n)$ be such that $\widetilde{T}(e_i) = b_i$ for all $i = 1, \ldots, n$. Clearly, $\widetilde{T} \in \mathrm{O}(n, k)$, because it send a g-orthonormal basis into another g-orthonormal basis; moreover, $\widetilde{T}\big(I\!R^k \oplus \{0\}\big) = S$. Replacing b_1 with $-b_1$ in the choice of an orthonormal basis of S has the effect of changing the sign of the determinant of the upper left block $k \times k$ of the matrix representation of $\widetilde{T}$ in the canonical basis, while replacing b_{k+1} with $-b_{k+1}$ has the effect of changing the sign of the lower right $(n - k) \times (n - k)$ block of such matrix. In conclusion, the appropriate choice for the sign of b_1 and b_{k+1} can be made to ensure that the corresponding operator $\widetilde{T}$ belongs to the connected component of the identity of $\mathrm{O}(n, k)$, and this concludes the proof. □

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[15](3) follows easily from (2). Namely, if $\exp(H) \in \mathrm{O}(n, k)$, then $\exp(qH) \in \mathrm{O}(n, k)$ for all $q \in \mathbb{Q}$, and since $\mathrm{O}(n, k)$ is closed, also $\exp(tH) \in \mathrm{O}(n, k)$ for all $t \in I\!R$. Differentiating $t \mapsto \exp(tH)$ at $t = 0$ we get that $H \in \mathrm{so}(n, k)$. The converse is trivial.

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