A Spherical Interpolation Algorithm Using Zonal Basis Functions

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Abstract

In this paper we present a new local algorithm for spherical interpolation of large scattered data sets. The method we implemented is a local Shepard's scheme using zonal basis functions as nodal functions. The algorithm is based on an optimized nearest-neighbour searching procedure. Experimental results show efficiency and accuracy of the algorithm.

Key words: zonal basis functions, spherical Shepard's formula, local algorithms, scattered data interpolation.
MSC 2000: 65D05, 65D15, 65D17

1 Introduction

Let $\mathbb{S}^{m-1} = \{x \in \mathbb{R}^m : \|x\|_2 = 1\}$ be the unit sphere in $\mathbb{R}^m$. We consider the problem of interpolating a function $f : \mathbb{S}^{m-1} \to \mathbb{R}$, $(m \geq 1)$, defined on a finite set $\mathcal{X}_n = \{x_i\}_{i=1}^n$ of distinct *data points* or *nodes* lying on $\mathbb{S}^{m-1}$. It consists of constructing a multivariate function F, which interpolates the *data values* or *function values* f_i at the nodes x_i, namely $F(x_i) = f_i$, $i = 1, \dots, n$. Possible applications include modeling closed surfaces in CAGD and representing scalar functions which estimate temperature, rainfall, pressure, ozone, gravitational forces, etc. at all points on the surface of the earth based on a discrete sample of values taken at arbitrary locations.

Recently, in [1], an efficient algorithm was proposed for the interpolation of large scattered data sets in bidimensional domains. It is based on a very fast *strip method*, consisting in the use of particular data structures named *strips* in the nearest neighbour searching procedure. It allows to obtain a very fast algorithm for bivariate interpolation. In this paper we extend the algorithm to the spherical setting. We found good results also in this case. At the moment investigations in the direction of comparison of this algorithm with the one proposed in [3] are under consideration.

The paper is organized as follows. In section 2 the zonal basis function (ZBF) method is briefly recalled. Section 3 is devoted to the local spherical interpolation

method with ZBFs. In section 4 the spherical algorithm used is explained and in section 5 numerical results are given.

2   ZBF Interpolation

Since we will propose a local interpolation scheme involving a *zonal basis function* (or *spherical radial basis function* (SRBF)) interpolant, here we focus both on theoretical and computational aspects of the ZBF method (see, e.g. [6, 4, 5]), recalling some basic mathematical interpolation tools [8].

Definition 2.1. *Given a set of distinct data points $\mathcal{X}_n = \{x_i\}_{i=1}^n$ arbitrarily distributed on $\mathbb{S}^{m-1}$, and the associated function values $\mathcal{F}_n = \{f_i\}_{i=1}^n$ of a function $f : \mathbb{S}^{m-1} \to \mathbb{R}$, a zonal basis function interpolant $s : \mathbb{S}^{m-1} \to \mathbb{R}$ has the form*

$$s(x) = \sum_{j=1}^n a_j \psi(d(x, x_j)), \quad x \in \mathbb{S}^{m-1}, \tag{1}$$

where $d(x, x_j) = \arccos\left(x^T x_j\right)$ denotes the geodesic distance, which is the length of the shorter part of the great circle arc joining x and x_j, $\psi : [0, \pi] \to \mathbb{R}$ is called zonal basis function, and s satisfies the interpolation conditions $s(x_i) = f_i$, $i = 1, \ldots, n$.

Although, as far as we know, there is no a complete characterization of the class of the functions ψ, a sufficient condition for nonsingularity is that the corresponding matrix

$$A_{i,j} = \psi(d(x_i, x_j)), \quad 1 \le i, j \le n, \tag{2}$$

be positive definite (see [6]).

Definition 2.2. *A continuous function $\psi : [0, \pi] \to \mathbb{R}$ is said positive definite of order n on $\mathbb{S}^{m-1}$, if*

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j \psi(d(x_i, x_j)) \ge 0, \tag{3}$$

for any set of distinct data points $\mathcal{X}_n = \{x_i\}_{i=1}^n$, and any $a = [a_1, \ldots, a_n]^T \in \mathbb{R}^n$. If the inequality (3) holds strictly for any nontrivial a, ψ is called strictly positive definite of order n. If ψ is (strictly) positive definite for any n, then it is called (strictly) positive definite.

Therefore, if ψ is (strictly) positive definite, the interpolant (1) is unique, since the corresponding interpolation matrix (2) is positive definite and hence nonsingular.

Generally, one requires that an interpolant reproduces the low degree spherical harmonics (as polynomials for RBFs in the multivariate setting), but this property is not satisfied. Hence, it is often convenient to add to s a spherical harmonic, which can be defined in the following way.

Definition 2.3. *Let $H_d \equiv H_d(\mathbb{S}^{m-1})$, $d \in \mathbb{Z}^+$, be the space of homogeneneous harmonics of degree d restricted to $\mathbb{S}^{m-1}$. The linear space H_d is called the space of spherical harmonics of exact degree d.*

It is well known that the dimension of H_d is given by

$$N_{m,d} = \dim(H_d) = \begin{cases} 1, & \text{if } d = 0, \\ \frac{2d+m-2}{d}\binom{d+m-3}{d-1}, & \text{if } d \geq 1, \end{cases}$$

and $N_{m,d} = O(d^{m-2})$, for $d \to \infty$. Moreover, the spherical harmonics of different degrees are orthogonal with respect to the L_2-inner product on $\mathbb{S}^{m-1}$

$$(f, g)_{L_2(\mathbb{S}^{m-1})} = \int_{\mathbb{S}^{m-1}} f(x)g(x)d\mu(x),$$

where $d\mu(x)$ is the standard measure on the sphere.

Now, denoting by $\{Y_{d,k} : k = 1, \ldots, N_{m,d}\}$ a (fixed) orthonormal basis of H_d, we have that $\mathcal{H}_d = \oplus_{j=0}^d H_j$, $d \in \mathbb{Z}^+$, is the space of spherical harmonics of degree at most d. Moreover, it is also known that $\{Y_{d,k} : k = 1, \ldots, N_{m,d}; d = 0, 1, \ldots\}$ is a complete orthonormal basis of $L_2(\mathbb{S}^{m-1})$. For more details, we refer to [8, 13].

Then, the drawback of the lacked reproduction of the low degree spherical harmonics can be overcome, adding to the ZBF interpolant s, given by (1), a spherical harmonic of degree d. It assumes the form

$$s(x) = \sum_{j=1}^n a_j \psi(d(x, x_j)) + \sum_{k=1}^V b_k Y_k(x), \quad x \in \mathbb{S}^{m-1}, \tag{4}$$

where $V = \dim \mathcal{H}_d(\mathbb{S}^{m-1})$, and $\{Y_1, \ldots, Y_V\}$ is a basis for $\mathbb{S}^{m-1}$.

The analytic solution (4) is obtained by requiring that s satisfies the interpolation conditions

$$s(x_i) = f_i, \quad i = 1, \ldots, n,$$

and the additional conditions

$$\sum_{i=1}^n a_i Y_k(x_i) = 0, \quad \text{for } k = 1, \ldots, V. \tag{5}$$

To compute the coefficients $a = [a_1, \ldots, a_n]^T$ and $b = [b_1, \ldots, b_V]^T$ in (4), it is required to solve the system of n linear equations in $n + V$ unknowns. Thus, supposing that $n \geq V$, we have the linear system

$$\begin{bmatrix} A & Y \\ Y^T & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{6}$$

where $A = \{\psi(d(x_i, x_j))\}_{i,j=1}^n$ is an $n \times n$ matrix (as in (2)), $Y = \{Y_k(x_i)\}$ is an $n \times V$ matrix, and f denotes the column vector of the k-th coordinate of the function f_i.

Definition 2.4. *A continuous function $\psi : [0,\pi] \to \mathbb{R}$ is said strictly conditionally positive definite of order $s \in \mathbb{N}$ on $\mathbb{S}^{m-1}$, if the quadratic form (3) is positive for any set of distinct data points $\mathcal{X}_n = \{x_i\}_{i=1}^n$ and any nonzero $a = [a_1, \ldots, a_n]^T \in \mathbb{R}^n$ satisfying (5).*

Definition 2.5. *Let s be a positive integer and let $V = \dim\mathcal{H}_{s-1}(\mathbb{S}^{m-1})$. A set of distinct data points $\{x_i\}_{i=1}^V$ is named $\mathcal{H}_{s-1}(\mathbb{S}^{m-1})$-unisolvent if the only element of $\mathcal{H}_{s-1}(\mathbb{S}^{m-1})$ to vanish at each x_i is the zero spherical harmonic.*

Any strictly conditionally positive function ψ of degree s can be used to provide an augmented ZBF interpolant (4) with $d = s - 1$. Nevertheless, in order to guarantee the solution uniqueness, we require also that the interpolation points contain an $\mathcal{H}_{s-1}(\mathbb{S}^{m-1})$-unisolvent subset. Then, the interpolant (4) is unique [10] (see also [9]).

Theorem 2.1. *Let ψ be a strictly conditionally positive definite on $\mathbb{S}^{m-1}$. Let $\mathcal{X}_n = \{x_i\}_{i=1}^n$ denote a set of n distinct data points in $\mathbb{S}^{m-1}$ such that $n \geq V = \dim\mathcal{H}_{s-1}(\mathbb{S}^{m-1})$, and $\mathcal{X}_n$ contains an $\mathcal{H}_{s-1}(\mathbb{S}^{m-1})$-unisolvent subset. Then the matrix of the linear system (6) is nonsingular.*

3 Local Spherical Interpolation by ZBFs

In this section we describe a local method for the multivariate interpolation of large scattered data sets lying on the sphere. The scheme is based on the local use of zonal basis functions, i.e. ZBF interpolants as nodal functions, and represents a further variant of the well-known modified Shepard's method. Hence, this local interpolation approach exploits the characteristic of accuracy of ZBFs, overcoming common disadvantages as the unstability due to the need of solving large linear systems (possibly, bad conditioned) and the inefficiency of the ZBF global interpolation method. A similar technique was already introduced at first by Pottmann and Eck [11] (MQ), and then by De Rossi [3] (ZBF).

As we present a local interpolation method, we need to define a ZBF interpolant of the form

$$Z(x) \equiv s_{|_{\mathcal{D}}}(x), \quad x \in \mathcal{D} \subset \mathcal{X}_n,$$

where $\mathcal{D}$ is the restriction of the data point set $\mathcal{X}_n$.

Therefore we consider the following definition of the modified spherical Shepard's method.

Definition 3.1. *Given a set of distinct data points $\mathcal{X}_n = \{x_i\}_{i=1}^n$, arbitrarily distributed on the sphere $\mathbb{S}^{m-1}$, with associated the corresponding set of real values $\mathcal{F}_n = \{f_i\}_{i=1}^n$ of an unknown function $f : \mathbb{S}^{m-1} \to \mathbb{R}$, the modified spherical Shepard's interpolant $F : \mathbb{S}^{m-1} \to \mathbb{R}$ takes the form*

$$F(x) = \sum_{j=1}^n Z_j(x)\bar{W}_j(x), \tag{7}$$

where the nodal functions $Z_j(x)$, $j = 1, \ldots, n$, are local approximants to f at x_j, relative to the subset $\mathcal{D}$ of the n_Z data points closest to x_j, satisfying the interpolation conditions $Z_j(x_j) = f_j$, and $\bar{W}_j(x)$, $j = 1, \ldots, n$, are the weight functions

$$\bar{W}_j(x) = \left[\frac{W_j(x)}{\sum_{k=1}^{n} W_k(x)} \right]^p, \quad j = 1, \ldots, n, \quad (p > 0),$$

with

$$W_j(x) = \tau(x, x_j)/\alpha(x, x_j).$$

The localizing function $\tau(x, x_j)$, often called step function, is

$$\tau(x, x_j) = \begin{cases} 1, & \text{if } x_j \in \mathcal{C}(x; s), \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{C}(x; s)$ is a hypercube of centre at x and side s, whereas $\alpha(x, x_j) = \arccos\left(x^T x_j\right)$.

To control the localization of the ZBFs, a certain number n_W of nodes close to x must be considered.

4   Spherical Interpolation Algorithm

In this section we propose an efficient algorithm for the interpolation on the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. In practice, this is typically the most interesting case, since it represents some physical phenomena in many areas, including e.g. geophysics and meteorology where the sphere is taken as model of the earth.

Thus, we consider the problem of approximating a function $f : D \to \mathbb{R}$, defined only on a finite set of distinct and scattered data points $\mathcal{X}_n = \{(x_i, y_i, z_i)\}_{i=1}^{n}$, where $D = \mathbb{S}^2 \subseteq [0,1] \times [0,1] \times [0,1] \subset \mathbb{R}^3$, and $\mathcal{F}_n = \{f_i\}_{i=1}^{n}$ is the set of corresponding values of the unknown function. We now describe a spherical interpolation algorithm, which is based on a strip searching procedure and a standard sorting procedure as quicksort routine. This requires on average a time complexity $O(M \log M)$, where M is the number of points to be sorted.

Moreover, the employment of such a strip structure allows some advantages: it optimizes the searching procedure of nodes making the interpolation algorithm efficient and guarantees a high parallelism.

Here is the algorithm in detail:

STEP 1. Let data points $\mathcal{X}_n$, data values $\mathcal{F}_n$, evaluation points $\mathcal{G}_s$, and localizing parameters n_Z and n_W be given.

STEP 2. Order the set $\mathcal{X}_n$ with respect to a common direction (e.g. the z-axis), by applying a *quicksort$_z$* procedure.

STEP 3. Construct a local (circle) neighbourhood, for each node (x_i, y_i, z_i), $i = 1, \ldots, n$. The half-size of the neighbourhoods depends on the dimension n, the considered value n_Z, and the positive integer k_1, i.e.

$$\delta_Z = \arccos\left(1 - 2\sqrt{k_1}\frac{n_Z}{n}\right), \quad k_1 = 1, 2 \ldots \tag{8}$$

STEP 4. Find the number of strips

$$q = \left\lceil \frac{\pi}{\delta_Z} \right\rceil ,$$

deriving directly by the length π of the shorter part of the great circle joining "north pole" and "south pole", and the neighbourhood half-size δ_Z.

STEP 5. Construct a suitable family of q strips of equal width $\delta_{s_1} \equiv \delta_Z$ (with possible exception of one of them) and parallel to xy-plane (either, equivalently, parallel to xz-plane or yz-plane) on the domain D. The set $\mathcal{X}_n$ of nodes is partitioned by the strip structure into q subsets $\mathcal{X}_{n_k}$, whose n_k elements are $(x_{k1}, y_{k1}, z_{k1}), \ldots, (x_{kn_k}, y_{kn_k}, z_{kn_k})$, $k = 1, \ldots, q$.

This lead to the following *strip partitioning procedure*:

Procedure 1.
```
count := 0;
zs := -1;
for (k from 1 to q)
{
n[k] := 0;
i := count + 1;
u[k] := k*delta_z;
v[k] := zs - (1 + cos(u[k]));
while (z[i] ≤ v[k] && i ≤ n)
{
n[k] := n[k] + 1;
count := count + 1;
i := i + 1;
BS[k] := count - n[k] + 1;
ES[k] := count;
OUTPUT(n[k] data of the k-th strip).
}
}
```

STEP 6. For each node of $\mathcal{X}_{n_k}$, $k = 1, \ldots, q$, define the strips to be examined for determining all data points belonging to a (local) neighbourhood centred at (x_i, y_i, z_i), $i = 1, \ldots, n$, applying the *strip searching procedure* described below. The number of nodes of the neighbourhood is counted and stored in m_i.

Procedure 2.
```
for (k from 1 to q)
{
B := k - 1;
E := k + 1;
if (B < 1)
  B := 1;
```

```
if (E > q)
  E := q;
for (h from BS[k] to ES[k])
{
 m[h] := 0;
 for (i from B to E)
 {
  for (j from BS[i] to ES[i])
  {
   if ((x[j],y[j],z[j]) belongs to the h-th neighbourhood of centre
       (x[h],y[h],z[h]) and spherical radius delta_z)
    m[h] := m[h] + 1;
    STORE[h][m[h]] ← (x[j],y[j],z[j],f[j]);
  }
 }
}
OUTPUT(data set belonging to the h-th neighbourhood of centre
      (x[h],y[h],z[h]) and spherical radius delta_z).
}
}
```

STEP 7. Order, and then reduce to n_Z, all the nodes belonging to a circle neighbourhood centred at x_i, $i = 1, \ldots, n$, by applying a based-distance sorting process, that is a $quicksort_d$ procedure.

STEP 8. For each node (x_j, y_j, z_j), find a local interpolant Z_j, $j = 1, \ldots, n$, constructed on the n_Z data points closest to it.

STEP 9. Order all the points of the set $\mathcal{G}_s$ with respect to the z-axis, by applying a $quicksort_z$ procedure.

STEP 10. For each evaluation point $(x, y, z) \in D$, construct a circle neighbourhood, whose half-size depends on the dimension n, the parameter value n_W, and the (positive integer) number k_2, that is,

$$\delta_W = \arccos \left(1 - 2\sqrt{k_2} \frac{n_W}{n} \right), \quad k_2 = 1, 2, \ldots \tag{9}$$

STEP 11. Determine the number of strips

$$r = \left\lceil \frac{\pi}{\delta_W} \right\rceil,$$

deriving directly by the length π of the shorter part of the great circle joining "north pole" and "south pole", and the neighbourhood half-size δ_W.

STEP 12. Construct a second (suitable) family of r strips of equal width $\delta_{s_2} \equiv \delta_W$ (with possible exception of one of them), again parallel to xy-plane on the domain D. The set $\mathcal{G}_s = \{(x_i, y_i, z_i)\}_{i=1}^s$ of evaluation points is partitioned into r subsets $\mathcal{G}_{p_k}$, whose p_k evaluation points are (x_{k1}, y_{k1}, z_{k1}), ..., $(x_{kp_k}, y_{kp_k}, z_{kp_k})$, $k = 1, \ldots, r$.

The strip structure is similar to that presented in Procedure 1; in particular, we have that q ct r, n[k] ct p[k], n ct s, and delta_z ct delta_w, where "ct" means "change to".

STEP 13. For each evaluation point of $\mathcal{G}_{p_k}$, $k = 1, \ldots, r$, search all data points belonging to a (local) neighbourhood of centre (x_i, y_i, z_i) and geodesic radius δ_W, by applying a procedure based on strips. The number of nodes of the neighbourhood is counted and stored in s_i, $i = 1, \ldots, s$.

As regard to Procedure 2, the following changes are required: q ct r, m[h] ct s[h], and delta_z ct delta_w.

STEP 14. Order, and then reduce to n_W, the nodes of each neighbourhood by applying a *quicksort_d* procedure.

STEP 15. Find a local weight function $\bar{W}_j(x, y, z)$, $j = 1, \ldots, n$, considering only the n_W points closest to the evaluation point (x, y, z), where (x, y, z) denotes the generic evaluation point.

STEP 16. Apply the modified spherical Shepard's formula (7), and evaluate the surface at each evaluation point $(x, y, z) \in D$.

Note that to localize the nodes closest to each strip point, we establish the minimal number of strips to be examined, which here is three, i.e. the strip on which the considered data point lies, the previous and the next strips. The reason of such a value follows from the choice of setting $\delta_{s_1} \equiv \delta_Z$ and $\delta_{s_2} \equiv \delta_W$. Indeed, a node belonging to a strip can be closer to data points that lie in nearby strips than those in the same strip. Therefore, the searching of nodes belonging to local neighbourhood must be extended to all the strips in which there is, at least, a possible candidate (point). Obviously, for all nodes of the "first" and "last" strip, we reduce the strips to be examined to two (see Procedure 2 in STEP 6, and STEP 13).

The size of circle neighbourhoods is carried out so that, supposing a uniform distribution of points on all the domain D, each local neighbourhood has a prefixed number of nodes. The condition is satisfied, taking into account the dimension n, the parameter n_Z (or n_W), and the positive integer k_1 (or k_2). In particular, the rule (8) in STEP 3 (or (9) in STEP 10) estimates for $k_1 = 1$ (or $k_2 = 1$), at least, n_Z (or n_W) points for each neighbourhood. However, the approch we propose is completely automatic, for which the procedure locates the minimal positive integer k_1 (or k_2) satisfying the request to having a sufficient number of data points on each neighbourhood. This means that the method works successfully also when the distribution of data points is not uniform.

5 Numerical Results

In this section we show the accuracy and efficiency of the proposed algorithm, which has been implemented in *C language*. All the numerical results we present are obtained on a Pentium IV computer (2.40 GHz). In particular, we are also interested to stress the effectiveness of the considered strip searching procedure which allow to reduce the execution CPU times. For this reason, we propose a comparison between the spherical

interpolation algorithm implemented by using the strip structure on the sphere (SA), and the classical algorithm (CA), where the sphere $\mathbb{S}^2$ is not partitioned in strips (see Table 1).

In the tests we consider a few data sets of $n = 2^i \cdot 500$, $i = 0, 1, \ldots, 7$, Halton points on the sphere [14] as scattered points to be interpolated, and a set of 600 spiral points as evaluation points which are generated by using the method of Saff and Kuijlaars [12]. Data values are taken by the restriction on $\mathbb{S}^2$ of the following four (trivariate) test functions [11, 7]:

$$f_1(x, y, z) = \frac{1 + 2x + 3y + 4z}{6},$$

$$f_2(x, y, z) = \frac{9x^3 - 2x^2y + 3xy^2 - 4y^3 + 2z^3 - xyz}{10},$$

$$f_3(x, y, z) = \frac{e^x + 2e^{y+z}}{10},$$

$$f_4(x, y, z) = \sin x \sin y \sin z.$$

	CPU Time	
n	SA	CA
500	0.400	0.511
1000	0.441	0.671
2000	0.901	2.103
4000	2.303	9.484
8000	5.969	34.139
16000	15.001	192.037
32000	38.565	807.241
64000	175.753	3717.986

Table 1: CPU times (in seconds) obtained by SA and CA using ψ_2 for f_1.

The choice of the appropriate numbers n_Z and n_W is a non trivial problem, since it determines the accuracy of the local ZBF scheme. Numerical investigations pointed out that "good" values for these parameters are $n_Z = 15$ and $n_W = 10$. Moreover, among several tested ZBFs, we take the spherical inverse multiquadric (IMQ) [4] and the logarithmic spline [7]:

$$\psi_1(t) = \frac{1}{\sqrt{1 + \gamma^2 - 2\gamma c}}, \qquad \text{spherical IMQ},$$

$$\psi_2(t) = \frac{1}{\beta} \log \left(1 + \frac{2\beta}{\sqrt{1 + \beta^2 - 2\beta c} + 1 - \beta} \right), \qquad \text{logarithmic spline},$$

where $\beta, \gamma \in (0, 1)$, $c = \cos(t)$, and t is the geodesic distance on the sphere, namely

n	f_1	f_2	f_3	f_4
500	9.4916E − 3 1.6234E − 3	9.1391E − 3 1.6347E − 3	6.1264E − 3 9.6098E − 4	3.7898E − 3 5.5405E − 4
1000	2.5101E − 3 5.0926E − 4	3.7769E − 3 5.1455E − 4	2.0022E − 3 3.3969E − 4	2.0022E − 3 3.3969E − 4
2000	1.2924E − 3 2.0323E − 4	1.0688E − 3 1.7905E − 4	8.5904E − 4 1.4012E − 4	5.8907E − 4 7.3540E − 5
4000	2.4042E − 4 4.7891E − 5	3.1177E − 4 4.9201E − 5	1.9583E − 4 3.3424E − 5	1.3363E − 4 1.6935E − 5
8000	7.4362E − 5 1.2049E − 5	6.4579E − 5 1.0369E − 5	4.0073E − 5 7.8723E − 6	2.4360E − 5 4.0642E − 6
16000	4.5552E − 5 3.3560E − 6	2.2178E − 5 2.8248E − 6	1.8007E − 5 1.9938E − 6	4.9581E − 6 9.0486E − 7
32000	6.4436E − 6 7.2957E − 7	5.1260E − 6 6.8387E − 7	6.0991E − 6 4.8144E − 7	2.1046E − 6 2.3501E − 7
64000	1.6482E − 6 1.5209E − 7	1.1047E − 6 1.4685E − 7	7.1330E − 7 9.3410E − 8	4.5072E − 7 4.7949E − 8

Table 2: MAEs and RMSEs by using ψ_1.

n	f_1	f_2	f_3	f_4
500	4.1540E − 3 7.4688E − 4	5.2372E − 3 8.7653E − 4	2.1841E − 3 3.7548E − 4	2.6918E − 3 3.3230E − 4
1000	1.2445E − 3 2.0671E − 4	1.9831E − 3 2.4179E − 4	4.9512E − 4 1.0690E − 4	4.8661E − 4 1.0567E − 4
2000	3.0853E − 4 7.1861E − 5	5.0038E − 4 7.8670E − 5	2.5460E − 4 4.2001E − 5	2.6100E − 4 3.7240E − 5
4000	9.7610E − 5 1.7739E − 5	1.3159E − 4 2.1183E − 5	5.8100E − 5 9.8421E − 6	5.3480E − 5 8.1228E − 6
8000	2.7372E − 5 4.4705E − 6	2.7909E − 5 4.3736E − 6	1.4293E − 5 2.3327E − 6	1.1469E − 5 2.0051E − 6
16000	1.8184E − 5 1.2999E − 6	8.0451E − 6 1.1528E − 6	6.9378E − 6 6.3696E − 7	2.9160E − 6 4.4485E − 7
32000	1.4535E − 6 2.4965E − 7	2.2620E − 6 2.8757E − 7	1.0116E − 6 1.2674E − 7	9.7478E − 7 1.0951E − 7
64000	4.9573E − 7 6.0061E − 8	1.0710E − 6 7.9300E − 8	3.0403E − 7 3.1628E − 8	1.7742E − 7 2.3408E − 8

Table 3: MAEs and RMSEs by using ψ_2.

$t \in [0, \pi]$. These two functions are both (strictly) positive definite on $\mathbb{S}^2$, and their values of the shape parameters are chosen to be $\gamma = \beta = 0.7$. This is a good trade-off between accuracy and stability by varying the dimension n, taking into account that in

a local approach the number of points to be interpolated is small; hence, the condition numbers of the interpolation matrices are relatively small. Regarding the value of p in the weight functions, we took $p = 1$.

Finally, in Tables 2 and 3 we show the maximum absolute errors (MAEs) and the root mean square errors (RMSEs) achieved by using ψ_1 and ψ_2, respectively.

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