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Abstract This paper analyzes preferences in the presence of ambiguity that are *rational* in the sense of satisfying the classical ordering condition as well as monotonicity. Under technical conditions that are natural in an Anscombe–Aumann environment, we show that even for such a general preference model, it is possible to identify a set of priors, as first envisioned by Ellsberg (Q J Econ 75:643–669, 1961). We then discuss ambiguity attitudes, as well as unambiguous acts and events, for the class of rational preferences we consider.

Section 6 of this paper subsumes an earlier paper of Ghirardato, Maccheroni, and Marinacci presented and circulated under the title "Revealed Ambiguity and Its Consequences: Unambiguous Acts and Events." We are grateful to the participants of RUD 2005 conference for comments on that paper, and to the participants of the Ellsberg Symposium (Vienna, May 2010) and a referee for comments on this paper. We thank Peter Klibanoff for very useful suggestions.

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1 Introduction

Ellsberg's seminal paper (1961) ignited a large and growing literature aimed at developing decision models that accommodate a concern for ambiguity, which is uncertainty about the precise stochastic nature of the problem a decision maker faces (for a recent survey, see Gilboa and Marinacci 2010). Among the first and most prominent contributions, Schmeidler's (1989) axiomatization of Choquet-expected utility (CEU) and Gilboa and Schmeidler's (1989) foundations for maxmin-expected utility (MEU) with multiple priors occupy a special place. Furthermore, applications in several areas of economics have demonstrated their usefulness.

More recently, several influential works have proposed decision models that overcome specific limitations of the CEU and MEU models. Two behavioral aspects have received special attention. First, both the CEU and the MEU model satisfy *Certainty Independence*: the main implication of this axiom is that preferences and, in particular, ambiguity attitudes are unaffected by changes in the "scale" and "location" of utilities. To fix ideas, suppose the decision maker has a linear utility, and assume that he is just indifferent between receiving \$3 dollars for sure, and participating in a bet that yields \$10 dollars if an (ambiguous) event obtains, and 0 otherwise. Then, Certainty Independence also implies that the decision maker would be indifferent: (i) between receiving \$300 for sure, and participating in a bet that yields \$1,000 if the event obtains and 0 otherwise and also (ii) between receiving \$1,003 for sure and participating in a bet that yields \$1,010 if the event obtains and \$1,000 otherwise. Analogies with choice under risk suggest that decision makers may reasonably violate either one or both of these conclusions.

Second, the MEU model is characterized by a specific form of dislike for ambiguity, formalized by the "Uncertainty Aversion" axiom due to Schmeidler's (1989). This axiom, which delivers quasi-concavity of the functional representing preferences, is an assumption shared by many classes of preferences discussed in the literature. It reveals a form of dislike for ambiguity even outside the MEU realm; see Cerreia-Vioglio et al. (2008, C3M henceforth). At the same time, Uncertainty Aversion imposes restrictions on preferences which one may want to dispense with; see Ghirardato and Marinacci (2002, GM henceforth) for a theoretical discussion or Baillon et al. (forthcoming) for an experimental perspective.

Recent decision-theoretic models relax the Certainty Independence and Uncertainty Aversion axioms in specific ways. For instance, Variational preferences (Maccheroni et al. 2006) relax invariance to the scale of utilities, but retain invariance to their location, as well as Uncertainty Aversion. The "Uncertainty Averse" preferences of C3M satisfy the eponymous axiom, but not necessarily Certainty Independence. Grant and Polak (2007) drop Certainty Independence, and weaken Uncertainty Aversion. Finally, Siniscalchi (2009) retains invariance to the location of utilities, but drops scale invariance, as well as Uncertainty Aversion.

In this paper, *both* Certainty Independence and Uncertainty Aversion are dropped. We consider preferences that only satisfy what in our view are the basic tenets of rationality under ambiguity: *weak order* and *monotonicity*. We call these preferences *rational*. Since they are a weak order, they are rational in the usual sense of utility theory. At the same time, the monotonicity assumption guarantees consistency with state-wise dominance, which is arguably not affected by considerations of ambiguity. All the models discussed above, and several others belong to this class of preferences.[1] In particular, this class includes both the Uncertainty Averse preferences of C3M and the MBC preferences introduced by Ghirardato and Siniscalchi (2010, GS henceforth).

We first argue that, for such preferences, a set of priors can be obtained following the approach of Ghirardato et al. (2004, GMM henceforth); i.e., as a representation of the derived unambiguous preference relation.[2] Thus, in a specific behavioral sense, one can identify probabilities that are significant for the decision maker's choices regardless of the representation of her preferences, which following GS we call "relevant priors." We carry on this task in an Anscombe–Aumann setting, and under two additional assumptions: Risk Independence and Archimedean continuity. We call the rational preferences satisfying these additional axioms *MBA preferences* (for *M*onotonic, *B*ernoullian, and *A*rchimedean). We thus directly generalize the results of GMM and Nehring (2002).

We then leverage this general representation result to analyze the individual's perception of ambiguity and her attitudes toward it. MBA preferences provide a relatively "neutral" ground for the study of these issues, precisely because they do not incorporate any *specific* assumption about invariance and/or attitudes toward ambiguity. First, we show that MBA preferences admit a "generalized Hurwicz" (or α-MEU) representation, thus extending an analogous result established by GMM for preferences satisfying Certainty Independence. This representation provides a useful tool to study, for instance, comparative ambiguity attitudes. Second, we discuss two different notions of ambiguity aversion and the relations between them. Third, we propose behavioral definitions of unambiguous acts and consequently unambiguous events and characterize them in terms of the set of priors we identify. Finally, we offer some consequences of the previous definitions. In particular, we extend a result of Marinacci (2002) on probabilistic sophistication with multiple priors to MBA preferences.

Related literature

As outlined above, the main contributions of this paper are the following: (1) showing that the (arguably) mild rationality assumptions for choice under ambiguity guarantee the existence of a set of priors, first envisioned by Ellsberg (1961) and modeled in the seminal papers of Gilboa and Schmeidler's (1989) and Bewley (2002); (2) the

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[1] Grant and Polak (2007) also consider a version of their model that relaxes monotonicity.

[2] Nehring (2001) and Gilboa et al. (2010) derive a set of priors from a *separate* relation, which they interpret as embodying "objective rationality," and impose consistency conditions between such relation and the decision maker's preference relation.

discussion of ambiguity attitudes in such general context; and (3) the characterization of unambiguous acts and events and consequences thereof.

With respect to the first contribution, our debt to the GMM paper is obvious. The added contribution here is clearly in observing how (most of) the representation results of that paper generalize to rational preferences which do not satisfy Certainty Independence, but only Risk Independence. The contributions of C3M and GS are on the other hand complementary to this paper. The main focus of C3M is the analysis of rational preferences that also satisfy Schmeidler's "Uncertainty Aversion" axiom. The main focus of the GS paper is the characterization of the set of relevant priors for popular preference models. Such characterizations hinge on a differential result that requires a stronger continuity condition and thus applies only to a subset of MBA preferences, which GS dub MBC (where C stands for "[Cauchy] continuous"). C3M also characterizes the set of relevant priors in several ways and provides a differential characterization different from the one in GS.

The discussion on ambiguity attitudes is also related to earlier work. We show how the ideas in Ghirardato and Marinacci (2002) can be extended to the MBA class of preferences. We refer to that paper for detailed discussion on the relation of such vision of ambiguity aversion to those espoused in other papers, in particular Schmeidler's (1989) and Epstein (1999).

As to this paper's third contribution, this paper fits within a well-established literature. Early attempts to characterize behaviorally ambiguity were focussed on ambiguity of events in specific preference models. Such is the case of Nehring (1999) and Zhang (2002), which consider CEU preferences. Subsequently, Epstein and Zhang (2001) and Nehring (2001) offered definitions of unambiguous event which apply in principle to any preference, providing a characterization over rich state spaces. Similarly, Klibanoff et al. (2005) propose a notion of unambiguous event and characterize it within the context of their "Smooth Ambiguity" model. We refer the reader to Sect. 6.3, and to Nehring (2006), Amarante and Filiz (2007), and Klibanoff et al. (2011) for discussion on the relations between the definitions of unambiguous event in the cited papers and the one presented here. To the best of our knowledge, the only previous paper that provides a definition of unambiguous *act* as primitive, and events as derivative, is GM. However, their definition only applies to preferences that are ambiguity *averse* (or loving) according to the definition in that paper. For such preferences, the definition of unambiguous act offered in the two papers can be shown to coincide.

Finally, some of the consequences that we draw from our definitions of ambiguity owe to previous work, and our debts and contributions are clearly identified in the respective sections.

2 Notation and preliminaries

We consider a state space S, endowed with an algebra Σ. $B_0(\Sigma, \Gamma)$ indicates the set of simple Σ-measurable functions on S with values in the interval $\Gamma \subset \mathbb{R}$. We endow $B_0(\Sigma, \Gamma)$ with the topology induced by the sup-norm. For simplicity, we write $B_0(\Sigma)$ instead of $B_0(\Sigma, \mathbb{R})$.

The set of finitely additive probabilities on Σ is denoted $ba_1(\Sigma)$. We endow $ba_1(\Sigma)$ with the (relative) weak* topology that is the topology induced by $B_0(\Sigma)$. Notice that $ba_1(\Sigma)$ is compact under this topology.

We say that a functional $I : B_0(\Sigma, \Gamma) \to \mathbb{R}$ is:

- **monotonic** if $I(a) \geq I(b)$ provided $a \geq b$
- **continuous** if it is sup-norm continuous
- **normalized** if $I(\alpha\, 1_S) = \alpha$ for all $\alpha \in \Gamma$

Let X be a convex subset of a vector space. (Simple) acts are Σ-measurable functions $f : S \to X$ such that $f(S) = \{f(s) : s \in S\}$ is finite. The set of all acts is denoted by $\mathscr{F}$. We define mixtures of acts pointwise; that is, for each $\alpha \in [0, 1]$, $\alpha f + (1-\alpha)g$ is the act that delivers the prize $\alpha f(s) + (1-\alpha)g(s)$ in state s. Given $f, g \in \mathscr{F}$ and $A \in \Sigma$, we denote by $f\,A\,g$ the act in $\mathscr{F}$, which yields $f(s)$ for $s \in A$ and $g(s)$ for $s \in A^c$.

3 Rational preferences and relevant priors: characterizations

In this section, we first briefly introduce our basic assumptions on preferences, characterizing what we earlier dubbed the "MBA" model. We refer the reader to C3M and GS for more detailed discussion of the axioms. We then argue that for MBA preferences, the unambiguous preference relation introduced by GMM can be used to obtain a set of relevant probabilistic models for the decision maker. This identifies a set of probabilities that we refer to as the set of *relevant priors*.

3.1 Axioms

The main object of study is a binary relation $\succsim$ on $\mathscr{F}$. As usual, $\succ$ (resp. $\sim$) denotes the asymmetric (resp. symmetric) component of $\succsim$. With a small abuse of notation, we denote with the same symbol the prize x and the constant act that delivers x for all s.

Axiom 1 (Weak Order) *The relation $\succsim$ is non-trivial, complete, and transitive on $\mathscr{F}$.*

Axiom 2 (Monotonicity) *If $f, g \in \mathscr{F}$ and $f(s) \succsim g(s)$ for all $s \in S$ then $f \succsim g$.*

The above axioms define the class of *rational preferences*. The next two axioms are tailored to the Anscombe–Aumann setup we consider.

Axiom 3 (Risk Independence) *If $x, y, z \in X$ and $\lambda \in (0, 1]$ then $x \succ y$ implies $\lambda x + (1-\lambda)z \succ \lambda y + (1-\lambda)z$.*

Axiom 4 (Archimedean) *If $f, g, h \in \mathscr{F}$ and $f \succ g \succ h$, then there are $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h$.*

The above four axioms imply the existence of:

- a *Bernoulli utility index* on X, that is, a utility function $u : X \to \mathbb{R}$ which is affine and represents the restriction of $\succsim$ to X;

- the existence of *certainty equivalents* x_f for all acts $f \in \mathscr{F}$.

A binary relation $\succsim$ on $\mathscr{F}$ that satisfies Axioms 1–4 will henceforth be called an **MBA preference** (for **M**onotonic, **B**ernoullian, **A**rchimedean).

We now provide a basic representation result for binary relations satisfying the above axioms. It generalizes previous results of Gilboa and Schmeidler's (1989), GMM, Maccheroni et al. (2006), C3M, and GS, which all impose more stringent axiomatic requirements on preferences.

Proposition 1 *A binary relation $\succsim$ satisfies Axioms 1–4 if and only if there exists a non-constant, affine function $u : X \to \mathbb{R}$ and a normalized, monotonic, continuous functional $I : B_0(\Sigma, u(X)) \to \mathbb{R}$ such that for each $f, g \in \mathscr{F}$*

$$f \succsim g \Longleftrightarrow I(u \circ f) \geq I(u \circ g). \tag{1}$$

Moreover, if (I_v, v) also satisfies Eq. (1) and $I_v : B_0(\Sigma, v(X)) \to \mathbb{R}$ is normalized, then there exist $\lambda, \mu \in \mathbb{R}$ with $\lambda > 0$ such that $v(x) = \lambda u(x) + \mu$ for all $x \in X$ and $I_v(b) = \lambda I(\lambda^{-1}[b - \mu]) + \mu$ for all $b \in B_0(\Sigma, v(X))$.

Observe that differently from Lemma 1 in GMM, the functional I is not necessarily constant-linear.[3] I therefore depends upon the choice of utility function (see Ghirardato et al. 2005). On the other hand, due to its normalization, I is uniquely determined by u and the equality $I(u(f)) = I\left(u\left(x_f\right) 1_S\right) = u\left(x_f\right)$.

3.2 Relevant priors and unambiguous preferences

We now recall GMM's notion of "unambiguous preference" relation (see also Nehring 2007). Although it is defined under more general assumptions on preferences, such relation has the same interpretation as in GMM: since ambiguity sensitivity may lead to violations of the Anscombe–Aumann independence axiom, we look for rankings that are not reversed by mixtures.

Definition 1 Let $f, g \in \mathscr{F}$. We say that f is **unambiguously preferred to** g, denoted by $f \succsim^* g$, if and only if, for each $h \in \mathscr{F}$ and each $\lambda \in (0, 1]$, $\lambda f + (1 - \lambda)h \succsim \lambda g + (1 - \lambda)h$.

It is not hard to verify that the relation $\succsim^*$ enjoys the properties identified by GMM (see GMM Propositions 4 and 5), and hence, as in GMM, it admits a representation à la Bewley (2002) (see GMM Proposition A.2):[4]

[3] A functional I on $B_0(\Sigma)$ is said to be **constant-linear** if $I(\alpha a + \beta 1_S) = \alpha I(a) + \beta$ for all $a \in B_0(\Sigma)$ and for all $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$.

[4] Here and henceforth, for results that are routine extension of existing results, we omit the proof and provide a reference to the existing result.

Proposition 2 (GMM, Propositions 4 and 5) *Let* $\succcurlyeq$ *be an MBA preference. Then, there exists a non-empty, unique, convex, and closed set* $C \subset ba_1(\Sigma)$ *such that for each* $f, g \in \mathscr{F}$,

$$f \succcurlyeq^* g \iff \int u \circ f \, dP \geq \int u \circ g \, \mathrm{d}P \quad \textit{for all } P \in C,$$

where u *is the function obtained in Proposition 1. Moreover,* C *is independent of the choice of normalization of* u.

The last sentence—which follows from the structure of the Bewley representation—shows that C is cardinally invariant even though I is not.

Thus, the unambiguous preference gives rise to a set of priors, which GMM interpret as the (subjective) ambiguity revealed by the decision maker's preferences. We refer the reader to that paper for discussion of the appropriateness of such interpretation.

GS propose a behavioral definition of the set of priors that are *relevant* for the individual's primitive preference relation $\succcurlyeq$. They then show that the resulting set is precisely C and also show that the arguments provided by GMM in support of their interpretation of C as revealed ambiguity extend to the preferences they study. We refer the interested reader to GS for details. We shall sometimes implicitly invoke GS' equivalence result and thus refer to C as the set of "relevant priors."

Henceforth, any MBA preference $\succcurlyeq$ is consistently assumed to be represented by the pair (I, u) as per Proposition 1, and by the relevant priors C as per Proposition 2.

4 A generalized Hurwicz representation

We now turn to the first consequence of the general representation results of the previous section. We show that the generalized α-MEU representation suggested by GMM, which is in the spirit of Hurwicz's "pessimism index" model (Hurwicz 1951), extends to MBA preferences, and so does its interpretation in terms of comparative ambiguity.

We first introduce convenient notation. For each probability $P \in ba_1(\Sigma)$ and function $a \in B_0(\Sigma)$, let $P(a) = \int a \, \mathrm{d}P$. Also, given a closed set $D \subset ba_1(\Sigma)$ and function $a \in B_0(\Sigma)$, let $\underline{D}(a) = \min_{P \in D} P(a)$ and $\overline{D}(a) = \max_{P \in D} P(a)$. Notice that $\underline{D}$ (resp. $\overline{D}$) is a normalized, monotonic, constant-linear, and concave (resp. convex) functional on $B_0(\Sigma)$. In light of the previous two propositions, we then get the following immediate Corollary.

Corollary 3 *Let* $\succcurlyeq$ *be an MBA preference. For each* $a \in B_0(\Sigma, u(X))$ *we have that*

$$\underline{C}(a) \equiv \min_{P \in C} P(a) \leq I(a) \leq \max_{P \in C} P(a) \equiv \overline{C}(a).$$

A second piece of terminology is useful. GMM deem an act *crisp* if, intuitively, it cannot be used to hedge the ambiguity of any other act. GMM formalize this intuition via a behavioral condition that indirectly relies upon Certainty Independence. Since MBA preferences do not necessarily satisfy this property, we require a slightly stronger definition. We say that an act crisp if and only if it is unambiguously indifferent to a constant.[5] Formally, denote by $\sim^*$ the symmetric component of $\succsim^*$. Then, act $f \in \mathscr{F}$ is **crisp** if there exists $x \in X$ such that $f \sim^* x$: for each $h \in \mathscr{F}$ and each $\lambda \in (0, 1]$,

$$\lambda f + (1 - \lambda)h \sim \lambda x + (1 - \lambda)h.$$

Intuitively, f cannot be used to hedge the ambiguity of any other act g. The characterization of crispness in terms of C immediately follows.

Corollary 4 *Let $\succsim$ be an MBA preference. Act $f \in \mathscr{F}$ is crisp if and only if $\underline{C}(u \circ f) = \overline{C}(u \circ f)$.*

We can now provide the sought generalized α-MEU representation. Given a normalized representation (I, u) of an MBA preference $\succsim$, define a function $\alpha : B_0(\Sigma, u(X)) \to \mathbb{R}$ by letting

$$\alpha(a) = \frac{\overline{C}(a) - I(a)}{\overline{C}(a) - \underline{C}(a)} \tag{2}$$

for all $a \in B_0(\Sigma, u(X))$ such that $\overline{C}(a) \neq \underline{C}(a)$, and $\alpha(a) = \frac{1}{2}$ otherwise. The following result is immediately proved.

Proposition 5 *If $\succsim$ is an MBA preference, then there exist a non-empty, closed, and convex set $C \subset ba_1(\Sigma)$, a non-constant, affine function $u : X \to \mathbb{R}$, and a function $\alpha : B_0(\Sigma, u(X)) \to [0, 1]$ such that*

(i) *for each $f, g \in \mathscr{F}$,*

$$\begin{aligned} f \succsim g \iff & \alpha\ (u \circ f)\ \underline{C}(u \circ f) + [1 - \alpha(u \circ f)]\ \overline{C}(u \circ f) \\ & \geq \alpha(u \circ g)\ \underline{C}(u \circ g) + [1 - \alpha(u \circ g)]\ \overline{C}(u \circ g); \end{aligned}$$

(ii) *u and C represent $\succsim^*$ in the sense of Proposition 2;*

(iii) *for each non-crisp $f, g \in \mathscr{F}$, if $P(u \circ f) = P(u \circ g)$ for all $P \in C$, then $\alpha(u \circ f) = \alpha(u \circ g)$. For each crisp $f \in \mathscr{F}$, $\alpha(u \circ f) = \frac{1}{2}$.*

Finally, if (u', C', α') also satisfy (i), (ii), and (iii) then $C' = C$, $u'(x) = \lambda u(x) + \mu$ for some $\lambda, \mu \in \mathbb{R}$ with $\lambda > 0$, and $\alpha'(\lambda\ u \circ f + \mu) = \alpha(u \circ f)$ for all non-crisp $f \in \mathscr{F}$.

Remark 4.1 The uniqueness part of Proposition 5 may be restated as follows: C is unique, u is cardinally unique, and the function $\alpha(\cdot)$ is also unique (for non-crisp acts) if viewed as a function of *acts* rather than utility profiles. More precisely, $\alpha(\cdot)$ is invariant to cardinal transformations of the utility function u. (It is worth recalling that GMM define $\alpha(\cdot)$ over equivalence classes of acts, rather than functions.) As we shall see, there is a sense in which $\alpha(\cdot)$ can be interpreted as capturing the ambiguity aversion of the decision maker; i.e., an ambiguity index.

[5] For GMM's preferences, the two conditions are equivalent: this follows immediately from Corollary 4 below and GMM's Proposition 10.

Since the functional I derived in Proposition 1 is not necessarily constant-linear, the functional α does not have the same structure as in GMM. There, it is shown that, for each two acts $f, g \in \mathscr{F}$, $\alpha(u \circ f) = \alpha(u \circ g)$ holds if

$$P(u \circ f) \geq Q(u \circ f) \Leftrightarrow P(u \circ g) \geq Q(u \circ g) \qquad \forall P, Q \in C.$$

For MBA preferences, $\alpha(u \circ f) = \alpha(u \circ g)$ it requires the more restrictive condition that $P(u \circ f) = P(u \circ g)$ for all $P \in C$.

4.1 Does B stand for biseparable?

MBA preferences share some of the properties of what Ghirardato and Marinacci (2001) call "biseparable" preferences. In our context, a preference $\succcurlyeq$ is **biseparable** if there exists a unique capacity $\rho : \Sigma \to \mathbb{R}$ such that, given any representation (I, u) of $\succcurlyeq$, with I normalized, we have for each binary act $x\,A\,y$ with $x \succ y$,

$$I(u \circ (x\,A\,y)) = u(x)\rho(A) + u(y)(1 - \rho(A)). \tag{3}$$

Biseparability thus requires that the "decision weight" attached to the event A in the evaluation of the bet $x\,A\,y$ be independent of the prizes x and y (provided $x \succ y$). Also observe that biseparability is a property of preferences, not of their representation: Eq. (3) is equivalent to the requirement that $x\,A\,y \sim \rho(A)x + [1 - \rho(A)]y$, where the r.h.s. of this indifference is a mixture of the prizes x and y. Hence, the capacity ρ is also independent of the choice of u.[6]

It is not hard to see that in general, MBA preferences may fail to be biseparable, even though they induce a cardinal and affine utility u. The following example illustrates such a case.

Example 1 Recall (Klibanoff et al. 2005) that a Smooth Ambiguity preference is an MBA preference whose representation (I, u) takes the form

$$I(u \circ f) = \phi^{-1}\left(\int\limits_{ba_1(\Sigma)} \phi\,(Q(u \circ f))\,\mathrm{d}\mu(Q) \right)$$

where u is a Bernoulli utility function, $\phi : u(X) \to \mathbb{R}$ is continuous and strictly increasing, and μ is a "second-order" probability defined over $ba_1(\Sigma)$. Consider an arbitrary state space S and $X = (0, \infty)$. Suppose $u(x) = x$, $\mu(\{Q_1\}) = \mu(\{Q_2\}) = \frac{1}{2}$ with $Q_1(A) = Q_2(A^c) = \frac{3}{4}$ for some event $A \in \Sigma$, and $\phi(y) = \log(y)$. In this case, ambiguity aversion is intuitively decreasing in y. It follows that $I(a) =$

[6] Consequently, under biseparability, the restriction of the normalized functional I to binary acts is also independent of u, even though, for general acts, this is not generally the case. As it is argued in Ghirardato et al. (2005), I is invariant with respect to u for all acts f in our Anscombe–Aumann framework only if we assume that the preference satisfies Certainty Independence.

$e^{\frac{1}{2}\log Q_1(a)+\frac{1}{2}\log Q_2(a)}$ for all $a \in B_0(\Sigma, (0, \infty))$. Consider now the bet 2 A 1 that pays 2 USD if A obtains and 1 otherwise. We have

$$I(2\ A\ 1) = e^{\frac{1}{2}\log(1+Q_1(A))+\frac{1}{2}\log(1+Q_2(A))} \approx 1.47902.$$

If, on the other hand, we consider the bet 3 A 2 then

$$I(3\ A\ 2) = e^{\frac{1}{2}\log(2+Q_1(A))+\frac{1}{2}\log(2+Q_2(A))} \approx 2.48746$$

Thus, if we apply Eq. (3) to the bet 2 A 1, we conclude that $\rho(A)$ equals 0.47902. If we consider the bet 3 A 2, Eq. (3) implies that $\rho(A)$ is 0.48746.

We therefore see that in this case, $\rho(A)$ cannot be defined independently of the choice of $x \succ y$. This is a violation of biseparability. Intuitively, since $\phi(y) = \log(y)$ displays decreasing absolute ambiguity aversion, as we increase the prizes involved, we get a less conservative willingness to bet on the ambiguous event A.

While invariance of I and ρ to transformations of the utility function does not obtain, for MBA preferences we can still obtain a "locally" biseparable representation of $\succcurlyeq$. Fix a pair (I, u) that represents $\succcurlyeq$ with I normalized and $x \succ y$. Given the bet $x\ A\ y$ on event $A \in \Sigma$, define

$$\rho_{x,y}(A) \equiv \alpha(u \circ x\ A\ y)(\underline{C}(1_A) - \overline{C}(1_A)) + \overline{C}(1_A). \tag{4}$$

The uniqueness properties of the ambiguity index $\alpha(\cdot)$ ensure that the quantity $\rho_{x,y}$ is independent of the utility function adopted (see Proposition 5). It is then easy to verify that, when restricted to binary acts (bets) of the form $x\ A\ y$ (for arbitrary $A \in \Sigma$), the preference $\succcurlyeq$ has the representation

$$I(u \circ (x\ A\ y)) = u(x)\rho_{x,y}(A) + u(y)(1 - \rho_{x,y}(A)). \tag{5}$$

With this notation, an MBA preference is biseparable if $\rho_{x,y}$ does not depend upon x and y; we call such a preference **MBis**, for **M**onotone and **Bis**eparable. It is natural to ask whether an additional axiom identifies the MBis subclass of MBA preferences. Ghirardato and Marinacci (2001) describe and axiomatize a model of preferences that turns out to have exactly the type of separability we need. The main axiom is the following; recall that an act $f \in \mathscr{F}$ is **binary** if it takes the form $f = x\ A\ y$ for some $A \in \Sigma$ and $x, y \in X$.

Axiom 5 (Binary Certainty Independence) *For all $f, g \in \mathscr{F}$, with f, g binary acts, $x \in X$, and $\lambda \in (0, 1]$: $f \succ g$ if and only if $\lambda f + (1-\lambda)x \succ \lambda g + (1-\lambda)x$.*

We then have the following characterization (see Theorem 9 in Ghirardato and Marinacci 2001).

Proposition 6 *Let $\succcurlyeq$ be an MBA preference. $\succcurlyeq$ satisfies Axiom 5 if and only if it is biseparable.*

5 Ambiguity aversion

Here, we consider the characterization of ambiguity attitudes for MBA preferences. We first show that, as suggested in Remark 4.1 and consistently with the analysis in GMM, the function α can be interpreted as an index of ambiguity aversion: The higher α, the more averse to ambiguity the decision maker.

More precisely, following GM, we say that a preference $\succcurlyeq_1$ is **more averse to ambiguity than** $\succcurlyeq_2$ if for each $f \in \mathscr{F}$ and each $x \in X$, $f \succcurlyeq_1 x$ implies $f \succcurlyeq_2 x$.[7] The comparison is made between preferences that display the same relevant priors C and utility u; equivalently preferences such that, for each $f, g \in \mathscr{F}$,

$$f \succcurlyeq_1^* g \Longleftrightarrow f \succcurlyeq_2^* g. \tag{6}$$

We immediately obtain:

Proposition 7 (GMM, Proposition 12) *Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be MBA preferences and let $\succcurlyeq_1$ and $\succcurlyeq_2$ reveal identical ambiguity in the sense of* Eq. (6). *The following conditions are equivalent:*

1. *$\succcurlyeq_1$ is more ambiguity averse than $\succcurlyeq_2$;*
2. *$\alpha_1(u \circ f) \geq \alpha_2(u \circ f)$ for all $f \in \mathscr{F}$.*

Notice that since as observed the function α may be dependent on the normalization of utility, we need to normalize the two utility functions, u_1 and u_2, to be identical before performing the comparison of the α functions.

Turning to an *absolute* notion of ambiguity aversion, we recall that GM (in this differing from Epstein 1999) suggest using subjective expected utility preferences as benchmarks for ambiguity neutrality. They propose the following axiomatic definition of ambiguity aversion:

Axiom 6 (Ambiguity Aversion) *There exists a non-trivial SEU preference $\geqslant$ such that for each $g \in \mathscr{F}$ and each $x \in X$*

$$x \succcurlyeq g \Longrightarrow x \geqslant g.$$

That is, a preference is ambiguity averse if it is more ambiguity averse than some SEU preference that displays the same risk attitudes.[8]

The characterization of ambiguity aversion given by GM immediately generalizes to MBA preferences. We first need some more terminology. Given an MBA preference with a representation (I, u), define

$$\begin{aligned} Core(I) &= \{P \in ba_1(\Sigma) : \forall a \in B_0(\Sigma, u(X)),\ I(a) \leq P(a)\} \quad \text{and} \\ Eroc(I) &= \{P \in ba_1(\Sigma) : \forall a \in B_0(\Sigma, u(X)),\ I(a) \geq P(a)\}. \end{aligned}$$

[7] This is equivalent to the definition found in GM. For when $\succcurlyeq_1$ and $\succcurlyeq_2$ are MBA preferences (in an Anscombe–Aumann setting), it can be seen that $f \succcurlyeq_1 x \Rightarrow f \succcurlyeq_2 x$ is equivalent to $f \succ_1 x \Rightarrow f \succ_2 x$ and to (its contra-positive) $x \succcurlyeq_2 f \Rightarrow x \succcurlyeq_1 f$.

[8] To further clarify, we consider SEU preferences à la Anscombe–Aumann, rather than à la Savage.

These notions correspond to the game-theoretic notions when the preference, $\succcurlyeq$, is CEU.

Even for MBA preferences, absolute ambiguity aversion corresponds to non-emptiness of $Core(I)$. (The symmetric property of ambiguity love is analogously characterized as non-emptiness of $Eroc(I)$.)

Proposition 8 (GM, Theorem 12) *Let $\succcurlyeq$ be an MBA preference. $\succcurlyeq$ is ambiguity averse if and only if $Core(I) \neq \emptyset$.*

It is also easy to extend Proposition 16 of GMM to show that such sets must be contained in the set of relevant priors C. This result follows from Proposition 5 and standard separation arguments.

Proposition 9 (GMM, Proposition 16) *Let $\succcurlyeq$ be an MBA preference. Then*

$$Core(I) \cup Eroc(I) \subset C.$$

The GM proposal is not the most popular definition of ambiguity aversion in the literature. The following notion, proposed by Schmeidler's (1989), claims that title. It imposes convexity of preferences.[9]

Axiom 7 (Convexity) *If $f, g \in \mathscr{F}$ and $\alpha \in (0, 1)$ then*

$$f \sim g \Longrightarrow \alpha f + (1 - \alpha) g \succcurlyeq f.$$

For MBA preferences, these two notions of aversion to ambiguity are different. Indeed, Example 5 below presents an ambiguity averse preference that does not satisfy Convexity. On the other hand, the following is an example of a preference that is not ambiguity averse but satisfies Convexity.

Example 2 Suppose $X = \mathbb{R}$ and consider $S = \{s_1, s_2\}$. Further, suppose Σ is the power set. Then, we can identify each element $P \in ba_1(\Sigma)$ with the number $P(\{s_1\})$. For this reason, without loss of generality, we denote by P both probability distribution and number. Next, consider the binary relation $\succcurlyeq$ over $\mathscr{F}$ represented by the functional $V : \mathscr{F} \to \mathbb{R}$ defined by

$$V(f) = \min_{P \in ba_1(\Sigma)} \left(\frac{\left(\int f dP\right)^+}{c_1(P)} - \frac{\left(\int f dP\right)^-}{c_2(P)} \right)$$

where $c_1, c_2 : ba_1(\Sigma) \to [0, \infty]$ are such that $c_1(P) = \frac{P+1}{2}$ and $c_2(P) = 1 + P$. It is immediate to see that c_1 is affine and continuous and c_2 is affine and continuous. Note also that u does not appear because it is chosen to be the identity. Moreover, $\min_{P \in ba_1(\Sigma)} c_1(P) = \frac{1}{2} > 0$ and $\max_{P \in ba_1(\Sigma)} c_1(P) = 1$ while $\min_{P \in ba_1(\Sigma)} c_2(P) = 1$. By C3M, $\succcurlyeq$ is an MBA preference that satisfies convexity. However, in light of the discussion in Cerreia-Vioglio et al. (2009), $\succcurlyeq$ is ambiguity averse only if $\arg\max c_1 \cap \arg\min c_2 \neq \emptyset$, which is clearly not satisfied in our case.

[9] Schmeidler calls this property "uncertainty aversion," while GM call it "ambiguity hedging."

However, the next result shows that a connection exists between the two notions of aversion to ambiguity: convexity amounts to ambiguity aversion holding "locally" for all acts. For convenience, we restrict attention to MBA preferences for which there is **no worst consequence**, that is, for each $x \in X$ there exists $y \in X$ such that $x \succ y$. Given a representation (I, u) as in Proposition 1, this is equivalent to the condition that $\inf_{x \in X} u(x) \notin u(X)$.[10]

Theorem 10 *Let $\succcurlyeq$ be an MBA preference that has no worst consequence. The following conditions are equivalent:*

(i) $\succcurlyeq$ is convex;
(ii) for each $f \in \mathscr{F}$ there exists a SEU preference $\geqslant_f$ such that for each $g \in \mathscr{F}$,

$$f \geqslant_f g \Longrightarrow f \succcurlyeq g.$$

In view of Theorem 10, Axiom 7 implies the following weak version of Axiom 6: For each $x \in X$, there exists a SEU preference $\geqslant_x$ such that for each $g \in \mathscr{F}$,

$$x \geqslant_x g \Longrightarrow x \succcurlyeq g.$$

Relative to Axiom 6, here the SEU preference $\geqslant_x$ depends on x. Hence, Axiom 7 implies Axiom 6 for all preferences for which this dependence can be removed. Consider the sets

$$S_{\succcurlyeq}(x) = \{\geqslant : \geqslant \text{ is SEU and for each } g \in \mathscr{F},\ x \geqslant g \Rightarrow x \succcurlyeq g\} \qquad \forall x \in X.$$

In other words, $S_{\succcurlyeq}(x)$ is the collection of all SEU preferences that are more ambiguity averse than $\succcurlyeq$ at x. Axiom 7 implies that $S_{\succcurlyeq}(x) \neq \emptyset$ for all x. On the other hand, Axiom 6 is equivalent to

$$\bigcap_{x \in X} S_{\succcurlyeq}(x) \neq \emptyset.$$

As shown by Maccheroni et al. (2006, Proposition 7), a sufficient condition for the above to hold is that the MBA preference relation satisfies their Weak Certainty Independence axiom (i.e., that the representing I be translation invariant) as well as Axiom 7.[11]

6 Ambiguity of acts and events

Having discussed ambiguity attitude, in this section we look at ambiguity itself. We first propose a notion of unambiguous act that strengthens that of crisp act (see Sect. 4). We

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[10] The result is true, with a lengthier proof, without this assumption. Observe that the latter does *not* imply that $u(X)$ must be unbounded below. For example, consider $X = (0, 1)$ and $u(x) = x$.

[11] A functional $I : B_0(\Sigma) \to \mathbb{R}$ is **translation-invariant** if $I(a + \alpha 1_S) = I(a) + \alpha$ for all $a \in B_0(\Sigma)$ and $\alpha \in \mathbb{R}$.

further characterize it for MBA preferences. Second, we employ this notion to define unambiguous events and provide some characterizations. Armed with the characterization of unambiguous acts and events for MBA preferences, we proceed to investigate some consequences of the characterizations. In particular, we observe how, in the spirit of Epstein and Zhang (2001), the derived set of unambiguous events can be used to provide a "fully subjective" theory of expected utility (different from the one they propose). We finally generalize Marinacci (2002) result on the consistency of probabilistic sophistication and ambiguity aversion to non (α-)MEU preferences.

Throughout this section, it is convenient to adopt explicit notation for simple acts. Fix a finite partition $\{E_1, \ldots, E_n\}$ of S in Σ and corresponding prizes $x_1, \ldots, x_n \in X$. The act that delivers prize x_i in states $s \in E_i$, for $i = 1, \ldots, n$, will be denoted by $\{x_1, E_1; \ldots; x_n, E_n\}$. As before, if $n = 2$, then $\{x_1, E; x_2, S \setminus E\}$ will be denoted simply by $x_1 \, E \, x_2$

6.1 Unambiguous acts

We begin by motivating our definition of unambiguous acts. In keeping with the intuition that ambiguity is revealed by non-neutral attitudes toward *hedging*, a starting point is to require that unambiguous acts be crisp. To elaborate, we surely want the set of unambiguous acts to include all constant acts. It seems plausible to require that this set also includes acts that, like constants, are revealed not to provide any hedging opportunities.

However, we would like the notion of unambiguous acts to capture an additional intuition. Consider the three-color Ellsberg urn, containing 30 red balls and 60 green and blue balls, in unspecified proportions. It is natural to regard a "bet on red" as an unambiguous act, because the partition it induces on the state space $S = \{r, g, b\}$ the winning event $\{r\}$ and the losing event $\{g, b\}$—consists of events whose relative likelihood is intuitively clear. But, by the same token, a "bet on *not* red" should also be regarded as unambiguous.

More broadly, if two acts f, g induce the same partition of the state space S, in the sense that for all states $s, s' \in S$, $f(s) = f(s')$ if and only if $g(s) = g(s')$, then either they are both ambiguous, or else they are both unambiguous. In other words, the property of being ambiguous or unambiguous depends upon the partition an act induces, rather than on the specific assignment of distinct prizes to different elements of the induced partition. The following example demonstrates that this additional, natural requirement has bite.

Example 3 Let $S = \{s_1, s_2, s_3\}$ and consider an MBA preference with set C equal to the convex hull of the two priors $P = [1/3, 1/4, 5/12]$ and $Q = [1/4, 5/12, 1/3]$. Consider the act $f = \{x, \{s_1\}; y, \{s_2\}; z, \{s_3\}\}$, with $u(x) = 1, u(y) = 4, u(z) = 7$. Observe that $P(u \circ f) = Q(u \circ f)$, so f is crisp (see Corollary 4). However, the act $g = \{y, \{s_1\}; z, \{s_2\}; x, \{s_3\}\}$, which "permutes" the payoffs delivered by f but is measurable with respect to the same partition, satisfies $P(u \circ g) \neq Q(u \circ g)$: hence, it is not crisp.

Now, if unambiguous acts must be crisp (as we wish to assume), then g must be deemed ambiguous. Since f and g induce the same partition of S, the preceding argument then implies that we must deem f ambiguous as well.

Observe that, in Example 3, the prizes delivered by the acts f and g are the same. This is the sense in which g is a "permutation" of f. We formalize this notion of permutation below.

The discussion so far suggests the following loose provisional definition: *an act is unambiguous if all its "permutations" are crisp*. However, a final difficulty must be overcome. Acts map states to *consequences*, but hedging considerations involve *utility* trade-offs. Hence, if we deem f unambiguous, and $f(s) \sim g(s)$ for all $s \in S$, we should deem g unambiguous, too. Indeed, it turns out that, in the approach we pursue, this is *necessary* in order to avoid paradoxical conclusions; see the next example.

Example 4 Consider again the 3-color Ellsberg urn, with $S = \{r, g, b\}$; consider prizes x, y, z with $x \neq y \neq z$ and $u(x) = 1 > 0 = u(y) = u(z)$, and let $f = \{x, \{r\}; y, \{g\}; z, \{b\}\}$, so f is, intuitively, a bet on red, even though strictly speaking it is not a binary act. Finally, consider the set C generated by $P = [1/3, 2/3, 0]$ and $Q = [1/3, 0, 2/3]$. In keeping with the Ellsbergian intuition, we wish to deem f unambiguous; however, consider the act $f' = \{y, \{r\}; x, \{g\}; z, \{b\}\}$, which delivers the same prizes as f and is measurable with respect to the same partition. Then $P(u \circ f') = \frac{2}{3} > 0 = Q(u \circ f')$, so f' is not crisp.

As in the previous example, f' must be deemed ambiguous, and hence the provisional definition would deem f ambiguous as well, which seems counterintuitive.

Our definition of unambiguous act takes care of the difficulty illustrated in Example 4 by defining permutations in terms of utility levels instead of payoffs.

Definition 2 An act $g \in \mathscr{F}$ is a **$\succcurlyeq$-permutation** of another act $f \in \mathscr{F}$ if:

(i) for each $s \in S$ there exists $s' \in S$ such that $f(s) \sim g(s')$;
(ii) for each $s \in S$ there exists $s' \in S$ such that $g(s) \sim f(s')$;
(iii) for each $s, s' \in S$, $f(s) \sim f(s')$ if and only if $g(s) \sim g(s')$.

An act $f \in \mathscr{F}$ is **unambiguous** if every $\succcurlyeq$-permutation of f is crisp. The class of all unambiguous acts is denoted by $\mathscr{U}$.

Note that, if preferences are represented by a Bernoulli utility u on X, then conditions (i) and (ii) above are equivalent to the statement that $u \circ f(S) = u \circ g(S)$.

The following result shows that the set $\mathscr{U}$ is the largest set of crisp acts which is "closed" with respect to $\succcurlyeq$-permutations.

Proposition 11 *Let $\succcurlyeq$ be an MBA preference. $\mathscr{U}$ is the largest set of crisp acts such that if $f \in \mathscr{U}$ and $g \in \mathscr{F}$ is a $\succcurlyeq$-permutation of f then $g \in \mathscr{U}$.*

The main result of this section shows that unambiguous acts have a sharp characterization in terms of their expected utility with respect to probabilities in the set C.

Theorem 12 *Let $\succcurlyeq$ be an MBA preference and f be an element of $\mathscr{F}$. The following conditions are equivalent:*

(i) $f \in \mathscr{U}$;
(ii) $P(\{s \in S : f(s) \sim x\}) = Q(\{s \in S : f(s) \sim x\})$ *for all* $x \in X$, $P, Q \in C$;

(iii) $P(\{s \in S : u \circ f(s) \geq \gamma\}) = Q(\{s \in S : u \circ f(s) \geq \gamma\})$ *for all* $\gamma \in \mathbb{R}$, $P, Q \in C$;

(iv) $P(\{s \in S : u \circ f(s) = \gamma\}) = Q(\{s \in S : u \circ f(s) = \gamma\})$ *for all* $\gamma \in \mathbb{R}$, $P, Q \in C$.

Statement (ii) is possibly the most useful, and powerful, characterization of unambiguous acts. In words, an act is unambiguous if and only if the events in the partition it induces have the same probability according to all members of the set C. In particular, this implies that if f is unambiguous and g induces the same partition as f (possibly delivering entirely different prizes), then g is also unambiguous.

6.2 Unambiguous events

It is natural to define unambiguous any event with respect to which unambiguous acts are measurable (a similar approach to defining unambiguous events was earlier advocated by GM).

Definition 3 The class of **unambiguous events** is

$$\Lambda = \{\{s : f(s) \sim x\} \;:\; f \in \mathscr{U},\, x \in X\}.$$

Analogously to what we had for unambiguous acts, we can offer two characterization results for unambiguous events. The first is a behavioral result:

Proposition 13 *For each* $A \in \Sigma$, $A \in \Lambda$ *if and only if for each* $x, y \in X$ *such that* $x \not\sim y$ *the act* $x\,A\,y$ *is crisp.*

By part (ii) of Theorem 12, arguing as we did after the statement of that theorem, the part "for each $x, y \in X$ such that $x \not\sim y$" could be changed to "for some $x, y \in X$ such that $x \succ y$" without invalidating the result (see Appendix D.2.1). This makes the behavioral identification of the set Λ conceptually easier, and it also conforms with our intuition that ambiguity is a property of the event partition the act is based on.

Thus, an event A is unambiguous if it is such that any *bet on* such event—i.e., any act of the form $x\,A\,y$ with $x \succ y$—cannot be used to hedge the ambiguity in another act [Nehring (2001) proposes a different definition which, under Certainty Independence, turns out to be equivalent to Definition 3, and hence also to an earlier one he presented in Nehring (1999)]. Conversely, A is ambiguous if $x\,A\,y \not\sim^* z$ for all $z \in X$. For example, this is the case if $x\,A\,y \sim z$ but there exist $\lambda \in (0, 1]$ and $h \in \mathscr{F}$ such that $\lambda x\,A\,y + (1-\lambda)h \not\sim \lambda z + (1-\lambda)h$.

The second result shows that unambiguous events have a simple and intuitive characterization in terms of the probabilities in C. Notice that this is independent of the normalization chosen for u. There is also a natural connection with the "local" willingness to bet $\rho_{x,y}$ defined in Eq. (5).

Proposition 14 *For each* $A \in \Sigma$, $A \in \Lambda$ *if and only if* $P(A) = Q(A) = \rho_{x,y}(A)$ *for all* $P, Q \in C$ *and all* $x, y \in X$ *with* $x \succ y$.

As a consequence, for all MBA preferences, the collection Λ has a simple and intuitive structure (see Zhang 2002; Nehring 1999).

Corollary 15 *Λ is a (finite) λ-system. That is:* (1) $S \in \Lambda$*;* (2) *if* $A \in \Lambda$ *then* $A^c \in \Lambda$*;* (3) *if* $A, B \in \Lambda$ *and* $A \cap B = \emptyset$ *then* $A \cup B \in \Lambda$.

It is natural to surmise that *any* act whose upper level sets are unambiguous events should be deemed unambiguous (see, e.g., Epstein and Zhang 2001). Proposition 14, paired with Theorem 11, allows us to show that this is indeed the case.

Corollary 16 *For each act* $f \in \mathscr{F}$, $f \in \mathscr{U}$ *if and only if its upper preference sets* $\{s \in S : f(s) \succcurlyeq x\}$ *belong to* Λ *for all* $x \in X$.

Nehring (1999) shows that, if S is finite and I is a Choquet integral (so that the set C can be simply characterized; see Example 17 in GMM), the set Λ can be further characterized as follows:

$$\Lambda = \{A \in \Sigma : \rho(B) = \rho(B \cap A) + \rho(B \cap A^c) \text{ for all } B \in \Sigma\},$$

where $\rho = \rho_{x,y}$, which in the CEU case is independent of the choice of x and y. It follows that for CEU preferences Λ is an algebra, a result that shows that such preferences *cannot* be used to model some potentially interesting ambiguity situations (see for instance the 4-color example in Zhang 2002).

Klibanoff et al. (2005) propose a behavioral notion of unambiguous event and characterize it in the context of their Smooth Ambiguity model. They discuss it further in Klibanoff et al. (2011). Under mild additional conditions (for example those required in Klibanoff et al. 2011, Theorem 2.1), it can be shown that for Smooth Ambiguity preferences, their notion coincides with ours.[12] Moreover, since Klibanoff et al. (2011, Theorem 3.2) show the equivalence of their approach to the one of GM (to be discussed below), for such preferences, the three approaches deliver the same set of unambiguous events. But then by Corollary 16, they also deliver the same set of unambiguous acts.

6.2.1 Ambiguity and willingness to bet

GM propose a behavioral notion of unambiguous event for a subclass of the biseparable preferences mentioned in Sect. 4.1. They show that it has a simple characterization in terms of the willingness to bet set function ρ of Eq. (3). An event B is unambiguous in their sense if and only if $\rho(B) + \rho(B^c) = 1$.

The definition given above enjoys two main advantages over this earlier proposal: it is more general because it applies to any MBA preference; more importantly, it is more accurate as it distinguishes between events that are truly perceived unambiguous and those that appear to be because of the behavior of the decision maker's ambiguity attitude. The following result illustrates this point. Here, $\rho_{x,y}(\cdot)$ is the local willingness to bet index defined in Eq. (5) and $\alpha(\cdot)$ is the ambiguity index of Eq. (2).

[12] A proof is available upon request.

Proposition 17 *Let $\succsim$ be an MBA preference and let $x, y \in X$ be such that $x \succ y$. The following conditions are equivalent for each $A \in \Sigma$:*

(i) $\rho_{x,y}(A) + \rho_{x,y}(A^c) = 1$ *($\rho_{x,y}$ is* ***complement-additive****);*
(ii) *either $A \in \Lambda$, or $A \in \Sigma \setminus \Lambda$ and $\alpha(u \circ x\ A\ y) + \alpha(u \circ x\ A^c\ y) = 1$.*

Moreover, if $\succsim$ is ambiguity averse, then (i) and (ii) are equivalent to

(iii) $\rho_{x,y}(A) = P(A)$ *for all $P \in Core(I)$ and all $A \in \Sigma$.*

To interpret, an event satisfies the condition $\rho_{x,y}(A) + \rho_{x,y}(A^c) = 1$ for some x and y in exactly two cases: either 1) A is unambiguous or 2) A is not unambiguous but the decision maker's ambiguity index in evaluating the bets $x\ A\ y$ and $x\ A^c\ y$ behaves so as to perfectly compensate the ambiguity aversion (resp. love) revealed in evaluating $x\ A\ y$ by evaluating the complementary bet $x\ A^c\ y$ in an ambiguity seeking (resp. averse) fashion. That is, $\rho_{x,y}(A) + \rho_{x,y}(A^c) = 1$ could be satisfied by pure mathematical accident, if the decision maker's ambiguity attitude is "inconsistent" in just the right way.

On the other hand, suppose that the preference satisfies, given x and y, for each $A \in \Sigma \setminus \Lambda$

$$\alpha(u \circ x\ A\ y) + \alpha(u \circ x\ A^c\ y) \neq 1 \tag{7}$$

Then, $\rho_{x,y}(A) + \rho_{x,y}(A^c) = 1$ if and only if A is unambiguous. For instance, this is the case of a decision maker for whom $\alpha > 1/2$ uniformly. The following example shows one case of such consistency of ambiguity aversion in a CEU preference.

Example 5 Consider the following variant of the Ellsberg "3-color" paradox. An urn contains 120 balls, 30 of which are red, while the remaining 90 are either blue, green, or yellow. Assume a decision maker facing this problem has a CEU preference $\succsim$ represented by a (non-constant and convex-ranged) utility u and a capacity ρ on $S = \{r, g, b, y\}$,[13] where

$$\rho\,(B) = \frac{1}{4}\upsilon_{\{r\}}\,(B) + \frac{3}{4}\nu\,(B \cap \{g, b, y\}) \text{ for all } B.$$

$\upsilon_{\{r\}}$ is such that $\upsilon_{\{r\}}\,(B) = 1$ if $r \in B$ and $\upsilon_{\{r\}}\,(B) = 0$ otherwise. On the other hand, ν is a capacity on $\{g, b, y\}$ defined as follows: $\nu(\emptyset) = 0$, $\nu(\{g, b, y\}) = 1$ and

$$\nu(\{g\}) = \nu(\{b\}) = \nu(\{y\}) = \frac{7}{24}, \quad \nu(\{g, b\}) = \nu(\{g, y\}) = \nu(\{b, y\}) = \frac{1}{2}.$$

Observe first that $Core(\rho)(= Core(I))$ contains (at least) the uniform probability on S and ρ is not supermodular. Therefore, $\succsim$ satisfies Ambiguity Aversion, but not Convexity. Observe next that $\rho(\{r\}) = 1/4$ and $\rho(\{g, b, y\}) = 3/4$. That is, $\{\{r\}, \{g, b, y\}\}$ is a candidate for being an unambiguous partition. According to Proposition 17, this

[13] Notice that such preference is biseparable, so that ρ does not depend on the choice of x and y and $\alpha(u \circ xAy) = \alpha(u \circ x'Ay') \equiv \alpha(A)$ for all $x \succ y$ and $x' \succ y'$.

will be the case if $\alpha(\{r\}) + \alpha(\{g, b, y\}) \neq 1$. Using Example 17 of GMM, it can be checked after some tedious calculation that for $\succcurlyeq$

$$C = \text{Conv}\left\{[1/4, x, y, z] \in \mathbb{R}^4 : [x, y, z] \in \text{Per}(\{5/32, 7/32, 12/32\})\right\}.$$

It follows that $\Lambda = \{\emptyset, S, \{r\}, \{g, b, y\}\}$ as expected. Moreover,

$$\alpha(\{y\}) = \alpha(\{b\}) = \alpha(\{g\}) = 5/7, \quad \alpha(\{r, g, b\}) = \alpha(\{r, g, y\}) = \alpha(\{r, b, y\}) = 1,$$
$$\alpha(\{r, g\}) = \alpha(\{r, b\}) = \alpha(\{r, y\}) = 5/8, \quad \alpha(\{b, y\}) = \alpha(\{g, y\}) = \alpha(\{g, b\}) = 1.$$

That is, $\succcurlyeq$ satisfies $\alpha(A) + \alpha(A^c) \neq 1$ for any $B \in \Sigma \setminus \Lambda$.

It turns out that Eq. (7) has a simple behavioral characterization:

Proposition 18 *Let $\succcurlyeq$ be an MBA preference and let $x, y \in X$ be such that $x \succ y$. Equation (7) holds for $A \in \Sigma \setminus \Lambda$ if and only if*

$$\frac{1}{2}c_{xAy} + \frac{1}{2}c_{xA^cy} \nsim \frac{1}{2}x + \frac{1}{2}y \tag{8}$$

where for each $f \in \mathscr{F}$ we denote by c_f one of its certainty equivalents.

We shall see that this result proves useful in characterizing situations in which complement additivity is a full "marker" for the lack of ambiguity (see Proposition 21).

The equivalence of (iii) with (i) in Proposition 17 is also interesting. It shows that for an ambiguity averse decision maker, the GM condition corresponds to agreement of $\rho_{x,y}$ with the members of $Core(I)$, rather than—see Proposition 14—with the members of the (larger) set C.

We conclude this discussion by observing that the definition of the set Λ and some of the notation and terminology introduced in the previous paragraphs allow us to provide an alternative characterization of MBis preferences, complementing Proposition 6. If there are "enough" unambiguous events, Savage's Postulate P4—which is in general weaker than Binary Certainty Independence—suffices to guarantee that the preference is biseparable. We need some notation first. Given a set $D \subseteq ba_1(\Sigma)$ and a collection $\Upsilon \subseteq \Sigma$, we denote $D(\Upsilon) \equiv \{P(A) : \exists P \in D, A \in \Upsilon\}$.

Proposition 19 *Let $\succcurlyeq$ be an MBA preference with unambiguous events Λ such that $C(\Lambda)$ is dense in* $(0, 1)$. *The following conditions are equivalent:*

(i) *there exists a unique capacity ρ such that Eq.* (3) *holds for all binary act $x\,A\,y$ and all normalized representation (I, u) of $\succcurlyeq$;*
(ii) *$\succcurlyeq$ satisfies Savage's P4 axiom. That is, for each $A, B \in \Sigma$ and each $x, y, x', y' \in X$ such that $x \succ y$ and $x' \succ y'$, $x\,A\,y \succcurlyeq x\,B\,y$ if and only if $x'\,A\,y' \succcurlyeq x'\,B\,y'$.*

6.3 A "fully subjective" expected utility model

As observed by Epstein and Zhang (2001), there is an important sense in which Savage's (1954) construction of subjective probability is not "fully subjective." In fact, Savage (and later Machina and Schmeidler 1992, in their extension of Savage's construction) assumes exogenously that the probability that represents the decision maker's beliefs is defined on the whole (σ-) algebra Σ. Examples like Ellsberg's paradox suggest that a natural extension of Savage's philosophy might be to define probabilities wherever the decision maker feels comfortable, and avoid doing so otherwise, thus making also the domain of the probability charge "subjective." Epstein and Zhang propose a definition of unambiguous event, and in the spirit of Machina and Schmeidler (1992) provide an axiomatization of preferences whose induced likelihood relations are represented by a probability charge on the set of unambiguous events—which under such axiomatic restrictions (with a minor amendment, see Kopylov 2007) is a λ-system. Kopylov (2007) provides an analogous result using a slightly different set of axioms, generating weaker structural restrictions on the set of unambiguous events (it is what he calls a "mosaic").

The results obtained thus far allow us to provide a different "fully subjective" version of Savage's model, summarized below (see also Nehring 2002, Proposition 1):[14]

Proposition 20 *If $\succcurlyeq$ is an MBA preference on $\mathscr{F}$, then there is a finite λ-system of events $\Lambda \subseteq \Sigma$ such that $\succcurlyeq$ has a SEU representation (with utility u) on the set $\mathscr{U}$ of the Λ-measurable acts. That is, there exists a probability charge $P : \Sigma \to [0, 1]$ such that for each $f, g \in \mathscr{U}$,*

$$f \succcurlyeq g \Longleftrightarrow \int u \circ f \, \mathrm{d}P \geq \int u \circ g \, \mathrm{d}P$$

Moreover, P can be chosen to be in C and in this case it is uniquely defined on Λ.

We thus conclude that the sets of unambiguous events and acts derived above provide us with natural "endogenous" domains for a theory of subjective expected utility maximization. The decision maker assigns sharply defined probabilities only to those events that are revealed unambiguous by his behavior, assigning interval-valued probabilities to all the other events. Observe that nothing in our analysis prevents the trivial case $\Lambda = \{\emptyset, S\}$, in which SEU maximization never really appears. This is a difference with Epstein and Zhang's analysis, in which the set of unambiguous events is very rich by axiomatic requirement on the preferences.

As it is apparent from the statement, there is a sense in which our requirement on preferences is more stringent than Epstein–Zhang's. We look for a set of acts on which the preference $\succcurlyeq$ satisfies the full-blown SEU model of Savage, rather than just being probabilistically sophisticated in the sense of Machina and Schmeidler. The difference

[14] As observed by Kopylov (2007), one can use Zhang's (2002) definition of unambiguous event to obtain a "fully subjective" SEU model, similarly to what we do here. The axiomatics and the sets of unambiguous events being different, the results are not equivalent.

has more than just theoretical significance. The Epstein–Zhang construction is based on a definition of unambiguous event which implies that $\Lambda = \Sigma$, i.e., every event is unambiguous, when the decision maker is probabilistically sophisticated. However, as discussed at length in Ghirardato and Marinacci (2002), a probabilistically sophisticated decision maker might still be reacting to the presence of ambiguity. The only way to make sure that he is not, is to have a (rich enough) collection of events which are *exogenously known* to be unambiguous as a calibration device. Therefore, the conclusion that all events are unambiguous to a probabilistically sophisticated decision maker hinges on an exogenous notion of ambiguity of events that we dispense with.

A problem that is common to all such "fully subjective" approaches is that the domain of the probability charge may be far from being unique. That is, while our set Λ is certainly unique, it is not true that one cannot find another set of events on which $\succcurlyeq$ has an SEU representation. Just to make a simple example, suppose that $\succcurlyeq$ is a CEU preference on a finite S, and consider any monotonic class like $\Gamma = \{\{s_1\}, \{s_1, s_2\}, \ldots, S\}$. Given the family of acts that are Γ-measurable, there is a probability P that represents $\succcurlyeq$, as all such acts are comonotonic. On the other hand, one would have a hard time arguing that Γ is a natural domain for a "fully subjective" theory. But even imposing structural requirements on the domain (e.g., that it be a λ-system) is not enough to uniquely identify it in general.[15] There might be a multiplicity of "endogenous domains" for subjective probability, so that the choice of one must be motivated by considerations other than identifying where the decision maker is capable of formulating sharp probabilities.

6.4 Unambiguous events and weak probabilistic sophistication

A result of Marinacci (2002) shows that preferences that 1) have an α-MEU representation (with constant $\alpha \neq 1/2$) and 2) are probabilistically sophisticated with respect to a non-atomic prior collapse to SEU as soon as the set of priors used in the representation induces a "non-trivial" Λ (see below). Indeed, the result requires an even weaker condition than probabilistic sophistication. We say that a probability $P \in ba_1(\Sigma)$ is **convex-ranged** on Σ if for each $B \in \Sigma$ and each $\alpha \in [0, P(B)]$ there exists $\Sigma \ni A \subseteq B$ such that $P(A) = \alpha$.

Definition 4 A binary relation $\succcurlyeq$ on $\mathscr{F}$ has **weak probabilistic beliefs** if there exists a convex-ranged $P^* \in ba_1(\Sigma)$ and $x \succ y$ such that for each $A, B \in \Sigma$,

$$P^*(A) = P^*(B) \Longrightarrow x\,A\,y \sim x\,B\,y.$$

Thus, a preference has weak probabilistic beliefs if the indifference sets of the likelihood relation obtained by considering bets on events (with fixed payoffs $x \succ y$) contain the level sets of the probability P^*. The condition is weaker than probabilistic

[15] A similar observation is made by Kopylov (2007) about his results, although he uses the weaker notion of mosaic.

sophistication, as it does not require full agreement between the ranking induced by P^* and the likelihood ordering.[16]

We show that Marinacci's result generalizes to a broad class of MBA preferences violating the constant ambiguity index assumption.[17] It is only needed that ambiguity attitudes over bets do not fluctuate in an "inconsistent" fashion; that is, that condition (8) holds.

Proposition 21 *Let $\succcurlyeq$ be an MBA preference with unambiguous events Λ and let Σ be a σ-algebra. Suppose that $\succcurlyeq$ satisfies condition (8) for all $A \in \Sigma \setminus \Lambda$ and that C only contains probability measures and satisfies $C(\Lambda) \neq \{0, 1\}$. The following conditions are equivalent:*

(i) *$\succcurlyeq$ has weak probabilistic beliefs;*
(ii) *$\succcurlyeq$ is a SEU preference whose beliefs are represented by a non-atomic probability measure P^*.*

Marinacci's original result is an impossibility statement: under the assumptions of his theorem, probabilistic sophistication is compatible with α-MEU preferences only in the degenerate case of EU preferences. Our extension shows that Marinacci's result is indeed much more sweeping than that. In particular, it applies also to CEU preferences. Of course, the discussion in Marinacci (2002) on the importance of the assumptions in the theorem still applies. In particular, we want to emphasize a simple example of a class of CEU preferences which is probabilistic sophisticated without being SEU.

Example 6 On an infinite state space (S, Σ), consider a non-atomic probability measure P and a strictly convex transformation function $\varphi : [0, 1] \to [0, 1]$, increasing and satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$. Then, a CEU preference $\succcurlyeq$ with (some utility u and) capacity $\rho = \varphi(P)$—a subjective Rank-Dependent EU preference—is probabilistically sophisticated and not SEU. Notice that $\succcurlyeq$ is MBA (indeed, invariant biseparable) and satisfies condition (7), since it has $\alpha \equiv 1$ by the strict convexity of φ. However, it can be checked that for $\succcurlyeq$ we have $\Lambda = \{\emptyset, S\}$, so that there is no non-trivial unambiguous event.

We close by recalling an axiom from GMM which can be employed to ensure that, as in the assumptions of Proposition 21, all the elements of the set C are probability *measures*, rather than charges:

Axiom 8 (Monotone Continuity) *For all $x, y, z \in X$ such that $y \succ z$, if $\{A_n\}_{n \in \mathbb{N}}$ is a sequence in Σ such that $A_n \downarrow \emptyset$, then there exists $\bar{n}$ such that $y \succcurlyeq^* x A_{\bar{n}} z$.*

It is immediate to see that Proposition B.1 in GMM extends to MBA preferences, showing that in the presence of the previous axioms, Monotone Continuity is necessary and sufficient for C to contain only probability measures.

[16] Moreover, probabilistic sophistication imposes further requirements beyond the existence of probabilistic beliefs. While the requirement that P^* be convex-ranged is not strictly speaking part of the definition of probabilistic sophistication, all the existing axiomatizations of probabilistic sophistication in a fully subjective setting—first and foremost Machina and Schmeidler (1992)—characterize preferences inducing convex-ranged beliefs.

[17] Similar results are proved by Cerreia-Vioglio et al. (2009) and Strzalecki (2010).

Appendix A: Proofs of the results in Section 3

A.1 Proof of Proposition 1

We just prove the necessity part of the statement. Sufficiency follows from routine arguments. Since $\succsim$ satisfies Weak Order, Risk Independence, Archimedean, by Kreps (1988, Theorem 5.11) it follows that there exists an affine function $u : X \to \mathbb{R}$ such that $x \succsim y$ if and only if $u(x) \geq u(y)$. Since $\succsim$ in non-trivial and satisfies Monotonicity, u is non-constant. We next show that each f in $\mathscr{F}$ admits a certainty equivalent.

Claim 1 For each $f \in \mathscr{F}$ there exists $x_f \in X$ such that $x_f \sim f$.

Proof of the Claim: Since $f(S)$ is a finite subset of X and since $\succsim$ is a Weak Order and it satisfies Monotonicity, it follows that there exist two consequences x_1 and x_0 in X such that $x_1 \succsim f \succsim x_0$. We denote by $x_\alpha = \alpha x_1 + (1-\alpha) x_0$ for all $\alpha \in [0, 1]$. If either $x_0 \sim f$ or $x_1 \sim f$ then the statement follows. Otherwise, we have that $x_1 \succ f \succ x_0$. Define

$$U = \{\alpha \in (0,1) : \alpha x_1 + (1-\alpha) x_0 \succ f\}$$
$$\textit{and}$$
$$L = \{\beta \in (0,1) : f \succ \beta x_1 + (1-\beta) x_0\}.$$

Since $\succsim$ satisfies Archimedean, it follows that U and L are non-empty. Moreover, since $\succsim$ satisfies Weak Order and u is affine, we have that

$$\alpha > \beta \quad \forall \alpha \in U, \forall \beta \in L. \tag{9}$$

Define $\bar{\alpha} = \inf_{\alpha \in U} \alpha$ and $\bar{\beta} = \sup_{\beta \in L} \beta$. By (9), it is immediate to see that $\bar{\alpha} \geq \bar{\beta}$. Since U and L are non-empty, we have that $1 > \bar{\alpha} \geq \bar{\beta} > 0$. Then, we have three cases:

1. $x_{\bar{\alpha}} \sim f$. The statement follows by imposing $x_f = x_{\bar{\alpha}}$.
2. $\bar{\alpha} \in U$. It follows that $x_{\bar{\alpha}} \succ f$. Since $\succsim$ satisfies Archimedean, it follows that there exists $\lambda \in (0, 1)$ such that
$$x_{\lambda\bar{\alpha}} = \lambda x_{\bar{\alpha}} + (1-\lambda) x_0 \succ f,$$
thus $\lambda\bar{\alpha} \in U$ and $\lambda\bar{\alpha} < \bar{\alpha}$. This is a contradiction with $\bar{\alpha} = \inf_{\alpha \in U} \alpha$.
3. $\bar{\alpha} \notin U$ and $x_{\bar{\alpha}} \not\sim f$. Since $\succsim$ satisfies Weak Order, it follows that $f \succ x_{\bar{\alpha}}$, that is, $\bar{\alpha} \in L$. Since $\bar{\alpha} \geq \bar{\beta} = \sup_{\beta \in L} \beta \geq \bar{\alpha}$, this implies that $\bar{\alpha} = \bar{\beta}$. Since $\succsim$ satisfies Archimedean, it follows that there exists $\lambda \in (0, 1)$ such that
$$f \succ \lambda x_1 + (1-\lambda) x_{\bar{\beta}} = x_{\lambda + (1-\lambda)\bar{\beta}},$$
thus $\lambda + (1-\lambda)\bar{\beta} \in L$ and $\bar{\beta} < \lambda + (1-\lambda)\bar{\beta}$. This is a contradiction with $\bar{\beta} = \sup_{\beta \in L} \beta$. □

Notice that $u(X)$ is an interval and $B_0(\Sigma, u(X)) = \{u \circ f : f \in \mathscr{F}\}$. We define $I : B_0(\Sigma, u(X)) \to \mathbb{R}$ by

$$I(a) = u\left(x_f\right) \text{ where } f \in \mathscr{F} \text{ and } u \circ f = a.$$

In light of the previous claim, I is well defined. Indeed, pick $a \in B_0(\Sigma, u(X))$. Consider $f, g \in \mathscr{F}$ such that $u \circ f = a = u \circ g$. It follows that $u(f(s)) = a(s) = u(g(s))$ for all $s \in S$. Since u represents $\succsim$ over X, it follows that $f(s) \sim g(s)$ for all $s \in S$. By Monotonicity, we can conclude that $f \sim g$. Since $\succsim$ satisfies Weak Order, it follows that $x_f \sim x_g$. Thus, we have that

$$u\left(x_f\right) = I(a) = u\left(x_g\right).$$

Consider $a, b \in B_0(\Sigma, u(X))$ such that $a(s) \geq b(s)$ for all $s \in S$. It follows that there exist $f, g \in \mathscr{F}$ such that $u \circ f = a$ and $u \circ g = b$. Since $a \geq b$ and $\succsim$ satisfies Monotonicity, it follows that $f \succsim g$. Since $\succsim$ satisfies Weak Order and u represents $\succsim$ on X, we thus obtain that

$$x_f \succsim x_g \text{ and } I(a) = u\left(x_f\right) \geq u\left(x_g\right) = I(b).$$

Next, we show that I is normalized. Pick $k \in u(X)$. By assumption, there exists $x \in X$ such that $u(x) = k$. Moreover, if $a = k1_S$ then $a = u \circ f$ where $f = x$. Notice that x_f can be chosen to be equal to x. By definition of I, it follows that

$$I(a) = u\left(x_f\right) = u(x) = k.$$

Pick $f, g \in \mathscr{F}$. Since $\succsim$ satisfies Weak Order and u represents $\succsim$ on X, we have that

$$f \succsim g \Leftrightarrow x_f \succsim x_g \Leftrightarrow u\left(x_f\right) \geq u\left(x_g\right) \Leftrightarrow I(u \circ f) \geq I(u \circ g). \tag{10}$$

Finally, we prove the continuity of I. Since I is normalized and monotonic, $I(B_0(\Sigma, u(X))) = u(X)$. Consider $a, b \in B_0(\Sigma, u(X))$ such that $a \leq b$ and $I(b) > k$ where $k \in \mathbb{R}$. It follows that there exist f and g in $\mathscr{F}$ such that $a = u \circ f$ and $b = u \circ g$. We have two cases:

1. $I(a) > k$. In this case, $B_0(\Sigma, u(X)) \ni \alpha b + (1-\alpha) a \geq a$ for all $\alpha \in (0, 1)$. Since I is monotonic, it follows that

$$I(\alpha b + (1-\alpha) a) \geq I(a) > k.$$

2. $I(a) \leq k$. Since $I(b) > k$, we have that there exists $k' \in u(X)$ such that $I(b) > k' > k \geq I(a)$. This implies that there exists $x' \in X$ such that $u(x') = k'$. By (10), we have that $g \succ x' \succ f$. Since $\succcurlyeq$ satisfies Archimedean, it follows that there exists $\alpha \in (0,1)$ such that $\alpha g + (1-\alpha) f \succ x'$. Since u is affine and by (10), we have that

$$I(\alpha b + (1-\alpha) a) = I(u \circ (\alpha g + (1-\alpha) f)) > I(u(x')) = u(x') = k' > k.$$

It follows that I satisfies condition (iv) in Lemma 45 of C3M. By Proposition 46 of C3M, I is lower semicontinuous. Upper semicontinuity follows by a symmetric argument.

The uniqueness part of the statement follows from routine arguments.

Appendix B: Proofs of the results in Section 4

B.1 Proof of Proposition 6

Suppose $\succcurlyeq$ is biseparable, so $\rho_{x,y}$ is independent of x, y. Then, for each $x, y \in X$ with $x \succ y$ and each $A \in \Sigma$, $I(u \circ x\,A\,y) = \rho(A)u(x) + [1-\rho(A)]u(y)$. Furthermore, if $x \sim y$, $I(u \circ x\,A\,y) = I(u(x)) = u(x) = \rho(A)u(x) + [1-\rho(A)]u(y)$ where the first equality follows from the normalization and monotonicity of I. Thus, $I(u \circ x\,A\,y) = \rho(A)u(x) + [1-\rho(A)]u(y)$ whenever $x \succcurlyeq y$.

Now, for any two binary acts f, g, we can always choose $A, A' \in \Sigma$ so that $f = x\,A\,y$ and $g = x'\,A'\,y'$, with $x \succcurlyeq y$ and $x' \succcurlyeq y'$. Then, for each $z \in X$ and $\lambda \in (0,1]$, $\lambda f + (1-\lambda)z = (\lambda x + (1-\lambda)z)\,A\,(\lambda y + (1-\lambda)z)$ and so $I(u \circ [\lambda f + (1-\lambda)z]) = \rho(A)u(\lambda x + (1-\lambda)z) + [1-\rho(A)]u(\lambda y + (1-\lambda)z) = \lambda I(u \circ f) + (1-\lambda)u(z)$ and similarly for $\lambda g + (1-\lambda)z$. Axiom 5 follows.

In the opposite direction, suppose Axiom 5 holds. Fix $A \in \Sigma$ and consider the fictitious state space $S_A = \{s, t\}$ acts $\mathscr{F}_A = X^{S_A}$, and preferences $\succcurlyeq_A$ on $\mathscr{F}_A$ defined by $f_A \succcurlyeq_A g_A$ if and only if $f_A(s)\,A\,f_A(t) \succcurlyeq g_A(s)\,A\,g_A(t)$ for all $f_A, g_A \in \mathscr{F}_A$. Then $\succcurlyeq_A$ satisfies the GMM axioms and admits a representation (I_A, u_A), with I_A normalized, monotonic, and, constant-linear. Furthermore, we can assume w.l.o.g. that $u_A = u$, because $\succcurlyeq_A$ and $\succcurlyeq$ agree on constant acts.

Now consider $x, y, x', y' \in X$ with $x \succ y$ and $x' \succ y'$. There exist $\alpha, \beta \in \mathbb{R}$, with $\alpha > 0$, such that $\alpha u(x) + \beta = u(x')$ and $\alpha u(y) + \beta = u(y')$: hence, if $f_A, g_A \in \mathscr{F}_A$ are defined by $f_A(s) = x$, $f_A(t) = y$, $g_A(s) = x'$ and $g_A(t) = y'$, we have $I_A(u \circ g_A) = \alpha I_A(u \circ f_A) + \beta$. Therefore, if $c_{f_A}, c_{g_A} \in X$ are the $\succcurlyeq_A$–certainty equivalents of f_A and g_A respectively then $u(c_{g_A}) = \alpha u(c_{f_A}) + \beta$ as well.

Now $c_{f_A} \sim_A f_A$ if and only if $c_{f_A} \sim x\,A\,y$ and similarly $c_{g_A} \sim_A g_A$ if and only if $c_{g_A} \sim x'\,A\,y'$. It follows that $I(u \circ x'\,A\,y') = u(c_{g_A}) = \alpha u(c_{f_A}) + \beta = \alpha I(u \circ x\,A\,y) + \beta$. Equation (5) and the fact that $\alpha u(x) + \beta = u(x')$ and $\alpha u(y) + \beta = u(y')$ then imply that $\rho_{x,y}(A) = \rho_{x',y'}(A)$.

Hence, a set function $\rho : \Sigma \to [0,1]$ that satisfies Eq. (3) can be uniquely defined. Since I is normalized and monotonic, it is straightforward to verify that ρ is in fact a capacity.

Appendix C: Proofs of the results in Section 5

C.1 Proof of Theorem 10

Assume that $\succsim$ satisfies Weak Order, Monotonicity, Risk Independence, and Archimedean (Continuity). By Proposition 1, it follows that $\succsim$ satisfies Continuity as defined in C3M. Recall that $\succsim$ has no worst consequence.

(i) implies (ii). By Theorem 3 of C3M, if $\succsim$ satisfies Convexity then there exists a non-constant affine function $u : X \to \mathbb{R}$ and a function $G^\star : u(X) \times ba_1(\Sigma) \to (-\infty, \infty]$ such that the functional $I : B_0(\Sigma, u(X)) \to \mathbb{R}$, defined by

$$I(a) = \min_{P \in ba_1(\Sigma)} G^\star \left(\int a \, \mathrm{d}P, P \right),$$

is normalized, well defined, and such that

$$f \succsim g \Leftrightarrow I(u \circ f) \geq I(u \circ g).$$

Moreover, $G^\star(t, P) = \sup_{h \in \mathscr{F}} \left\{ u(x_h) : \int u \circ h \, dP \leq t \right\}$ for all $(t, P) \in u(X) \times ba_1(\Sigma)$. Fix an act $f \in \mathscr{F}$. Consider $P_f \in ba_1(\Sigma)$ such that $G^\star \left(\int u \circ f \, dP_f, P_f \right) = I(u \cdot f)$. Define $t = \int u \circ f \, dP$. Assume that $g \in \mathscr{F}$ is such that $\int u \circ f \, dP_f \geq \int u \circ g \, dP_f$. By the definition of $G^\star(t, P_f)$ and since $t \geq \int u \circ g \, dP_f$, it follows that $u(x_g) \leq G^\star(t, P_f) = I(u \circ f)$. Since I is normalized, it follows that $I(u \circ g) = I(u(x_g)) \leq I(u \circ f)$, that is, $f \succsim g$. Summing up, if we define the binary relation $\geq_f$ on $\mathscr{F}$ by

$$f_1 \geq_f f_2 \Leftrightarrow \int u \circ f_1 \, \mathrm{d}P \geq \int u \circ f_2 \, dP$$

then we have that $f \geq_f g$ implies that $f \succsim g$. Since f was arbitrarily chosen, the statement follows.

(ii) implies (i). By Proposition 1, it follows that there exists a non-constant affine function $u : X \to \mathbb{R}$ and a normalized, monotonic, and continuous functional $I : B_0(\Sigma, u(X)) \to \mathbb{R}$ such that $f \succsim g$ if and only if $I(u \circ f) \geq I(u \circ g)$. We define $G^\star : u(X) \times ba_1(\Sigma) \to (-\infty, \infty]$ by

$$G^\star(t, P) = \sup_{h \in \mathscr{F}} \left\{ u(x_h) : \int u \circ h \, \mathrm{d}P \leq t \right\} \qquad \forall (t, P) \in u(X) \times ba_1(\Sigma).$$

Notice that $G^\star(\cdot, P) : \mathbb{R} \to (-\infty, \infty]$ is an increasing function for all $P \in ba_1(\Sigma)$. Moreover, observe that $I(u \circ f) = u(x_f) \leq G^\star \left(\int u \circ f \, dP, P \right)$ for all $f \in \mathscr{F}$ and for all $P \in ba_1(\Sigma)$. It follows that

$$I(u \circ f) \leq \inf_{P \in ba_1(\Sigma)} G^\star \left(\int u \circ f \mathrm{d}P, P \right) \qquad \forall f \in \mathscr{F}.$$

Pick $f \in \mathscr{F}$. By assumption, there exists a non-trivial SEU preference $\geq_f$ such that

$$f \geq_f g \Rightarrow f \succsim g.$$

In other words, we have that there exists $\bar{P} \in ba_1(\Sigma)$ such that

$$\int u \circ f \, \mathrm{d}\bar{P} \geq \int u \circ g \, \mathrm{d}\bar{P} \Rightarrow I(u \circ f) \geq I(u \circ g).$$

By definition of $G^\star$, this implies that

$$G^\star\left(\int u \circ f \, \mathrm{d}\bar{P}, \bar{P}\right) = I(u \circ f).$$

Since f was arbitrarily chosen, we can conclude that

$$I(u \circ f) = \min_{P \in ba_1(\Sigma)} G^\star\left(\int u \circ f \, \mathrm{d}P, P\right) \qquad \forall f \in \mathscr{F}. \tag{11}$$

Consider $f, g \in \mathscr{F}$ such that $f \sim g$. Define $k = I(u \circ f) = I(u \circ g)$. Define

$$U_P(k) = \left\{h \in \mathscr{F} : G^\star\left(\int u \circ h \, \mathrm{d}P, P\right) \geq k\right\}.$$

Since $G^\star(\cdot, P)$ is an increasing function for all $P \in ba_1(\Sigma)$, it follows that $U_P(k)$ is closed under convex combinations for all $P \in ba_1(\Sigma)$. By (11), it follows that $f, g \in U_P(k)$ for all $P \in ba_1(\Sigma)$. This implies that

$$G^\star\left(\int u \circ (\alpha f + (1-\alpha) g) \, \mathrm{d}P, P\right) \geq k \qquad \forall \alpha \in (0,1), \forall P \in ba_1(\Sigma).$$

By (11), we can conclude that $I(u \circ (\alpha f + (1-\alpha) g)) \geq I(u \circ f)$, that is, $\alpha f + (1-\alpha) g \succsim f$. Since f and g were arbitrarily chosen, it follows that $\succsim$ satisfies Convexity.

Appendix D: Proofs of the results in Section 6

Throughout this appendix we write $\underline{C}(A)$ (resp. $\overline{C}(A)$) in place of $\underline{C}(1_A)$ (resp. $\overline{C}(1_A)$). We also write $\alpha_u \circ f$ in lieu of $\alpha(u \circ f)$. Notice that for expositional reasons, the results are proved in a different order than that in the main text.

We also make a useful observation. Call **reduced** an act f such that $f(s) \sim f(s')$ implies $f(s) = f(s')$. Given any non-reduced act f, we observe that there is a reduced act which, while being state-by-state indifferent to f, "simplifies" it by restricting its range so that it only contains non-indifferent payoffs. A **$\succsim$-reduction** g of f is a reduced act $g = \{x_1, A_1; \ldots; x_n, A_n\}$, with $x_1 \succ x_2 \succ \ldots \succ x_n$ and $\{A_1, \ldots, A_n\}$ a partition of S in Σ such that $g(s) \sim f(s)$ for all $s \in S$. Finally, given a reduced act $f = \{x_1, A_1; \ldots; x_n, A_n\}$, with $x_1 \succ x_2 \succ \ldots \succ x_n$ and $\{A_1, \ldots, A_n\}$ a partition of S in Σ, and a permutation σ of $\{1, 2, \ldots, n\}$. Define the **permuted act** f_σ as $f_\sigma = \{x_{\sigma(1)}, A_1; \ldots; x_{\sigma(n)}, A_n\}$. The following lemma is immediately verified.

Lemma 22 *Let $\succcurlyeq$ be an MBA preference. f is unambiguous if and only if there exists some $\succcurlyeq$-reduction g of f for which g_σ is crisp for all permutation σ of g's payoffs.*

Proof Note that a $\succcurlyeq$-reduction of an act f is a $\succcurlyeq$-permutation according to Definition 2. Hence, if f is unambiguous and g is a $\succcurlyeq$-reduction of g , each permutation of g is a $\succcurlyeq$-permutation of f, and therefore, it is crisp. Conversely, let $\bar{f}$ be a $\succcurlyeq$ -permutation of f, and let g be a $\succcurlyeq$-reduction of f for which g_σ is crisp for all permutation σ. In particular, there exists a permutation $\bar{\sigma}$ such that $g_{\bar{\sigma}}(s) \sim \bar{f}(s)$ for all s. By assumption, $g_{\bar{\sigma}}$ is crisp, so $g_{\bar{\sigma}} \sim^* x$ for some $x \in X$. But then, by monotonicity of $\succcurlyeq^*$, also $\bar{f} \sim^* x$, that is, $\bar{f}$ is crisp. Thus, f is unambiguous.

D.1 Proof of Proposition 11

Let $\mathscr{U}'$ be the set defined in the statement of the proposition. More precisely, let $\mathscr{U}'$ be the union of all sets $\mathscr{V}$ of crisp acts that are closed under $\succcurlyeq$ -permutations. Notice that, if f is crisp, the set of all $\succcurlyeq$ -permutations of f is one such set $\mathscr{V}$, because the $\succcurlyeq$-permutation relation is an equivalence. Furthermore, all constants are crisp; thus, $\mathscr{U}'$ is both well defined and non-empty.

We will prove that $\mathscr{U} = \mathscr{U}'$. We begin with the observation that any act f whose $\succcurlyeq$-permutations are all crisp must belong to $\mathscr{U}'$. In fact, if $f \notin \mathscr{U}'$, one could add f and all its $\succcurlyeq$-permutations to $\mathscr{U}'$, thus obtaining a larger set and contradicting the definition of $\mathscr{U}'$. Conversely, if $f \in \mathscr{U}'$, then any $\succcurlyeq$-permutation of f must be in $\mathscr{U}'$, hence crisp. This proves that f is unambiguous.

D.2 Proofs of Propositions 13 and 14, and of Corollary 15

We first prove a useful lemma:

Lemma 23 *Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c \in \mathbb{R}$ be such that $\sum_{h=1}^n a_h b_{\sigma(h)} = c$ for all permutations $\sigma \in Per(n)$. Then either $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.*

Proof By contradiction, assume that there exist $i, j \in \{1, \ldots, n\}$ such that $a_i \neq a_j$ and $k, l \in \{1, \ldots, n\}$ such that $b_k \neq b_l$. Consider a permutation σ such that $\sigma(i) = k$ and $\sigma(j) = l$, and the permutation $\sigma' = \sigma(kl)$ obtained applying σ and then switching around k and l. It follows that

$$a_i b_k + a_j b_l + \sum_{h \neq i,j} a_h b_{\sigma(h)} = \sum_{h=1}^n a_h b_{\sigma(h)} = c = \sum_{h=1}^n a_h b_{\sigma'(h)}$$
$$= a_i b_l + a_j b_k + \sum_{h \neq i,j} a_h b_{\sigma(h)},$$

whence $a_i b_k + a_j b_l = a_i b_l + a_j b_k$. That is, $a_i(b_k - b_l) = a_j(b_k - b_l)$ which implies $a_i = a_j$, a contradiction.

D.2.1 Proofs of Propositions 13 and 14

We prove the two Propositions by showing that the following statements are equivalent for each $A \in \Sigma$:

(i) $A \in \Lambda$;
(ii) $P(A) = Q(A) = \rho_{x,y}(A)$ for all $P, Q \in C$ and $x \succ y$;
(iii) For each $x \nsim y$, the act $x\ A\ y$ is crisp;
(iv) For some $x \succ y$, the act $x\ A\ y$ is crisp.

Proposition 13 follows from point (i) being equivalent to point (iii). Proposition 14 basically follows from point (i) being equivalent to point (ii).

(i) ⇒ (ii): Suppose that $A \in \Lambda$. Therefore, there exists $f \in \mathscr{U}$ and $x \in X$ such that $A = \{s \in S : f(s) \sim x\}$. Since $f \in \mathscr{U}$ there exists a reduction $\{x_i, A_i\}_{i=1}^n$ of f (with $x_i \nsim x_j$ for every $i \neq j$) such that for every permutation $\sigma \in \text{Per}(n)$, $\{x_{\sigma(i)}, A_i\}_{i=1}^n$ is crisp. This implies that for each $P, Q \in C$

$$\sum_{i=1}^{n} u\left(x_{\sigma(i)}\right) P(A_i) = \sum_{i=1}^{n} u\left(x_{\sigma(i)}\right) Q(A_i),$$

that is,

$$\sum_{i=1}^{n} [P(A_i) - Q(A_i)]\, u\left(x_{\sigma(i)}\right) = 0. \tag{12}$$

By previous lemma, either $P(A_1) - Q(A_1) = P(A_2) - Q(A_2) = \cdots = P(A_n) - Q(A_n) = b$ or $u(x_1) = u(x_2) = \cdots = u(x_n)$. In the former case $1 = \sum_{i=1}^{n} P(A_i) = \sum_{i=1}^{n} Q(A_i) + nb = 1 + nb$. This implies that $b = 0$ and A_i satisfies condition (ii) for all $i = 1, 2, \ldots, n$. As $A \in \{A_i : i = 1, \ldots, n\}$, the conclusion follows. In the latter case, $n = 1$ and $A = \{f \sim x\}$ is then either S or $\emptyset$ (depending on whether $x \sim x_1$ or not). Clearly $P(S) = Q(S) = 1$ and $P(\emptyset) = Q(\emptyset) = 0$ for all $P, Q \in C$, so that once again (ii) follows. Notice finally that if $P(A) = Q(A)$ for all $P, Q \in C$, it then follows from the definition of $\rho_{x,y}$ that $\rho_{x,y}(A) = P(A) = Q(A)$.

(ii)⇒(iii): Let $x \nsim y$. If $x \succ y$ then

$$\begin{aligned} P(u(x\ A\ y)) &= (u(x) - u(y))\, P(A) + u(y) = (u(x) - u(y))\, Q(A) + u(y) \\ &= Q(u(x\ A\ y)) \end{aligned}$$

for all $P, Q \in C$. That is, $x\ A\ y$ is crisp. If $y \succ x$ then $x\ A\ y = y\ A^c\ x$. By the previous part of the proof, it follows that

$$P(u(x\,A\,y)) = P(u(y\,A^c\,x)) = Q(u(y\,A^c\,x)) = Q(u(x\,A\,y)),$$

proving the statement.

(iii)⇒(iv): Obvious.

(iv)⇒(i): Let $x \succ y$ be such that $x\,A\,y$ is crisp. We want to show that $f = x\,A\,y \in \mathscr{U}$. This is the case if f has a $\succcurlyeq$-reduction whose permutations are all crisp. But f is a reduced act, and the only permutation of f is $g = x\,A^c\,y$. Since f is crisp,

$$\begin{aligned} P(u(x\,A\,y)) &= (u(x) - u(y))\,P(A) + u(y) = (u(x) - u(y))\,Q(A) + u(y) \\ &= Q(u(x\,A\,y)) \end{aligned}$$

which implies that $P(A) = Q(A)$. In turn, this implies $P(A^c) = Q(A^c)$, so that

$$\begin{aligned} P(u(x\,A^c\,y)) &= (u(x) - u(y))\,P(A^c) + u(y) = (u(x) - u(y))\,Q(A^c) + u(y) \\ &= Q(u(x\,A^c\,y)) \end{aligned}$$

and g is also crisp.

D.2.2 Proof of Corollary 15

Since constant acts belong to X, it is immediate to check that $S \in \Lambda$. On the other hand, by Proposition 14, it follows that if $A \in \Lambda$ then $A^c \in \Lambda$. Finally, again by Proposition 14, if $A, B \in \Lambda$ and $A \cap B = \emptyset$ then for each $P, Q \in C$

$$P(A \cup B) = P(A) + P(B) = Q(A) + Q(B) = Q(A \cup B).$$

It is immediate to check that this implies that $\rho_{x,y}(A \cup B) = P(A \cup B)$ for all $P \in C$, proving that $A \cup B \in \Lambda$.

D.3 Proofs of Theorem 12 and Corollary 16

Using the definition of Λ and the characterizations of Propositions 13 and 14, the statements to be shown equivalent are reformulated as follows:

(i) $f \in \mathscr{U}$;
(ii) $\{s \in S : f(s) \succcurlyeq x\} \in \Lambda$ for all $x \in X$;
(iii) $\{s \in S : f(s) \sim x\} \in \Lambda$ for all $x \in X$;
(iv) $\{s \in S : u \circ f(s) \geq \gamma\} \in \Lambda$ for all $\gamma \in \mathbb{R}$;
(v) $\{s \in S : u \circ f(s) = \gamma\} \in \Lambda$ for all $\gamma \in \mathbb{R}$;
(vi) For each $\succcurlyeq$-reduction $\{x_i, A_i\}_{i=1}^n$ of f (with $x_i \not\sim x_j$ if $i \neq j$), $\{A_1, A_2, \ldots, A_n\}$ is a partition of S in Λ;
(vii) There exists a $\succcurlyeq$-reduction $\{x_i, A_i\}_{i=1}^n$ of f, with $\{A_1, A_2, \ldots, A_n\}$ a partition of S in Λ (and $x_i \not\sim x_j$ if $i \neq j$).

The equivalence of (i) and (vii) follows immediately from the argument used to show (i)⇒(ii) in Appendix D.2.1 and from Proposition 14. We shall now prove that statements (ii)–(vii) are equivalent.

(vii)⇒(ii): Given f, let $g = \{x_i, A_i\}_{i=1}^n$ be its $\succsim$-reduction with $\{A_1, A_2, \ldots, A_n\}$ a partition of S in Λ (and $x_i \not\sim x_j$ if $i \neq j$), so that $u \circ f = u \circ g = \Sigma_{i=1}^n u(x_i) 1_{A_i}$ and, in particular, $u \circ f(s) = u \circ g(s)$ for all $s \in S$. Since u represents $\succsim$ on X, it follows that $\{s \in S : f(s) \succsim x\} = \{s \in S : u \circ f(s) \geq u(x)\}$ for all $x \in X$. Hence, $\{s \in S : f(s) \succsim x\}$ is a disjoint union of elements of Λ, which is a finite λ-system.

(ii)⇒(iv): Notice that $u(X)$ is an interval. Let $\gamma \in \mathbb{R}$. If $\gamma \in u(X)$, say $\gamma = u(x')$, then $\{s \in S : u \circ f(s) \geq \gamma\} = \{s \in S : f(s) \succsim x'\} \in \Lambda$. Else, either $\gamma < t$ for all $t \in u(X)$, and then $\{s \in S : u \circ f(s) \geq \gamma\} = S \in \Lambda$, or $\gamma > t$ for all $t \in u(X)$, and then $\{s \in S : u \circ f(s) \geq \gamma\} = \emptyset \in \Lambda$.

(iv)⇒(v): Let $u \circ f = \Sigma_{i=1}^n \gamma_i 1_{A_i}$, with $\{A_1, A_2, \ldots, A_n\}$ a partition of S in Σ and $\gamma_1 > \gamma_2 > \ldots > \gamma_n$. If $\gamma \notin \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, then $\{s \in S : u \circ f(s) = \gamma\} = \emptyset \in \Lambda$. The set $A_1 = \{s \in S : u \circ f(s) = \gamma_1\} = \{s \in S : u \circ f(s) \geq \gamma_1\} \in \Lambda$. For each $i \geq 2$ then $\Lambda \ni \{s \in S : u \circ f(s) \geq \gamma_i\} = \{s \in S : u \circ f(s) \in \{\gamma_1, \gamma_2, \ldots, \gamma_i\}\} = \bigcup_{j=1}^i \left\{s \in S : u \circ f(s) = \gamma_j\right\} = A_1 \cup A_2 \cup \ldots \cup A_i$. Therefore, for each $i \geq 2$, $\{s \in S : u \circ f(s) = \gamma_i\} = A_i = (A_1 \cup A_2 \cup \ldots \cup A_i) \setminus (A_1 \cup A_2 \cup \ldots \cup A_{i-1}) \in \Lambda$. Indeed, remember that if Λ is a λ-system, $B, C \in \Lambda$, and $C \subseteq B$ then $B \setminus C \in \Lambda$.

(v)⇒(iii): For each $x \in X, \{s \in S : f(s) \sim x\} = \{s \in S : u \circ f(s) = u(x)\} \in \Lambda$.

(iii)⇒(vi): Consider $f \in \mathscr{F}$. let $g = \{x_i, A_i\}_{i=1}^n$ be any one of its $\succsim$-reductions, with $\{A_1, A_2, \ldots, A_n\}$ a partition of S in Σ (and $x_i \not\sim x_j$ if $i \neq j$). It follows that $u \circ f = u \circ g = \Sigma_{i=1}^n u(x_i) 1_{A_i}$. Therefore, $A_i = \{s \in S : u \circ f(s) = u(x_i)\} = \{s \in S : f(s) \sim x_i\} \in \Lambda$ for all $i = 1, \ldots, n$.

(vi)⇒(vii): Trivial.

D.4 Proofs of Propositions 17, 18 and 19

D.4.1 Proposition 17

Given $x \succ y$, define $\rho_{x,y}$ via Eq. (4). Then $\rho_{x,y}(A) + \rho_{x,y}(A^c) = 1$ if and only if

$$[\alpha_u(x\, A\, y)(\underline{C}(A) - \overline{C}(A)) + \overline{C}(A)] + [\alpha_u(x\, A^c\, y)(\underline{C}(A^c) - \overline{C}(A^c)) + \overline{C}(A^c)] = 1$$

which, since $\overline{C}(A^c) = 1 - \underline{C}(A)$ and $\underline{C}(A^c) = 1 - \overline{C}(A)$, is equivalent to

$$\alpha_u(x\, A\, y)(\underline{C}(A) - \overline{C}(A)) + \alpha_u(x\, A^c\, y)(\underline{C}(A) - \overline{C}(A)) + (\overline{C}(A) - \underline{C}(A)) = 0.$$

In turn, this is equivalent to

$$(\overline{C}(A) - \underline{C}(A)) = (\alpha_u(x\, A\, y) + \alpha_u(x\, A^c\, y))(\overline{C}(A) - \underline{C}(A))$$

Therefore, $\rho_{x,y}(A) + \rho_{x,y}(A^c) = 1$ if and only if either $\overline{C}(A) = \underline{C}(A)$ or $\alpha_u(x\, A\, y) + \alpha_u(x\, A^c\, y) = 1$.

Next, under ambiguity aversion, we show the equivalence of (iii) with (ii), and thus with (i). By Eq. (5) and given $x \succ y$, $I(u \circ (x\, A\, y)) + I(u \circ (x\, A^c\, y))$ can be rewritten as follows:

$$2u(y) + [u(x) - u(y)]\big[\alpha_u(x\,A\,y)(\underline{C}(A) - \overline{C}(A)) + \overline{C}(A) + \alpha_u(x\,A^c\,y)(\underline{C}(A^c) - \overline{C}(A^c)) + \overline{C}(A^c)\big].$$

After further rewriting and using the shorthand $\Delta C(A) = \overline{C}(A) - \underline{C}(A)$ and $\Delta u = u(x) - u(y)$,[18] we obtain

$$\begin{aligned} I(u\circ(x\,A\,y)) + I(u\circ(x\,A^c\,y)) &= 2u(y) + \Delta u\,\big[(\alpha_u(x\,A\,y) + \alpha_u(x\,A^c\,y) - 1)(-\Delta C(A)) + 1\big] \\ &= 2u(y) + \Delta u\,\big[(1 - \alpha_u(x\,A\,y) - \alpha_u(x\,A^c\,y))\Delta C(A) + 1\big] \\ &= u(x) + u(y) + \Delta u\,\big[(1 - \alpha_u(x\,A\,y) - \alpha_u(x\,A^c\,y))\Delta C(A)\big] \end{aligned} \tag{13}$$

If the decision maker is ambiguity averse then for each $A \in \Sigma$ and $P \in Core(I)$, $I(u\circ(x\,A\,y)) \le P(u\circ(x\,A\,y))$ and $I(u\circ(x\,A^c\,y)) \le P(u\circ(x\,A^c\,y))$, implying

$$I(u\circ(x\,A\,y)) + I(u\circ(x\,A^c\,y)) \le P(u\circ(x\,A\,y)) + P(u\circ(x\,A^c\,y)) = u(x) + u(y).$$

If condition (ii) holds, then we have $(1 - \alpha_u(x\,A\,y) - \alpha_u(x\,A^c\,y))\,\Delta C(A) = 0$ and hence $I(u\circ(x\,A\,y)) + I(u\circ(x\,A^c\,y)) = u(x) + u(y)$. We can then conclude that $I(u\circ(x\,A\,y)) = P(u\circ(x\,A\,y))$ and hence $\rho_{x,y}(A) = P(A)$, as required.

Conversely, suppose that $\rho_{x,y}(A) = P(A)$ hence $I(u\circ(x\,A\,y)) = P(u\circ(x\,A\,y))$ for all $A \in \Sigma$ and $P \in Core(I)$. Then

$$I(u\circ(x\,A\,y)) + I(u\circ(x\,A^c\,y)) = P(u\circ(x\,A\,y)) + P(u\circ(x\,A^c\,y)) = u(x) + u(y)$$

which implies $\Delta u\,[(1 - \alpha_u(x\,A\,y) - \alpha_u(x\,A^c\,y))\Delta C(A)] = 0$; that is, condition (ii).

D.4.2 Proposition 18

By Eq. (13), we see that for the given $x \succ y$, Eq. (8) holds if and only if

$$\frac{1}{2}u(x) + \frac{1}{2}u(y) \neq \frac{1}{2}u(x) + \frac{1}{2}u(y) + \frac{1}{2}\Delta u\left[\left(1 - \alpha_u(x\,A\,y) - \alpha_u(x\,A^c\,y)\right)\Delta C(A)\right]$$

[18] Notice that $x \succ y$ implies $u(x) > u(y)$ which in turn delivers $\Delta u > 0$.

which, in turn, holds if and only if

$$\big(1-\alpha_u(x\,A\,y)-\alpha_u(x\,A^c\,y)\big)\,\Delta C(A)\neq 0.$$

which proves the statement.

D.4.3 Proposition 19

We begin by recalling that, given a normalized representation (I,u) and $x \succ y$, $x\,A\,y \succcurlyeq x\,B\,y$ if and only if

$$\alpha_u(x\,A\,y)(\underline{C}(A)-\overline{C}(A))+\overline{C}(A)\geq\alpha_u(x\,B\,y)(\underline{C}(B)-\overline{C}(B))+\overline{C}(B)$$

with the left-hand (resp. right-hand) side collapsing to $P(A)=\underline{C}(A)=\overline{C}(A)$ (resp. $P(B)=\underline{C}(B)=\overline{C}(B)$) if $A\in\Lambda$ (resp. $B\in\Lambda$). Clearly, if there is a unique ρ for which Eq. (3) holds, $\alpha_u(x\,A\,y)$ does not depend on x or y. Hence, the implication (i) $\Rightarrow$ (ii) is trivial. We prove that (ii) $\Rightarrow$ (i).

It is enough to show that $\alpha_u(x\,A\,y)=\alpha_u(x'\,A\,y')$ for each u and $x\succ y, x'\succ y'$: this implies that $\rho_{x,y}(A)=\rho_{x',y'}(A)$ whenever $x\succ y, x'\succ y'$, so a set function $\rho:\Sigma\to[0,1]$ that satisfies Eq. (3) can be uniquely defined; it is then straightforward to verify that ρ is a capacity.

Thus, argue by contradiction and suppose w.l.o.g. that $\alpha_u(x\,A\,y)>\alpha_u(x'\,A\,y')$. By the richness assumption on $C(\Lambda)$, there exists $B\in\Lambda$ such that $P(B)=\underline{C}(B)=\overline{C}(B)$ satisfies

$$\alpha_u(x\,A\,y)(\underline{C}(A)-\overline{C}(A))+\overline{C}(A)<P(B)<\alpha_u(x'\,A\,y')(\underline{C}(A)-\overline{C}(A))+\overline{C}(A)$$

but then we have a violation of P4, since the first inequality implies $x\,B\,y\succ x\,A\,y$, and the second implies $x'\,A\,y'\succ x'\,B\,y'$. Thus, we must have $\alpha_u(x\,A\,y)=\alpha_u(x'\,A\,y')$. This completes the proof.

D.5 Proof of Proposition 21

The implication (ii) $\Rightarrow$ (i) is trivial. We prove (i) $\Rightarrow$ (ii). By weak probabilistic beliefs (assumption (i)), there exists $x\succ y$ and a convex-ranged probability charge P^* such that for all $A,B\in\Sigma$

$$P^*(A)=P^*(B)\Longrightarrow\rho_{x,y}(A)=\rho_{x,y}(B)$$

Consider now $A\in\Lambda$ such that $\underline{C}(A)=\overline{C}(A)=\rho_{x,y}(A)\in(0,1)$. It follows that $P^*(A)\in(0,1)$, since $P^*(A)=0$ (resp. $P^*(A)=1$) implies $P^*(A)=P^*(\emptyset)$ (resp. $P^*(A)=P^*(S)$), which in turn implies by (i) that $\rho_{x,y}(A)=\rho_{x,y}(\emptyset)=0$ (resp. $\rho_{x,y}(A)=\rho_{x,y}(S)=1$), a contradiction.

Let $B \in \Sigma$ be such that $P^*(B) = P^*(A)$, so that (i) implies $\rho_{x,y}(B) = \rho_{x,y}(A)$ and (since $P^*(B^c) = P^*(A^c)$ as well) $\rho_{x,y}(B^c) = \rho_{x,y}(A^c)$. It follows that

$$\rho_{x,y}(B) + \rho_{x,y}(B^c) = \rho_{x,y}(A) + \rho_{x,y}(A^c) = 1$$

where the last equality follows from Proposition 17.

We also know that $\succcurlyeq$ satisfies condition (8) for all $A \in \Sigma \setminus \Lambda$. It therefore follows from Propositions 17 and 18 that $\rho_{x,y}(B) + \rho_{x,y}(B^c) = 1$ implies $B \in \Lambda$, so that $\rho_{x,y}(B) = P(B)$ for all $P \in C$. We can thus conclude that with the chosen $A \in \Lambda$ we have for each $B \in \Sigma$ and each $P \in C$,

$$P^*(B) = P^*(A) \Longrightarrow P(B) = P(A)$$

so that $P^* = P$ follows from Theorem 2 of Marinacci (2002). Since this is true for each $P \in C$—that is, $C = \{P^*\}$—we conclude that $\succcurlyeq$ is a SEU preference with probability P^*.

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