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B-spline surfaces on criss-cross triangulations for curve network interpolation

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Abstract

This paper deals with the problem of interpolating a B-spline curve network, in order to create a surface satisfying such a constraint and defined by using bivariate C^1 quadratic splines on criss-cross triangulations as blending functions. Such a surface exists and is unique.

Key words: interpolation, B-spline surface, curve network
MSC 2000: 65D05, 65D07, 65D17

1 Introduction

In the literature, the interpolation of a curve network by a smooth parametric surface has been a well-studied problem since the second half of last century. Indeed it is easier and more intuitive to describe a complicated 3D free-form surface by creating a curve network than to directly manage surface control points.

Such an interpolation problem is faced up by considering different approaches, for example classical blending-function methods and interpolation of the curve mesh either by a smooth regularly parametrized surface with one polynomial piece per facet or by subdivision schemes.

We recall that, in general, most surface fitting methods fall into one of two categories: global or local methods. A global method represents a surface by a ready available expression on the whole parametric domain and it is generally obtained either by solving a system of equations or by other schemes, for instance the ones based on the quasi-interpolation, that do not require to solve any system of equations. Instead, local methods are more geometric in nature, constructing the surface patch-wise, using only local data for each step and imposing the required smoothness patch by patch.

Moreover, it is well known that NURBS, based on rectangular patches, are a widely used computational geometry technology in CAGD, thanks to their many good properties, and tensor product B-spline surfaces are a specific kind of NURBS [2]. However, sometimes the above surfaces can create unwanted oscillations and some inflection points on the surface, due to their high coordinate degree. An example of tensor product B-spline surface interpolating a B-spline curve network is proposed in [2, Chap. 10]: it is obtained from a blending-function method, constructing three different surfaces expressed by means of B-spline functions belonging to different spaces, so that the bivariate resulting one is not defined on the rectangular grid associated to the knot vectors of the curve net, since a knot refinement is required.

NURBS on triangulations can avoid such unwanted oscillations, since they have a total degree lower than the one of NURBS based on rectangular patches. Therefore they can be very useful in CAGD (see e.g. [6]).

In this paper we want to combine the advantageous features of the above cited techniques, proposing a new global method, to generate a surface interpolating a B-spline curve network and using as blending functions bivariate B-splines on a criss-cross triangulation T_{mn} of the parametric domain. In particular, we consider C^1 quadratic B-spline curve networks and we define a parametric surface that satisfies the above interpolation constraint and it is based on the B-spline functions spanning the space $S_2^1(T_{mn})$ of all quadratic C^1 splines on T_{mn}.

2 Construction of the C^1 quadratic B-spline surface interpolating the curve network

Let $\{P_j^{(r)}\}_{j=0}^{n+1}$, $r = 0, \ldots, m$ and $\{Q_i^{(s)}\}_{i=0}^{m+1}$, $s = 0, \ldots, n$, $P_j^{(r)}$, $Q_i^{(s)} \in \mathbb{R}^3$, be the control points of the two sets of C^1 quadratic B-spline curves

$$\begin{aligned} \phi_r(v) &= \sum_{j=0}^{n+1} P_j^{(r)} B_j(v), \qquad r = 0, \ldots, m, \qquad v \in [c, d] \subset \mathbb{R}, \\ \psi_s(u) &= \sum_{i=0}^{m+1} Q_i^{(s)} B_i(u), \qquad s = 0, \ldots, n, \qquad u \in [a, b] \subset \mathbb{R}. \end{aligned} \tag{1}$$

We assume that the curves (1) satisfy the following compatibility conditions:

C1. as independent sets, they are compatible in the B-spline sense, that is, all the $\psi_s(u)$

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are defined on a common parametric knot vector

$$U = \{a = u_{-2} \equiv u_{-1} \equiv u_0 < u_1 < \cdots < u_m \equiv u_{m+1} \equiv u_{m+2} = b\} \quad (2)$$

and all the $\phi_r(v)$ are defined on a common parametric knot vector

$$V = \{c = v_{-2} \equiv v_{-1} \equiv v_0 < v_1 < \cdots < v_n \equiv v_{n+1} \equiv v_{n+2} = d\}. \quad (3)$$

In our notation the B-splines $B_i(u)$ and $B_j(v)$ have supports $[u_{i-2}, u_{i+1}]$ and $[v_{j-2}, v_{j+1}]$, respectively [3, 4].

Moreover, we assume (ξ_i, v_s), with

$$\xi_i = \frac{u_{i-1} + u_i}{2}, \; i = 0, \ldots, m+1 \quad (4)$$

and (u_r, η_j), with

$$\eta_j = \frac{v_{j-1} + v_j}{2}, \; j = 0, \ldots, n+1 \quad (5)$$

as the pre-image of $Q_i^{(s)}$ for all s and $P_j^{(r)}$ for all r, respectively. We recall that ξ_i and η_j are known as Greville abscissae;

C2. $\phi_r(v_s) = \psi_s(u_r)$, $r = 0, \ldots, m$, $s = 0, \ldots, n$. We remark that, from (1), the number of curve network control points is $(m+1)(n+2) + (m+2)(n+1)$ and the number of constraints is $(m+1)(n+1)$. Therefore, a curve network satisfying such a condition always exists and several control points can be arbitrarily chosen to define it.

Let T_{mn} be the criss-cross triangulation of the parameter domain $\Omega = [a, b] \times [c, d]$, based on the knots $U \times V$, given in (2) and (3) and let $\mathcal{B}_{mn} = \{B_{ij}(u, v), (i, j) \in K_{mn}\}$, with $K_{mn} = \{(i, j) : 0 \leq i \leq m+1, \; 0 \leq j \leq n+1\}$ be the collection of $(m+2)(n+2)$ bivariate B-splines [1, 3, 4, 5], spanning the space $S_2^1(T_{mn})$, i.e. the space of all C^1 quadratic splines whose restriction to each triangle of T_{mn} is a bivariate polynomial of total degree two. It is well known that dim $S_2^1(T_{mn}) = (m+2)(n+2) - 1$ [5].

By means of $\mathcal{B}_{mn}$, we can define a C^1 quadratic B-spline parametric surface of the form

$$\mathcal{S}(u, v) = (\mathcal{S}_x(u, v), \mathcal{S}_y(u, v), \mathcal{S}_z(u, v))^{\mathrm{T}} = \sum_{i=0}^{m+1} \sum_{j=0}^{n+1} C_{ij} B_{ij}(u, v), \quad (u, v) \in \Omega, \quad (6)$$

interpolating the curve network (1). In order to do it, we have to get the control points $C_{ij} \in \mathbb{R}^3$, $i = 0, \ldots, m+1$, $j = 0, \ldots, n+1$, whose pre-images in Ω are the points (ξ_i, η_j), defined in (4) and (5), respectively. Thus, we interpret the curves (1) as isoparametric curves of $\mathcal{S}$, i.e. we impose

$$\mathcal{S}(u_r, v) = \phi_r(v), \; r = 0, \ldots, m,$$
$$\mathcal{S}(u, v_s) = \psi_s(u), \; s = 0, \ldots, n$$

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and we deduce the following constraints that have to be satisfied by the surface control points $\{C_{ij}\}$:

$$\begin{array}{l}\sigma_{r+1}C_{r,j}+\sigma'_{r+1}C_{r+1,j}=P_j^{(r)},\ r=1,\dots,m-1,\ j=1,\dots,n,\\ \tau_{s+1}C_{i,s}+\tau'_{s+1}C_{i,s+1}=Q_i^{(s)},\ s=1,\dots,n-1,\ i=1,\dots,m,\\ C_{i0}=Q_i^{(0)},\ C_{i,n+1}=Q_i^{(n)},\quad i=0,\dots,m+1,\\ C_{0j}=P_j^{(0)},\ C_{m+1,j}=P_j^{(m)},\quad j=0,\dots,n+1,\end{array}$$

where, for $0\le r\le m+1$ and $0\le s\le n+1$,

$$\begin{array}{ll}\sigma_{r+1}=\frac{h_{r+1}}{h_r+h_{r+1}}, & \sigma'_r=\frac{h_{r-1}}{h_{r-1}+h_r},\\ \tau_{s+1}=\frac{k_{s+1}}{k_s+k_{s+1}}, & \tau'_s=\frac{k_{s-1}}{k_{s-1}+k_s},\end{array} \tag{7}$$

with $h_r=u_r-u_{r-1}$, $k_s=v_s-v_{s-1}$ and $h_{-1}=h_{m+2}=k_{-1}=k_{n+2}=0$ (when in (7) we have $\frac{0}{0}$, we set the corresponding value equal to zero).

Moreover, since the B_{ij}'s are linearly dependent, by using the dependence relationship [5], it is possible to prove that the surface $\mathcal{S}$ is unique and it can be expressed as a linear combination of the B_{ij}'s, with coefficients depending only on the curve network control points $\{P_j^{(r)}\}_{j=0}^{n+1}$ and $\{Q_i^{(s)}\}_{i=0}^{m+1}$. Indeed, it can be written in the form (6) with

$$C_{ij}=\Gamma_{ij}+(-1)^{i+j}C_{11}\frac{h_i}{h_1}\frac{k_j}{k_1}, \tag{8}$$

$i=1,\dots,m$, $j=1,\dots,n$, for any $C_{11}\in\mathbb{R}^3$, where

$$\Gamma_{ij}=\sum_{r=1}^{i-1}(-1)^{r+1}\frac{(h_{i-r}+h_{i-r+1})h_i}{h_{i-r}h_{i-r+1}}P_j^{(i-r)}+(-1)^i\frac{h_i}{h_1}\sum_{s=1}^{j-1}(-1)^s\frac{(k_{j-s}+k_{j-s+1})k_j}{k_{j-s}k_{j-s+1}}Q_1^{(j-s)} \tag{9}$$

and $\sum_{\ell=1}^{0}\cdot\ =0$. Furthermore, it is independent of the C_{11} choice and it also has the following expression:

$$\mathcal{S}(u,v)=\mathcal{S}_b(u,v)+\mathcal{S}_\Gamma(u,v),$$

where

$$\mathcal{S}_b(u,v)=\sum_{j=0}^{n+1}\left(P_j^{(0)}B_{0j}(u,v)+P_j^{(m)}B_{m+1,j}(u,v)\right)+\sum_{i=1}^{m}\left(Q_i^{(0)}B_{i0}(u,v)+Q_i^{(n)}B_{i,n+1}(u,v)\right),$$

$$\mathcal{S}_\Gamma(u,v)=\sum_{i=1}^{m}\sum_{j=1}^{n}\Gamma_{ij}B_{ij}(u,v),$$

with Γ_{ij} defined in (9).

Finally, we remark that, if the sequence of partitions $\{U\times V\}$ of Ω is A-quasi uniform (i.e. there exists a constant $A\ge 1$ such that $0<\max_{i,j}\{h_i,k_j\}/\min_{i,j}\{h_i,k_j\}\le A$), then the C_{ij} generation process (8)-(9) is stable, because the round-off error growth is linear.

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3 An application

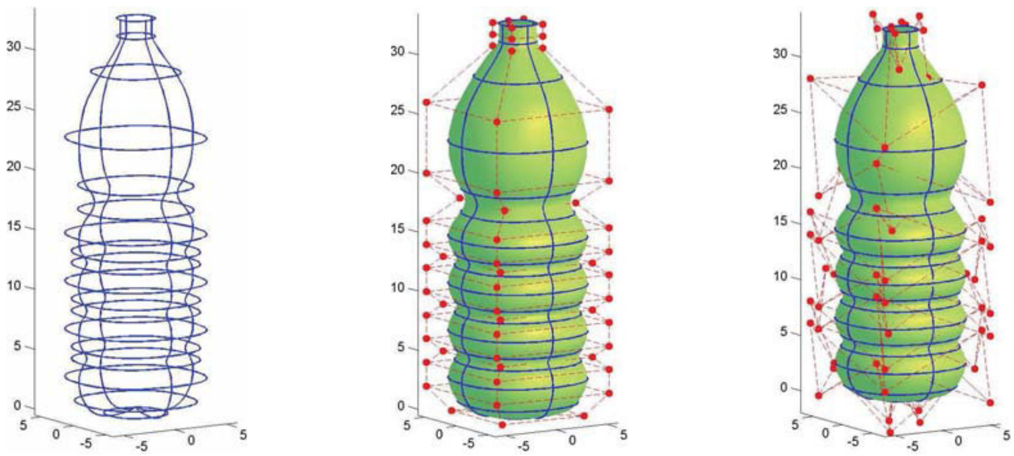
In this section we present an application and we verify that the surface (6) is unique and it is independent of C_{11} choice.

It is well known that the control points of a spline surface usually should give an idea of its shape and its possible symmetries. In (8)-(9) we obtain all control points of the spline surface interpolating the network (1) depending on the curve control points and on C_{11}. Then different choices of C_{11} lead to different surface control points, some of which do not respect such shape property. However in this application we are not interested in the surface control point shape, because here we just want to underline the unicity of the surface (6), interpolating (1). For this reason in the following example we take two different C_{11} that provide two quite different control point sets, but the same surface interpolating the curve network.

We consider $m = 18$, $n = 4$, the two knot vectors $U = \{u_i\}_{i=-2}^{20}$, $V = \{v_j\}_{j=-2}^{6}$, as in (2) and (3), with $u_i = i$, $i = 0, \ldots, 18$ and $v_j = j$, $j = 0, \ldots, 4$ and the B-spline curve network of type (1) shown in Fig. 1(a) with given control points $\{P_j^{(r)}\}_{j=0}^{5}$, $r = 0, \ldots, 18$ and $\{Q_i^{(s)}\}_{i=0}^{19}$, $s = 0, \ldots, 4$, $P_j^{(r)}$, $Q_i^{(s)} \in \mathbb{R}^3$.

The same quadratic spline surface (6), interpolating the above curve net and defined by the bivariate B-splines spanning $S_2^1(T_{18,4})$, is obtained both assuming $C_{11} = (1.5, 1.5, 32)$ (Fig. 1(b)) and assuming $C_{11} = (1, 0, 30)$ (Fig. 1(c)), as we have also numerically verified.

Moreover, we can note how very different are the surface control points corresponding to the different choiches of C_{11}.



(a) (b) (c)

Figure 1: (a) The curve network and the surfaces $\mathcal{S}$ interpolating such a curve network with (b) $C_{11} = (1.5, 1.5, 32)$ and (c) $C_{11} = (1, 0, 30)$.

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