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Published version:

DOI:10.1080/19452829.2014.956709

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INEQUALITY AVERSION AND THE EXTENDED GINI IN THE LIGHT OF A TWO-PERSON CAKE-SHARING PROBLEM

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Forthcoming in 2014 *Journal of Human Development and Capabilities*

Abstract:
This brief note aims to communicate, in simple terms, the 'meaning' of the family of 'Extended' Gini coefficients of inequality, in terms of the shares accruing to the agents in an elementary two-person cake-sharing problem. In the process, a natural notion of the 'potential fairness' of a distribution, as well as the notion of 'distribution-sensitivity', are sought to be explicated in easily accessible terms.

Key words: *Extended Gini coefficient; Inequality aversion; Equally distributed equivalent income; Cake-sharing problem; Distributional fairness; Distribution-Sensitivity*

Acknowledgement:
The authors are indebted to two anonymous referees of this Journal, for their very helpful comments on earlier versions of the paper.

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1. INTRODUCTION

The measurement of inequality is a key feature of the study of human development, and the present note is an interpretive account of an aspect of the subject. The meaning of social inequality is most transparently understood as the share of a cake of given size going to the poorer of two individuals in a classical two-person cake-sharing problem. Attempts at linking the Gini coefficient for a general n-person income distribution to the shares in a two-person cake-division exercise have been made earlier by Subramanian (2002) and Shorrocks (2005). A similar exercise is carried out here for the family of extended Gini coefficients.

A brief history of the class of generalized Ginis is available in the following extraordinarily helpful summary (which we reproduce verbatim) supplied to us by one of the referees of this paper. The family of extended Gini coefficients was introduced in a technically demanding paper by Donaldson and Weymark (1980), using discrete mathematics, attributed by these authors to a 1979 University of British Columbia discussion paper of John Weymark (itself later appearing as Weymark, 1981, which is also technically demanding). It can, however, be found in more readable form, as a by-product in Kakwani's (1980) poverty paper, and is introduced and explored in a thorough and focused but not-too-technical manner by Yitzhaki (1983); these latter two formulations are based on the Lorenz curve, and are expressed using a continuous distribution.

A *family* of inequality indices allows for the possibility of plural attitudes of aversion to inequality. The notion of the 'potential fairness of a distribution' is sought to be related, in this note, to such heterogeneous attitudes, and to be linked - in simple and natural terms – to the division of a cake between two persons. Specifically, for any two-person distribution, it seems reasonable to suggest that the distribution is 'potentially fair' if the amount of cake required to be transferred from the richer person to the poorer person in order to secure equality is exactly the same in the perceptions of both the richer person and the poorer person. If the required transfer is lesser/greater in the perception of the richer person than in that of the poorer person, we have a case of 'potential unfairness'/ 'potential super-fairness'. These issues are explicated in simple terms in what follows.

It is useful here, as pointed out by a referee, to draw reference to a possible limitation of this paper. In particular, it is conceivable that the cake-sharing analogy is not necessarily the most appropriate interpretive account of attitudes to inequality that one can think of (and this could be particularly true for Gini-type measures which, according to Shorrocks [2005], could yield an exaggerated impression of inequality in terms of cake-shares). Arguably, the '*leaky-bucket thought experiment*' offers a more persuasive interpretation of the subject: the idea here is to assess inequality aversion in terms of the amount of a permissible progressive transfer that can be 'leaked away' while yet preserving inequality-invariance. This, however, is not the route pursued in the present essay: those interested in this alternative approach are referred to (among others) Amiel, Creedy and Hurn (1999) Duclos (2000), and Seidl (2001). Briefly, our claim is not that the 'cake-sharing analogy' is *the single most* helpful interpretive device to employ in the context of discussion, but rather that it does have its uses.

The notion of 'transfer-sensitivity', which is a proximate motivation for the family of extended Gini coefficients, is also sought, in this note, to be explained in easily understood terms as the cake-shares in a two-person cake-sharing problem.

2. THE EXTENDED GINI FAMILY

Underlying the class of extended or generalized Gini coefficients is a class of extended or generalized 'Rank-Order Weighted' social welfare functions. An income distribution will be understood as an n-vector $\mathbf{x} = (x_1,...,x_i,...,x_n)$ (where n is an integer not less than 2), with x_i standing for the income of the ith person in a community of n individuals, and *non-decreasingly* ordered, i.e. $x_i \leq x_{i+1}, i = 1,...,n-1$. An *extended Rank-Order Weighted social welfare function* W_δ defined on any income vector $\mathbf{x}$ is written as a weighted sum of individual incomes, the weights being the incomes' rank-orders raised to some positive integer $\delta \geq 1$ (note that income-ties can be broken arbitrarily):

(1) $W_\delta(\mathbf{x}) = \sum_{i=1}^{n}(n+1-i)^\delta x_i$; $\delta \geq 1$.

The welfare function W_δ respects equity by employing an income weighting structure that assigns a higher weight to a lower income; further, the parameter δ captures one's aversion to inequality: the larger the value of δ, the greater the relative weight placed on smaller incomes.

Given any ordered n-vector $\mathbf{x}$, the equally distributed equivalent (EDE) income is defined as that level of income x^e such that if it is shared by all members of the community, then the resulting welfare is the same as that which obtains with the currently unequal distribution $\mathbf{x}$. That is, x^e is obtained from the equation $W_\delta(x^e,...,x^e) = W_\delta(\mathbf{x})$; given (1), it is easy to verify that

(2) $x^e = \sum_{i=1}^{n}(n+1-i)^\delta x_i / \sum_{i=1}^{n}(n+1-i)^\delta$.

Following Subramanian (2002), define x_0^e to be that level of income such that if it is equally distributed, then aggregate welfare is the same as what would obtain if a single person appropriated the entire income: x_0^e can be derived from the equation $W_\delta(x_0^e,...,x_0^e) = W_\delta(0,...,0,n\mu)$, where μ is the mean of the distribution $\mathbf{x}$. Given (1), again it can be verified that

(3) $x_0^e = n\mu / \sum_{i=1}^{n}(n+1-i)^\delta$.

The extended class of Gini coefficients can now be obtained as the ratio of the shortfall of x^e from μ to the maximum such shortfall, which is the difference between μ and x_0^e:

(4) $G(\delta) = (\mu - x^e)/(\mu - x_0^e)$, $\delta \geq 1$.

Given the expressions for x^e and x_0^e in (2) and (3), and making the appropriate substitutions in (4), we can - with a bit of manipulation - obtain the following expression for the extended class of Gini coefficients:

(5) $$G(\delta) = \frac{\sum_{i=1}^{n}(n+1-i)^\delta(\mu - x_i)}{\sum_{i=1}^{n}\left((n+1-i)^\delta - 1\right)\mu}, \; \delta \geq 1.$$

Note that $G(\delta)$ is an increasing function of inequality-aversion (the proof of the positivity of $G'(\delta)$ for $n \geq 3$ is available on request from the authors). It is important to note here that the expression we have derived for $G(\delta)$ is slightly different from that derived by Donaldson and Weymark (1980), hence the cumbersome requirement of a separate demonstration of the increasingness of the $G(\delta)$ function: for the corresponding result on the canonical Donaldson-Weymark formulation, the reader is referred to Yitzhaki (1983), Lambert (1989) and Aaberge (2000). Specifically, our indices are *normalized* (to reflect a value of unity for all perfectly concentrated distributions, irrespective of the dimensionality of the distribution): this is essential, as we shall see, for ensuring that the inequality coefficients can be 'translated' into two-person shares that add up to unity. Normalized measures, it may be added, are not replication-invariant (that is, they are not invariant with respect to integral replications of an income distribution). Shorrocks (2005) preserves the property of replication invariance (at the cost of normalization) in the Gini coefficient: the outcome is that corresponding shares in the cake-cutting exercise can add up to more than unity. From a practical point of view, normalization does not distort the value of the Gini coefficient in any serious way when the number of individuals in a society is 'large': that is to say, the value of the expression for $G(\delta)$ in (5) will converge, asymptotically, on that corresponding to the conventional (Donaldson-Weymark) formulation.

3. THE EXTENDED GINI AND TWO-PERSON CAKE-SHARES

Following Subramanian (2002), define – given any distribution **x** - the *dichotomously allocated equivalent distribution* (DAED) as a two-person ordered income vector $\mathbf{x}^* = (x_1^*, x_2^*)$ such that $\mathbf{x}^*$ has both the same mean μ and the same extended Gini coefficient $G(\delta)$ as the distribution $\mathbf{x}$. Then, given the definitions of the mean and of the extended Gini, and applying these to the DAED $\mathbf{x}^*$, one obtains:

(6) $(x_1^* + x_2^*)/2 = \mu$; and

(7) $[\mu(2^\delta + 1) - (2^\delta x_1^* + x_2^*)]/(2^\delta - 1)\mu = G(\delta)$.

Solving for x_1^* yields:

(8) $x_1^* = \mu[1 - G(\delta)]$.

Recall that x_1^* is the income of the poorer person in the DAED $\mathbf{x}^*$. It follows that the income-share or cake-share of the poorer person is then given by $\sigma \equiv x_1^*/2\mu$, or, in view of (8), by

(9) $\sigma(\delta) = [1 - G(\delta)]/2$.

(9) is a straightforward generalization of the relation for the 'original Gini' $G(1)$ established in Subramanian (2002), namely that $\sigma(1) = \mu[1 - G(1)]/2$.

Equation (9) suggests the following.

First, the extended Gini for any arbitrary *n*-person distribution can be helpfully interpreted in terms of the share of cake going to the poorer person in a two-person cake-sharing problem.

Second, the quantum of cake x_1^* going to the poorer person turns out to be, quite simply, the extended version of Sen's welfare index, $W_\delta \equiv \mu[1 - G(\delta)]$ which , in turn, is just Atkinson's equally distributed equivalent income. Indeed, the vector $\mathbf{x}$ could also be interpreted as an uncertain prospect of outcomes

$\mathbf{x}=(x_1,...,x_n)$ with expected value μ and dispersion $G(\delta)$: x_1^*, then, can be interpreted as the 'certainty equivalent' of the uncertain prospect. $G(\delta)$ itself, then, can be interpreted as the proportion of total income/output that one is willing to sacrifice in order to eliminate the unpleasantness of inequality/risk.

Third, note that since $\sigma(\delta)=[1-G(\delta)]/2$ (equation (9)) and $G(\delta)$ is an increasing function of δ, one must conclude that the perceived cake-share of the poorer of the two recipients shrinks as the aversion to inequality increases.

The above observations suggest that there are interpretational advantages to be had from linking the extended Gini to the notion of the amount or share of a cake of given size going to the poorer of two individuals. Sen (1978) has pointed out that when we deal with 'ethical' indices of inequality which are explicitly derived from underlying social welfare functions in terms of the loss in welfare experienced owing to the presence of inequality, then we allow for the possibilities of both (a) assessing inequality in a given distribution differently, depending on how averse we are to inequality; and (b) assessing inequality identically in two different distributions, because of an unvarying disposition to inequality. These issues are spelt out briefly in the context of the $G(\delta)$ family of inequality indices in the following two sections.

4. SAME DISTRIBUTION, DIFFERENT DEGREES OF INEQUALITY AVERSION: ON THE 'POTENTIAL FAIRNESS' OF A DISTRIBUTION

. We first consider the case of 'same distribution, different degrees of inequality aversion', which throws some light on the notions of 'heterogeneity' and 'potential fairness'. Heterogeneity may be said to obtain when people in a society entertain more than one attitude to inequality, as manifested in the possession of different degrees of inequality aversion, which is captured in the parameter δ. Consider a particularly simple example of dichotomous preferences with respect to inequality-aversion, in which, say, the poorest q individuals' attitude to inequality is captured in an inequality-aversion measure of δ_1, while the richest $(n-q)$ individuals share an inequality-aversion measure of δ_2, with $\delta_1<\delta_2$. That is, the relatively poor, here, are assumed to be *less* inequality-averse than the relatively rich. Carrying this over to the analogy of the cake-sharing problem, it is 'as though' the poorer of the two individuals had an inequality-aversion measure of δ_1, and the richer person an inequality-aversion measure of δ_2, with $1 \le \delta_1 < \delta_2$.

Let us designate by $\sigma^1 \equiv \sigma(\delta_1)$ the cake-share received by person 1 (the poorer person), *in her own perception*, and by $\sigma^2 \equiv \sigma(\delta_2)$ the cake-share received by person 1 in *the perception of person 2* (the richer person). To summarize: the perceived cake-shares received by persons 1 and 2 (in terms of the respective individuals' own respective perceptions), are σ^1 and $1-\sigma^2$, respectively. It follows that the sum of the perceived shares is

$s = \sigma^1 + (1-\sigma^2)$.

One can see now that since $\sigma(\cdot)$ is a declining function of δ, $\sigma^1(\equiv \sigma(\delta_1)) > \sigma^2(\equiv \sigma(\delta_2))$ (because $\delta_1 < \delta_2$ *ex hypothesi*), whence $s \equiv \sigma^1 - \sigma^2 + 1 > 1$. When s exceeds unity, we have a case of 'super-fairness', in the following sense.

What would equality call for in the perception of the two persons? Person 2 would believe that equality requires her to transfer $(0.5-\sigma^2)$ of the cake to person 1; if person 1 receives this amount, it would be more than what person 1 expects in the cause of equality, which is $(0.5-\sigma^1)$ (recall that $\sigma^1 > \sigma^2$). The possibility therefore exists of 2 transferring less than she thinks she needs to, and of 1 receiving more than she thinks she needs to, for equality in their respective perceptions to be realized. If $\delta_1 < \delta_2$ spells the possibility of 'potential super-fairness', one presumes that, by symmetric reasoning, $\delta_1 > \delta_2$ - which is empirically more plausible - would spell the possibility of 'potential unfairness'. Of course, in a homogeneous population, where all individuals share the same attitude to inequality (as captured in a uniquely shared δ), neither 'super-fairness' nor 'unfairness' emerges as a possibility. The extended Gini coefficient, by allowing for heterogeneity in the choice of δ, thus also allows for an interpretation of alternative conceptions of fairness within the framework of the canonical cake-sharing problem.

5. DIFFERENT DISTRIBUTIONS, SAME DEGREE OF INEQUALITY AVERSION: ON THE 'TRANSFER-SENSITIVITY' OF A DISTRIBUTION

By considering the case of the generalized Gini index in the context of an assessment of different distributions under a shared unique perception of δ, we can throw some light on the notion of 'transfer-sensitivity', which draws on Kolm's (1976) 'principle of diminishing transfers', a principle that requires, loosely speaking, that an inequality index should display greater sensitivity to income-transfers at the lower than at the upper end of an income distribution. More precise content to this principle is afforded by the transfer-sensitivity requirements formulated by Kakwani (1980) and Foster (1985). The two formulations are conveniently summarized by Foster (1985).

One formulation requires (Foster 1985, p.229) that '… a transfer of a fixed amount of income between … persons a fixed number of ranks apart must have a larger effect on inequality the lower the income ranking of the pair', and the other formulation requires (Foster, *ibid.*) that '… a transfer of a fixed amount of income between two persons whose incomes differ by a fixed amount must have a larger effect on inequality the lower the incomes of the pair.' Call these two formulations of transfer-sensitivity Formulation 1 and Formulation 2 respectively. It is clear that the two Formulations are motivated by two considerations – call these, respectively, Consideration 1 and Consideration 2. Consideration 1, which drives Formulation 1, suggests that our sensitivity to transfers at different income levels is mediated by the inter-personal *positional distance* (see Subramanian, 1987) between the two persons in each pair of persons involved in the transfers. Consideration 2, which drives Formulation 2, suggests that our sensitivity to transfers at different income levels is mediated by the inter-personal *income gap* between the two persons in each pair of persons involved in the transfers. Taken together, the suggestion is that transfer-sensitivity ought to be informed by both Consideration 1 and Consideration 2.

Notice, however, that Formulation 1 ignores Consideration 2, while Formulation 2 ignores Consideration 1. As a consequence, each Formulation, by itself, can be seen to be altogether too strong. A reasonably weaker notion of transfer-sensitivity is obtained by combining Formulations 1 and 2, to yield what one may call a principle of 'discriminating distribution-sensitivity'. Let us say that an inequality index is '*discriminatingly distribution-sensitive*' if a progressive rank-preserving transfer of income between two individuals who are both a fixed income and a fixed number of individuals apart causes a greater reduction in inequality the poorer is the pair of individuals. It is well known that the extended Gini coefficients, for values of δ not less than two, are discriminatingly distribution-sensitive in the above sense, although the 'regular' Gini coefficient $G(1)$ is not. How is this reflected in the values of the cake shares $\sigma(\delta)$?

To see what is involved, consider two equi-dimensional distributions $\mathbf{x}$ and $\mathbf{y}$ with the same mean, such that the Lorenz curve for $\mathbf{x}$ is skewed toward (1,1) of the unit square (so the Lorenz curve 'bulges at the bottom'), and the Lorenz curve for $\mathbf{y}$ is skewed toward (0,0) (so the Lorenz curve 'bulges at the top'). Furthermore, assume that the areas enclosed by the two Lorenz curves and the diagonal of the unit square are the same (see Figure 1). Suppose we measure inequality, in turn, in terms of $G(1)$ and $G(2)$. Then, it will be the case that $\sigma_{\mathbf{x}}(1) = \sigma_{\mathbf{y}}(1)$, but $\sigma_{\mathbf{x}}(2) < \sigma_{\mathbf{y}}(2)$. Two different distributions $\mathbf{x}$ and $\mathbf{y}$ will thus display the same 'cake-share' σ when inequality is measured by the Gini coefficient; but the same two distributions will display different inequality values, and therefore corresponding 'cake-shares', even when aversion to inequality is unvarying, for higher values of inequality-aversion $(\delta \geq 2)$.

[Figure 1 here]

A simple numerical example will illustrate the above proposition. Suppose $\mathbf{x}$ and $\mathbf{y}$ are given by the respective 5-distributions (10,20,30,45,45) and (10,25,25,40,50). Both distributions have the same mean (of 30) and they can be seen to have been derived from the distribution $\mathbf{a} =$ (10,20,30,40,50) in the following ways: $\mathbf{x}$ is derived from $\mathbf{a}$ by a transfer of 5 units of income from the richest person to the next richest person, while $\mathbf{y}$ is derived from $\mathbf{a}$ by the transfer of an identical amount of 5 units of income from the third poorest person to the second poorest person. The persons involved in the two transfers are both a fixed income (10 units) and a fixed number of individuals (1) apart. By the requirement of discriminating distribution sensitivity, we should expect $\sigma_{\mathbf{x}}$ to be smaller than $\sigma_{\mathbf{y}}$. This requirement, we find, is satisfied for $\delta = 2$, but not for $\delta = 1$: as it happens, and as can be verified, $\sigma_{\mathbf{x}}(1) = \sigma_{\mathbf{y}}(1) = 0.3417$, whereas $\sigma_{\mathbf{x}}(2)(= 0.3227) < \sigma_{\mathbf{y}}(2)(= 0.3261)$.

6. CONCLUDING REMARKS

This has been an essentially quick and simple note aimed at facilitating an interpretation of the family of extended Gini coefficients of inequality, in terms of a canonical, two-person cake-sharing problem, such that issues of variability in inequality aversion toward a given distribution and variability in distributions for a given degree of inequality aversion can be relatively easily comprehended.

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Figure 1: Intersecting Lorenz Curves with Differing Skewness

And the Same Area Under the Curves