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Reconstruction of separatrix curves and surfaces in squirrels competition models with niche

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Abstract

In dynamical systems saddle points partition the domain into basins of attractions of the remaining locally stable equilibria. This problem is rather common especially in population dynamics models, like prey-predator or competition systems. In this paper we focus on squirrels population models with niche and we design algorithms for the detection and the refinement of points lying on the separatrix curve in the 2D setting and on the separatrix surface in 3D, thus partitioning the respective domains. Then, in order to reconstruct the separatrix curve and surface, we apply the Partition of Unity method, which makes use of Wendland's functions as local approximants.

Key words: population dynamical systems, population models, scattered data interpolation, Partition of Unity method, Wendland's functions
MSC 2000: AMS codes 65D05, 65D17, 92D25

1 Introduction

The competition of two or more species that live in the same environment can be modelled mathematically by a differential system, whose unknowns are the populations as functions of time and their interactions are described by a number of parameters (see [1, 6, 8]). To obtain a particular solution of the system, we have to add the initial conditions of the populations that establish their states, prior to any interaction. Then, given a model of this type, we can observe that, under some conditions imposed on biological parameters and according to the initial data, the trajectories, namely the model solutions, at the end of the observation period stabilize at certain points, called the stable equilibria. Moreover, we may also note that, depending on the initial states, the trend of solution trajectories stabilizes at different equilibria, each representing the prevalence of one or more species over the other one(s). So we can imagine to divide the points of the domain (e.g., plane or space), seen as initial conditions of the populations, in different regions depending on where the trajectory originating in them will ultimately stabilize. Thus, the aim of this work is to construct an approximation curve or an approximation surface, which divides the considered domain in two or more regions, called the basins of attraction of each equilibrium.

In particular, in this article we discuss two specific population models with niche, which investigate squirrels competition of two and three different populations (see also [4, 5]). The former considers a 2D model with competition between red native and grey exotic squirrels, while the latter involves a 3D model with competition among red native, red indigenous and grey exotic squirrels. At first, we carry out an analytical study of the two models, aimed at finding the location of equilibrium points and at establishing conditions to be imposed on the parameters so that the behavior described above in fact occurs and the separatrix curve and surface exist. Then, after choosing parameters which satisfy these assumptions for feasibility and stability of the equilibria, we study their numerical versions, and then we proceed to approximate the separatrix curve and surface. For this purpose we have implemented several MATLAB functions for the approximation of the points, obtained by a bisection algorithm, and the graphical representation of the separating curve and surface. Hence, after detecting the points lying on the separatrix curve and surface, first we proceed by applying a refinement algorithm in order to reduce the number of points to interpolate, and then we approximate the curve and surface using the Partition of Unity method with local approximants given by compactly supported Wendland's functions (see, e.g., [3, 7]). This method is an effective and efficient tool in approximation theory, since it allows us to interpolate a large number of scattered data in an accurate and stable way. We point out that a different refinement algorithm was already proposed by some of the authors for a 2D model in [2].

The paper is organized as follows. In Sections 2 and 3 we consider the 2D and 3D models respectively, carrying out an analytical study of each competition model. Section 4 is devoted to present the designed algorithms for the detection and the refinement of points lying on the separatrix curve and surface. In Section 5 we describe the Partition of Unity method used for approximating such curves and surfaces. Section 6 shows some numerical results in both 2D and 3D cases. Finally, Section 7 deals with conclusions and future work.

2 The two populations model

Let us consider the following competition model, with N and E denoting the red native and the grey exotic squirrels, respectively,

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$$\frac{dN}{dt} = p\left(1 - \frac{N}{u}\right)N - aE(1-b)N,$$
$$\frac{dE}{dt} = r\left(1 - \frac{E}{z}\right)E - cN(1-b)E, \tag{1}$$

where p and r are the growth rates of N and E, respectively, a and c are the relative coefficients of competition, u and z are the relative carrying capacities of the two populations and b denotes the fraction of red squirrels which hide in a niche. We remark that the model (1) describes the interaction of the two different populations of squirrels within the same environment.

The resolution of the system imposing that the derivates are equal to zero and the analytical study of the model show that there are four equilibrium points associated with the model, which are given by

$$E_0 = (0,0); \quad E_1 = (0,z); \quad E_2 = (u,0);$$

$$E_3 = \left(\frac{-ur(p+abz-az)}{-pr+aucz-2aubcz+azub^2c}, \frac{-zp(r-cu+cub)}{-pr+aucz-2aubcz+azub^2c}\right).$$

The study of stability points out that for some choices of the parameters, namely

$$p < az(1-b), \quad r < cu(1-b),$$

the origin E_0 is an unstable equilibrium, E_1 and E_2 are stable equilibria, and E_3 is an unstable equilibrium, specifically a saddle point. This occurs, for example, for $r=1$, $p=2$, $b=0.5$, $u=1$, $c=3$, $a=2$, $z=3$. This suggests the existence of a separating curve that divides the model domain into two subregions, called basins of attraction of each respective equilibrium, each containing paths ending in E_1 or E_2.

In Figure 1 we show trajectories starting from the initial conditions $x_1 = (0.5,3)$, $x_2 = (2,2)$, $x_3 = (1,3)$, $x_4 = (2,1)$, $x_5 = (3,1)$, $x_6 = (2,4)$, $x_7 = (1.5,3)$ and $x_8 = (3,3)$, and converging to the point E_1 of coordinates $(0,3)$ and to the point E_2 of coordinates $(1,0)$. Here the parameters have been chosen as above.

3 The three populations model

Let us now consider the 3D competition model, with N, A and E denoting the red native, the red indigenous and the grey exotic squirrels, respectively,

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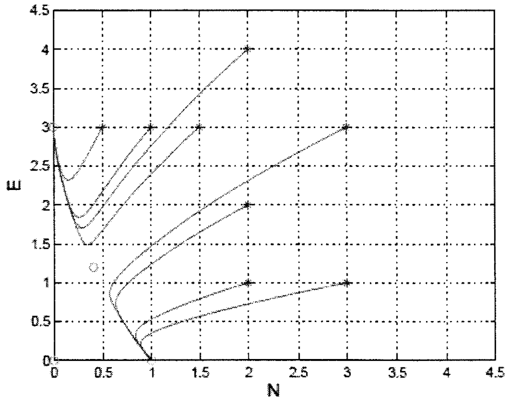

Figure 1: Example of initial conditions and trajectories converging to equilibria for the model problem (1).

$$\frac{dN}{dt} = p\left(1 - \frac{N}{u}\right)N - aE(1-b)N,$$
$$\frac{dA}{dt} = q\left(1 - \frac{A}{v}\right)A - cE(1-e)A, \qquad (2)$$
$$\frac{dE}{dt} = r\left(1 - \frac{E}{z}\right)E - fN(1-b)E - gA(1-e)E,$$

where p, q and r are the growth rates of N, A and E, respectively, a, c, f and g are the relative coefficients of competition, u, v and z are the relative carrying capacities of the three populations, b and e denote the fraction of the populations N and A, respectively, which hide in a niche. Also in this case, we remark that the model (2) describes the interaction of the three different populations of squirrels within the same environment.

The resolution of the system imposing that the derivates are equal to zero, as in the

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2D case, and the analytical study of the model show that critical points are given by

$$\begin{aligned}
E_0 = \; & (0,0,0);\;\; E_1 = (u,0,0);\;\; E_2 = (0,v,0);\;\; E_3 = (u,v,0);\;\; E_4 = (0,0,z);\\
E_5 = \; & \left(\frac{ur[az(1-b)-p]}{azuf(1-b)^2-pr}, 0, \frac{zp[fu(1-b)-r]}{azuf(1-b)^2-pr}\right);\\
E_6 = \; & \left(0, \frac{vr[cz(1-e)-q]}{czvg(1-e)^2-qr}, \frac{zq[vg(1-e)-r]}{czvg(1-e)^2-qr}\right);\\
E_7 = \; & \left(\frac{u[\alpha-prq-zaq(1-b)(vg(1-e)-r)]}{\alpha+\beta-prq},\right.\\
& \frac{v[\beta-prq-pcz(1-e)(fu(1-b)-r)]}{\alpha+\beta-prq},\\
& \left.\frac{zpq[fu(1-b)+vg(1-e)]}{\alpha+\beta-prq}\right).
\end{aligned}$$

where, for brevity, we indicated for E_7,

$$\alpha = pczvg(1-e)^2, \;\; \beta = azufq(1-b)^2.$$

The study of stability shows that under some conditions, for example with the choices of the parameters $r = 9$, $q = 0.6$, $p = 0.6$, $b = 1/2$, $u = 1.5$, $c = 8$, $a = 8$, $z = 3$, $v = 2$, $e = 0.5$, $f = 6$, $g = 5$, the points E_3 and E_4 are stable equilibria, E_1 and E_2 are unstable, E_5 and E_6 are not admissible and E_7 is a saddle point. This suggests the existence of a separating surface that divides the model domain into two basins, each of them containing one path ending in E_3 or E_4.

In Figure 2 we show trajectories starting from the initial conditions $x_1 = (5,8,1)$, $x_2 = (6,6.5,2)$, $x_3 = (7.5,7,4)$, $x_4 = (4,8,3)$, $x_5 = (3,8,5)$, $x_6 = (2,4,4)$, $x_7 = (4,3,3)$ and $x_8 = (7,6,6)$, and converging to the point E_3 of coordinates $(1.5,2,0)$ and to the point E_4 of coordinates $(0,0,3)$.

4 Detection and refinement of separatrix points

At first, to determine the separatrix curve and surface for (1) and (2), respectively, we need to consider a set of points as initial conditions in a square domain $[0,\gamma]^2$, where $\gamma \in \mathbb{R}^+$, and in a cube domain $[0,\gamma]^3$ (in the following we will fix $\gamma = 10$). Then we take points in pairs and we check if trajectories of the two points converge to different equilibria. If this the case, then we proceed with a bisection algorithm to determine a separatrix point. When each point of the set has been compared with all the other ones, we perform a refinement of the set of separatrix points. In fact, in general we find a large number of separatrix points. In the following we propose a refinement process which computes a smaller set of points. The set of the refined points is then interpolated using a suitable method (see Section 5).

More precisely, we start considering, in the 2D case, n initial conditions equispaced in the interval $[0,10]$ on the x-axis and n on the y-axis. Performing the bisection algorithm,

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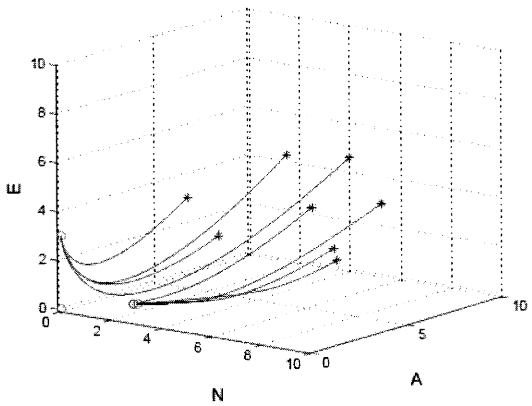

Figure 2: Example of initial conditions and trajectories converging to equilibria for the model problem (2).

a certain number of points, in general large, is found on the separatrix curve. To obtain a smaller set of nodes well distributed on the separatrix curve, we can proceed as follows. We divide the interval $[0, 10]$ in l subintervals and we compute an average of the found points in each subinterval. Given a vector of equispaced points $x(k)$, $k = 1, \ldots, l+1$, in the interval $[0, 10]$, we define

$$m_k = \min\{i : a_{i,1} \in [x(k), x(k+1))\}, \quad k = 1, \ldots, l,$$

and

$$M_k = \max\{i : a_{i,1} \in [x(k), x(k+1))\}, \quad k = 1, \ldots, l.$$

Starting from the matrix $A = (a_{i,j})$, $i = 1, \ldots, N$, $j = 1, 2$, we define the matrix of the refined points $A'' = (a''_{i,j})$, $i = 1, \ldots, l+2$, $j = 1, 2$, of entries:

$$a''_{1,j} = a_{1,j}, \quad j = 1, 2,$$

$$a''_{i,1} = \frac{\sum_{i=m_k}^{M_k} a_{i,1}}{M_k - m_k + 1}, \quad k = 1, \ldots, l,$$

$$a''_{i,2} = \frac{\sum_{i=m_k}^{M_k} a_{i,2}}{M_k - m_k + 1}, \quad k = 1, \ldots, l,$$

$$a''_{l+2,j} = a_{N,j}, \quad j = 1, 2.$$

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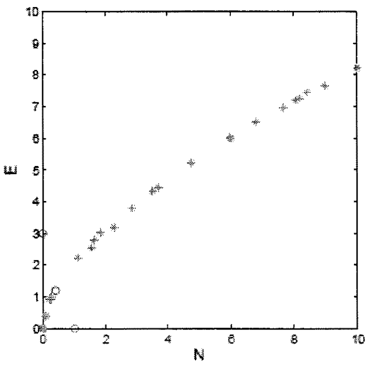
An advantage of this technique is given by the fact that we can choose a small number n of initial conditions, without taking into account of the number of points lying on the separatrix curve. For example, Figure 3 (left) shows the points found using $n = 4$. Dividing the interval $[0, 10]$ in $l = 10$ subintervals and considering the $N = 27$ points picked up on the separatrix curve, the refinement process provides us the $l + 2 = 12$ points reported in Figure 3 (right).



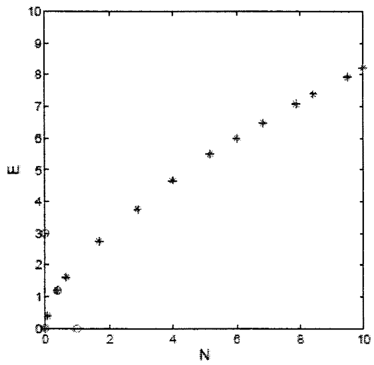


Figure 3: Set of points detected by the bisection algorithm (left) and set of points found by the refinement algorithm (right) in the 2D case.

For the 3D case, we use a similar technique. At first we construct a grid on the faces of the cube and the bisection algorithm is applied with the following initial conditions

$$(x(i), y(j), 0) \quad \text{and} \quad (x(i), y(j), 10), \quad i = 1, \dots n, \quad j = 1, \dots n,$$

$$(x(i), 0, z(j)) \quad \text{and} \quad (x(i), 10, z(j)), \quad i = 1, \dots n, \quad j = 1, \dots n,$$

$$(0, y(i), z(j)) \quad \text{and} \quad (10, y(i), z(j)), \quad i = 1, \dots n, \quad j = 1, \dots n.$$

This choice permits us to find well distributed points on the separatrix surface. The N points found by the bisection algorithm are organized in a matrix $A = (a_{i,j})$, $i = 1, \dots, N$, $j = 1, 2, 3$, and then refined. We define

$$M_x = \max_i (a_{i,1}), \quad i = 1, \dots, N,$$

$$M_y = \max_i (a_{i,2}), \quad i = 1, \dots, N,$$

and we divide the interval $[0, M_x]$ in L subintervals and $[0, M_y]$ in H and we make an average

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of the points in each subinterval. Given a vector of equispaced points $x(l)$, $l = 1, \ldots, L+1$, in $[0, M_x]$, and a vector $y(h)$, $h = 1, \ldots, H+1$, in $[0, M_y]$, let us define

$$I_{lh} = \{i : a_{i,1} \in [x(l), x(l+1)] \quad \text{and} \quad a_{i,2} \in [y(h), y(h+1)]\},$$

with $l = 1, \ldots, L, \quad h = 1, \ldots, H$. Starting from the matrix $A = (a_{i,j})$ we find the matrix of the refined points $A' = (a'_{i,j})$, whose entries are given by

$$a'_{i,1} = \frac{\sum_{i \in I_{lh}} a_{i,1}}{Card(I_{lh})}, \quad l = 1, \ldots, L, \quad h = 1, \ldots, H,$$

$$a'_{i,2} = \frac{\sum_{i \in I_{lh}} a_{i,2}}{Card(I_{lh})}, \quad l = 1, \ldots, L, \quad h = 1, \ldots, H,$$

$$a'_{i,3} = \frac{\sum_{i \in I_{lh}} a_{i,3}}{Card(I_{lh})}, \quad l = 1, \ldots, L, \quad h = 1, \ldots, H,$$

and $i = 1, \ldots, K$, where K is the number of subintervals containing at least a point.

As an example, in Figure 4 (left) we show the found points choosing $n = 10$. The $N = 182$ points have been refined taking $H = 10$ and $L = 10$. In this way, as shown in Figure 4 (right), we obtain $K = L \cdot H = 100$ points.

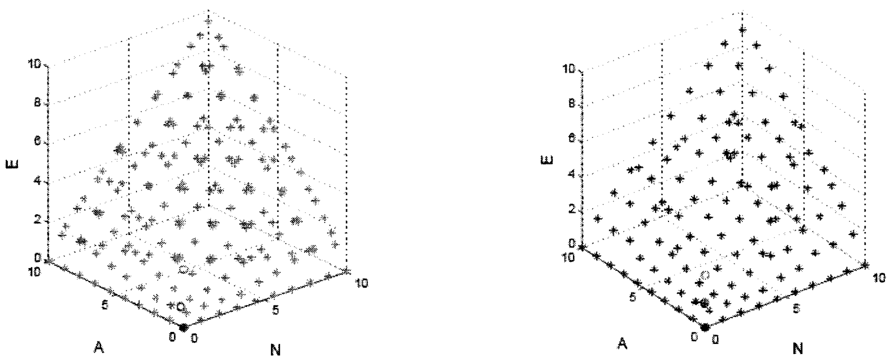

Figure 4: Set of points detected by the bisection algorithm (left) and set of points found by the refinement algorithm (right) in the 3D case.

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5 Reconstruction of separatrix curves and surfaces

In this section we present the interpolation method we use to connect the found points applying the refinement algorithm.

Let us consider a set $\mathcal{Q} = \{x_i, i = 1, \ldots, n\}$ of distinct data points arbitrarily distributed on $\Omega \subseteq \mathbb{R}^m$, and an associated set $\mathcal{F} = \{f_i, i = 1, \ldots, n\}$ of data values.

The basic idea of the Partition of Unity method is to start with a partition of the open and bounded domain $\Omega \subseteq \mathbb{R}^m$ into d cells (subdomains) Ω_j such that $\Omega \subseteq \bigcup_{j=1}^{d} \Omega_j$ with some mild overlap among the cells. At first, we choose a partition of unity, i.e. a family of compactly supported, non-negative, continuous functions W_j with $\text{supp}(W_j) \subseteq \Omega_j$ such that $\sum_{j=1}^{d} W_j(x) = 1$, for all $x \in \Omega$. Then, for each cell Ω_j we consider a local approximant R_j and form the global approximant given by

$$\mathcal{I}(x) = \sum_{j=1}^{d} R_j(x) W_j(x), \qquad x \in \Omega. \tag{3}$$

Note that if the local approximants satisfy the interpolation conditions at data point x_i, i.e. $R_j(x_i) = f_i$, then the global approximant also interpolates at this node, i.e. $\mathcal{I}(x_i) = f_i$, for $i = 1, \ldots, n$ (see [3, 7] for further details).

As a local approximant we can take a radial basis function interpolant $R_j : \Omega \to \mathbb{R}$, which has the form

$$R_j(x) = \sum_{j=1}^{n} \alpha_j \phi(||x - x_j||_2), \quad x \in \Omega, \tag{4}$$

where $\phi : [0, \infty) \to \mathbb{R}$ is called *radial basis function*, $|| \cdot ||_2$ is the Euclidean norm, and $\{\alpha_j\}$ are the coefficients to be determined by solving the linear system generated by radial basis functions. Moreover, R_j satisfies the interpolation conditions $R_j(x_i) = f_i$, $i = 1, \ldots, n$ (see [3]).

Usually, it can be highly advantageous to work with locally supported functions since they lead to sparse linear systems. Wendland found a class of radial basis functions which are smooth, locally supported, and strictly positive definite on $\mathbb{R}$ (see, e.g, [7]). They consist of a product of a truncated power function and a low degree polynomial. For example, here we take the Wendland $C2$ function

$$\phi(r) = (1 - \beta r)_+^4 (4\beta r + 1),$$

where $r = ||x - x_j||_2$, $\beta \in \mathbb{R}^+$ is the shape parameter, and $(\cdot)_+$ denotes the truncated power function. This means that the function $\phi(r)$ is nonnegative; in fact, $(1 - \beta r)_+$ is defined as $(1 - \beta r)$ for $r \in [0, 1/\beta]$, and 0 for $r > 1/\beta$.

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Note that the Partition of Unity method preserves the local approximation order for the global fit. Hence, we can efficiently compute large radial basis function interpolants by solving small radial basis functions interpolation problems (in parallel as well) and then combine them together with the global partition of unity $\{W_j\}_{j=1}^{d}$. This approach enables us to decompose a large problem into many small problems, and at the same time ensures that the accuracy obtained for the local fits is carried over to the global fit. In particular, the Partition of Unity method can be thought as a Shepard's method with higher-order data, since local approximations R_j instead of data values f_j are used. Moreover, the use of Wendland's functions guarantees a good compromise between accuracy and stability.

6 Numerical experiments

In this section we summarize the extensive experiments to test our detection and approximation techniques. Here, we refer to the dynamical systems (1) and (2), taking $r = 1$, $p = 2$, $b = 0.5$, $u = 1$, $c = 3$, $a = 2$, $z = 3$, and $r = 9$, $q = 0.6$, $p = 0.6$, $b = 1/2$, $u = 1.5$, $c = 8$, $a = 8$, $z = 3$, $v = 2$, $e = 0.5$, $f = 6$, $g = 5$, respectively, as values of biological parameters and integrating the models in the interval $[0, 10]$.

As of accuracy of the Partition of Unity method a crucial task concerns the choice of the shape parameter β of Wendland's function. In fact, it can significantly affect the approximation result and, therefore, the quality of the separatrix curves and surfaces. From our study we found that good shape parameter values are given for $0.01 \leq \beta \leq 0.05$ (case 2D) and $0.001 \leq \beta \leq 0.01$ (case 3D). In Figure 5 (left to right) we show curves obtained approximating the refined data set when we consider the value $\beta = 0.1$ and $\beta = 0.015$, respectively, as shape parameters for the Wendland $C2$ function and a number $d = 2$ of partitions of Ω. Figure 6 shows the case 3D: separating surfaces are reconstructed using $\beta = 0.02$ (left) and $\beta = 0.005$ (right), respectively, and a number $d = 4$ of partitions of Ω.

7 Conclusions and future work

In this paper we presented an approximation method for the detection of points lying on the separatrix curve for the model (1) and the separatrix surface for the model (2), that are the curve and the surface which partition the respective domains into basins of attractions of the locally stable equilibria. The problem is rather common for population dynamics systems. An efficient algorithm based on the Partition of Unity method, which uses Wendland's functions as local approximants, was used for the reconstruction of separatrix curve and surface. Work in progress considers finding an approximation scheme also for a dynamical system of dimension three, which has three equilibrium points and basins of attractions.

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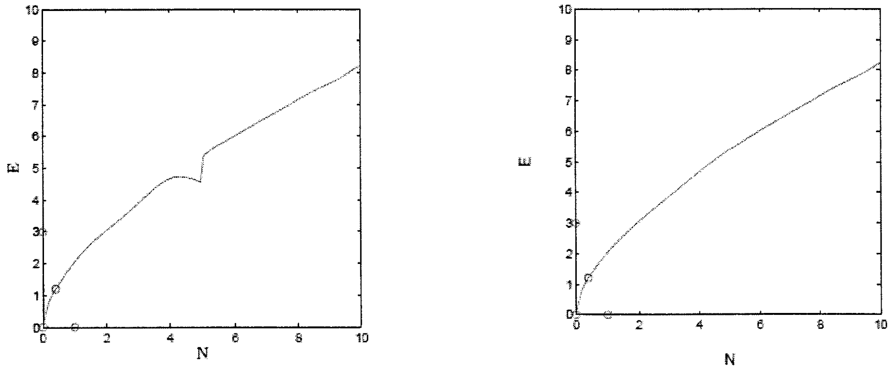

Figure 5: Approximation of separatrix curve: approximated curves using $\beta = 0.1$ (left) and $\beta = 0.015$ (right).

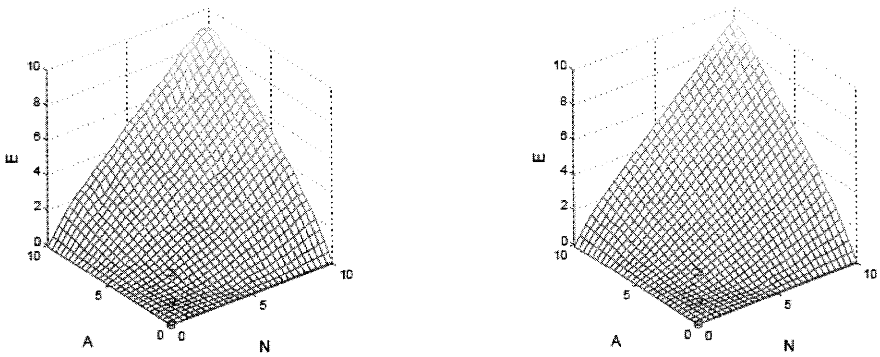

Figure 6: Reconstruction of separatrix surfaces: approximated surfaces using $\beta = 0.02$ (left) and $\beta = 0.005$ (right).

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