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Abstract

In this paper we analyze time-frequency representations in the Cohen class, i.e., quadratic forms expressed as a convolution between the classical Wigner transform and a kernel, with respect to uncertainty principles of local type. More precisely the results we obtain concerning the energy distribution of these representations show that a "too large" amount of energy cannot be concentrated in a "too small" set of the time-frequency plane. In particular, for a signal $f \in L^2(\mathbb{R}^d)$, the energy of a time-frequency representation contained in a measurable set M must be controlled by the standard deviations of $|f|^2$ and $|\hat{f}|^2$, and by suitable quantities measuring the size of M.

Keywords: Time-Frequency representations, Wigner sesquilinear and quadratic form, local uncertainty principles.

Mathematics Subject Classification: 42B10.

1 Introduction

In this paper we prove local uncertainty principles for time-frequency representations in the Cohen class, i.e. quadratic forms of the kind

$$Q_\sigma f(x,\omega) := (\sigma * \operatorname{Wig} f)(x,\omega), \tag{1.1}$$

where $\operatorname{Wig} f$ is the classical Wigner transform, defined as

$$\operatorname{Wig} f(x,\omega) = \int e^{-2\pi i t\omega} f\left(x+\frac{t}{2}\right) \overline{f\left(x-\frac{t}{2}\right)}\, dt,$$

and σ is a function or distribution on $\mathbb{R}^{2d}$. This class appears both as a widely used set of time-frequency representations, as well as in connection with some theoretical aspects of harmonic analysis, for example Weyl symbols of localization operators belong to this class. References can be found in [6], [7], [8], [14], [18], [13], [19].

Our purpose is to provide a reformulation, in the framework of the time-frequency representations, of the local uncertainty principle for the Fourier transform introduced by Price in [17], [12], [16].

In order to motivate the main results of this paper, which are contained in Sections 2, 3, 4, we begin by reviewing some basic facts on the Cohen class; we recall then the local uncertainty principle of Price and compare it with the classical Heisenberg uncertainty principle.

The expression (1.1) makes sense for $f \in \mathcal{S}(\mathbb{R}^d)$ and $\sigma \in \mathcal{S}(\mathbb{R}^{2d})$, and in this case $Q_\sigma f \in \mathcal{S}(\mathbb{R}^{2d})$. Other more general functional frameworks are however possible in the Schwartz distribution space $\mathcal{S}'$, in such a way that the convolution in (1.1) is well defined. For example, when $f \in L^2(\mathbb{R}^d)$ and $\sigma \in L^1(\mathbb{R}^{2d})$ we obtain that $Q_\sigma f$ is a well defined element of $L^2(\mathbb{R}^{2d})$.

From the point of view of time-frequency analysis, when considering separately a function f and its Fourier transform $\hat{f}$ we analyze separately the energy distribution of the "signal" f with respect to time, represented by $|f(x)|^2$, and the energy distribution of f with respect to frequency, represented by $|\hat{f}(\omega)|^2$. A time-frequency representations $Q_\sigma f(x,\omega)$ gives the energy distribution of a signal f with respect to time x and frequency ω at the same time, and in fact it doubles the dimension of its domain, being $(x,\omega) \in \mathbb{R}^{2d}$. The Cohen class contains the most important covariant representations, and moreover gives the freedom to design the kernel σ in order that the corresponding form Q_σ has specific features. In this framework we refer for example to [7, Chapter 11], [1], [2], [15]. As particular cases of the representation Q_σ we recover the Wigner transform when σ is the Dirac distribution δ; moreover, if $\tau \in [0,1]$, $\tau \neq 1/2$, and $\sigma(x,\omega) = \frac{2^d}{|2\tau-1|^d} e^{2\pi i \frac{2}{2\tau-1} x\omega}$, the corresponding representation Q_σ becomes the τ-Wigner transform

$$\mathrm{Wig}_\tau f(x,\omega) = \int e^{-2\pi i t\omega} f(x+\tau t)\, \overline{f(x-(1-\tau)t)}\, dt, \tag{1.2}$$

see for example [4] (the classical Wigner transform is obtained by letting $\tau = 1/2$). For a deep investigation of the Winger representations in connection with symplectic geometry and quantization we refer to [9], [10] and [11]. In the cases $\tau = 0$ and $\tau = 1$ we get the Rihaczek and conjugate Rihaczek forms, given by

$$Rf(x,\omega) = e^{-2\pi i x\omega} f(x)\overline{\hat{f}(\omega)} \quad \text{and} \quad R^*f(x,\omega) = e^{2\pi i x\omega}\overline{f(x)}\hat{f}(\omega),$$

respectively. Another relevant class of time-frequency representations contained in the Cohen class is the Spectrogram, defined as follows. Given a "window" function $\phi \in \mathcal{S}(\mathbb{R}^d)$, the Gabor transform of $f \in \mathcal{S}(\mathbb{R}^d)$ is given by $V_\phi f(x,\omega) = \int e^{-2\pi i t\omega} f(t)\overline{\phi(t-x)}\, dt$. Then, for $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$ (with possible generalizations to larger functional settings) the (generalized) spectrogram is defined as follows:

$$\mathrm{Sp}_{\phi,\psi} f(x,\omega) = (V_\phi f \cdot \overline{V_\psi f})(x,\omega). \tag{1.3}$$

For $\phi = \psi$ we have in particular the classical spectrogram $|V_\phi f(x,\omega)|^2$. We refer to [3], [14] for a treatment of (1.3) and for further references. Here we just recall that the generalized spectrogram belongs to the Cohen class, and the corresponding kernel is $\sigma = \mathrm{Wig}(\tilde{\psi}, \tilde{\phi})$, where $\tilde{g}(t) := g(-t)$.

In this paper we prove local uncertainty principles for representations in the Cohen class, transferring to the time-frequency frame the idea of local uncertainty principle of Price [17] for the Fourier transform. We set $\|\cdot\|_{L^p(E)}$ for the usual L^p-norm on $E \subset \mathbb{R}^d$ or $E \subset \mathbb{R}^{2d}$ (if $E = \mathbb{R}^d$ or $E = \mathbb{R}^{2d}$ we simply write $\|\cdot\|_p$). Moreover, the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$ is given by $\hat{f}(\omega) = \int e^{-2\pi i t\omega} f(t)\, dt$, with standard extensions to $L^2(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$.

For a function $f \in L^2(\mathbb{R})$ (for simplicity we consider here $d = 1$) we define $\overline{x} = \int x|f(x)|^2\, dx$ and $\overline{\omega} = \int \omega|\hat{f}(\omega)|^2\, d\omega$. Then the corresponding standard deviations are $\sigma_f = \|(x-\overline{x})f(x)\|_2$ and $\sigma_{\hat{f}} = \|(\omega - \overline{\omega})\hat{f}(\omega)\|_2$. The classical Heisenberg uncertainty principle states that, for every $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$ we have

$$\sigma_f \sigma_{\hat{f}} \geq \frac{1}{4\pi}. \tag{1.4}$$

The local uncertainty principle, cf. [17], states that, for every $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$ and for every measurable set $E \subset \mathbb{R}$ we have

$$\|\hat{f}\|_{L^2(E)} \leq 2\pi\, \mathfrak{m}(E)\sigma_f, \tag{1.5}$$

where $\mathfrak{m}(E)$ is the Lebesgue measure of E (see the subsequent Theorem 2.1 for a more general formulation). Since σ_f and $\sigma_{\hat{f}}$ measure how much the function f and its Fourier transform are concentrated, (1.4) tells us that if a function is very concentrated (i.e., σ_f is small), then $\sigma_{\hat{f}}$ must be large, i.e., the Fourier transform of f must be sufficiently spread out. From the Heisenberg uncertainty principle however we do not have information on the admissible ways $\hat{f}$ may be spread out; for example it could be spread out in a uniform way, or it could be concentrated in small intervals sufficiently far away from one another. The latter possibility is excluded by (1.5), which tells us that the energy of the Fourier transform $\hat{f}$ in a measurable set E must be small as σ_f and $\mathfrak{m}(E)$ are small.

Observe that in (1.5) the energy of $\hat{f}(\omega)$ is estimated on a set $E \subset \mathbb{R}^d$, which in the time-frequency space $\mathbb{R}^{2d}_{(x,\omega)}$ would correspond to a set M of the form $\mathbb{R}^d \times E$, i.e., an horizontal strip. In this paper we prove estimates of the energy of a time-frequency distribution in a general set $M \subset \mathbb{R}^{2d}_{(x,\omega)}$, not necessarily an horizontal (or vertical) strip, obtaining that if M is sufficiently "small" and the function f or its Fourier transform $\hat{f}$ are not too spread out then the energy of the time-frequency distribution $Q_\sigma f$ in M must be small. One of the main points is to specify the meaning of "small" for the set M, eventually depending on the kernel σ of the representation. We leave to the next sections the precise definitions, presenting here the main results and some examples.

Given a time-frequency representation Q_σ in the Cohen class, we prove that there exists a positive constant C such that for every measurable set $M \subset \mathbb{R}^{2d}$, every $f \in L^2(\mathbb{R})$ and every $\alpha, \alpha_1, \alpha_2 > d/2$ the following inequalities hold:

$$\|Q_\sigma f\|^2_{L^2(M)} \leq C\langle M\rangle_1 \|f\|_2^{4-d/\alpha} \||t-\bar{t}|^\alpha f\|_2^{d/\alpha},$$
$$\|Q_\sigma f\|^2_{L^2(M)} \leq C\langle M\rangle_2 \|\hat{f}\|_2^{4-d/\alpha} \||\omega-\overline{\omega}|^\alpha \hat{f}\|_2^{d/\alpha},$$
$$\|Q_\sigma f\|^2_{L^2(M)} \leq C\langle M\rangle_3 \|f\|_2^{4-d/\alpha_1-d/\alpha_2} \||t-\bar{t}|^{\alpha_1} f\|_2^{d/\alpha_1} \||\omega-\overline{\omega}|^{\alpha_2} \hat{f}\|_2^{d/\alpha_2},$$

where $\langle M\rangle_j$, $j = 1, 2, 3$, are real non negative functions of the set M, which in a suitable sense measure the size of M; they will be precisely defined in the subsequent sections for different (classes of) kernels σ. In particular, the kernels that we are able to treat are $\sigma = \delta$ (Dirac distribution, corresponding to the classical Wigner), $\sigma = \frac{2^d}{|2\tau-1|^d} e^{2\pi i \frac{2}{2\tau-1} x\omega}$ (that corresponds to the τ-Wigner distributions for $\tau \neq 1/2$, and in particular to the Rihaczek and conjugate Rihaczek representations for $\tau = 0$ and $\tau = 1$, respectively), and then a generic $\sigma \in L^1(\mathbb{R}^{2d}) \cap L^2(\mathbb{R}^{2d})$. We give now some examples, in order to better explain the results and compare the previous inequalities with the local uncertainty principle (1.5). We consider now $d = 1$, $\alpha = \alpha_1 = \alpha_2 = 1$ and $\|f\|_2 = 1$, and observe that in this particular case our results read in the following way:

$$\|Q_\sigma f\|^2_{L^2(M)} \leq C\langle M\rangle_1 \sigma_f, \tag{1.6}$$
$$\|Q_\sigma f\|^2_{L^2(M)} \leq C\langle M\rangle_2 \sigma_{\hat{f}}, \tag{1.7}$$
$$\|Q_\sigma f\|^2_{L^2(M)} \leq C\langle M\rangle_3 \sigma_f \sigma_{\hat{f}}, \tag{1.8}$$

for $M \subset \mathbb{R}^2$ measurable. In the case $M = \mathbb{R} \times E$, for a measurable set $E \subset \mathbb{R}$, we will see that $\langle M\rangle_1 = \mathfrak{m}(E)$. The inequality (1.6) then becomes $\|Q_\sigma f\|^2_{L^2(\mathbb{R}\times E)} \leq C\,\mathfrak{m}(E)\sigma_f$; it is the closest generalization of (1.5), since it tells that in an horizontal strip having E as the projection on the ω-axis, the energy of the time-frequency distribution has to be small proportionally to the measure of E and the standard deviation of f. Analogously, for strips of the kind $M = F \times \mathbb{R}$, $F \subset \mathbb{R}$ measurable, we will see that (1.7) becomes $\|Q_\sigma f\|^2_{L^2(F\times\mathbb{R})} \leq C\,\mathfrak{m}(F)\sigma_{\hat{f}}$. On the other hand, we have here much more freedom in the choice of the set M. For example, we can consider oblique strips, i.e. sets of the kind $M = \{(x,\omega) \in \mathbb{R}^2 : kx+a \leq \omega \leq kx+b\}$ for $a < b$ and $k \neq 0$ (otherwise we are in one of the previous cases). For such sets, we will see that (1.6) and (1.7) read as $\|Q_\sigma f\|^2_{L^2(M)} \leq C(b-a)\sigma_f$ and $\|Q_\sigma f\|^2_{L^2(M)} \leq C\frac{b-a}{|k|}\sigma_{\hat{f}}$, respectively. This tells us that a time-frequency representation applied to a function f such that at least one between σ_f and $\sigma_{\hat{f}}$ is finite, cannot contain a great amount of energy in a narrow strip of the time-frequency plane. Till now we have analyzed (1.6) and (1.7). Concerning the last estimate (1.8), it becomes significative in general when M has finite measure (as subset of $\mathbb{R}^{2d}$), while both its orthogonal projections on the x and ω axes have infinite measure (as subsets of $\mathbb{R}^d$). In this case (1.6) and (1.7) shall not give any information, while (1.8) becomes $\|Q_\sigma f\|^2_{L^2(M)} \leq C\,\mathfrak{m}(M)\sigma_f\sigma_{\hat{f}}$, saying that if M has small Lebesgue measure and both σ_f and $\sigma_{\hat{f}}$ are finite, then $Q_\sigma f$ must show small energy in the set M, even if M is very spread in the time-frequency plane.

The paper is organized as follows. We prove local uncertainty principles for the Rihaczek and conjugate Rihaczek in Section 2, for τ-Wigner representations in Section 3, and for representations in the Cohen class with kernel belonging to $L^1(\mathbb{R}^{2d}) \cap L^2(\mathbb{R}^{2d})$ in Section 4. The reason for keeping separate these cases is that both the proofs and the results are different. We present examples and comparisons between the various results, in particular for what concerns the bounding functions $\langle M \rangle_j$, and consequently for what concerns the sets M for which our results become significative.

2 Local uncertainty principle for Rihaczek transform

We start this Section recalling the local uncertainty principle for the Fourier transform of Price [17], see also [12], [16] in a slight generalized form obtained by using translations and modulations of the signal. We study then the case of the Rihaczek and conjugate Rihaczek representations.

Theorem 2.1. *Let $E \subset \mathbb{R}^d$ a measurable set and $\alpha > d/2$. Then for every $f \in L^2(\mathbb{R}^d)$ and $\bar{t}, \overline{\omega} \in \mathbb{R}^d$ we have*

$$\int_E |\hat{f}(\omega)|^2 \, d\omega < K \, \mathfrak{m}(E) \|f\|_2^{2-d/\alpha} \||t - \bar{t}|^\alpha f\|_2^{d/\alpha} \tag{2.1}$$

and

$$\int_E |f(t)|^2 \, dt < K \, \mathfrak{m}(E) \|\hat{f}\|_2^{2-d/\alpha} \||\omega - \overline{\omega}|^\alpha \hat{f}\|_2^{d/\alpha}, \tag{2.2}$$

where $\mathfrak{m}(E)$ the Lebesgue measure of the set E,

$$K = K(d, \alpha) = \frac{\pi^{d/2}}{\alpha} \left(\Gamma \left(\frac{d}{2} \right) \right)^{-1} \Gamma \left(\frac{d}{2\alpha} \right) \Gamma \left(1 - \frac{d}{2\alpha} \right) \left(\frac{2\alpha}{d} - 1 \right)^{\frac{d}{2\alpha}} \left(1 - \frac{d}{2\alpha} \right)^{-1} \tag{2.3}$$

and Γ is the Euler function given by $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} \, dt$. Moreover, the constant K is optimal, and equality in (2.1)-(2.2) *is never attained when $f \neq 0$.*

The main tool used in [17] to prove Theorem 2.1 is the following proposition, that we recall here since we shall use it in the following for time-frequency representations.

Proposition 2.2 (Price [17, Proposition 2.1']). *For every $\alpha > d/2$, $f \in L^2(\mathbb{R}^d)$ and $\bar{t} \in \mathbb{R}^d$ we have*

$$\|f\|_1 \leq \sqrt{K} \|f\|_2^{1-\frac{d}{2\alpha}} \||t - \bar{t}|^\alpha f\|_2^{\frac{d}{2\alpha}} \tag{2.4}$$

where K is given by (2.3)*. In particular, if $f \in L^2(\mathbb{R}^d)$ and $\||t - \bar{t}|^\alpha f\|_2 < \infty$ for some $\alpha > d/2$ and $\bar{t} \in \mathbb{R}^d$, then $f \in L^1(\mathbb{R}^d)$.*

We observe that in [17] the estimate (2.4) is proved only for $\bar{t} = 0$; the case $\bar{t} \neq 0$ can be easily deduced by considering a translation of f instead of f itself. Furthermore, we remark that in [17] a more general version of Proposition 2.2 is proved, involving in (2.4) L^p and L^q

norms. This can be used, with techniques analogous to those of this paper, to obtain local uncertainty principles in the Cohen class involving general Lebesgue norms. However this would not lead to a much deeper insight, therefore, as in Price [17], we choose to limit our attention to the L^2 framework.

In order to state the local uncertainty principle for the Rihaczek and conjugate Rihaczek representations we need the following definition.

Definition 2.3. *Fix a measurable set $M \subset \mathbb{R}^{2d}$. We set $M_x = \{\omega \in \mathbb{R}^d : (x, \omega) \in M\}$ for $x \in \mathbb{R}^d$, and $M_\omega = \{x \in \mathbb{R}^d : (x, \omega) \in M\}$ for $\omega \in \mathbb{R}^d$. Then we know that $M_x \subset \mathbb{R}^d$ is measurable for almost every $x \in \mathbb{R}^d$, and $M_\omega \subset \mathbb{R}^d$ is measurable for almost every $\omega \in \mathbb{R}^d$. We define*

$$\mathfrak{m}_\omega(M) = \| \mathfrak{m}(M_x) \|_{L^\infty(\mathbb{R}^d_x)}$$

and

$$\mathfrak{m}_x(M) = \| \mathfrak{m}(M_\omega) \|_{L^\infty(\mathbb{R}^d_\omega)},$$

where the Lebesgue measure in the norm in the right-hand sides is the d-dimensional one.

Remark 2.4. *Observe that $\mathfrak{m}_\omega(M)$ and $\mathfrak{m}_x(M)$ are well defined (either finite or infinite) for every measurable set $M \subset \mathbb{R}^{2d}$.*

We can now state the main result of this section.

Theorem 2.5. *Let $M \subset \mathbb{R}^{2d}$ a measurable set, and $\alpha, \alpha_1, \alpha_2 > d/2$. Then for every $f \in L^2(\mathbb{R}^d)$ and for every fixed $\bar{t}, \overline{\omega} \in \mathbb{R}^d$ we have:*

$$\begin{aligned} \|Rf\|^2_{L^2(M)} \leq \min\big\{ & K\, \mathfrak{m}_\omega(M) \|f\|_2^{4-\frac{d}{\alpha}} \| |t-\bar{t}|^\alpha f\|_2^{\frac{d}{\alpha}}, \\ & K\, \mathfrak{m}_x(M) \|\hat{f}\|_2^{4-\frac{d}{\alpha}} \| |\omega-\overline{\omega}|^\alpha \hat{f}\|_2^{\frac{d}{\alpha}}, \\ & K_1 K_2\, \mathfrak{m}(M) \|f\|_2^{4-\frac{d}{\alpha_1}-\frac{d}{\alpha_2}} \| |t-\bar{t}|^{\alpha_1} f\|_2^{\frac{d}{\alpha_1}} \| |\omega-\overline{\omega}|^{\alpha_2} \hat{f}\|_2^{\frac{d}{\alpha_2}} \big\}, \end{aligned} \tag{2.5}$$

where K, K_1, K_2 are given by (2.3) in correspondence to α, α_1 and α_2 respectively, and Rf is the Rihaczek representation. The same estimates hold for the conjugate Rihaczek form R^*f.

Proof. We observe at first that for every measurable set $M \subset \mathbb{R}^{2d}$ we have $\|Rf\|_{L^2(M)} = \|R^*f\|_{L^2(M)}$, and so we can limit ourselves to prove (2.5) in the case of the Rihaczek.

(i) We start by proving the first estimate in (2.5). From Fubini Theorem, writing $\Pi_x M$ for the orthogonal projection of M on the x-space, we have

$$\|Rf\|_{L^2(M)} = \int_{\Pi_x M} \int_{M_x} |f(x)\hat{f}(\omega)|^2 \, d\omega \, dx = \int_{\Pi_x M} |f(x)|^2 \|\hat{f}\|^2_{L^2(M_x)} \, dx.$$

Since M_x is measurable for almost every x, we can apply Theorem 2.1 to $\|\hat{f}\|^2_{L^2(M_x)}$ almost everywhere in x, obtaining

$$\begin{aligned}\|Rf\|_{L^2(M)} &\leq K\|f\|_2^{2-\frac{d}{\alpha}}\||t-\bar{t}|^{\alpha}f\|_2^{\frac{d}{\alpha}}\int_{\Pi_x M}|f(x)|^2\,\mathfrak{m}(M_x)\,dx\\ &\leq K\|f\|_2^{2-\frac{d}{\alpha}}\||t-\bar{t}|^{\alpha}f\|_2^{\frac{d}{\alpha}}\,\mathfrak{m}_{\omega}(M)\|f\|_2^2.\end{aligned}$$

(ii) As in the previous case, we have

$$\|Rf\|_{L^2(M)} = \int_{\Pi_\omega M}|\hat{f}(\omega)|^2\|f\|^2_{L^2(M_\omega)}\,d\omega. \tag{2.6}$$

We can then complete the proof by applying Theorem 2.1 and the same procedure as point (i).

(iii) From (2.6) and Theorem 2.1 we have that

$$\|Rf\|_{L^2(M)} \leq K_2\|\hat{f}\|^2_\infty\|\hat{f}\|_2^{2-\frac{d}{\alpha_2}}\||\omega-\overline{\omega}|^{\alpha_2}\hat{f}\|_2^{\frac{d}{\alpha_2}}\int_{\Pi_\omega M}\mathfrak{m}(M_\omega)\,d\omega.$$

Now, $\int_{\Pi_\omega M}\mathfrak{m}(M_\omega)\,d\omega = \mathfrak{m}(M)$. By the mapping properties of the Fourier transform we then have

$$\|Rf\|_{L^2(M)} \leq K_2\,\mathfrak{m}(M)\|f\|_1^2\|f\|_2^{2-\frac{d}{\alpha_2}}\||\omega-\overline{\omega}|^{\alpha_2}\hat{f}\|_2^{\frac{d}{\alpha_2}}.$$

We can then apply Proposition 2.2 with α_1 instead of α and we get the third estimate in (2.5).

□

Remark 2.6. *Since $M_x \subset \Pi_\omega M$ for every $x \in \mathbb{R}^d$ and $M_\omega \subset \Pi_x M$ for every $\omega \in \mathbb{R}^d$, we have that*

$$\mathfrak{m}_\omega(M) \leq \mathfrak{m}(\Pi_\omega M) \quad \text{and} \quad \mathfrak{m}_x(M) \leq \mathfrak{m}(\Pi_x M), \tag{2.7}$$

and so, under the hypotheses of Theorem 2.5, the estimates (2.5) hold with $\mathfrak{m}(\Pi_\omega M)$ and $\mathfrak{m}(\Pi_x M)$ instead of $\mathfrak{m}_\omega(M)$ and $\mathfrak{m}_x(M)$, respectively. On the other hand, in general we do not have equality in (2.7); we may even have $\mathfrak{m}_\omega(M) < +\infty$ and $\mathfrak{m}(\Pi_\omega M) = +\infty$, so the expressions $\mathfrak{m}_\omega(M)$ and $\mathfrak{m}_x(M)$ become significative in many cases.

An immediate consequence of the previous theorem is the following estimate. It is trivially obtained multiplying the first and the second estimate in (2.5), but actually yields new information about the local concentration in the particular case where M is rectangle. Analogues corollaries will hold with regard to Theorems 3.6 and 4.3 and they will be omitted.

Corollary 2.7. *In the same hypotheses of Theorem* 2.5*, if $M = E\times F$, with E, F measurable sets in $\mathbb{R}^d$, then we have*

$$\|Rf\|^2_{L^2(M)} \leq \sqrt{K_1K_2}\sqrt{\mathfrak{m}(M)}\|f\|_2^{4-\frac{d}{2\alpha_1}-\frac{d}{2\alpha_2}}\||t-\bar{t}|^{\alpha_1}f\|^{\frac{d}{2\alpha_1}}\||\omega-\overline{\omega}|^{\alpha_2}\widehat{f}\|^{\frac{d}{2\alpha_2}}.$$

We want now to analyze some examples, in the case $d = 1$ (i.e., $M \subset \mathbb{R}^2$), in order to clarify the meaning of Theorem 2.5, in particular from the point of view of the classes of sets M for which the result becomes significative.

First of all we remark that our results generalize the local uncertainty principle of Price in the following sense. If $M = E \times F$, with E, F measurable sets in $\mathbb{R}^d$, then $\|Rf\|_{L^2(M)} = \|f\|_{L^2(E)}\|\widehat{f}\|_{L^2(F)}$ and $\mathfrak{m}(M) = \mathfrak{m}(E)\,\mathfrak{m}(F)$. Therefore, by simply multiplying (2.1) and (2.2) we get the third case of (2.5). The fact that estimate (2.5) involves a minimum over three different cases and is valid for general measurable sets $M \subseteq \mathbb{R}^{2d}$ indicates the extention of the obtained generalization. We illustrate this now in some more details.

Example 2.8. *Theorem* 2.5 *tells us which are the functions f and the sets M such that the* (*conjugate*) *Rihaczek representation of f must contain a small percentage of energy in M. This happens when the right-hand side of* (2.5) *is small, and this in turn depends on a combination of some features, namely, the concentrations of f and $\hat{f}$, and the size of the set M. The concentrations of f and $\hat{f}$ are measured by*

$$\inf_{\bar{t}\in\mathbb{R}^d,\ \alpha>d/2}\||t-\bar{t}|^{\alpha}f\|_2 \tag{2.8}$$

and

$$\inf_{\overline{\omega}\in\mathbb{R}^d,\ \alpha>d/2}\||\omega-\overline{\omega}|^{\alpha}\hat{f}\|_2, \tag{2.9}$$

respectively. The size of M is measured by one of the quantities $\mathfrak{m}_\omega(M)$, $\mathfrak{m}_x(M)$, $\mathfrak{m}(M)$.

We analyze now some particular classes of $M \subset \mathbb{R}^2$.

(i) *Let $M = F \times E$, for measurable sets $F, E \subset \mathbb{R}$. Then $\mathfrak{m}_\omega(M) = \mathfrak{m}(E)$ and $\mathfrak{m}_x(M) = \mathfrak{m}(F)$. Fix a function f such that both f and $\hat{f}$ are not spread out* (*in the sense that the quantities* (2.8) *and* (2.9) *are finite*)*. Then Rf and R^*f contain a small percentage of energy in M when one between E and F have small measure. In particular, this is true in small horizontal and vertical strips $\mathbb{R}\times E$ and $F\times\mathbb{R}$. On the other hand, if for example E has finite* (*but not necessarily small*) *measure and f is very concentrated* (*in the sense that* (2.8) *is very small*)*, then the first inequality in* (2.5) *tells us that Rf and R^*f must show small energy in $\mathbb{R}\times E$. Further information can be similarly deduced from the other inequalities.*

(ii) *Consider a set of the kind $M = \{(x,\omega) \in \mathbb{R}^2 : x + a \leq \omega \leq x + b\}$ with $a < b$; we have $\mathfrak{m}_\omega(M) = \mathfrak{m}_x(M) = b - a$. Observe that $\mathfrak{m}(\Pi_x M) = \mathfrak{m}(\Pi_\omega M) = \mathfrak{m}(M) = +\infty$, so this is a simple example such that the inequalities in* (2.7) *are strict. In this case only the* *first two inequalities in* (2.5) *are significative, and they tell that if f is such that one between* (2.8) *and* (2.9) *is finite, then the time-frequency representations Rf and R^*f must contain in M an amount of energy as small as $b-a$ is small,* i.e., *as small as the strip is narrow. On the other hand, if we fix a (not necessarily narrow) oblique strip M, the energy of the representations in M must be small if one between f and $\hat{f}$ is very concentrated.*

Observe that these arguments easily apply to strips not parallel to the bisector. In fact, consider $M = \{(x,\omega) \in \mathbb{R}^2 : cx + a \leq \omega \leq cx + b\}$, with $a < b$ and $c \neq 0$, and write $\alpha = \min\{-a/c, -b/c\}$ and $\beta = \max\{-a/c, -b/c\}$. We have that $\mathfrak{m}_\omega(M) = b - a$ and $\mathfrak{m}_x(M) = \beta - \alpha$, giving again that $\mathfrak{m}_\omega(M)$ and $\mathfrak{m}_x(M)$ are as small as the strip is narrow.

(iii) *We can generalize the case* (ii)*, by considering, for a measurable function $\mu : \mathbb{R} \to \mathbb{R}$, sets of the kind*

$$M_1 = \{(x,\omega) \in \mathbb{R}^d : \mu(x) + a \leq \omega \leq \mu(x) + b\},$$

or

$$M_2 = \{(x,\omega) \in \mathbb{R}^d : \mu(\omega) + a \leq x \leq \mu(\omega) + b\}.$$

We have in these cases $\mathfrak{m}_\omega(M_1) = \mathfrak{m}_x(M_2) = b - a$. Then such sets are "small" as $b - a$ is small, in the sense of $\mathfrak{m}_\omega(M_1)$ and $\mathfrak{m}_x(M_2)$, respectively, and we have similar information as in case (ii)*. The same happens also with more general sets; consider for example*

$$M_3 = \{(x,\omega) \in \mathbb{R}^2 : r^2 \leq x^2 + \omega^2 \leq R^2\}$$

for $0 < r < R$. Observe that $\mathfrak{m}_\omega(M_3) = \mathfrak{m}_x(M_3) = 2\sqrt{R^2 - r^2}$, and $\mathfrak{m}(M) = \pi(R^2 - r^2)$. Then the smallness of M_3 is proportional to the smallness of $R - r$, and reasoning as in case (ii) *we have that a good combination of concentration of f (or $\hat{f}$) and smallness of $R - r$ gives that Rf and R^*f must show small energy in M_3.*

3 Local uncertainty principle for τ-Wigner representations

In this section we prove a result, analogous to Theorem 2.5, for the τ-Wigner transforms, cf. (1.2). The result is slightly different from the point of view of the quantities that measure the size of the set M. As a consequence, we shall be able to treat, for the τ-Wigner representations, sets of the kind of Example 2.8, (i) and (ii), but not of the kind of Example 2.8, (iii). We start by proving a preliminary local uncertainty principle for the τ-Wigner representations, that constitutes the basic result for proving in the following part of the section a stronger version of it. Recall that, for every $f \in L^2(\mathbb{R}^d)$, we have $\operatorname{Wig}_\tau f \in L^2(\mathbb{R}^{2d})$, cf. [4], and so, for a measurable set $M \subset \mathbb{R}^{2d}$, the norm $\|\operatorname{Wig}_\tau f\|_{L^2(M)}$ is finite.

Proposition 3.1. *Let $M \subset \mathbb{R}^{2d}$ be a measurable set, and $\alpha, \alpha_1, \alpha_2 > d/2$. Then for every $f \in L^2(\mathbb{R}^d)$ and for every fixed $\bar{t}, \overline{\omega} \in \mathbb{R}^d$ we have:*

$$\begin{aligned}\|\operatorname{Wig}_\tau f\|^2_{L^2(M)} \leq \min\{&c_\tau K\, \mathfrak{m}(\Pi_\omega M)\|f\|_2^{4-\frac{d}{\alpha}} \||t-\bar{t}|^\alpha f\|_2^{\frac{d}{\alpha}}, \\ &c_\tau K\, \mathfrak{m}(\Pi_x M)\|\hat{f}\|_2^{4-\frac{d}{\alpha}} \||\omega-\overline{\omega}|^\alpha \hat{f}\|_2^{\frac{d}{\alpha}}, \\ &c_\tau K_1 K_2\, \mathfrak{m}(M)\|f\|_2^{4-\frac{d}{\alpha_1}-\frac{d}{\alpha_2}} \||t-\bar{t}|^{\alpha_1} f\|_2^{\frac{d}{\alpha_1}} \||\omega-\overline{\omega}|^{\alpha_2} \hat{f}\|_2^{\frac{d}{\alpha_2}}\},\end{aligned} \tag{3.1}$$

for every $\tau \in [0,1]$, where K, K_1, K_2 are given by (2.3) *corresponding to $\alpha, \alpha_1, \alpha_2$, and*

$$c_\tau = \begin{cases} \min\left\{\frac{1}{\tau^d}, \frac{1}{(1-\tau)^d}\right\}, & \tau \in (0,1) \\ 1, & \tau = 0,1 \end{cases} \tag{3.2}$$

Remark 3.2. *Since the τ-Wigner representations contain the Rihaczek and conjugate Rihaczek for $\tau = 0$ and $\tau = 1$, respectively, we can compare Proposition* 3.1*, for $\tau = 0,1$, with Theorem* 2.5*, and observe that the latter is stronger, in the sense that the constants involving the set M in the right-hand side are better, as we can deduce from Remark* 2.6*. In the sequel of the section we shall improve the estimates* (3.1)*, even tough we shall not obtain the same constants as in* (2.5)*.*

Proof of Proposition 3.1. From Proposition 2.2 we have that in the first estimate in (3.1) we can assume $f \in L^1(\mathbb{R}^d)$, otherwise the right-hand side is infinity and the estimate is trivial. Analogously we can assume $\hat{f} \in L^1(\mathbb{R}^d)$ in the second estimate, and $f, \hat{f} \in L^1(\mathbb{R}^d)$ in the third.

(i) We start by proving the second estimate of (3.1). By Fubini Theorem we have

$$\begin{aligned}\|\operatorname{Wig}_\tau f\|^2_{L^2(M)} &\leq \int_{\Pi_x M}\left(\int_{\Pi_\omega M} |\operatorname{Wig}_\tau f(x,\omega)|^2\, d\omega\right) dx \\ &\leq \mathfrak{m}(\Pi_x M) \sup_{x\in\mathbb{R}^d} \|\operatorname{Wig}_\tau f(x,\omega)\|^2_{L^2_\omega(\mathbb{R}^d)}.\end{aligned}$$

Now, since $\operatorname{Wig}_\tau f(x,\omega) = \mathcal{F}_{t\to\omega}\left[f(x+\tau t)\overline{f(x-(1-\tau)t)}\right]$ and the Fourier transform is an isomorphism on L^2 we obtain

$$\|\operatorname{Wig}_\tau f\|^2_{L^2(M)} \leq \mathfrak{m}(\Pi_x M) \sup_{x\in\mathbb{R}^d} \int \left|f(x+\tau t)\overline{f(x-(1-\tau)t)}\right|^2 dt.$$

Recall that we can assume that $\hat{f} \in L^1(\mathbb{R}^d)$, so that $f \in L^\infty(\mathbb{R}^d)$. Suppose now that $\tau \in [0, 1/2]$; we can proceed as follows:

$$\begin{aligned}\|\operatorname{Wig}_\tau f\|^2_{L^2(M)} &\leq \mathfrak{m}(\Pi_x M) \sup_{x\in\mathbb{R}^d} \|f\|^2_\infty \int \left|f(x-(1-\tau)t)\right|^2 dt \\ &= \frac{1}{(1-\tau)^d}\, \mathfrak{m}(\Pi_x M)\|f\|^2_\infty \|f\|^2_2.\end{aligned} \tag{3.3}$$

If $\tau \in [1/2, 1]$ we can leave $f(x + \tau t)$ instead of $f(x - (1 - \tau)t)$ in the integral in (3.3), and we obtain the same estimate with $1/\tau^d$ instead of $1/(1 - \tau)^d$. Since $\|f\|_2 = \|\hat{f}\|_2$ and $\|f\|_\infty \le \|\hat{f}\|_1$, we have

$$\| \operatorname{Wig}_\tau f\|^2_{L^2(M)} \le c_\tau \, \mathfrak{m}(\Pi_x M) \|\hat{f}\|_1^2 \|\hat{f}\|_2^2.$$

The conclusion is then and application of Proposition 2.2.

(ii) Recall that, for every $\tau \in [0, 1]$,

$$\operatorname{Wig}_\tau f(x, \omega) = \operatorname{Wig}_{1-\tau} \hat{f}(\omega, -x), \tag{3.4}$$

cf. for example [5]. Let $M_1 = \{(x, \omega) \in \mathbb{R}^{2d} : (-x, \omega) \in M\}$. We have

$$\| \operatorname{Wig}_\tau f\|_{L^2(M)} = \| \operatorname{Wig}_{1-\tau} \hat{f}(\omega, -x)\|_{L^2(M)} = \| \operatorname{Wig}_{1-\tau} \hat{f}(\omega, x)\|_{L^2(M_1)}.$$

We can then repeat the same procedure as in point (i), with x and ω interchanged, $1 - \tau$ in place of τ and $\hat{f}$ instead of f, obtaining:

$$\| \operatorname{Wig}_\tau f\|^2_{L^2(M)} \le c_{1-\tau} \, \mathfrak{m}(\Pi_\omega M_1) \|\hat{\hat{f}}\|_1^2 \|\hat{\hat{f}}\|_2^2.$$

Now we observe that $c_{1-\tau} = c_\tau$, and $\mathfrak{m}(\Pi_\omega M_1) = \mathfrak{m}(\Pi_\omega M)$. Since $\hat{\hat{f}}(t) = f(-t)$ we have

$$\| \operatorname{Wig}_\tau f\|^2_{L^2(M)} \le c_\tau \, \mathfrak{m}(\Pi_\omega M) \|f\|_1^2 \|f\|_2^2,$$

and we conclude by applying Proposition 2.2.

(iii) The third estimate in (3.1) can be proved by using the continuity properties of the τ-Wigner transform, in particular the fact that, for $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we have $\operatorname{Wig}_\tau f \in L^\infty(\mathbb{R}^{2d})$ and $\| \operatorname{Wig}_\tau f\|_\infty \le c_\tau \|f\|_1 \|f\|_\infty$; this can be easily proved by a direct estimate, and is also part of a general study of the continuity properties of Wig_τ in Lebesgue spaces, that can be found in [4]. From the observations at the beginning of the proof, we can assume without loss of generality that $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and so we have

$$\| \operatorname{Wig}_\tau f\|^2_{L^2(M)} \le \mathfrak{m}(M) \| \operatorname{Wig}_\tau f\|^2_\infty \le c_\tau \, \mathfrak{m}(M) \|f\|_1^2 \|f\|_\infty^2 \le c_\tau \, \mathfrak{m}(M) \|f\|_1^2 \|\hat{f}\|_1^2.$$

The conclusion is then an application of Proposition 2.2 to $\|f\|_1$ and $\|\hat{f}\|_1$, with α_1 and α_2 respectively.

□

Remark 3.3. *In the following we shall need a slightly more general version of the first inequality in* (3.1). *Fix M, α and f as in Proposition* 3.1, *and let $g, h \in L^2(\mathbb{R}^d)$ be such that $|g(t)| = |h(t)| = |f(t)|$ for almost every $t \in \mathbb{R}^d$. Then for every $\bar{t} \in \mathbb{R}^d$ we have*

$$\| \operatorname{Wig}_\tau (g, h)\|^2_{L^2(M)} \le c_\tau K \, \mathfrak{m}(\Pi_\omega M) \|f\|_2^{4-\frac{d}{\alpha}} \| |t - \bar{t}|^\alpha f\|_2^{\frac{d}{\alpha}},$$

for every $\tau \in [0,1]$, where c_τ and K are as in Proposition 3.1, and $\mathrm{Wig}_\tau(f,g)$ is the polarized sesquilinear form corresponding to the τ-Wigner, i.e.

$$\mathrm{Wig}_\tau(f,g)(x,\omega) = \int e^{-2\pi i t\omega} f(x+\tau t)\,\overline{g(x-(1-\tau)t)}\,dt.$$

We can in fact use the identity

$$\mathrm{Wig}_\tau(g,h)(x,\omega) = \mathrm{Wig}_{1-\tau}(\hat{g},\hat{h})(\omega,-x),$$

cf. [5]*, and since the L^p-norms of g and h coincide with the L^p-norm of f for every p, we can repeat the same procedure as in the proof of Proposition* 3.1.

Proposition 3.1 gives non trivial information in the cases M, or at least one of its projections, has finite measure (i.e., as basic examples, sets with finite measure, and horizontal and vertical strips); we want now to generalize this result, in order to get information also when M is of the kind of an oblique strip. In order to do this, we need some preliminary results. We shall consider the case $\tau \in (0,1)$, since for $\tau = 0,1$ (corresponding to Rihaczek and conjugate Rihaczek) we already have better information from Section 2.

Lemma 3.4. *Let $\tau \in (0,1)$ and $f,g \in L^2(\mathbb{R}^d)$. Fix a real symmetric $d\times d$ matrix C, and write (Cx,x) for the corresponding quadratic form. We have*

$$\mathrm{Wig}_\tau(f,g)(x,\omega) = e^{\frac{1-2\tau}{\tau(1-\tau)}\pi i(Cx,x)}\,\mathrm{Wig}_\tau\left(e^{-\pi i\frac{1-\tau}{\tau}(Ct,t)}f(t), e^{-\pi i\frac{\tau}{1-\tau}(Ct,t)}g(t)\right)(x,\omega-Cx). \tag{3.5}$$

Proof. Since C is symmetric, we have that $(Ct,x) = (Cx,t)$ for every $t,x \in \mathbb{R}^d$. Then

$$\begin{aligned}\frac{1-\tau}{\tau}\big(C(x+\tau t), x+\tau t\big) - \frac{\tau}{1-\tau}\big(C(x-(1-\tau)t), x-(1-\tau)t\big) = \\ = \frac{1-2\tau}{\tau(1-\tau)}(Cx,x) + 2(Cx,t).\end{aligned}$$

Using this fact, we have that

$$\begin{aligned}&\mathrm{Wig}_\tau\left(e^{-\pi i\frac{1-\tau}{\tau}(Ct,t)}f(t), e^{-\pi i\frac{\tau}{1-\tau}(Ct,t)}g(t)\right)(x,\omega) = \\ &= \int e^{-2\pi i t\omega} e^{-\frac{1-2\tau}{\tau(1-\tau)}\pi i(Cx,x)} e^{-2\pi i(Cx,t)} f(x+\tau t)\overline{g(x-(1-\tau)t)}\,dt \\ &= e^{-\frac{1-2\tau}{\tau(1-\tau)}\pi i(Cx,x)} \int e^{-2\pi i t(\omega+Cx)} f(x+\tau t)\overline{g(x-(1-\tau)t)}\,dt \\ &= e^{-\frac{1-2\tau}{\tau(1-\tau)}\pi i(Cx,x)}\,\mathrm{Wig}_\tau(f,g)(x,\omega+Cx),\end{aligned}$$

that is equivalent to (3.5). □

In order to state the main result of this section, we need to define new quantities for measuring the size of the set M.

Definition 3.5. *Consider, for real symmetric $d \times d$ matrices C and D, the following transformations:*

$$\Phi_C : (x, \omega) \in \mathbb{R}^{2d} \quad \longmapsto \quad (X, W) = (x, \omega - Cx) \in \mathbb{R}^{2d}$$

and

$$\Psi_D : (x, \omega) \in \mathbb{R}^{2d} \quad \longmapsto \quad (X, W) = (x + D\omega, \omega) \in \mathbb{R}^{2d}.$$

We define

$$\mathfrak{m}_{\omega,\Phi}(M) = \inf \left\{ \mathfrak{m}\big(\Pi_W(\Phi_C(M))\big),\ C \text{ real symmetric } d \times d \right\} \tag{3.6}$$

and

$$\mathfrak{m}_{x,\Psi}(M) = \inf \left\{ \mathfrak{m}\big(\Pi_X(\Psi_D(M))\big),\ D \text{ real symmetric } d \times d \right\}, \tag{3.7}$$

for every measurable set $M \subset \mathbb{R}^{2d}$.

We have the following local uncertainty principle for the τ-Wigner representations.

Theorem 3.6. *Let $M \subset \mathbb{R}^{2d}$ be a measurable set, and $\alpha, \alpha_1, \alpha_2 > d/2$. Then for every $f \in L^2(\mathbb{R}^d)$ and for every fixed $\bar{t}, \overline{\omega} \in \mathbb{R}^d$ we have:*

$$\begin{aligned} \| \operatorname{Wig}_\tau f\|^2_{L^2(M)} \leq \min\big\{ & c_\tau K\, \mathfrak{m}_{\omega,\Phi}(M) \|f\|_2^{4-\frac{d}{\alpha}} \||t-\bar{t}|^\alpha f\|_2^{\frac{d}{\alpha}}, \\ & c_\tau K\, \mathfrak{m}_{x,\Psi}(M) \|\hat{f}\|_2^{4-\frac{d}{\alpha}} \||\omega-\overline{\omega}|^\alpha \hat{f}\|_2^{\frac{d}{\alpha}}, \\ & c_\tau K_1 K_2\, \mathfrak{m}(M) \|f\|_2^{4-\frac{d}{\alpha_1}-\frac{d}{\alpha_2}} \||t-\bar{t}|^{\alpha_1} f\|_2^{\frac{d}{\alpha_1}} \||\omega-\overline{\omega}|^{\alpha_2} \hat{f}\|_2^{\frac{d}{\alpha_2}} \big\}, \end{aligned} \tag{3.8}$$

where K, K_1, K_2 are given by (2.3) corresponding to $\alpha, \alpha_1, \alpha_2$ respectively, and c_τ by (3.2).

Proof. The third inequality of (3.8) has already been proved in Proposition 3.1. We then have to prove the other two estimates.

(i) Concerning the first inequality, from Lemma 3.4 we have that for every real symmetric $d \times d$ matrix C,

$$\| \operatorname{Wig}_\tau f\|^2_{L^2(M)} = \int_M |\operatorname{Wig}_\tau(g,h)(x, \omega - Cx)|^2\, dx\, d\omega,$$

where $g(t) = e^{-\pi i \frac{1-\tau}{\tau}(Ct,t)} f(t)$ and $h(t) = e^{-\pi i \frac{\tau}{1-\tau}(Ct,t)} f(t)$. Then, by the change of variables $X = x$, $W = \omega - Cx$, we get

$$\| \operatorname{Wig}_\tau f\|^2_{L^2(M)} = \int_{\Phi_C(M)} |\operatorname{Wig}_\tau(g,h)(X,W)|^2\, dX\, dW = \| \operatorname{Wig}_\tau(g,h)\|^2_{L^2(\Phi_C(M))}.$$

Since $|g(t)| = |h(t)| = |f(t)|$ for every $t \in \mathbb{R}^d$, we can then apply Remark 3.3 and conclude that

$$\| \operatorname{Wig}_\tau f\|^2_{L^2(M)} \le c_\tau K\, \mathfrak{m}\big(\Pi_W(\Phi_C(M))\big) \|f\|_2^{4-\frac{d}{\alpha}} \||t-\bar{t}|^\alpha f\|_2^{\frac{d}{\alpha}},$$

for every $\alpha > d/2$ and $\bar{t} \in \mathbb{R}^d$. Then, taking the inf over all C in the right-hand side, we get the first inequality of (3.8).

(ii) Using (3.4) and Lemma 3.4 we have

$$\begin{aligned} \| \operatorname{Wig}_\tau f\|^2_{L^2(M)} &= \int_M |\operatorname{Wig}_{1-\tau} \hat{f}(\omega,-x)|^2\, dx\, d\omega \\ &= \int_M |\operatorname{Wig}_{1-\tau}(g_1,h_1)(\omega,-x-D\omega)|^2\, dx\, d\omega, \end{aligned}$$

where $g_1(t) = e^{-\pi i \frac{1-\tau}{\tau}(Dt,t)} \hat{f}(t)$ and $h_1(t) = e^{-\pi i \frac{tau}{1-\tau}(Dt,t)} \hat{f}(t)$. Now, by the change of variables $X = x + D\omega$, $W = \omega$, we get

$$\| \operatorname{Wig}_\tau f\|^2_{L^2(M)} \le \int_{M_D} |\operatorname{Wig}_{1-\tau}(g_1,h_1)(W,X)|^2\, dX\, dW, \tag{3.9}$$

where $M_D = \{(X,W) \in \mathbb{R}^{2d} : (-X,W) \in \Psi_D(M)\}$. Now we have $|g_1(t)| = |h_1(t)| = |\hat{f}(t)|$, and so we can apply Remark 3.3; since in the integral in (3.9) the variables X and W are interchanged, we have

$$\| \operatorname{Wig}_\tau f\|^2_{L^2(M)} \le c_{1-\tau} K\, \mathfrak{m}\big(\Pi_X(M_D)\big) \|\hat{f}|_2^{4-\frac{d}{\alpha}} \||\omega-\overline{\omega}|^\alpha \hat{f}\|_2^{\frac{d}{\alpha}},$$

for every $\overline{\omega} \in \mathbb{R}^d$. The conclusion then follows from the fact that $c_{1-\tau} = c_\tau$ and $\mathfrak{m}\big(\Pi_X(M_D)\big) = \mathfrak{m}\big(\Pi_X(\Psi_D(M))\big)$.

□

Remark 3.7. *For every measurable set $M \subset \mathbb{R}^{2d}$ we have*

$$\mathfrak{m}_{\omega,\Phi}(M) \le \mathfrak{m}(\Pi_\omega M), \quad \mathfrak{m}_{x,\Psi}(M) \le \mathfrak{m}(\Pi_x M),$$

and the inequalities can be strict; it can even happen that both the projections have infinite measure and both $\mathfrak{m}_{\omega,\Phi}(M)$ and $\mathfrak{m}_{x,\Psi}(M)$ are finite, see Example 3.8 below. So Theorem 3.6 is stronger than Proposition 3.1.

As we have observed in Section 2, the local uncertainty principles (3.8) gives information on how small the energy of $\operatorname{Wig}_\tau f$ must be in the set M, depending on the concentration of f and $\hat{f}$ (in the sense of (2.8) and (2.9), respectively), and on the size of M (here in the sense of $\mathfrak{m}_{\omega,\Phi}(M)$, $\mathfrak{m}_{x,\Psi}(M)$, $\mathfrak{m}(M)$). We observe that the quantities (3.6) and (3.7) are finite when the image of the set M through a transformation of the kind of Φ_C or Ψ_D is contained in an horizontal or vertical strip. In particular this is the case when M is an oblique strip, as in Example 2.8, (ii). We now run through Example 2.8 and see what happens in the case of Theorem 3.6.

Example 3.8. *We analyze now the sets $M \subset \mathbb{R}^2$ of Example 2.8; we want to compare the quantities $\mathfrak{m}_{\omega,\Phi}(M)$, $\mathfrak{m}_{x,\Psi}(M)$ of Theorem 3.6, with $\mathfrak{m}_\omega(M)$, $\mathfrak{m}_x(M)$ of Theorem 2.5, respectively.*

(i) *Let $M = F \times E$, for measurable sets $F, E \subset \mathbb{R}$. In this case, we have $\mathfrak{m}_{\omega,\Phi}(M) = \mathfrak{m}_\omega(M) = \mathfrak{m}(E)$ and $\mathfrak{m}_{x,\Psi}(M) = \mathfrak{m}_x(M) = \mathfrak{m}(F)$, and so the situation is the same as for the Rihaczek and conjugate Rihaczek.*

(ii) *Consider now the case of an oblique strip $M = \{(x,\omega) \in \mathbb{R}^2 : cx + a \leq \omega \leq cx + b\}$, with $a < b$ and $c \neq 0$. We write $\alpha = \min\{-a/c, -b/c\}$ and $\beta = \max\{-a/c, -b/c\}$. In this case the matrices C and D in Definition 3.5 are real constants, and, for the set M, the* inf *in (3.6) and (3.7) are realized for $C = c$ and $D = -1/c$, respectively. For these values of C and D we have*

$$\Phi_C(M) = \{(X,W) \in \mathbb{R}^2 : a \leq W \leq b\}$$

and

$$\Psi_D(M) = \{(X,W) \in \mathbb{R}^2 : \alpha \leq X \leq \beta\},$$

so that $\mathfrak{m}_{\omega,\Phi}(M) = b - a$ and $\mathfrak{m}_{x,\Psi}(M) = \beta - \alpha$. Comparing with Example 2.8 we observe that also in this case we have $\mathfrak{m}_{\omega,\Phi}(M) = \mathfrak{m}_\omega(M)$ and $\mathfrak{m}_{x,\Psi}(M) = \mathfrak{m}_x(M)$. Then the observations we made for Rihaczek and conjugate Rihaczek apply also to the τ-Wigner; if the strip is narrow and/or one between f and $\hat{f}$ are very concentrated, Wig_τ *must contain a small amount of energy in M.*

(iii) *The case of Example 2.8, (iii) cannot be treated in general for the τ-Wigner, unless M is contained in a strip, and even in this case, $\mathfrak{m}_{\omega,\Phi}(M)$ and $\mathfrak{m}_{x,\Psi}(M)$ can be strictly bigger than the corresponding $\mathfrak{m}_\omega(M)$ and $\mathfrak{m}_x(M)$ of the Rihaczek case. As particular cases, let us compare $\mathfrak{m}_\omega(M)$ with $\mathfrak{m}_{\omega,\Phi}(M)$ for the sets*

$$M_1 = \{(x,\omega) \in \mathbb{R}^2 : x^2 + a \leq \omega \leq x^2 + b\}$$

and

$$M_2 = \{(x,\omega) \in \mathbb{R}^2 : x + \sin x + a \leq \omega \leq x + \sin x + b\},$$

for $a < b$. As observed in Example 2.8 we have $\mathfrak{m}_\omega(M_1) = \mathfrak{m}_\omega(M_2) = b - a$. On the other hand, $\mathfrak{m}_{\omega,\Phi}(M_1) = +\infty$ and $\mathfrak{m}_{\omega,\Phi}(M_2) = b - a + 2$, that means that on M_1 the first estimate in (3.8) gives no information for Wig_τ*, while the corresponding one in (2.5) is not trivial for R and R^*; moreover, on M_2 the first estimate in (3.8) is weaker than the corresponding one in (2.5).*

Remark 3.9. *It is natural to compare the constants appearing in estimates (2.5) for the Rihaczek representation with those in the corresponding estimate (3.8) for the* Wig_τ*. To this aim we observe that the transformations of the type Φ_C send "vertical" sections of the set M*

into "vertical" sections of the set $\phi_C(M)$, i.e. $\Phi_C(M_x) = (\Phi_C M)_x$. As Φ_C are isometries and $(\Phi_C M)_x \subseteq \Pi_\omega \Phi_C M$, we have, for every x and for every C:

$$\mathfrak{m}(M_x) = \mathfrak{m}(\Phi_C(M_x)) \leq \mathfrak{m}(\Pi_\omega \Phi_C M).$$

Taking the supremum in x on the left-hand side and infimum in C on the right-hand side, we have therefore

$$\mathfrak{m}_\omega(M) = \sup_x \mathfrak{m}(M_x) \leq \inf_C \mathfrak{m}(\Pi_\omega \Phi_C M) = \mathfrak{m}_{\omega,\Phi}(M).$$

In an analogous way we have $\mathfrak{m}_x(M) \leq \mathfrak{m}_{x,\Psi}(M)$. The estimates for the Rihaczek contain therefore "better" constants for measuring the "size" of M then those for the $\tau-$Wigner representation.

4 Local uncertainty principle in the Cohen class

In this section we shall prove that results of the kind of Proposition 3.1 and Theorem 3.6 also apply to representations of the form $Q_\sigma f(x,\omega) = (\sigma * \operatorname{Wig} f)(x,\omega)$ belonging to the Cohen class. However, we consider here the case of kernels $\sigma \in L^1(\mathbb{R}^{2d}) \cap L^2(\mathbb{R}^{2d})$ and, as none of the kernels of Wig_τ, R and R^* satisfy this property, the results of this Section are independent of the local uncertainty principles of Sections 2 and 3. Moreover, the proofs in this section make use of the previous results on the classical Wigner representation, but present non trivial differences with respect to the ones of Section 3.

We start by proving the analogue of Proposition 3.1 for the Cohen class.

Proposition 4.1. *Let $M \subset \mathbb{R}^{2d}$ be a measurable set, and $\sigma \in L^1(\mathbb{R}^{2d}) \cap L^2(\mathbb{R}^{2d})$. Then for every $\alpha, \alpha_1, \alpha_2 > d/2$, $\bar{t}, \overline{\omega} \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$ we have*

$$\begin{aligned}\|Q_\sigma f\|^2_{L^2(M)} \leq \min\big\{&K\,\mathfrak{m}(\Pi_\omega M)\|\sigma\|_2^2\|f\|_2^{4-\frac{d}{\alpha}}\||t-\bar{t}|^\alpha f\|_2^{\frac{d}{\alpha}},\\ &K\,\mathfrak{m}(\Pi_x M)\|\sigma\|_2^2\|\hat{f}\|_2^{4-\frac{d}{\alpha}}\||\omega-\overline{\omega}|^\alpha \hat{f}\|_2^{\frac{d}{\alpha}},\\ &2^d K_1 K_2\,\mathfrak{m}(M)\|\sigma\|_1^2\|f\|_2^{4-\frac{d}{\alpha_1}-\frac{d}{\alpha_2}}\||t-\bar{t}|^{\alpha_1} f\|_2^{\frac{d}{\alpha_1}}\||\omega-\overline{\omega}|^{\alpha_2}\hat{f}\|_2^{\frac{d}{\alpha_2}}\big\},\end{aligned} \tag{4.1}$$

where K, K_1, K_2 are given as usual by (2.3). More precisely, the first two estimates are valid for every $\sigma \in L^2(\mathbb{R}^{2d})$ and the third one holds for every $\sigma \in L^1(\mathbb{R}^{2d})$.

Proof. Using the same arguments as at the beginning of the proof of Proposition 3.1, we observe that we can assume $f \in L^1(\mathbb{R}^d)$ in the first estimate of (4.1), $\hat{f} \in L^1(\mathbb{R}^d)$ in the second one, and $f, \hat{f} \in L^1(\mathbb{R}^d)$ in the third one, in order that the corresponding right-hand sides are finite.

(i) We start by proving the second estimate, under the hypothesis $\sigma \in L^1(\mathbb{R}^{2d}) \cap L^2(\mathbb{R}^{2d})$ (that ensures that $Q_\sigma f$ is well defined for every $f \in L^2(\mathbb{R}^d)$). We observe that

$$\begin{aligned} Q_\sigma f(x,\omega) &= (\sigma * \operatorname{Wig} f)(x,\omega) \\ &= \int \sigma(y,\eta) e^{-2\pi i t(\omega-\eta)} f\Big(x-y+\frac{t}{2}\Big)\overline{f\Big(x-y-\frac{t}{2}\Big)}\, dt\, dy\, d\eta, \end{aligned}$$

and in this expression we can interchange the order of integration as we want, since the integrand belongs to $L^1(\mathbb{R}^d_t \times \mathbb{R}^d_y \times \mathbb{R}^d_\eta)$. We then have

$$\begin{aligned} &\|Q_\sigma f\|^2_{L^2(M)} \le \\ &\le \int_{\Pi_x M}\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^{3d}} e^{-2\pi i t\omega} e^{2\pi i t\eta} f\Big(x-y+\frac{t}{2}\Big)\overline{f\Big(x-y-\frac{t}{2}\Big)}\sigma(y,\eta)\, dy\, d\eta\, dt \right|^2 d\omega\, dx \\ &\le \mathfrak{m}(\Pi_x M) \sup_{x\in\Pi_x M} \left\| \mathcal{F}_{t\to\omega}\left[\int_{\mathbb{R}^{2d}} e^{2\pi i t\eta} f\Big(x-y+\frac{t}{2}\Big)\overline{f\Big(x-y-\frac{t}{2}\Big)}\sigma(y,\eta)\, dy\, d\eta \right] \right\|^2_{L^2(\mathbb{R}^d_\omega)}. \end{aligned}$$

Now, writing $\sigma_1 = \mathcal{F}_2^{-1}\sigma$, where $\mathcal{F}_2^{-1}$ means the inverse Fourier transform with respect to the second $\mathbb{R}^d$ variable, we have:

$$\begin{aligned} \|Q_\sigma f\|^2_{L^2(M)} &\le \mathfrak{m}(\Pi_x M) \sup_{x\in\mathbb{R}^d} \left\| \int \sigma_1(y,t) f\Big(x-y+\frac{t}{2}\Big)\overline{f\Big(x-y-\frac{t}{2}\Big)}\, dy \right\|^2_{L^2(\mathbb{R}^d_t)} \\ &\le \mathfrak{m}(\Pi_x M) \left\| \left\| \int \sigma_1(y,t) f\Big(x-y+\frac{t}{2}\Big)\overline{f\Big(x-y-\frac{t}{2}\Big)}\, dy \right\|_{L^\infty(\mathbb{R}^d_x)} \right\|^2_{L^2(\mathbb{R}^d_t)} \\ &= \mathfrak{m}(\Pi_x M) \left\| \left\| \left[\sigma_1(\cdot,t) * \left(f\Big(\cdot+\frac{t}{2}\Big)\overline{f\Big(\cdot-\frac{t}{2}\Big)} \right) \right](x) \right\|_{L^\infty(\mathbb{R}^d_x)} \right\|^2_{L^2(\mathbb{R}^d_t)}. \end{aligned}$$

In the L^∞ norm we can apply Young inequality, in the form $\|g * h\|_\infty \le \|g\|_2\|h\|_2$, obtaining

$$\|Q_\sigma f\|^2_{L^2(M)} \le \mathfrak{m}(\Pi_x M) \int \left| \sigma_1(y,t) f\Big(s+\frac{t}{2}\Big)\overline{f\Big(s-\frac{t}{2}\Big)} \right|^2 dy\, ds\, dt.$$

Since we can assume without loss of generality that $f \in L^\infty(\mathbb{R}^d)$, we can estimate as follows

$$\|Q_\sigma f\|^2_{L^2(M)} \le \mathfrak{m}(\Pi_x M)\|\sigma_1\|_2^2\|f\|_\infty^2\|f\|_2^2 = \mathfrak{m}(\Pi_x M)\|\sigma\|_2^2\|\hat f\|_1^2\|\hat f\|_2^2,$$

since $\sigma_1 = \mathcal{F}_2^{-1}\sigma$. The conclusion is then an application of Proposition 2.2.

(ii) By the formula (3.4) for $\tau = 1/2$ we easily get

$$Q_\sigma f(x,\omega) = Q_{\sigma_2}\hat f(\omega,-x), \tag{4.2}$$

where $\sigma_2(s,\zeta)=\sigma(-\zeta,s)$. Writing $M_1=\{(x,\omega)\in\mathbb{R}^{2d}:(-x,\omega)\in M\}$ we then have

$$\|Q_\sigma f\|^2_{L^2(M)}=\int_M |Q_\sigma f(x,\omega)|^2\,dx\,d\omega=\int_{M_1}|Q_{\sigma_2}\hat{f}(\omega,x)|^2\,dx\,d\omega;$$

by point (i) we obtain that for every $\bar{t}\in\mathbb{R}^d$ and for every $\alpha>d/2$

$$\|Q_\sigma f\|^2_{L^2(M)}\le K\,\mathfrak{m}(\Pi_\omega M_1)\|\sigma_2\|_2^2\|\hat{\hat{f}}\|_2^{4-\frac{d}{\alpha}}\||t-\bar{t}|^\alpha\hat{\hat{f}}\|_2^{\frac{d}{\alpha}}.$$

Now, $\mathfrak{m}(\Pi_\omega M_1)=\mathfrak{m}(\Pi_\omega M)$, $\|\sigma_2\|_2=\|\sigma\|_2$, and moreover $\hat{\hat{f}}(t)=f(-t)$, so since $\bar{t}$ is arbitrary we have the desired estimate, in the case $\sigma\in L^1(\mathbb{R}^{2d})\cap L^2(\mathbb{R}^{2d})$.

(iii) Suppose now $\sigma\in L^1(\mathbb{R}^{2d})$. Since $\|Q_\sigma f\|^2_{L^2(M)}\le\mathfrak{m}(M)\|Q_\sigma f\|^2_\infty$, we have by Young inequality

$$\begin{aligned}\|Q_\sigma f\|^2_{L^2(M)}&\le\mathfrak{m}(M)\|\sigma\|_1^2\|\operatorname{Wig}f\|^2_\infty\\&\le\mathfrak{m}(M)\|\sigma\|_1^2\sup_{(x,\omega)\in\mathbb{R}^{2d}}\int\left|f\Big(x+\frac{t}{2}\Big)\overline{f\Big(x-\frac{t}{2}\Big)}\right|dt\\&\le 2^d\,\mathfrak{m}(M)\|\sigma\|_1^2\|f\|^2_\infty\|f\|_1^2\\&\le 2^d\,\mathfrak{m}(M)\|\sigma\|_1^2\|\hat{f}\|_1^2\|f\|_1^2.\end{aligned}$$

We can then apply Proposition 2.2, with two different α_1 and α_2 for f and $\hat{f}$, respectively, and obtain the conclusion.

Now we want to extend the validity of the first two estimates in (4.1) to the case of kernels $\sigma\in L^2(\mathbb{R}^{2d})$ but not necessarily in $L^1(\mathbb{R}^{2d})$. We prove for example the second estimate in (4.1), the first is analogous.

(a) Suppose that $\mathfrak{m}(M)<+\infty$. For $f\in L^2(\mathbb{R}^d)$ we have $\operatorname{Wig}f\in L^2(\mathbb{R}^{2d})$ and, from the density of $L^1(\mathbb{R}^{2d})\cap L^2(\mathbb{R}^{2d})$ in $L^2(\mathbb{R}^{2d})$, σ is the limit in $L^2(\mathbb{R}^{2d})$ of a sequence of functions $\sigma_n\in L^1(\mathbb{R}^{2d})\cap L^2(\mathbb{R}^{2d})$. The continuity of the convolution $L^2*L^2\longrightarrow L^\infty$ implies then that $Q_{\sigma_n}f=\sigma_n*\operatorname{Wig}f\longrightarrow\sigma*\operatorname{Wig}f=Q_\sigma f$ in $L^\infty(\mathbb{R}^{2d})$ and therefore also in $L^\infty(M)$. The continuous immersion $L^\infty(M)\hookrightarrow L^2(M)$, valid under the assumption $\mathfrak{m}(M)<+\infty$, implies now that $Q_{\sigma_n}f\longrightarrow Q_\sigma f$ in $L^2(M)$. Proposition 4.1 applied to σ_n yields the estimates

$$\|Q_{\sigma_n}f\|^2_{L^2(M)}\le K\,\mathfrak{m}(\Pi_x M)\|\sigma_n\|_2^2\|\hat{f}\|_2^{4-\frac{d}{\alpha}}\||\omega-\overline{\omega}|^\alpha\hat{f}\|_2^{\frac{d}{\alpha}},$$

which for $n\to+\infty$ gives

$$\|Q_\sigma f\|^2_{L^2(M)}\le K\,\mathfrak{m}(\Pi_x M)\|\sigma\|_2^2\|\hat{f}\|_2^{4-\frac{d}{\alpha}}\||\omega-\overline{\omega}|^\alpha\hat{f}\|_2^{\frac{d}{\alpha}}.$$

(b) Suppose now that $\mathfrak{m}(M) = +\infty$. If $\mathfrak{m}(\Pi_x M) = +\infty$ then the assertion is trivially true. Suppose then $\mathfrak{m}(\Pi_x M) < +\infty$ and set $M_n = M \cap B^n$, $n \in \mathbb{N}$, with $B^n = \{Z \in \mathbb{R}^{2d} : |Z| \leq n\}$. From part (a) applied to M_n we have

$$\|Q_\sigma f\|^2_{L^2(M_n)} \leq K\, \mathfrak{m}(\Pi_x M_n) \|\sigma\|_2^2 \|\hat{f}\|_2^{4-\frac{d}{\alpha}} \||\omega - \overline{\omega}|^\alpha \hat{f}\|_2^{\frac{d}{\alpha}}.$$

From $M_n \subseteq M$ we have that the sequences $\|Q_\sigma f\|^2_{L^2(M_n)}$ and $\mathfrak{m}(\Pi_x M_n)$ are increasingly convergent to $\|Q_\sigma f\|^2_{L^2(M)}$ and $\mathfrak{m}(\Pi_x M)$ respectively which proves the estimate

$$\|Q_\sigma f\|^2_{L^2(M)} \leq K\, \mathfrak{m}(\Pi_x M) \|\sigma\|_2^2 \|\hat{f}\|_2^{4-\frac{d}{\alpha}} \||\omega - \overline{\omega}|^\alpha \hat{f}\|_2^{\frac{d}{\alpha}}.$$

□

Remark 4.2. *We know that in general for $f \in L^2(\mathbb{R}^d)$ and $\sigma \in L^2(\mathbb{R}^{2d})$, Young's inequality only yields $Q_\sigma f \in L^\infty(\mathbb{R}^{2d})$. Therefore, although $Q_\sigma f$ needs not to be in $L^2(\mathbb{R}^{2d})$, the last part of the proof of Proposition* 4.1 *shows that $Q_\sigma f \in L^2(M)$ for every measurable set M for which either $\mathfrak{m}(\Pi_x M)$ or $\mathfrak{m}(\Pi_\omega M)$ are finite.*

We can now state the following local uncertainty principle for the Cohen class, corresponding of Theorem 3.6 for a generic representation Q_σ with σ as in Proposition 4.1.

Theorem 4.3. *Let $M \subset \mathbb{R}^{2d}$ be a measurable set, and $\alpha, \alpha_1, \alpha_2 > d/2$. Then for every $f \in L^2(\mathbb{R}^d)$ and for every fixed $\overline{t}, \overline{\omega} \in \mathbb{R}^d$ we have:*

$$\begin{aligned}\|Q_\sigma f\|^2_{L^2(M)} \leq \min\big\{& K\, \mathfrak{m}_{\omega,\Phi}(M) \|\sigma\|_2^2 \|f\|_2^{4-\frac{d}{\alpha}} \||t - \overline{t}|^\alpha f\|_2^{\frac{d}{\alpha}}, \\ & K\, \mathfrak{m}_{x,\Psi}(M) \|\sigma\|_2^2 \|\hat{f}\|_2^{4-\frac{d}{\alpha}} \||\omega - \overline{\omega}|^\alpha \hat{f}\|_2^{\frac{d}{\alpha}}, \\ & 2^d K_1 K_2\, \mathfrak{m}(M) \|\sigma\|_1^2 \|f\|_2^{4-\frac{d}{\alpha_1}-\frac{d}{\alpha_2}} \||t - \overline{t}|^{\alpha_1} f\|_2^{\frac{d}{\alpha_1}} \||\omega - \overline{\omega}|^{\alpha_2} \hat{f}\|_2^{\frac{d}{\alpha_2}}\big\},\end{aligned} \tag{4.3}$$

where K, K_1, K_2 are given by (2.3) *and $\mathfrak{m}_{\omega,\Phi}(M)$, $\mathfrak{m}_{x,\Psi}(M)$ are defined in Definition* 3.5. *The first two estimates in* (4.3) *are valid for every $\sigma \in L^2(\mathbb{R}^{2d})$ and the third one holds for every $\sigma \in L^1(\mathbb{R}^{2d})$.*

Proof. The third inequality of (4.3) has already been proved in Proposition 4.1, so we only have to prove the other two estimates. From Lemma 3.4, applied for $\tau = 1/2$, we get

$$\begin{aligned}Q_\sigma f(x,\omega) &= \int \sigma(y,\eta)\, \mathrm{Wig}\, f(x-y, \omega-\eta)\, dy\, d\eta \\ &= \int \sigma(y,\eta)\, \mathrm{Wig}\big(e^{-\pi i (Ct,t)} f(t)\big)(x-y, \omega-\eta-C(x-y))\, dy\, d\eta\end{aligned}$$

for every real symmetric $d \times d$ matrix C. Then, by the change of variables $\eta - Cy = \zeta$, we obtain

$$Q_\sigma f(x,\omega) = \int \sigma(y, \zeta + Cy)\, \mathrm{Wig}\big(e^{-\pi i (Ct,t)} f(t)\big)(x-y, \omega - Cx - \zeta)\, dy\, d\zeta.$$

Writing $\sigma_3(y,\zeta) = \sigma(y, \zeta + Cy)$ we have

$$Q_\sigma f(x,\omega) = Q_{\sigma_3}\big(e^{-\pi i (Ct,t)} f(t)\big)(x, \omega - Cx). \tag{4.4}$$

(i) We prove now the first estimate of (4.3). From (4.4) we have

$$\|Q_\sigma f\|^2_{L^2(M)} = \int_M \Big| Q_{\sigma_3}\big(e^{-\pi i (Ct,t)} f(t)\big)(x, \omega - Cx)\Big|^2 \, dx\, d\omega,$$

for every real symmetric $d \times d$ matrix C. By the change of variables $X = x$, $W = \omega - Cx$ we obtain

$$\|Q_\sigma f\|^2_{L^2(M)} = \big\|Q_{\sigma_3}\big(e^{-\pi i (Ct,t)} f(t)\big)\big\|^2_{L^2(\Phi_C(M))},$$

where Φ_C is given in Definition 3.5. We can then apply the first estimate in (4.1), obtaining that

$$\|Q_\sigma f\|^2_{L^2(M)} \leq K\, \mathfrak{m}\big(\Pi_W(\Phi_C(M))\big) \|\sigma_3\|_2^2 \|e^{-\pi i (Ct,t)} f(t)\|_2^{4-\frac{d}{\alpha}} \||t - \bar{t}|^\alpha e^{-\pi i (Ct,t)} f(t)\|_2^{\frac{d}{\alpha}}$$

for every $\alpha > d/2$ and $\bar{t} \in \mathbb{R}^d$. Now, we observe that $e^{-\pi i (Ct,t)}$ can be deleted from the norms, and $\|\sigma_3\|_2 = \|\sigma\|_2$. Then we can take the inf over all C in the right-hand side, and we have the desired estimate.

(ii) By (4.2) and the same procedure that we used to obtain (4.4), we have

$$Q_\sigma f(x,\omega) = Q_{\sigma_2} \hat{f}(\omega, -x) = Q_{\sigma_4}\big(e^{-\pi i (Dt,t)} \hat{f}(t)\big)(\omega, -x - D\omega),$$

for every real symmetric $d \times d$ matrix D, where $\sigma_4(y,\zeta) = \sigma(-\zeta - Dy, y)$. By the change of variables $X = x + D\omega$, $W = \omega$ we then have

$$\|Q_\sigma f\|^2_{L^2(M)} = \int_{\Psi_D(M)} \big|Q_{\sigma_4}\big(e^{-\pi i (Dt,t)} \hat{f}(t)\big)(W, -X)\big|^2 dX\, dW.$$

We now write $M_D = \{(X,W) \in \mathbb{R}^{2d} : (-X, W) \in \Psi_D(M)\}$ and, using again the first estimate of (4.1), we obtain

$$\begin{aligned} \|Q_\sigma f\|^2_{L^2(M)} &= \int_{M_D} \big|Q_{\sigma_4}\big(e^{-\pi i (Dt,t)} \hat{f}(t)\big)(W, X)\big|^2 dX\, dW \\ &\leq K\, \mathfrak{m}\big(\Pi_X(M_D)\big) \|\sigma_4\|_2^2 \|e^{-\pi i (Dt,t)} \hat{f}(t)\|_2^{4-\frac{d}{\alpha}} \||\omega - \overline{\omega}|^\alpha e^{-\pi i (D\omega,\omega)} \hat{f}(\omega)\|_2^{\frac{d}{\alpha}}, \end{aligned}$$

for every $\alpha > d/2$ and $\overline{\omega} \in \mathbb{R}^d$. Observe now that, as before, $e^{-\pi i (D\omega,\omega)}$ can be deleted; moreover, $\mathfrak{m}\big(\Pi_X(M_D)\big) = \mathfrak{m}\big(\Pi_X(\Psi_D(M))\big)$ and $\|\sigma_4\|_2 = \|\sigma\|_2$. The conclusion then follows by taking the inf over all D in the right-hand side, and the proof is complete.

□

We complete the description with some remarks about Theorem 4.3.

1) Taking for simplicity the case $d = 1$, the two sets $M = \{(x, \omega) \in \mathbb{R}^2 : 0 \le ax + by \le c\}$ and $M = \{(x, \omega) \in \mathbb{R}^2 :$ either $|\omega| \le e^{-x^2}$ or $|x| \le e^{-\omega^2}\}$ show that the finiteness of $\mathfrak{m}_{\omega,\Phi}(M)$ and $\mathfrak{m}_{x,\Psi}(M)$ are actually independent from that of $\mathfrak{m}(M)$, and that each of the estimates (4.3) can be optimal.

2) The estimates in Theorem 4.3 can be easily adapted to particular regions M for which $\mathfrak{m}_{\omega,\Phi}(M) = \mathfrak{m}_{x,\Psi}(M) = \mathfrak{m}(M) = +\infty$. Suppose in fact that M can be decomposed into a finite (disjoint) union $M = M_1 \cup ... \cup M_N$ of regions M_j for which one of the previous three quantities is finite, then $\|Q_\sigma f\|^2_{L^2(M)} = \sum_{j=1}^N \|Q_\sigma f\|^2_{L^2(M_j)}$ and a suitable estimate can be applied to each term. For example in $\mathbb{R}^2$ we can consider $M = \{(x, \omega) : |x| \le a \vee |\omega| \le b;\ a, b \ge 0\}$ obtaining

$$\|Q_\sigma f\|^2_{L^2(M)} \le 2K\|\sigma\|_2^2 \|f\|_2^{4-\frac{d}{\alpha}} \big(a\ \||t - \bar{t}|^\alpha f\|_2^{\frac{d}{\alpha}} + b\ \||\omega - \overline{\omega}|^\alpha \hat{f}\|_2^{\frac{d}{\alpha}}\big). \tag{4.5}$$

More generally, we have then proved that, even if $Q_\sigma f$ needs not to be in $L^2(\mathbb{R}^2)$, it is always square-integrable e.g. on every finite union of "strips" of the type $c_0 \le ax + by \le c_1$.

As application of the result of Theorem 4.3 we prove the following "local" boundedness property which pursues the direction of Remark 4.2. As already observed, under the hypothesis $\sigma \in L^2(\mathbb{R}^{2d})$ we do not have in general a bounded map $Q_\sigma : f \in L^2(\mathbb{R}^d) \longrightarrow Q_\sigma f \in L^2(\mathbb{R}^{2d})$. This includes for example the important case of spectrograms $|V_g f|^2$ with window $g \in L^2(\mathbb{R}^d)$ for which one has $\sigma = \mathrm{Wig}(\tilde{g}) \in L^2(\mathbb{R}^{2d})$. The following property is then of practical interest in applications as it permits to still consider only the L^2 functional framework for f and $Q_\sigma f$ also in this case, under reasonable limitations on f, $\widehat{f}$, $Q_\sigma f$.

Proposition 4.4. *Let $\sigma \in L^2(\mathbb{R}^{2d})$, $M \subset \mathbb{R}^{2d}$ and $B_0^R = \{x \in \mathbb{R}^d : |x| \le R\}$ (when necessary we implicitly extend $f \in L^2(B_0^R)$ by zero outside B_0^R), then Q_σ defines the following quadratic bounded maps:*
(i) *If $\mathfrak{m}_{\omega,\Phi}(M) < +\infty$ then $Q_\sigma : f \in L^2(B_0^R) \longrightarrow Q_\sigma f \in L^2(M)$ and*

$$\|Q_\sigma f\|_{L^2(M)} \le \sqrt{\frac{2}{d}}\ \frac{\pi^{d/4} R^{d/2}}{\Gamma(d/2)^{1/2}}\, \mathfrak{m}_{\omega,\Phi}(M)^{1/2} \|\sigma\|_2 \|f\|_2^2.$$

(ii) *If $\mathfrak{m}_{x,\Psi}(M) < +\infty$ then $Q_\sigma : f \in \mathcal{F}L^2(B_0^R) \longrightarrow Q_\sigma f \in L^2(M)$ and*

$$\|Q_\sigma f\|_{L^2(M)} \le \sqrt{\frac{2}{d}}\ \frac{\pi^{d/4} R^{d/2}}{\Gamma(d/2)^{1/2}}\, \mathfrak{m}_{\omega,\Phi}(M)^{1/2} \|\sigma\|_2 \|f\|_2^2.$$

(iii) *If $\mathfrak{m}(M) < +\infty$ then $Q_\sigma : f \in L^2(\mathbb{R}^d) \longrightarrow Q_\sigma f \in L^2(M)$ and*

$$\|Q_\sigma f\|_{L^2(M)} \le \mathfrak{m}(M)^{1/2} \|\sigma\|_2 \|f\|_2^2.$$

Proof. (i) Suppose that $\mathfrak{m}_{\omega,\Phi}(M) < +\infty$. Choosing $\bar{t} = 0$ the first estimate (4.3) is

$$\|Q_\sigma f\|^2_{L^2(M)} \leq K_\alpha \, \mathfrak{m}_{\omega,\Phi}(M)\|\sigma\|_2^2\|f\|_2^{4-\frac{d}{\alpha}}\||t|^\alpha f\|_2^{\frac{d}{\alpha}}.$$

As supp$f \subseteq B_0^R$ the right-hand side is well defined for every α and

$$\|f\|_2^{4-\frac{d}{\alpha}}\||t|^\alpha f\|_2^{\frac{d}{\alpha}} \leq \|f\|_2^{4-\frac{d}{\alpha}} R^d \|f\|_2^{\frac{d}{\alpha}} = \|f\|_2^4 R^d. \tag{4.6}$$

Rewriting the constant K_α using Euler's formula for the Gamma function $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ we have

$$\begin{aligned} K_\alpha &= \frac{\pi^{d/2}}{\alpha}\left(\Gamma\left(\frac{d}{2}\right)\right)^{-1}\Gamma\left(\frac{d}{2\alpha}\right)\Gamma\left(1-\frac{d}{2\alpha}\right)\left(\frac{2\alpha}{d}-1\right)^{\frac{d}{2\alpha}}\left(1-\frac{d}{2\alpha}\right)^{-1} \\ &= \frac{2\pi^{d/2}}{d}\left(\Gamma\left(\frac{d}{2}\right)\right)^{-1}\frac{\frac{\pi d}{2\alpha}}{\sin(\frac{\pi d}{2\alpha})}\left(\frac{2\alpha}{d}-1\right)^{\frac{d}{2\alpha}}\left(1-\frac{d}{2\alpha}\right)^{-1}. \end{aligned}$$

As the function K_α is decreasing in α, the best estimate is obtained by letting $\alpha \to +\infty$ and we have:

$$\lim_{\alpha\to\infty} K_\alpha = \frac{2\pi^{d/2}}{d}\left(\Gamma\left(\frac{d}{2}\right)\right)^{-1}. \tag{4.7}$$

The thesis follows then immediately from (4.6) and (4.7).
(ii) It is analogous to (i) using now the second estimate (4.3) $\widehat{f}$ has compact support.
(iii) Suppose that $\mathfrak{m}(M) < +\infty$. Then

$$\begin{aligned} \|Q_\sigma f\|_{L^2(M)} &\leq \mathfrak{m}(M)^{\frac{1}{2}}\|Q_\sigma f\|_\infty \leq \mathfrak{m}(M)^{\frac{1}{2}}\|\widehat{Q_\sigma f}\|_1 \\ &= \mathfrak{m}(M)^{\frac{1}{2}}\|\widehat{\sigma}\widehat{\mathrm{Wig}f}\|_1 \leq \mathfrak{m}(M)^{\frac{1}{2}}\|\widehat{\sigma}\|_2\|\widehat{\mathrm{Wig}f}\|_2 = \mathfrak{m}(M)^{\frac{1}{2}}\|\sigma\|_2\|f\|_2^2. \end{aligned}$$

□

In an analogous way the boundedness of the sesquiliner map $Q_\sigma(f,g)$ can be proved.

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