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Published version:

DOI:10.1112/blms/bdt038

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Abstract

In this paper, after introducing a natural generalization of the classical Wigner transform, namely the $\tau-$Wigner transforms, depending on the parameter $\tau \in [0, 1]$, we study the problem of its positivity. In particular we prove two theorems of Hudson type considering the action of the $\tau-$Wigner transforms on functions and on distributions respectively. We give then an application of our results concerning Weyl and localization pseudo-differential operators.

1 Introduction

The Wigner transform

$$Wig(f)(x, \omega) = \int_{\mathbb{R}^d} e^{-2\pi it\omega} f(x + t/2)\overline{f(x - t/2)}\, dt, \qquad f \in L^2(\mathbb{R}^d), \tag{1.1}$$

was proposed in 1932 by Wigner [16] in the context of quantum mechanics as quasi-probability distribution on the phase-space and subsequently introduced by Ville as a time-frequency representation of the energy of a signal f. Since then it has become one of the most used tools of harmonic analysis in quantum mechanics and in signal theory.

For what signal processing is concerned, it satisfies many of the properties one expects to be fulfilled by an energy distribution with respect to time and frequency. However one main drawbacks is that it fails to be positive. More precisely a famous theorem of Hudson [9] asserts that it is positive only on functions of gaussian type (Theorem 2.1 (ii) below, see also [7] and [6]). Many other quadratic (and some non quadratic) forms have been defined in the attempt to obtain satisfying representations of energy distribution of signals. In this context the Cohen class, defined as the class of time-frequency representations of the form $Q(f) = \sigma * Wig(f)$, for $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, covers essentially all most used representations, see for instance [4], [5]. The question of positivity within the Cohen class is highly non trivial and far from having found a general characterization. Interesting condition for positivity of bilinear forms can be found in [10], in [11] the results on the Wigner are deduced from L^p estimates, and in [15] a direct proof of the positivity and applications to rank one operator are presented.

The associated sesquilinear form

$$(f, g) \longrightarrow Wig(f, g)(x, \omega) = \int_{\mathbb{R}^d} e^{-2\pi i t\omega} f(x + t/2)\overline{g(x - t/2)\, dt}$$

defines a continuous map in many different functional settings, for example its acts continuously from $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^{2d})$, and is extendable to tempered distributions. Moreover the Paserval relation

$$\|Wig(f, g)\|_{L^2} = \|f\|_{L^2}\|g\|_{L^2}.$$

yields a bounded map $Wig: \ L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^{2d})$. For further properties see e.g. [6], [1].

In this paper we consider a subclass of the Cohen class, namely the τ-Wigner representations Wig_τ, with $\tau \in [0, 1]$ (see Def. (2.1)), which were introduced in [10] and have been studied in [3] in connection with pseudodifferential operators and the problem of interferences. The effect of the parameter τ is shown in [3] to consist in a shift of the so-called "ghost frequencies" and it is proved that the Born-Jordan representation is actually an integral of $\tau-$ Wigner representations over the interval $[0, 1]$, a fact that explains the better behavior of the Born-Jordan representation with respect to interferences.

In view of their use in signal analysis, the question of the positivity of the Wig_τ representations becomes then of considerable interest.

Besides this, it was proved in [3] that the Wig_τ sesquilinear form is related to the $\tau-$Weyl pseudo-differential quantization (see Shubin [12]) in a way which is analogous to the connection between Wigner form and Weyl operators.

This paper is dedicated to the study of the positivity of the Wig_τ representations and, as application, some consequences on $\tau-$Weyl pseudo-differential operators are presented in the last section. More precisely the paper is organized as follows. In section 2 we give the exact definitions and we state our results about positivity. More precisely they consist of an extension to the τ-Wigner forms of Hudson theorem at first in the case of L^2 functions (Thm. 2.1). Considering then the action on $\mathcal{S}'(\mathbb{R}^d)$, we give a suitable characterization of positivity also extended to this distributional setting (Thm. 2.3).

Preliminarily to the proof of our results we need some lemmas presented in section 3. The proof of the two main results of positivity, obtained with techniques similar to those in [15] and [6], are presented in section 4. In section 5 we give some applications of our previous results presenting how they produce counter-examples which show the absence of connections between positivity of an operator and that of its $\tau-$Weyl symbols.

2 Positivity of $\tau-$Wigner forms

For $\tau \in [0,1]$ and $f,g \in \mathcal{S}(\mathbb{R}^d)$, the τ-Wigner transform is defined as

$$Wig_\tau(f,g)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i t\omega} f(x+\tau t)\overline{g(x-(1-\tau)t)}\, dt. \tag{2.1}$$

It is therefore a natural generalization of the Wigner transform and it has revealed to be a useful tool in various aspects of time-frequency analysis (see [10] and [3]).

By standard density arguments the domain of (2.1) can be extended to more general spaces, for example the Lebesgue spaces. Moreover, given a function $F(x,t)$ let us define the linear change of variables S_τ as:

$$S_\tau(F(x,t)) := F(x+\tau t, x-(1-\tau)t);$$

then, for $f,g \in \mathcal{S}(\mathbb{R}^d)$, we can re-write $Wig_\tau(f,g)(x,\omega)$ as composition of three operators, namely:

$$Wig_\tau(f,g) = \mathcal{F}_2 S_\tau(f \otimes \overline{g}), \tag{2.2}$$

where $\mathcal{F}_2$ stands for the Fourier transform with respect to the second variable. In this way the τ-Wigner transform makes sense for $f,g \in \mathcal{S}'(\mathbb{R}^d)$ defining a continuous map:

$$Wig_\tau : \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^{2d}).$$

We state now the two main results of positivity of $Wig_\tau(f,g)$ in the settings of square integrable functions and tempered distributions respectively.

Theorem 2.1. *Let us suppose that $f,g \in L^2(\mathbb{R}^d)$. Then, for every $\tau \in [0,1]$, $Wig_\tau(f,g) \in L^2(\mathbb{R}^{2d})$, and moreover:*

(i) *For $\tau \in (0,1)$, $\tau \neq \frac{1}{2}$, we have that $Wig_\tau(f,g)(x,\omega) > 0$ a.e. in $\mathbb{R}^{2d}$ if and only if there exists a positive definite matrix $A \in GL(d,\mathbb{R})$, two vectors $\alpha,\beta \in \mathbb{R}^d$ and constants $c,d,a \in \mathbb{R}$ such that*

$$\begin{aligned} f(t) &= e^{-t^{\mathrm{tr}}At+\alpha t+i\beta t+c+ia} \\ g(t) &= e^{-\frac{\tau}{1-\tau}t^{\mathrm{tr}}At+\frac{\tau}{1-\tau}\alpha t+i\beta t+d+ia} \end{aligned} \tag{2.3}$$

where as usual

$$t^{\mathrm{tr}}At = \begin{pmatrix} t_1 & \cdots & t_n \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$$

(ii) *(Hudson Theorem) For* $\tau = \frac{1}{2}$*, we have* $Wig_{\frac{1}{2}}(f,g)(x,\omega) = Wig(f,g)(x,\omega) > 0$ *a.e. in* $\mathbb{R}^{2d}$ *if and only if there exists a complex* $d \times d$ *matrix* A *with positive definite real part, a vector* $\gamma \in \mathbb{C}^d$ *and a constant* $c \in \mathbb{C}$ *such that*

$$f(t) = e^{-t^{\mathrm{tr}}At+\gamma t+c}, \qquad g(t) = \lambda f(t)$$

for an arbitrary positive real constant λ*.*

Remark 2.2. *The new result in Theorem* 2.1 *is the case* $\tau \neq \frac{1}{2}$*; for completeness we have stated also the case* $\tau = \frac{1}{2}$*, (cf. [9]). Note that the matrix* A*, which has generally complex entries for* $\tau = 1/2$*, is real in the case* $\tau \neq 1/2$*.*

In order to state the Hudson's theorem for the τ-Wigner transform in the frame of tempered distributions let us define, for $f \in \mathcal{S}(\mathbb{R}^d)$, $x, \omega \in \mathbb{R}^d$ and $B \in GL(d, \mathbb{R})$, the following operators:

$$\begin{aligned} T_x f(f) &= f(t-x) \\ M_\omega f(t) &= e^{2\pi i \omega t} f(t) \\ \mathcal{U}_B f(t) &= |\det B|^{1/2} f(Bt) \end{aligned} \tag{2.4}$$

with obvious extensions to the case $f \in \mathcal{S}'(\mathbb{R}^d)$.

Theorem 2.3. *Let us suppose that* $f, g \in \mathcal{S}'(\mathbb{R}^d)$*. We consider the "splitting" of variable* $t = \big(t_{[1]}, t_{[2]}, t_{[3]}\big)$ *with* $t_{[1]} = (t_1, \ldots, t_h)$*,* $t_{[2]} = (t_{h+1}, \ldots, t_k)$*,* $t_{[3]} = (t_{k+1}, \ldots, t_d)$ *with* $0 \le h \le k \le d$ *(in the cases* $h = 0$*,* $h = k$ *and* $k = d$ *we respectively mean that* $t_{[1]}$*,* $t_{[2]}$ *and* $t_{[3]}$ *are empty). We then have:*

(i) *For* $\tau \in (0,1)$*,* $\tau \neq \frac{1}{2}$*, we have that* $Wig_\tau(f,g)(x,\omega) \in \mathcal{S}'(\mathbb{R}^{2d})$ *is a positive distribution if and only if there exist* $c_f, c_g, a \in \mathbb{R}$*,* $\theta, \sigma \in \mathbb{R}^d$*,* $D \in GL(d, \mathbb{R})$ *and a splitting of variables of the kind described above such that*

$$f(t) = e^{c_f + ia} \mathcal{U}_D \, T_\theta \, M_\sigma \big(e^{-t_{[1]}^2} \otimes \delta_{t_{[2]}} \otimes \mathbf{1}_{t_{[3]}}\big) \tag{2.5}$$

and

$$g(t) = e^{c_g + ia} \mathcal{U}_D \, T_\theta \, M_\sigma \big(e^{-\frac{\tau}{1-\tau} t_{[1]}^2} \otimes \delta_{t_{[2]}} \otimes \mathbf{1}_{t_{[3]}}\big), \tag{2.6}$$

where $\delta_{t_{[2]}}$ *is the Dirac distribution in the* $t_{[2]}$*-variable and* $\mathbf{1}_{t_{[3]}}$ *stands for the function identically* 1 *in the* $t_{[3]}$*-variable.*

(ii) $Wig(f,g) = Wig_{\frac{1}{2}}(f,g)$ *is a positive distribution if and only if there exist* $c, a \in \mathbb{R}$*,* $\theta, \sigma \in \mathbb{R}^d$*,* $D \in GL(d, \mathbb{R})$*,* $A \in GL(h, \mathbb{C})$ *with* $\Re A$ *positive definite, and a splitting of variables as before such that*

$$f(t) = e^{c+ia} \mathcal{U}_D \, T_\theta \, M_\sigma \big(e^{-t_{[1]}^{\mathrm{tr}} A t_{[1]}} \otimes \delta_{t_{[2]}} \otimes \mathbf{1}_{t_{[3]}}\big), \qquad g(t) = \lambda f(t) \tag{2.7}$$

for a positive constant λ*.*

3 Some properties of the τ-Wigner representation

In this section we study some important properties of the τ-Wigner representation and prove some technical results that will be used in the proof of the generalized Hudson's Theorems 2.1 and 2.3. We begin by proving the orthogonality relation for the τ-Wigner transform.

Proposition 3.1. *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$, and $\tau \in [0,1]$. Then we have that $Wig_\tau(f_j, g_j)(x,\omega) \in L^2(\mathbb{R}^{2d})$ for $j = 1,2$ and moreover*

$$\big(Wig_\tau(f_1, g_1), Wig_\tau(f_2, g_2)\big)_{L^2(\mathbb{R}^{2d})} = (f_1, f_2)_{L^2(\mathbb{R}^d)}\overline{(g_1, g_2)}_{L^2(\mathbb{R}^d)} \tag{3.1}$$

Proof. By (2.2) and the Parseval formula we have

$$\big(Wig_\tau(f_1, g_1), Wig_\tau(f_2, g_2)\big)_{L^2(\mathbb{R}^{2d})} = \big(S_\tau(f_1 \otimes \overline{g}_1), S_\tau(f_2 \otimes \overline{g}_2)\big)_{L^2(\mathbb{R}^{2d})};$$

we can then apply S_τ^{-1}, and since S_τ is a unitary linear change of variables and

$$(f_1 \otimes \overline{g}_1, f_2 \otimes \overline{g}_2)_{L^2(\mathbb{R}^{2d})} = (f_1, f_2)_{L^2(\mathbb{R}^d)}(\overline{g}_1, \overline{g}_2)_{L^2(\mathbb{R}^d)},$$

the proof is complete. □

Remark 3.2. *This means that every Wig_τ preserves the energy of a signal. In particular, for every $\tau, \sigma \in [0,1]$ and $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$ we have*

$$\big(Wig_\tau(f_1, g_1), Wig_\tau(f_2, g_2)\big)_{L^2(\mathbb{R}^{2d})} = \big(Wig_\sigma(f_1, g_1), Wig_\sigma(f_2, g_2)\big)_{L^2(\mathbb{R}^{2d})}.$$

Next we analyze how the τ-Wigner behaves with respect to translation, modulation and linear change of variables, cf. (2.4).

Proposition 3.3. *Let us fix $f, g \in \mathcal{S}'(\mathbb{R}^d)$; for every $\tau \in [0,1]$, $y, z, \beta, \gamma \in \mathbb{R}^d$ and $B \in GL(d, \mathbb{R})$ we have:*

$$Wig_\tau(T_y f, T_z g) = e^{-2\pi i(y-z)\omega}\, Wig_\tau(f, g)(x - (1-\tau)y - \tau z, \omega) \tag{3.2}$$

$$Wig_\tau(M_\beta f, M_\gamma g) = e^{2\pi i(\beta-\gamma)x}\, Wig_\tau(f, g)(x, \omega - \tau\beta - (1-\tau)\gamma) \tag{3.3}$$

$$Wig_\tau(\mathcal{U}_B f, \mathcal{U}_B g) = Wig_\tau(f, g)(Bx, (B^{-1})^{\text{tr}}\omega), \tag{3.4}$$

where B^{tr} is the transposed of the matrix B.

Proof. It is enough to prove the statement for $f, g \in \mathcal{S}(\mathbb{R}^d)$; the case when f and g are tempered distributions shall follow by density. By the change of variables $t - y + z = s$ we have

$$\begin{aligned} Wig_\tau(T_y f, T_z g)(x,\omega) &= \int_{\mathbb{R}^d} e^{-2\pi i t\omega} f(x - y + \tau t)\overline{g(x - z - (1-\tau)t)}\, dt \\ &= \int_{\mathbb{R}^d} e^{-2\pi i(s+y-z)\omega} f(x - (1-\tau)y - \tau z + \tau s)\overline{g(x - (1-\tau)y - \tau z - (1-\tau)s)}\, ds \\ &= e^{-2\pi i(y-z)\omega} Wig_\tau(f, g)(x - (1-\tau)y - \tau z, \omega). \end{aligned}$$

The relation concerning modulation is trivial; regarding the linear change of variables, we have:

$$\begin{aligned} Wig_\tau(\mathcal{U}_B f, \mathcal{U}_B g) &= \int_{\mathbb{R}^d} e^{-2\pi i t\omega} |\det B|^{\frac{1}{2}} f(Bx+\tau Bt) |\det B|^{\frac{1}{2}} \overline{g(Bx-(1-\tau)Bt)}\, dt \\ &= \int_{\mathbb{R}^d} e^{-2\pi i (B^{-1}s)\omega} f(Bx+\tau s) \overline{g(Bx-(1-\tau)s)}\, ds \\ &= Wig_\tau(f,g)(Bx, (B^{-1})^{\mathrm{tr}}\omega), \end{aligned}$$

since $(B^{-1}s)\omega = s[(B^{-1})^{\mathrm{tr}}\omega]$. □

We specify now the relation between Wig_τ and the Fourier transform.

Proposition 3.4. *For every $f, g \in \mathcal{S}'(\mathbb{R}^d)$ and $\tau \in [0,1]$ we have*

$$Wig_\tau(\hat{f}, \hat{g})(x,\omega) = Wig_{1-\tau}(f,g)(-\omega, x). \tag{3.5}$$

Proof. The conclusion is obvious for $\tau = 0$ and $\tau = 1$, since we have $Wig_0(f,g) = e^{-2\pi i x\omega} f(x)\overline{\hat{g}(\omega)}$ and $Wig_1(f,g) = e^{2\pi i x\omega}\hat{f}(\omega)\overline{g(x)}$. For $\tau \in (0,1)$ we shall prove the following formula, that is equivalent to (3.5):

$$Wig_\tau(f,g)(x,\omega) = Wig_\tau(\overline{\hat{g}}, \overline{\hat{f}})(\omega, x). \tag{3.6}$$

As in the proof of Proposition 3.3 we can limit our attention to $f, g \in \mathcal{S}(\mathbb{R}^d)$. Since $g(x-(1-\tau)t) = \int e^{2\pi i (x-(1-\tau)t)\eta}\hat{g}(\eta)\, d\eta$, we have by a change of variables that

$$\begin{aligned} Wig_\tau(f,g)(x,\omega) &= \int_{\mathbb{R}^d} e^{-2\pi i t\omega} f(x+\tau t)\overline{g(x-(1-\tau)t)}\, dt \\ &= \int e^{-2\pi i t\omega - 2\pi i x\eta + 2\pi i (1-\tau)t\eta} f(x+\tau t)\overline{\hat{g}(\eta)}\, d\eta\, dt \\ &= \tau^{-d} \int e^{-2\pi i s \frac{\omega-(1-\tau)\eta}{\tau}} e^{2\pi i x \frac{\omega-\eta}{\tau}} f(s)\overline{\hat{g}(\eta)}\, d\eta\, ds; \end{aligned}$$

by interchanging the order of integration and by a linear change of variables we then get

$$\begin{aligned} Wig_\tau(f,g)(x,\omega) &= \tau^{-d} \int e^{2\pi i x \frac{\omega-\eta}{\tau}} \hat{f}\left(\frac{\omega-(1-\tau)\eta}{\tau}\right)\overline{\hat{g}(\eta)}\, d\eta \\ &= \int e^{-2\pi i x t} \hat{f}(\omega-(1-\tau)t)\overline{\hat{g}(\omega+\tau t)}\, dt \\ &= Wig_\tau(\overline{\hat{g}}, \overline{\hat{f}})(\omega, x) \end{aligned}$$

and so the proof is complete. □

For the proof of the generalized Hudson's Theorem in the frame of tempered distributions we need the following technical lemma.

Lemma 3.5. *Let $\boldsymbol{\varrho}(x)$ and $\boldsymbol{\kappa}(\omega)$ be real positive definite quadratic forms on $\mathbb{R}^d$, and let us define, for $\phi \in \mathcal{S}'(\mathbb{R}^d)$,*

$$\phi_{\boldsymbol{\varrho},\boldsymbol{\kappa}}(x) = e^{-\boldsymbol{\varrho}(x)} e^{-\boldsymbol{\kappa}(D)} \phi(x), \tag{3.7}$$

where $e^{-\boldsymbol{\kappa}(D)}\phi(x) = \mathcal{F}^{-1}_{\omega\to x}\big(e^{-\boldsymbol{\kappa}(\omega)}\hat{\phi}(\omega)\big)$. For every $f, g \in \mathcal{S}'(\mathbb{R}^d)$ we then have

$$Wig_\tau(f_{\boldsymbol{\varrho},\frac{\tau}{1-\tau}\boldsymbol{\kappa}}, g_{\frac{\tau}{1-\tau}\boldsymbol{\varrho},\boldsymbol{\kappa}}) = e^{-\frac{1}{1-\tau}[\boldsymbol{\varrho}(x)+\boldsymbol{\kappa}(\omega)]-\tau[\boldsymbol{\varrho}(D_\omega)+\boldsymbol{\kappa}(D_x)]} Wig_\tau(f,g)(x,\omega). \tag{3.8}$$

Proof. It is enough to prove (3.8) for $f, g \in \mathcal{S}(\mathbb{R}^d)$, since the case when f and g are tempered distributions shall follow by standard density arguments. We start by considering the case $\boldsymbol{\kappa} = 0$; writing $\phi_{\boldsymbol{\varrho}}(x) = e^{-\boldsymbol{\varrho}(x)}\phi(x)$, we have:

$$Wig_\tau(f_{\boldsymbol{\varrho}}, g_{\frac{\tau}{1-\tau}\boldsymbol{\varrho}}) = \int e^{-2\pi it\omega} e^{-\boldsymbol{\varrho}(x+\tau t)} f(x+\tau t) e^{-\frac{\tau}{1-\tau}\boldsymbol{\varrho}(x-(1-\tau)t)} \overline{g(x-(1-\tau)t)}\, dt;$$

by simple computations we obtain

$$e^{-\boldsymbol{\varrho}(x+\tau t)} e^{-\frac{\tau}{1-\tau}\boldsymbol{\varrho}(x-(1-\tau)t)} = e^{\frac{1}{1-\tau}\boldsymbol{\varrho}(x)+\tau\boldsymbol{\varrho}(t)}.$$

We then have, by the change of variables $s = -t$, that

$$Wig_\tau(f_{\boldsymbol{\varrho}}, g_{\frac{\tau}{1-\tau}\boldsymbol{\varrho}}) = e^{-\frac{1}{1-\tau}\boldsymbol{\varrho}(x)} \int e^{2\pi is\omega} e^{-\tau\boldsymbol{\varrho}(-s)} f(x-\tau s)\overline{g(x+(1-\tau)s)}\, ds.$$

We now observe that $f(x-\tau s)\overline{g(x+(1-\tau)s)} = \mathcal{F}_{\eta\to s}\left(Wig_\tau(f,g)(x,\eta)\right)$; then, since $\boldsymbol{\varrho}(-s) = \boldsymbol{\varrho}(s)$ we get:

$$\begin{aligned} Wig_\tau(f_{\boldsymbol{\varrho}}, g_{\frac{\tau}{1-\tau}\boldsymbol{\varrho}}) &= e^{-\frac{1}{1-\tau}\boldsymbol{\varrho}(x)} \int e^{2\pi is\omega} e^{-\tau\boldsymbol{\varrho}(s)} \mathcal{F}_{\eta\to s}\left(Wig_\tau(f,g)(x,\eta)\right)\, ds \\ &= e^{-\frac{1}{1-\tau}\boldsymbol{\varrho}(x)} e^{-\tau\boldsymbol{\varrho}(D_\omega)} Wig_\tau(f,g)(x,\omega). \end{aligned} \tag{3.9}$$

We consider now $Wig_\tau\big(e^{-\frac{\tau}{1-\tau}\boldsymbol{\kappa}(D)}f, e^{-\boldsymbol{\kappa}(D)}g\big)(x,\omega)$; by (3.6), since $\mathcal{F}\big(e^{-\frac{\tau}{1-\tau}\boldsymbol{\kappa}(D)}f\big) = e^{-\frac{\tau}{1-\tau}\boldsymbol{\kappa}(y)}\hat{f}(y)$ and $\mathcal{F}\big(e^{-\boldsymbol{\kappa}(D)}g\big) = e^{-\boldsymbol{\kappa}(y)}\hat{g}(y)$, we have that

$$Wig_\tau\big(e^{-\frac{\tau}{1-\tau}\boldsymbol{\kappa}(D)}f, e^{-\boldsymbol{\kappa}(D)}g\big)(x,\omega) = Wig_\tau\big(\bar{\hat{g}}_{\boldsymbol{\kappa}}, \bar{\hat{f}}_{\frac{\tau}{1-\tau}\boldsymbol{\kappa}}\big)(\omega, x);$$

by (3.9) and (3.6) we obtain

$$\begin{aligned} Wig_\tau\big(e^{-\frac{\tau}{1-\tau}\boldsymbol{\kappa}(D)}f, e^{-\boldsymbol{\kappa}(D)}g\big)(x,\omega) &= e^{-\frac{1}{1-\tau}\boldsymbol{\kappa}(\omega)} e^{-\tau\boldsymbol{\kappa}(D_x)} Wig_\tau(\bar{\hat{g}}, \bar{\hat{f}})(\omega, x) \\ &= e^{-\frac{1}{1-\tau}\boldsymbol{\kappa}(\omega)} e^{-\tau\boldsymbol{\kappa}(D_x)} Wig_\tau(f,g)(x,\omega). \end{aligned} \tag{3.10}$$

Then (3.8) follows immediately from (3.9) and (3.10). ∎

Finally we shall need the τ-Wigner transform of some particular functions.

Lemma 3.6. *Let us consider*

$$\varphi(t) = e^{-t^{\mathrm{tr}}At+\alpha t}, \qquad \psi(t) = e^{-\frac{\tau}{1-\tau}t^{\mathrm{tr}}At+\frac{\tau}{1-\tau}\alpha t}, \tag{3.11}$$

where A is a real $d\times d$ positive definite matrix, and $\alpha \in \mathbb{R}^d$. Then, there exists $C \in GL(d,\mathbb{R})$ such that

$$Wig_\tau(\varphi,\psi)(x,\omega) = |\det C|\,\pi^{d/2}e^{-\frac{1}{1-\tau}(x^{\mathrm{tr}}Ax)+\frac{1}{1-\tau}\alpha x}e^{-\pi^2(C\omega)^2}$$

where by $(C\omega)^2$ we mean the inner product between $C\omega$ and itself. In particular, the τ-Wigner transform $Wig_\tau(\varphi,\psi)(x,\omega)$ is positive for every $(x,\omega)\in\mathbb{R}^{2d}$.

Proof. By definition of τ-Wigner we have that

$$Wig_\tau(\varphi,\psi)(x,\omega) = \\ = \int e^{-2\pi it\omega}e^{-(x+\tau t)^{\mathrm{tr}}A(x+\tau t)}e^{\alpha(x+\tau t)}e^{-\frac{\tau}{1-\tau}(x-(1-\tau)t)^{\mathrm{tr}}A(x-(1-\tau)t)}e^{\frac{\tau}{1-\tau}\alpha(x-(1-\tau)t)}\,dt;$$

since $(x+\tau t)^{\mathrm{tr}}A(x+\tau t)+\frac{\tau}{1-\tau}\,(x-(1-\tau)t)^{\mathrm{tr}}A(x-(1-\tau)t) = \frac{1}{1-\tau}\,x^{\mathrm{tr}}Ax+\tau\,t^{\mathrm{tr}}At$ and $\alpha(x+\tau t)+\frac{\tau}{1-\tau}\alpha(x-(1-\tau)t) = \frac{1}{1-\tau}\alpha x$ we then get:

$$Wig_\tau(\varphi,\psi)(x,\omega) = e^{-\frac{1}{1-\tau}x^{\mathrm{tr}}Ax+\frac{1}{1-\tau}\alpha x}\int e^{-2\pi it\omega}e^{-\tau\,t^{\mathrm{tr}}At}\,dt. \tag{3.12}$$

Now, since A is positive definite we can diagonalize it, finding $A = V^{\mathrm{tr}}\operatorname{diag}(\lambda_j)V$ for a matrix $V\in GL(d,\mathbb{R})$, where $\lambda_j>0$ are the eigenvalues of A. We then get

$$A = (\operatorname{diag}(\sqrt{\lambda_j})V)^{\mathrm{tr}}\cdot(\operatorname{diag}(\sqrt{\lambda_j})V)$$

and so

$$t^{\mathrm{tr}}At = (\operatorname{diag}(\sqrt{\lambda_j})Vt)^{\mathrm{tr}}\cdot(\operatorname{diag}(\sqrt{\lambda_j})Vt).$$

Now we make the change of variables $s=\sqrt{\tau}\operatorname{diag}(\sqrt{\lambda_j})Vt$ in the integral appearing in (3.12); setting for convenience $B=\sqrt{\tau}\operatorname{diag}(\sqrt{\lambda_j})V$ we have

$$\begin{aligned} Wig_\tau(\varphi,\psi)(x,\omega) &= e^{-\frac{1}{1-\tau}x^{\mathrm{tr}}Ax+\frac{1}{1-\tau}\alpha x}\frac{1}{|\det B|}\int e^{-2\pi iB^{-1}s\omega}e^{-s^2}\,ds \\ &= \frac{1}{|\det B|}e^{-\frac{1}{1-\tau}x^{\mathrm{tr}}Ax+\frac{1}{1-\tau}\alpha x}\pi^{d/2}e^{-\pi^2((B^{-1})^{\mathrm{tr}}\omega)^2}; \end{aligned}$$

the conclusion follows just by setting $C=(B^{-1})^{\mathrm{tr}}$. ☐

4 Proof of Hudson's Theorems for τ-Wigner representations

In this section we give the proof of Theorems 2.1 and 2.3. We start by considering the L^2 frame. We are just interested in the case $\tau \neq \frac{1}{2}$, since the result on the positivity of $Wig(f,g)$ is known; we remark however that the same proof remains valid also in the case $\tau = \frac{1}{2}$.

Proof of Theorem 2.1. We start by proving that if the τ-Wigner is positive then f and g must be as in (2.3). We observe at first that

$$Wig_\tau(e^{-\pi t^2}, e^{-\pi \frac{\tau}{1-\tau} t^2}) > 0,$$

as we can deduce from Lemma 3.6 with $A = \pi I$ and $\beta = 0$, where $I \in GL(d,\mathbb{R})$ is the identity. Moreover, from (3.2) with $z = y$ we have that

$$Wig_\tau(e^{-\pi (t-y)^2}, e^{-\pi \frac{\tau}{1-\tau} (t-y)^2})(x,\omega) = Wig_\tau(e^{-\pi t^2}, e^{-\pi \frac{\tau}{1-\tau} t^2})(x-y,\omega) > 0 \quad (4.1)$$

for every $y \in \mathbb{R}^d$, $(x,\omega) \in \mathbb{R}^{2d}$. On the other hand, we have

$$Wig_\tau(e^{-\pi (t-y)^2}, e^{-\pi \frac{\tau}{1-\tau} (t-y)^2})(x,\omega) = e^{-\pi \frac{1}{1-\tau} y^2} Wig_\tau(e^{-\pi t^2 + 2\pi t y}, e^{-\pi \frac{\tau}{1-\tau} t^2 + 2\pi \frac{\tau}{1-\tau} t y}),$$

and then by (4.1) we get

$$Wig_\tau(e^{-\pi t^2 + 2\pi t y}, e^{-\pi \frac{\tau}{1-\tau} t^2 + 2\pi \frac{\tau}{1-\tau} t y}) > 0$$

for every $y \in \mathbb{R}^d$, $(x,\omega) \in \mathbb{R}^{2d}$. Now, by (3.3) with $\beta = \gamma := y'$ we obtain immediately that

$$Wig_\tau\big(M_{-y'}(e^{-\pi t^2 + 2\pi t y}), M_{-y'}(e^{-\pi \frac{\tau}{1-\tau} t^2 + 2\pi \frac{\tau}{1-\tau} t y})\big) \quad (4.2)$$

is positive for every y, y', x, ω. Observe that

$$M_{-y'}(e^{-\pi t^2 + 2\pi t y}) = e^{-\pi t^2 - 2\pi i t z_1} \quad \text{and} \quad M_{-y'}(e^{-\pi \frac{\tau}{1-\tau} t^2 + 2\pi \frac{\tau}{1-\tau} t y}) e^{-\pi \frac{\tau}{1-\tau} t^2 - 2\pi i t z_2}$$

for $z_1, z_2 \in \mathbb{C}^d$ with the relation

$$z_1 = y' + iy, \qquad z_2 = y' + i\frac{\tau}{1-\tau} y; \quad (4.3)$$

we can then rewrite (4.2) in the following way:

$$Wig_\tau(e^{-\pi t^2 - 2\pi i t z_1}, e^{-\pi \frac{\tau}{1-\tau} t^2 - 2\pi i t z_2})(x,\omega) > 0 \quad (4.4)$$

for every $(x,\omega) \in \mathbb{R}^{2d}$ and $z_1, z_2 \in \mathbb{C}^d$ as in (4.3). Since $Wig_\tau(f,g)(x,\omega) > 0$ for every $(x,\omega) \in \mathbb{R}^{2d}$ by hypothesis, we get from (4.4)

$$\big(Wig_\tau(f,g), Wig_\tau(e^{-\pi t^2 - 2\pi i t z_1}, e^{-\pi \frac{\tau}{1-\tau} t^2 - 2\pi i t z_2})\big)_{L^2(\mathbb{R}^{2d})} > 0; \quad (4.5)$$

then from the orthogonality relation (3.1) we get

$$(f, e^{-\pi t^2 - 2\pi i t z_1})\overline{(g, e^{-\pi \frac{\tau}{1-\tau} t^2 - 2\pi i t z_2})} > 0 \tag{4.6}$$

for every z_1, z_2 as in (4.3). Now we observe that by a simple change of variables we have

$$\|e^{-\pi t^2 - 2\pi i t z_1}\|^2_{L^2(\mathbb{R}^d_t)} = \int_{\mathbb{R}^d} e^{-2\pi t^2 + 4\pi t \Im z_1}\, dt = e^{2\pi (\Im z_1)^2} \int_{\mathbb{R}^d} e^{-2\pi s^2}\, ds = \frac{e^{2\pi(\Im z_1)^2}}{2^{d/2}}.$$

Let us consider now the function

$$G(z_1) = (f, e^{-\pi t^2 - 2\pi i t z_1});$$

we have

$$|G(z_1)| \leq \|f\|_{L^2} \|e^{-\pi t^2 - 2\pi i t z_1}\|_{L^2} = \|f\|_{L^2} \frac{e^{\pi(\Im z_1)^2}}{2^{d/4}} \leq c e^{\pi |z_1|^2}$$

for a constant $c > 0$. Moreover, $G(z_1)$ is an entire function and it never vanishes, as we can deduce from (4.6); then, reasoning as in [6] we obtain that G is of the form

$$G(z_1) = e^{z_1^{\mathrm{tr}} A z_1 + b z_1 + c}. \tag{4.7}$$

We observe now that we can write

$$G(z_1) = \int_{\mathbb{R}^d} e^{2\pi i t \Re z_1} e^{-\pi t^2 + 2\pi t \Im z_1} f(t)\, dt,$$

and so $\mathcal{F}_{\Re z_1 \to t}\big(G(z_1)|_{\Im z_1 = 0}\big) = e^{-\pi t^2} f(t)$; then, by (4.7) we have that $e^{-\pi t^2} f(t)$ is a generalized Gaussian, i.e. an exponential whose exponent is a a polynomial of degree 2, cf. [6], [7, Lemma 4.4.2]. This implies that $f(t)$ is of the same form, i.e. there exist a $d \times d$ complex matrix A', $\beta' \in \mathbb{C}^d$ and $c' \in \mathbb{C}$ such that

$$f(t) = e^{-t^{\mathrm{tr}} A' t + \beta' t + c'}. \tag{4.8}$$

Reasoning in the same way on the expression

$$(g, e^{-\pi \frac{\tau}{1-\tau} t^2 - 2\pi i t z_2}),$$

cf. (4.6), we obtain that $g(t)$ must be of the same form as $f(t)$, so

$$g(t) = e^{-t^{\mathrm{tr}} A'' t + \beta'' t + c''} \tag{4.9}$$

for a $d \times d$ complex matrix A'', $\beta'' \in \mathbb{C}^d$ and $c'' \in \mathbb{C}$. Now we want to find the relations that must occur between $A', \beta', c', A'', \beta'', c''$. To this aim let us first observe that

$$Wig_\tau(f, g) = \mathcal{F}_{t \to \omega}\big(f(x + \tau t)\overline{g(x - (1-\tau)t)}\big),$$

and recall that by hypothesis $Wig_\tau(f,g) > 0$; then, since f and g are of the form (4.8)-(4.9) (in particular they are continuous) we have that the function

$$t \mapsto f(x+\tau t)\overline{g(x-(1-\tau)t)}$$

is positive definite, as we can deduce for example from [15, Lemma 1.4]. Then

$$f(x+\tau t)\overline{g(x-(1-\tau)t)} = \overline{f(x-\tau t)}g(x+(1-\tau)t). \tag{4.10}$$

Since we already have an explicit expression for f and g, we simply substitute (4.8) and (4.9) in (4.10), obtaining

$$\begin{cases} A' + \overline{A''} = \overline{A'} + A'' \\ (1-\tau)\overline{A''} - \tau A' = \tau\overline{A'} - (1-\tau)A'' \\ \tau^2 A' + (1-\tau)^2\overline{A''} = \tau^2\overline{A'} + (1-\tau)^2 A'' \\ \beta' + \overline{\beta''} = \overline{\beta'} + \beta'' \\ \tau\beta' - (1-\tau)\overline{\beta''} = -\tau\overline{\beta'} + (1-\tau)\beta'' \\ c' + \overline{c''} = \overline{c'} + c'' + 2k\pi i, \quad \text{for every } k \in \mathbb{Z} \end{cases}$$

The conditions above are equivalent to

$$\begin{cases} \Im A'' = \Im A' \\ \Re A'' = \frac{\tau}{1-\tau}\Re A' \\ \Im A'' = \left(\frac{\tau}{1-\tau}\right)^2 \Im A' \\ \Im\beta'' = \Im\beta' \\ \Re\beta'' = \frac{\tau}{1-\tau}\Re\beta' \\ \Im c'' = \Im c' \end{cases} \tag{4.11}$$

where we forgot about $2k\pi i$ in the condition on c', c'' because of the particular form of f and g. We then have different conclusions depending on τ. If $\tau \neq \frac{1}{2}$ we must have $\Im A' = \Im A'' = 0$, and so by (4.11) and (4.8)-(4.9) we deduce that the functions f and g are of the form stated in Theorem 2.1, where the matrix A in (2.3) must be positive definite because f and g are supposed to be in $L^2(\mathbb{R}^d)$. If $\tau = \frac{1}{2}$ the conditions (4.11) become $A' = A''$, $\beta' = \beta''$, $\Im c' = \Im c''$, and so we have the point (ii) of Theorem 2.1, where as before the condition $\Re A$ positive definite is required in order to ensure that f and g are in $L^2(\mathbb{R}^d)$.

Now we have to prove the converse: let us suppose that f and g are of the form (2.3), and prove that the corresponding $Wig_\tau(f,g)$ is positive (we consider only the case $\tau \neq \frac{1}{2}$ since for the usual Wigner transform it is already known). Let us consider the functions (3.11), and observe that

$$f(t) = e^c e^{ia} M_{\frac{\beta}{2\pi}}\varphi(t), \quad g(t) = e^d e^{ia} M_{\frac{\beta}{2\pi}}\psi(t),$$

with the notations (2.4); then, by the skew-linearity of the τ-Wigner transform and by Proposition 3.3 we have

$$Wig_\tau(f,g)(x,\omega) = e^{c+d} Wig_\tau(\varphi,\psi)\Big(x,\omega - \frac{\beta}{2\pi}\Big);$$

then, from Lemma 3.6, $Wig_\tau(f,g)(x,\omega) > 0$ for every $(x,\omega) \in \mathbb{R}^{2d}$. The proof is then complete. □

Remark 4.1. *We observe that for $f,g \in L^2(\mathbb{R}^d)$, $f,g \not\equiv 0$, and $\tau \in (0,1)$ we have*

$$Wig_\tau(f,g)(x,\omega) > 0 \;\; \Leftrightarrow Wig_\tau(f,g)(x,\omega) \geq 0;$$

in fact, if $Wig_\tau(f,g)(x,\omega) \geq 0$ we still have strict inequality in (4.5)*, and so the same proof of Theorem* 2.1 *works in the case $Wig_\tau(f,g)(x,\omega) \geq 0$.*

Now we want to prove the characterization of Theorem 2.3 on tempered distributions that make the τ-Wigner transform positive.

Proof of Theorem 2.3. Let $\tau \neq \frac{1}{2}$; we suppose at first that $Wig_\tau(f,g)$ is a positive distribution, and we want to prove that the tempered distributions f and g are as in the statement of the theorem. We consider two real quadratic form $\boldsymbol{\varrho}(t)$ and $\boldsymbol{\kappa}(t)$ on $\mathbb{R}^d$:

$$\boldsymbol{\varrho}(t) = t^{\mathrm{tr}} B t, \quad \boldsymbol{\kappa}(t) = t^{\mathrm{tr}} C t, \tag{4.12}$$

with two real $d \times d$ matrices B and C; we suppose that $\boldsymbol{\varrho}(t)$ and $\boldsymbol{\kappa}(t)$ are positive definite. Now we observe that $f_{\boldsymbol{\varrho},\frac{\tau}{1-\tau}\boldsymbol{\kappa}}$ and $g_{\frac{\tau}{1-\tau}\boldsymbol{\varrho},\boldsymbol{\kappa}}$, cf. (3.7), belong to $L^2(\mathbb{R}^d)$ (they are in fact C^∞ functions with good decay at infinity); indeed, since $\mathcal{F}^{-1}\big(e^{-\boldsymbol{\kappa}(t)}\big)$ is still a gaussian, for any $\phi \in \mathcal{S}'(\mathbb{R}^d)$ we have that $e^{-\boldsymbol{\kappa}(D)}\phi = \mathcal{F}^{-1}\big(e^{-\boldsymbol{\kappa}(t)}\big) * \phi$ is a C^∞ slowly increasing function, in particular it is dominated at infinity by a polynomial; then for any positive definite quadratic form $\boldsymbol{\varrho}(t)$ we have $e^{-\boldsymbol{\varrho}(t)} e^{-\boldsymbol{\kappa}(D)}\phi = \phi_{\boldsymbol{\varrho},\boldsymbol{\kappa}} \in L^2(\mathbb{R}^d)$. Now, since by hypothesis $Wig_\tau(f,g)(x,\omega)$ is a positive distribution, by Lemma 3.5 we have that

$$Wig_\tau\big(f_{\boldsymbol{\varrho},\frac{\tau}{1-\tau}\boldsymbol{\kappa}}, g_{\frac{\tau}{1-\tau}\boldsymbol{\varrho},\boldsymbol{\kappa}}\big)(x,\omega) > 0$$

for every $(x,\omega) \in \mathbb{R}^{2d}$ (observe that $Wig_\tau\big(f_{\boldsymbol{\varrho},\frac{\tau}{1-\tau}\boldsymbol{\kappa}}, g_{\frac{\tau}{1-\tau}\boldsymbol{\varrho},\boldsymbol{\kappa}}\big)$ is a smooth function with exponential decay at infinity, cf. (3.8)). Then, since $f_{\boldsymbol{\varrho},\frac{\tau}{1-\tau}\boldsymbol{\kappa}}$ and $g_{\frac{\tau}{1-\tau}\boldsymbol{\varrho},\boldsymbol{\kappa}}$ belong to $L^2(\mathbb{R}^d)$ we can apply Theorem 2.1 and deduce that

$$f_{\boldsymbol{\varrho},\frac{\tau}{1-\tau}\boldsymbol{\kappa}}(t) = e^{-t^{\mathrm{tr}}At+\alpha t+i\beta t+c+ia} \tag{4.13}$$

$$g_{\frac{\tau}{1-\tau}\boldsymbol{\varrho},\boldsymbol{\kappa}}(t) = e^{-\frac{\tau}{1-\tau}t^{\mathrm{tr}}At+\frac{\tau}{1-\tau}\alpha t+i\beta t+d+ia} \tag{4.14}$$

where A is a real positive definite $d\times d$ matrix, $\alpha,\beta\in\mathbb{R}^d$ and $c,d,a\in\mathbb{R}$. Now we want to compute f and g. For every $\phi\in\mathcal{S}'(\mathbb{R}^d)$ and for all real positive definite quadratic forms $\boldsymbol{\varrho}(t)$ and $\boldsymbol{\kappa}(t)$ we have

$$\phi = e^{\boldsymbol{\kappa}(D)}\big(e^{\boldsymbol{\varrho}(t)}\phi_{\boldsymbol{\varrho},\boldsymbol{\kappa}}(t)\big);$$

then by (4.13) we obtain

$$\begin{aligned} f(t) &= e^{\frac{\tau}{1-\tau}\boldsymbol{\kappa}(D)}\big(e^{\boldsymbol{\varrho}(t)-t^{\mathrm{tr}}At+\alpha t+i\beta t+c+ia}\big)\\ &= e^{c+ia}\mathcal{F}^{-1}_{\omega\to t}\big(e^{\frac{\tau}{1-\tau}\boldsymbol{\kappa}(\omega)}\mathcal{F}_{y\to\omega}\big(e^{-y^{\mathrm{tr}}(A-B)y+\alpha y+i\beta y}\big)\big). \end{aligned} \tag{4.15}$$

We start by computing $\mathcal{F}_{y\to\omega}\big(e^{-y^{\mathrm{tr}}(A-B)y+\alpha y+i\beta y}\big)$. As we have already remarked, since $f\in\mathcal{S}'(\mathbb{R}^d)$, we have that $e^{-y^{\mathrm{tr}}(A-B)y+\alpha y+i\beta y}=e^{-c-ia}e^{-\frac{\tau}{1-\tau}\boldsymbol{\kappa}(D)}f(y)$ is a C^∞ slowly increasing function; then $A-B$ must be positive semidefinite. So we can diagonalize it, obtaining $A-B=V^{\mathrm{tr}}\operatorname{diag}(\lambda_j)V$, where $\lambda_j\geq 0$ are the eigenvalues of $A-B$. Then, by the change of variables $s=Vy$ we obtain:

$$\begin{aligned} \mathcal{F}_{y\to\omega}\big(e^{-y^{\mathrm{tr}}(A-B)y+\alpha y+i\beta y}\big) &= \int e^{-2\pi iy\omega}e^{-(Vy)^{\mathrm{tr}}\operatorname{diag}(\lambda_j)(Vy)+\alpha y+i\beta y}\,dy\\ &= \frac{1}{|\det V|}\int e^{-2\pi is(V\omega)}e^{-s^{\mathrm{tr}}\operatorname{diag}(\lambda_j)s+\mu s+i\nu s}\,ds, \end{aligned} \tag{4.16}$$

where $\mu=\alpha V^{-1}$ and $\nu=\beta V^{-1}$, with $\mu,\nu\in\mathbb{R}^d$. Now let us observe that $s^{\mathrm{tr}}\operatorname{diag}(\lambda_j)s=\lambda_1 s_1^2+\cdots+\lambda_d s_d^2$; we can suppose without loss of regularity that $\lambda_j=0$ implies that $\lambda_n=0$ for every $n\geq j$, otherwise it is enough to make a change of variables in the integral in (4.16). Then there exists $k\in\mathbb{N}$, $0\leq k\leq d$, such that $\lambda_j>0$ for every $j=1,\ldots,k$ and $\lambda_{k+1}=\cdots=\lambda_d=0$ (where we mean that all λ_j vanish if $k=0$ and all λ_j are strictly positive if $k=d$). Now, since the left-hand side in (4.16) is a tempered distribution we must have $\mu_{k+1}=\cdots=\mu_d=0$, too; we then obtain

$$\begin{aligned} &\mathcal{F}_{y\to\omega}\big(e^{-y^{\mathrm{tr}}(A-B)y+\alpha y+i\beta y}\big) =\\ &\quad= \frac{1}{|\det V|}\int e^{-2\pi is(V\omega-\frac{\nu}{2\pi})}e^{-(\lambda_1 s_1^2+\cdots+\lambda_k s_k^2)+\mu_1 s_1+\cdots+\mu_k s_k}\,ds. \end{aligned}$$

We now introduce the following notation: for $\omega=(\omega_1,\ldots,\omega_d)\in\mathbb{R}^d$ we split

$$\omega=\big(\omega_{(1)},\omega_{(2)}\big) \text{ with } \omega_{(1)}=(\omega_1,\ldots,\omega_k) \text{ and } \omega_{(2)}=(\omega_{k+1},\ldots,\omega_d). \tag{4.17}$$

Then, since $\int e^{-2\pi it\omega}e^{-t^2}\,dt=\pi^{d/2}e^{-\pi^2\omega^2}$, we obtain (with the notations (2.4))

$$\mathcal{F}_{y\to\omega}\big(e^{-y^{\mathrm{tr}}(A-B)y+\alpha y+i\beta y}\big)=c_1\,\mathcal{U}_V\,T_{\frac{\nu}{2\pi}}\big(F(\omega_{(1)})\otimes\delta_{\omega_{(2)}}\big), \tag{4.18}$$

where

$$F(\omega_{(1)}) = e^{-\pi i(\frac{\mu_1}{\lambda_1}\omega_1+\cdots+\frac{\mu_k}{\lambda_k}\omega_k)}e^{-\pi^2(\frac{\omega_1^2}{\lambda_1}+\cdots+\frac{\omega_k^2}{\lambda_k})},$$

$\delta_{\omega_{(2)}}$ is the Dirac distribution in the $\omega_{(2)}$-variable and c_1 is a real positive constant given by $c_1 = \frac{1}{|\det V|^{3/2}}\frac{\pi^{k/2}}{\sqrt{\lambda_1\ldots\lambda_k}}e^{\frac{1}{4}(\frac{\mu_1^2}{\lambda_1}+\cdots+\frac{\mu_k^2}{\lambda_k})}$. We then have by (4.15) and (4.18) that

$$f(t) = c_2\int e^{2\pi it\omega}e^{\frac{\tau}{1-\tau}\boldsymbol{\kappa}(\omega)}\Big[\mathcal{U}_V\, T_{\frac{\nu}{2\pi}}\big(F(\omega_{(1)})\otimes\delta_{\omega_{(2)}}\big)\Big]\,d\omega,$$

where $c_2 = c_1e^{c+ia}$. We now make the change of variables $V\omega - \frac{\nu}{2\pi} = \zeta$ in the integral; since $F(\omega)\cdot\delta_{\omega_{(2)}} = F(\omega_{(1)},0)\otimes\delta_{\omega_{(2)}}$ for every C^∞ function $F(\omega)$ we then obtain

$$f(t) = c_3e^{2\pi i\gamma t}\int e^{2\pi iVt\zeta}\Big[e^{-\zeta_{(1)}^{\mathrm{tr}}R\zeta_{(1)}+\tilde{\mu}\zeta_{(1)}}e^{-2\pi i\tilde{\nu}\zeta_{(1)}}\otimes\delta_{\zeta_{(2)}}\Big]\,d\zeta, \tag{4.19}$$

where $\gamma\in\mathbb{R}^d$, $\tilde{\mu},\tilde{\nu}\in\mathbb{R}^k$, c_3 is a constant of the kind $c_3 = c_4e^{ia}$ with $c_4>0$, and R is a $k\times k$ matrix which depends on the quadratic form $\boldsymbol{\kappa}(\omega)$ in (4.15). Now, since $f\in\mathcal{S}'(\mathbb{R}^d)$, R in (4.19) must be positive semidefinite. Then we can diagonalize R, finding $R = S^{\mathrm{tr}}\,\mathrm{diag}(\tilde{\lambda}_j)S$, where $\tilde{\lambda}_j$, $j=1,\ldots,k$ are the eigenvalues of R, with $\tilde{\lambda}_j\geq 0$ for every $j=1,\ldots,k$. Then we can proceed as in the computation of (4.16); assuming that $\tilde{\lambda}_1,\ldots,\tilde{\lambda}_h>0$, $\tilde{\lambda}_{h+1},\ldots,\tilde{\lambda}_k=0$ we then split the variable $\zeta_{(1)}$ similarly as in (4.17). By convenience we then consider the following notation: for $t\in\mathbb{R}^d$ we split

$$t = \big(t_{[1]},t_{[2]},t_{[3]}\big)\text{ with } t_{[1]} = (t_1,\ldots,t_h),\ t_{[2]} = (t_{h+1},\ldots,t_k),\ t_{[3]} = (t_{k+1},\ldots,t_d),$$

where we have just renamed $t_{(2)}$ as $t_{[3]}$ and we have split $t_{(1)} = \big(t_{[1]},t_{[2]}\big)$. From (4.19) and the same computations as before we then get

$$f(t) = c_5\,\mathcal{U}_D\,T_\theta\big[e^{2\pi i\sigma t}\big(e^{-t_{[1]}^2}\otimes\delta_{t_{[2]}}\otimes\mathbf{1}_{t_{[3]}}\big)\big] \tag{4.20}$$

where $D\in GL(d,\mathbb{R})$, $\theta,\sigma\in\mathbb{R}^d$ and $c_5 = c_fe^{ia}$ with $c_f\geq 0$ and $a\in\mathbb{R}$; the notation $\mathbf{1}_{t_{[3]}}$ stands for the function identically 1 in the $t_{[3]}$-variable. Then we have proved that f is of the form (2.5). As for the function g, by (4.14) and the same computations as above we get (2.6).

In order to complete the proof of Theorem 2.3 it remains to show that if f and g are of the form (2.5)-(2.6) then the corresponding τ-Wigner transform $Wig_\tau(f,g)$ is a positive distribution. By Proposition 3.3 it is enough to prove that

$$Wig_\tau\big(e^{c_f+ia}e^{-t_{[1]}^2}\otimes\delta_{t_{[2]}}\otimes\mathbf{1}_{t_{[3]}},e^{c_g+ia}e^{-\frac{\tau}{1-\tau}t_{[1]}^2}\otimes\delta_{t_{[2]}}\otimes\mathbf{1}_{t_{[3]}}\big) \tag{4.21}$$

is a positive distribution. We have:

$$\begin{aligned}&Wig_\tau\big(e^{c_f+ia}e^{-t_{[1]}^2}\otimes\delta_{t_{[2]}}\otimes\mathbf{1}_{t_{[3]}},e^{c_g+ia}e^{-\frac{\tau}{1-\tau}t_{[1]}^2}\otimes\delta_{t_{[2]}}\otimes\mathbf{1}_{t_{[3]}}\big)=\\&\quad=e^{c_f+c_g}Wig_\tau\big(e^{-t_{[1]}^2},e^{-\frac{\tau}{1-\tau}t_{[1]}^2}\big)\otimes Wig_\tau\big(\delta_{t_{[2]}},\delta_{t_{[2]}}\big)\otimes Wig_\tau\big(\mathbf{1}_{t_{[3]}},\mathbf{1}_{t_{[3]}}\big),\end{aligned}$$

where the Wig_τ are intended as distributions in the $(x_{[j]},\omega_{[j]})$ variables, $j=1,2,3$, respectively. Now by (2.2) we have immediately

$$Wig_\tau\big(\mathbf{1}_{t_{[3]}},\mathbf{1}_{t_{[3]}}\big)=\mathbf{1}_{x_{[3]}}\otimes\delta_{\omega_{[3]}}; \tag{4.22}$$

moreover, from Proposition 3.4 and (4.22) it follows that

$$Wig_\tau\big(\delta_{t_{[2]}},\delta_{t_{[2]}}\big)(x_{[2]},\omega_{[2]})=Wig_{1-\tau}\big(\mathbf{1}_{t_{[2]}},\mathbf{1}_{t_{[2]}}\big)(-\omega_{[2]},x_{[2]})=\delta_{x_{[2]}}\otimes\mathbf{1}_{\omega_{[2]}}.$$

By Lemma 3.6 we finally have

$$\begin{aligned}&Wig_\tau\big(e^{c_f+ia}e^{-t_{[1]}^2}\otimes\delta_{t_{[2]}}\otimes\mathbf{1}_{t_{[3]}},e^{c_g+ia}e^{-\frac{\tau}{1-\tau}t_{[1]}^2}\otimes\delta_{t_{[2]}}\otimes\mathbf{1}_{t_{[3]}}\big)=\\&\quad=e^{c_f+c_g}|\det C|\pi^{h/2}e^{-\frac{1}{1-\tau}x_{[1]}^2}e^{-\pi^2(C\omega_{[1]})^2}\otimes\delta_{x_{[2]}}\otimes\mathbf{1}_{\omega_{[2]}}\otimes\mathbf{1}_{x_{[3]}}\otimes\delta_{\omega_{[3]}},\end{aligned}\tag{4.23}$$

and so (4.21) is a positive distribution. □

By Proposition 3.3 we have that $Wig_\tau(f,g)$, for f,g as in (2.5), (2.6), is obtained by applying translation and linear change of variables to the expression appearing in (4.23). This observation, together with (2.5), (2.6) and (2.7), gives rise to the following corollaries of Theorem 2.3.

Corollary 4.2. *Let us suppose that $f,g\in\mathcal{S}'(\mathbb{R}^d)$. We have:*

(a) *$Wig_\tau(f,g)$ belongs to $L^p(\mathbb{R}^{2d})$, $p\in[1,\infty]$, and satisfies $Wig_\tau(f,g)(x,\omega)\geq 0$ in $\mathbb{R}^{2d}$, if and only if f and g are as in Theorem* 2.3 *with $h=k=d$ (which means that $t_{[2]}$ and $t_{[3]}$ are empty, i.e. f and g are as in Theorem* 2.1*).*

(b) *The same conclusion as in* (a) *holds if we require that $Wig_\tau(f,g)$ belongs to $L^1_{\text{loc}}(\mathbb{R}^{2d})$ and satisfies $Wig_\tau(f,g)(x,\omega)\geq 0$.*

Corollary 4.3. *If $f,g\in L^p(\mathbb{R}^d)$, $1\leq p<\infty$, we have that $Wig_\tau(f,g)$ is a positive distribution if and only if f and g are as in Theorem* 2.3 *with $h=k=d$ (which means, as before, that f and g are in fact as in Theorem* 2.1*).*

Corollary 4.4. *If $f,g\in L^\infty(\mathbb{R}^d)$, or $f,g\in L^1_{\text{loc}}(\mathbb{R}^d)\cap\mathcal{S}'(\mathbb{R}^d)$, we have that the corresponding $Wig_\tau(f,g)$ is a positive distribution if and only if f and g are as in Theorem* 2.3 *with $h=k$ (which means that $t_{[2]}$ is empty, while both $t_{[1]}$ and $t_{[3]}$ may be non empty).*

Corollary 4.5. *Let us consider again $f, g \in \mathcal{S}'(\mathbb{R}^d)$. Then $Wig_\tau(f,g)$ is a strictly positive distribution if and only if f and g are as in Theorem* 2.3 *with $h = k = d$.*

Observe in particular that, in the case of tempered distributions, $Wig_\tau(f,g)$ positive is not any more equivalent to $Wig_\tau(f,g)$ strictly positive, as it was in the case of L^2, cf. Remark 4.1; in particular the only tempered distributions f, g that make the τ-Wigner transform strictly positive are the gaussians (2.3).

5 Applications to Pseudo-differential Operators

The connections between the Wigner form and the pseudo-differential calculus, in particular the Weyl operators, have their roots in two basic formulas which shall be the starting point of our observations. The first of them is the equality

$$(W^a f, g) = (a, Wig(g,f)), \tag{5.1}$$

(for simplicity suppose $f, g \in S(\mathbb{R}^d)$, $a \in S(\mathbb{R}^{2d})$ but many generalizations are possible) where W^a is the pseudo-differential operator with Weyl symbol $\sigma^W = a$ defined by

$$f \in \mathcal{S}(\mathbb{R}^d) \longrightarrow W^a f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\omega} a\left(\frac{x+y}{2}, \omega\right) f(y)\, dy\, d\omega \in \mathcal{S}(\mathbb{R}^d),$$

(The literature related to this subject is very vast, see e.g. [8], [12], [13], [14], [17]).

In [2] it is showed how equality (5.1) is actually a particular case of a more general correspondence between sesquilinear forms and pseudo-differential quantizations, involving many well-known types of time-frequency representations and classes of pseudo-differential operators.

The second basic formula is a link between rank one operators and their Weyl symbol. Namely suppose that $P : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is a rank one operator, i.e. there exist $\phi, \psi \in L^2(\mathbb{R}^d)$ such that $Pf = (f, \phi)\psi \in L^2(\mathbb{R}^d)$, then its Weyl symbol $\sigma^W(P)$ is the Wigner transform of ψ and ϕ, i.e.

$$\sigma^W(P) = Wig(\psi, \phi). \tag{5.2}$$

This formula gives a reason of the fact that *localization operators* (see e.g. [18]), which are "means" of rank one operators weighted with respect to a symbol $a(x, \omega)$, i.e. maps of the type

$$f \in \mathcal{S}(\mathbb{R}^d) \longrightarrow L^a_{\phi,\psi} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega)(f, \phi_{x,\omega})\psi_{x,\omega}(t)\, dt \in \mathcal{S}(\mathbb{R}^d), \tag{5.3}$$

with $\phi_{x,\omega} = e^{2\pi i x,\omega}\phi(t-x)$, $\psi_{x,\omega} = e^{2\pi i x,\omega}\psi(t-x)$, have as Weyl symbol the convolution

$$\sigma^W(L^a_{\phi,\psi}) = a * Wig(\psi, \phi). \tag{5.4}$$

These operators are used as *filters* in signal analysis and are a particular type of pseudo-differetial operator of Weyl type, see e.g. [2] for more details.

We shall rely on (5.1), (5.2) and (5.4) for our observations. We start however with an immediate consequence of Theorem 2.1, which does not have a correspondence for the classical Wigner form:

Corollary 5.1. *If $\tau \neq 1/2$ then there exist no functions $f \in L^2(\mathbb{R}^d)$ for which $Wig_\tau(f)$ is everywhere positive.*

Proof. If we assume that $Wig_\tau(f)$ is positive, then, setting $f = g$ in (2.3) for every $t \in \mathbb{R}^d$, it implies $\tau = 1/2$. □

We examine now some applications of the results of section 2 to pseudo-differential operators. We begin by recalling a natural generalization of the Weyl quantization, namely the $\tau-$Weyl quantization, which associates the operator

$$W_\tau^a : f \in \mathcal{S}(\mathbb{R}^d) \to W_\tau^a f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\omega} a\left(\tau x + (1-\tau)y, \omega\right) f(y)\, dy\, d\omega \in \mathcal{S}(\mathbb{R}^d)$$

with a "$\tau-$symbol" $a \in S(\mathbb{R}^{2d})$. Extensions to more general domains and symbol classes are defined as usual (see [12] for a standard reference).

Consider now operators of rank one in $L^2(\mathbb{R}^d)$, i.e. operators of the form $P_{\phi,\psi}u = (u,\phi)\psi$ for fixed $\phi, \psi \in L^2(\mathbb{R}^d)$. A straightforward computation and the use of the orthogonality formula (3.1) for $\tau-$Wigner forms leads to the equality:

$$(P_{\phi,\psi}f, g) = (Wig_\tau(\psi,\phi), Wig_\tau(g,f)), \qquad f, g \in L^2(\mathbb{R}^d). \tag{5.5}$$

This formula, generalizing (5.2), shows that the $\tau-$symbol of a rank one operator $P_{\phi,\psi}$ is the $\tau-$Wigner transform of ϕ and ψ, i.e.

$$\sigma_\tau(P_{\phi,\psi}) = Wig_\tau(\psi,\phi).$$

We use this fact in a few considerations about positivity of operators and symbols. Namely, for the $\tau-$Weyl quantization, just as for the classical Weyl quantization, there are no connections between positivity of the symbol and positivity of the operator. We point out this fact even with rank one operator in the following proposition whose proof is an immediate consequence of our results on the positivity of the $\tau-$Wigner. We recall that some complementary consequence of Hudson type theorems for rank one operators in the case of Weyl quantization $\tau = 1/2$ can be found in [15].

Proposition 5.2. *i) For $f, g \in L^2(\mathbb{R}^d)$ as in (2.3), let $P_{f,g}u = (u,f)g$ be the corresponding rank one operator. As f and g are linearly independent, the operator P is not positive, however for every $\tau \neq 1/2$ we have $\sigma_\tau(P_{f,g}) > 0$ everywhere on $\mathbb{R}^{2d}$.*

ii) Let $\phi \in L^2(\mathbb{R}^d)$, $\|\phi\|_2 = 1$, then $P_{\phi,\phi}$ is an orthogonal projection and therefore a positive operator, however for every $\tau \neq 1/2$ there does not exits $\phi \in L^2(\mathbb{R}^d)$ for which $\sigma_\tau(P_{\phi,\phi})$ is everywhere positive on $\mathbb{R}^{2d}$.

We consider next *localization operators* defined as in (5.3). As usual the spaces of symbols and windows, and the domain of the operators can be suitably generalized. It is not our aim to present here the most general setting but we observe that if symbol and windows are square integrable then we have bounded operators on L^2.

The connection between the localization operators and the $\tau-$Weyl quantization, well-known for $\tau = 1/2$, is generalized as follows (for simplicity we suppose that every function is in L^2).

Proposition 5.3. *A localization operator $L^a_{\phi,\psi}$ with symbol a and windows ϕ, ψ can be expressed as τ-Weyl operator according to the formula*

$$L^a_{\phi,\psi} = W^b_\tau$$

where $b = a * Wig_\tau(\psi, \phi)$.

Proof. First of all remark that, as Wig_τ is L^2-bounded, b is well-defined as a function of $L^\infty(\mathbb{R}^{2d})$. We use now some results from [2] and [3] connecting pseudo-differential operators and time-frequency representations, namely we have that for $u, v \in L^2(\mathbb{R}^d)$ the following equalities hold

$$\begin{aligned} (W^b_\tau u, v) &= (b, Wig_\tau(v, u)), \\ (L^a_{\phi,\psi} u, v) &= (a, Sp_{\psi,\phi}(v, u)) \end{aligned}$$

where the the *double-window spectrogram* $Sp_{\psi,\phi}(v, u)(x, \omega) = (v, \psi_{x,\omega})\overline{(u, \phi_{x,\omega})}$ can be expressed as element of the Cohen class by

$$Sp_{\psi,\phi}(v, u) = Wig_{1-\tau}(\tilde{\psi}, \tilde{\phi}) * Wig_\tau(v, u)$$

(with $\tilde{F}(x) = F(-x)$), see [2], Prop. 2.5 (iii).

We have then

$$\begin{aligned} (W^b_\tau u, v) &= (a, \widetilde{\overline{Wig_\tau}}(\psi, \phi) * Wig_\tau(v, u)) \\ &= (a, \widetilde{Wig_{1-\tau}(\phi, \psi)} * Wig_\tau(v, u)) \\ &= (a, Wig_{1-\tau}(\tilde{\phi}, \tilde{\psi}) * Wig_\tau(v, u)) \\ &= (a, Sp_{\psi,\phi}(v, u)) \quad = \quad (L^a_{\phi,\psi} u, v). \end{aligned}$$

which proves the thesis.

□

We conclude by showing how suitable localization operators yield examples of positive operators with non positive $\tau-$Weyl symbols.

Proposition 5.4. *For every $\tau \neq 1/2$ and every $\phi \in L^2(\mathbb{R}^d)\backslash\{0\}$ there exists a positive localization operators $L^a_\phi := L^a_{\phi,\phi}$, with window ϕ and suitable symbol $a \in L^2(\mathbb{R}^{2d})$, such that the $\tau-$Weyl symbol σ_τ of L^a_ϕ is not everywhere positive.*

Proof. It is well-known and easy to verify that localization operators with one window

$$L^a_\phi u = \int_{\mathbb{R}^{2d}} a(z)(f, \phi_z)_{L^2}\, \phi_z\, dz$$

are positive if $a(z) \geq 0$ a.e.

Let $\phi \in L^2(\mathbb{R}^d)$, $\phi \neq 0$, then from Theorem 2.1 we have that $Wig_\tau(\phi)$ is not everywhere positive. Suppose that $z_0 = (x_0, \omega_0)$ is a point such that $Wig_\tau(\phi)(z_0)$ is not positive. Then, as $Wig_\tau(\phi)$ is a continuous function, either

$$\Re(Wig_\tau(\phi))(z) < 0, \quad \text{or} \quad \Im(Wig_\tau(\phi))(z) \neq 0 \tag{5.6}$$

in a neighborhood $z_0 + B^\epsilon_0$ of z_0, (B^ϵ ball of radius ϵ). Suppose that $a \in L^2(\mathbb{R}^{2d})\backslash\{0\}$ satisfies $a \geq 0$ and supp $a \subseteq -B^\epsilon_0$. From Proposition 5.3 the operator L^a_ϕ has $\tau-$Weyl symbol $b = a * Wig_\tau(\phi)$ and we have

$$(a * Wig_\tau(\phi))(z) = \int_{B^\epsilon_0} Wig_\tau(\phi)(z + w)a(-w)\, dw$$

and therefore condition (5.6) holds for b, which proves the assertion.

□

(From the proof it is clear that the same happens with the Weyl symbol with the unique exclusion of the case where ϕ is of gaussian type as specified in Theorem 2.1, (ii))

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