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Abstract

In supervised work groups many factors concur to determine productivity. Some of them may be economical and some psychological. According to the literature, the heterogeneity in terms of individual capacity seems to be one of the principal causes for chaotic dynamics in a work group. May sorting groups of people with same capacity for effort be a solution? In the organizational psychology literature an important factor is the engagement in the task, while expectations are central in the economics literature. Therefore, we propose a dynamical model which takes into account both engagement in the task and expectations. An important lesson emerges. The intolerance deriving from the exposure to inequity may not be only caused by differences in individual capacities, but also by these factors combined. Consequently, solutions have to be found in this new direction.

1. Introduction

The importance of the perception of inequity cannot be underestimated when determining employees responses in the workplace. In fact, according to [8] concepts such as justice, fairness, and equity are of fundamental importance. In particular, according to [9] in modern justice research, the most widely studied allocation rule is that of equity: one's outcome should be proportional to one's inputs, that is, those who contribute more to a common task should reap most of the benefits. Among the different theories, Adam's contribution [1] is important in terms of explicitness [7] and rigor [24]. A model which extends Adam's theory considering past inequity is the one presented in [8]. This model allows us to consider situations resembling the pattern of the proverbial "straw that broke the camel's back". These kinds of situation can be modeled as a discontinuous function, i.e., an individual who so far has been tolerant to inequity, suddenly exhibits a rather intolerant behavior. Finally, equity theory has been extensively considered in customer–supplier relationships (see [33] and the references cited therein). Therefore, when considering authors such as [4], who advocate bringing the market inside the firms, the importance of equity theory assumes a new perspective and further relevance.

Recently, [10] has considered the effects of inequity in a work group. The authors assume that, as a result of the history of inequity individuals had been exposed to, the subordinates allocated their efforts in order to reduce the inequity. The analysis showed that having subordinate with the same capacities is quite different from having identical capacity subordinates. In fact, in the second case they were able to find chaotic dynamics in the subordinates' effort allocation.

Chaotic dynamics have several consequences: first of all, they make production of the team unpredictable; secondly, the work group production is not optimal; and finally, they make it extremely difficult for the supervisor to find incentives to increase production. Therefore, from the findings of [10] it seems natural to assume that a first step in this direction would be to have subordinates with the same capacity for effort. The literature has examined the problem of employees with different capacity from different perspectives. For example, [26] used participants' observation to examine quota restrictions in a work group of industrial workers. On the other hand, Economics also considers this problem; in fact, it is possible to find treatment of a principal delegating production to agents with different marginal costs (see for example [20]).

In this paper we examine the case of subordinates with the same capacity and propose a different reaction function. This function allow us to consider a wide range of behaviors such as imitation, compensation and also intolerance. Furthermore, by this function we are able to consider some other aspects such as engagement in the task and different expectations from the colleagues, see [32]. Our analysis allows us to confirm the results presented in [10], but also to find situations in which there may be chaotic dynamics even when the subordinates' capacity is identical.

In [10], besides heterogeneity in agents' capacity, only their intolerance to inequity was considered; there, the most interesting dynamics were the result of subordinates having different capacities. In the model we present in this paper several other aspects on which – according to [5] – the economic framework is silent, are considered. In particular, we consider not only intolerance to inequity but also engagement in the task and beliefs about the focal allocation; this allows us to study different reactions to the effort of the colleague and to show how disfunctional dynamics may occur also when subordinates have the same capacity.

The structure of the paper is the following. In Section 2 we describe the model of work group we consider, discuss the optimal incentive scheme when subordinates do not respond to inequity, and introduce how subordinates' reply is modeled according to how they react to the effort allocation by the colleague. Section 3 provides an exhaustive analysis of the equilibria when subordinates with different behavior are matched. On the other hand, Section 4 examines some example of dynamics; among the other there are examples in which subordinates with identical capacity exhibit chaotic dynamics. Finally, in the last section we discuss our results and examine future lines of research.

2. Model

The model of a supervised group we consider is the one analyzed in [10]. In this model a supervisor (acting as the principal) and two subordinates (acting as agents) cooperate. Agent *i* , (i=1,2) allocates his effort u_i with the supervisor and the effort l_i with the partner. The joint production function for agents is $\Gamma(u_1+u_2)^\alpha(l_1+l_2)^\beta$, where $\Gamma>0$ is a constant factor,[1] and $\alpha,\beta\in(0,1)$ are, respectively, the output elasticity with respect to the joint effort with the supervisor and with the partner. The agents have to decide both how much effort to exert, and how to partition it in the two complementary tasks.[2] Each agent *i* bears a cost for effort: agent *i* 's cost function $c_i : \mathbb{R}^2_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ will be denoted with $c_i(u_i,l_i)$; cost functions are private information. Furthermore, agent *i* can observe the level effort l_{-i} his partner provides with him,[3] but not the one which is provided with the supervisor. Conversely, the supervisor can only observe the joint output and the effort each agent provides with her. The supervisor's profit is a share $\gamma\in(0,1)$ of the supervised work group production minus the incentives she pays to her subordinates. In the following, we assume that the output is sold on market at unitary price, and that the production constant Γ and the sharing constant γ are such that $\Gamma\gamma=1$; this is not restrictive, it simplifies the notation, and allows us to simply consider monetary payoffs. Finally, agents' retribution consists of a fixed wage $w\geqslant 0$ plus a performance-contingent reward. In economic terms we assume that the participation constraint is met. From a different perspective, this means that the fixed wage is sufficient to meet physiological and safety needs in terms of the hierarchy of needs theory [23]. Although Maslow's theory has been examined and discussed (see [27] and [2]), recent studies found evidence that individuals tend to achieve basic and safety needs before other needs [31]. The performance-contingent reward is a linear incentive $\bar{b}_g$ proportional to the joint output of the team, and a linear incentive $\bar{b}_i$ on the effort each agent exerts with the supervisor. Therefore, the problem can be formalized as a bilevel programming problem:

$$\max_{\bar{b}_g, \bar{b}_1, \bar{b}_2} (1 - 2\bar{b}_g)(u_1 + u_2)^{\alpha}(l_1 + l_2)^{\beta} - \bar{b}_1 u_1 - \bar{b}_2 u_2 \tag{1}$$

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such that, given $\bar{b}_g, \bar{b}_1, \bar{b}_2$ subordinates solve

$$\max_{u_1, l_1} w + \bar{b}_g (u_1 + u_2)^{\alpha}(l_1 + l_2)^{\beta} + \bar{b}_1 u_1 - c_1(u_1, l_1) \tag{2}$$

$$\max_{u_2, l_2} w + \bar{b}_g (u_1 + u_2)^{\alpha}(l_1 + l_2)^{\beta} + \bar{b}_2 u_2 - c_2(u_2, l_2)$$

It is not restrictive to assume w=0, this simplify the analysis.

As in [10] the cost function we consider is the following:

$$c_i(u_i, l_i) = \begin{cases} 0 & \text{if } u_i + l_i \leqslant \bar{c}_i \\ +\infty & \text{if } u_i + l_i > \bar{c}_i \end{cases} \tag{3}$$

First observe that this cost function is non decreasing with respect to the aggregated effort. This kind of cost function assumes that each subordinate has a physical capacity $\bar{c}_i$ under which the effort has zero cost, or, alternatively, that at some exertion level the effort becomes unpleasant enough to lead the individual to conclude that it is not worth working any harder independently of the reward. In this case we assume that each individual knows his individual capacity and uses it without goldbricking. This cost function assumes that the cost for discretionary effort is null and is appropriate when modeling well motivated subordinates with high self-efficacy as discussed in [11].

As proved in [11], when considering fully rational agents the optimal incentive scheme is

$$\begin{cases} \bar{b}_g = \varepsilon > 0 \\ \bar{b}_1 = 0 \\ \bar{b}_2 = 0 \end{cases} \tag{4}$$

with effort allocation[4] of subordinates

$$(u_i, l_i) = \left(\frac{\alpha}{\alpha + \beta} \bar{c}_i, \frac{\beta}{\alpha + \beta} \bar{c}_i \right) \tag{5}$$

for i=1,2.

2.1. Modeling the behavior of same capacity agents

Rational economic agents decide their effort allocation in terms of best reply to the colleague's effort allocation. As discussed by several authors, when considering employment relations the economic framework is silent on many aspects, see for example [5]. Furthermore, [16, p. 10] state that "Many decisions seem to be made on the basis of factors other than cognitive ones, that is, factors other than estimations of probabilities, gains, costs, and the like".

According to the organizational psychology literature, several factors influence organizational behavior. The relationship between attitudes and employees' behavior are analyzed in [25]. Among the major job attitudes, it is well known that employee engagement plays an important effect of productivity, outcomes and profit (see for instance [19]). According to [25], employee engagement can be defined as the individual's involvement with, satisfaction with, and enthusiasm for, the work he/she does. In our model we consider subordinate *i* 's engagement to the common task and model it with a parameter $e_i \in [0,1]$. The larger the value of e_i, the more the engagement. It should be noted that, when for both subordinates the engagement in the common task is equal to $\beta/(\alpha+\beta)$, then the situation is similar to the one described in[10], where the engagement was implicitly assumed to be $\beta/(\alpha+\beta)$ for both subordinates. On the other hand, it must be observed that when for both subordinates the engagement is smaller than this threshold it is impossible to maximize the production.

Each agent *i* has capacity $\bar{c}_i$ to be allocated in the two different tasks. Obviously, the sum of the effort exerted with the colleague and the effort exerted with the supervisor must not exceed the capacity. Since each agent can observe only the effort that the partner exerts in the common task, and not the effort exerted with the supervisor, we assume that agents may have different reactions when observing the colleague exerting an effort which is not the focal one (see for example [1], [8] and [10]).

According to [8], the perceived inequity accumulates over time, and when a certain threshold is reached the agent may overreact when facing what appears to be a relatively minor inequity; see section "The straw that broke the camel's back" in [8]. Following [10], we consider situations in which such a threshold has already been exceeded and, in order to reduce tension, subordinates reallocate their efforts on the two tasks, altering their inputs in Adams' formulation.

In this paper we consider subordinates with the same capacity $\bar{c} = \bar{c}_1 = \bar{c}_2$. Therefore, when capacity $\bar{c}$ is normalized, the focal effort allocation (5) becomes

$$(u_1, l_1) = (u_2, l_2) = \left(\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}\right). \tag{6}$$

In particular, the reaction of an agent who observes a colleague exerting an effort lower than the focal level $\frac{\beta}{\alpha+\beta}$ can be of (Fig. 1(a)):

•retaliative imitation – the agent reduces his effort with the colleague;

•tolerance – the agent keeps the focal allocation;

•compensation – the agent compensates the colleague's lower effort in the common task increasing his effort with the colleague.

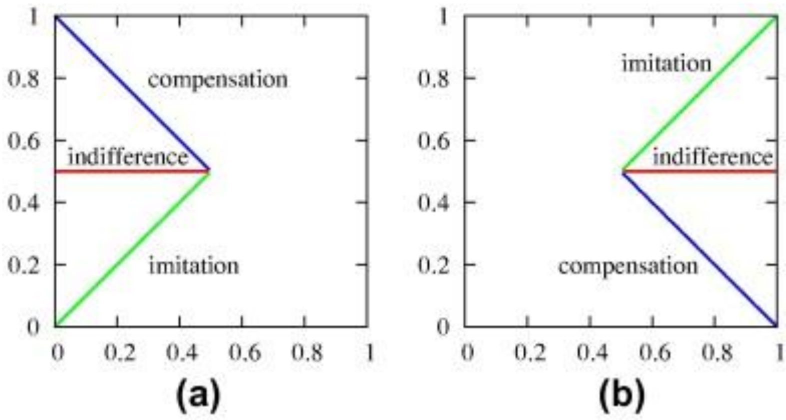

Fig. 1 Possible reactions of agent 2 when observing the colleague effort being different from the focal – here atl_i=0.5. (a) l_1<0.5; (b) l_1>0.5.

Similarly, an agent who observes a colleague exerting an effort higher than the focal level $\frac{\beta}{\alpha+\beta}$ can be of (Fig. 1(b)):

•compensation – the agent lowers his effort with the colleague in order to compensate the colleague's higher effort in the common task;

•tolerance – the agent keeps the focal allocation;

•imitation – the agent too increases his effort in the common task.

In [10] the reaction to a colleague exerting an effort different from the focal one was modeled by two factors. They respectively describe the reaction to colleague's effort when the latter is either lower or larger than the focal one. In the first case, the effort in the common task is reduced, while in the second case the agent adjusts his effort allocation in order to optimize the production. Finally, the focal allocation is chosen when observing a colleague whose effort on the common task corresponds to the focal allocation. For details on the derivation the reader is referred to [10]; the analytical form was

$$l_i(t+1) = \frac{\beta \bar{c}_i}{\alpha+\beta}\left(\frac{(\alpha+\beta)l_{-i}(t)}{\beta \bar{c}_i}\right)^{k_i-1} \exp\left[(k_i-1)\left(1-\frac{(\alpha+\beta)l_{-i}(t)}{\beta \bar{c}_i}\right)\right]. \tag{7}$$

where $\bar{c}_i$ is agent *i*'s capacity.

We now modify this reaction function to take into account that the domain is a finite set identical for both subordinates and that the capacity is the same for each subordinate, normalized, and common knowledge ($\bar{c}_1 = \bar{c}_2 = 1$). In this case the reaction function considered in [10] becomes

$$l_i(t+1) = \frac{\beta}{\alpha+\beta}\left(\frac{(\alpha+\beta)l_{-i}(t)}{\beta}\right)^{k_i-1} \exp\left[(k_i-1)\left(1-\frac{(\alpha+\beta)l_{-i}(t)}{\beta}\right)\right]. \tag{8}$$

It has the form of the product of a power times an exponential function where k_i is the parameter modeling intolerance. These two factors model the reactions to colleague's effort which are respectively lower or higher than the focal one.

In the formalization we introduce here, we consider subordinates engagement e_i and modify the two factors in order to take into account that the capacity is finite, identical and normalized to 1 for both subordinates. The first factor is then replaced by $(l_{-i}(t))^{a_i-1}$ and maintains the same functional form. The second one is replaced by factor $(1-l_{-i}(t))^{b_i-1}$ in order to consider the fact that both subordinates have the same finite capacity. In this case the second factor changes the functional form: it is no longer an exponential but becomes a power function. Therefore, the two factors now are $(l_{-i}(t))^{a_i-1}$ and $(1-l_{-i}(t))^{b_i-1}$ where $a_i,b_i \geqslant 1$ are two parameters which determine the shape of reaction as described above, and replace the tolerance parameter k_i in the formulation considered in[10]. Furthermore, in order to consider allocations different from the focal one, the reaction function needs to drop factor $(\alpha+\beta)/\beta$; this way its domain is [0,1], the set of common feasible efforts as $\bar{c}_1 = \bar{c}_2 = 1$. Summarizing, we observe that while the reaction function formally differs from the one considered in [10] for the second factor, it introduces several aspects which provide a finer modelization of the subordinates' behavior.

When the two factors are combined, and we introduce a constant θ_i in order to have the maximum effort equal to the engagement e_i, the reaction functions are

$$l_i(t+1) = \begin{cases} \frac{e_i}{\theta_i}(l_{-i}(t))^{a_i-1}(1-l_{-i}(t))^{b_i-1} & \text{if } a_i > 1 \text{ and } b_i > 1 \\ e_i(1-l_{-i}(t))^{b_i-1} & \text{if } a_i = 1 \text{ and } b_i > 1 \\ e_i(l_{-i}(t))^{a_i-1} & \text{if } a_i > 1 \text{ and } b_i = 1 \\ e_i & \text{if } a_i = b_i = 1 \end{cases} \qquad (9)$$

Where

$$\theta_i = \left(\frac{a_i-1}{a_i+b_i-2}\right)^{a_i-1}\left(1-\frac{a_i-1}{a_i+b_i-2}\right)^{b_i-1}, \quad i=1,2 \qquad (10)$$

The piecewise definition of the map in Eq. (9) comes from considering the extremes of interval [0,1] which, when either a_i=1 or b_i=1 or both could otherwise cause an indecision form.[5]

Summarizing, we do not derive function (9) from (8). Rather, we replace the reaction function (8) with a brand new one, in order to take into account the aforementioned assumptions: introducing subordinates' engagement; considering subordinates with identical capacity; modeling tolerance; and, finally, allowing the allocation to take values different from the focal one.

Fig. 2 illustrates how the reactions to the colleague's effort can be combined in order to have different kinds of behavior such as tolerance, focal retaliation, perfect imitation and various degrees of intolerance. We observe that the reaction function formal expression is equivalent to a beta probability density function; in this case we do not consider parameter values which give a U shape, as this shape is not realistic in this context.

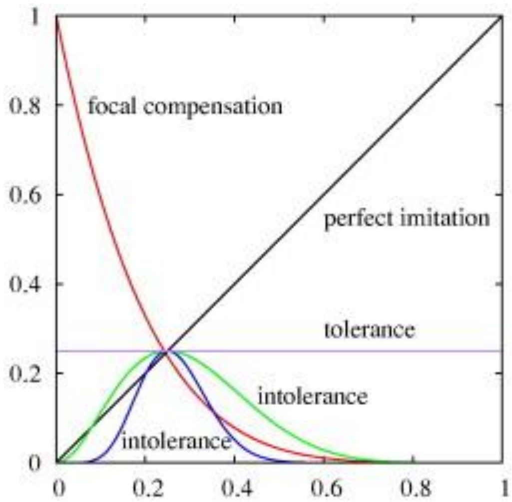

Fig. 2 Reaction function of subordinate 2 with: tolerance, $a_2 = 1,\ b_2 = 1,\ e_2 = .25$; intolerance, $a_2 = 3.1666,\ b_2 = 7.5,\ e_2 = .25$, and $a_2 = 9,\ b_2 = 25,\ e_2 = .25$; focal compensation, $a_2 = 1,\ b_2 = 6,\ e_2 = 1$; perfect imitation, $a_2 = 2,\ b_2 = 1,\ e_2 = 1$.

Coefficients a_i and b_i determine the shape of reaction curves and, therefore, model how subordinate react to colleague's effort. We observe that we have tolerant behavior whena_i=b_i=1, imitative behavior when $a_i > 1,\ b_i = 1$, compensative behavior when $a_i = 1,\ b_i > 1$ and intolerant behavior when $a_i > 1,\ b_i > 1$.

We can observe that in the cases of tolerance, focal retaliation and perfect imitation, the reply function is monotonic in the observed effort; by contrast, in the case of intolerance the reaction is non-monotonic and unimodal. It is also possible to find the analytic expression of the effort v_i which maximizes the reaction function when $(a_i, b_i) \neq (1,1)$:

$$v_i = \frac{a_i - 1}{a_i + b_i - 2} \qquad (11)$$

The effort v_i has two interpretations. On one side it may correspond to the belief the subordinate has about the focal allocation; on the other, it can be interpreted as the expectation he has about the other subordinates' effort in the common task. Table 1summarizes the model parameters and variables.

Symbol	Definition
$\alpha \in (0,1)$	Output elasticity w.r.t. joint effort with the supervisor
$\beta \in (0,1)$	Output elasticity w.r.t. joint effort with the partner
$e_i \in [0,1]$	Subordinate *i*'s engagement to the common task
$\bar{c}_i > 0$	Subordinate *i*'s capacity
$\bar{c} = 1$	Normalized common subordinates' capacity
$u_i \in [0,1]$	Subordinate *i*'s fraction of capacity allocated to task with the supervisor
$l_i \in [0,1]$	Subordinate *i*'s fraction of capacity allocated to the common task
$k_i \geqslant 1$	Intolerance parameter
$a_i, b_i \geqslant 1$	Shape parameters of the subordinate *i*'s reaction function
$v_i \geqslant 0$	Subordinate *i*'s fraction of capacity allocated to the common
	task which maximizes the reaction function

Table 1 Model parameters and variables.

Finally, it is possible to obtain a measure of agent's intolerance as in [10]. In fact, it is well known that

$$\int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x = B(a,b)$$

where B(a,b) is the Beta function. Therefore, it follows that

$$\int_0^1 \frac{e_i}{\theta_i}(l_{-i})^{a_i-1}(1-l_{-i})^{b_i-1}\mathrm{d}l_{-i} = \frac{e_i}{\theta_i}B(a_i,b_i) \tag{12}$$

In particular, when $a_i=b_i=1$ the integral value is e_i. Now, fixing value e_i, we can define agent *i*'s intolerance as

$$k_i(a_i,b_i) = \frac{\theta_i}{B(a_i,b_i)}$$

In fact, for a tolerant subordinate with unitary engagement the area under the reaction curve is one; as the subordinate's intolerance increases the area decreases. This way, given the subordinates parameters a_i and b_i, it is possible to measure his intolerance with a single parameter[6] as illustrated in Fig. 3(a). Therefore, given a fixed value of intolerance $\bar{k}$, it is possible to determine a_i and b_i such as the vertex of the reply function corresponds to a given $v \in [0,1]$ as illustrated in Fig. 3(b). This allows us to study what happens when intolerant subordinates have different beliefs about the focal allocation.

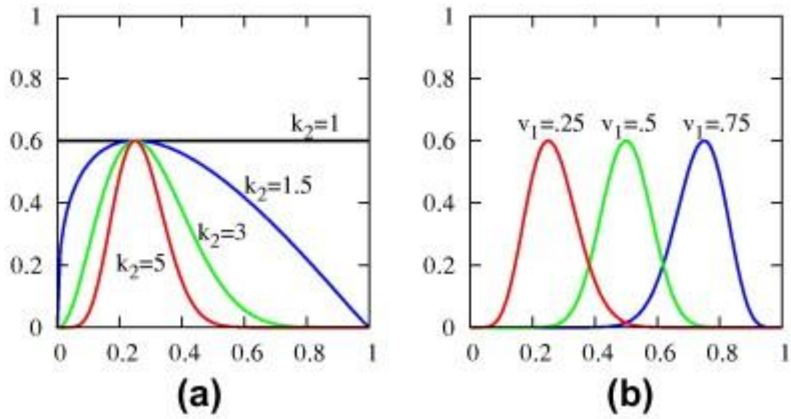

Fig. 3 Reaction function of subordinate 2 with engagement e_2=.60. (a) With different values of intolerance parameter $k_2 = 1,\ k_2 = 1.5,\ k_2 = 3$ and k_2=5; (b) with intolerance k_2=5 and different beliefs about focal allocation in $v = .25,\ v = .5$ and v=.75.

To summarize the definitions so far introduced we report them in Table 2.

Term	Definition
Focal effort allocation	The effort allocation in the two tasks proportional
	to the elasticities α and β (see [28])
Best reply	The strategy which produces the most favorable
	outcome for a player, given the other player's
	strategy
Tolerant	A subordinate who keeps the focal effort
	allocation despite what his partner does
Imitator	A subordinate who imitates the partner effort
	allocation in the common task
Compensator	A subordinate who tries to compensate the
	colleague's effort in the common task in order to
	maximize the production
Intolerant	Any behavior different from tolerant

Table 2 Terminology.

3. Basic properties of the dynamic adjustment

As in [10] we assume that both subordinates have naive expectations, that is, they expect that colleague's allocation remains the same as in the current period and react accordingly. The time evolution can be modeled by the iteration of a map $T:(l_1,l_2)\rightarrow(r_1(l_2),r_2(l_1))$ where $r_1,\ r_2$ are the reaction functions of subordinates 1 and 2 respectively. For the sake of simplicity they can be rewritten as $T(l_1(t),l_2(t))=(l_1(t+1),l_2(t+1))$ defined by:

$$l_1(t+1) = r_1(l_2(t)) \qquad (13)$$
$$l_2(t+1) = r_2(l_1(t))$$

The equilibria and cycles of the model only depend on parameters $a_i, b_i, e_i,\ i = 1, 2$. We recall that parameters θ_i also depend on a_i and b_i. These parameters directly affect and shape the dynamics. In fact, it is possible to observe cycles of any period and also chaotic behavior as it will be proved in the following.

The reaction functions depend on the behavior of the subordinates; therefore, the Jacobian expression depends on the subordinates in the work group. As a consequence, although the Jacobian has the general structure

$$J(l_1^*, l_2^*) = \begin{pmatrix} 0 & \frac{\mathrm{d}r_1}{\mathrm{d}l_2|l_2^*} \\ \frac{\mathrm{d}r_2}{\mathrm{d}l_1|l_1^*} & 0 \end{pmatrix} \qquad (14)$$

and eigenvalues

$$\begin{cases} \lambda_1 = -\sqrt{\frac{\mathrm{d}r_1}{\mathrm{d}l_2|l_2^*}\frac{\mathrm{d}r_2}{\mathrm{d}l_1|l_1^*}} \\ \lambda_2 = \sqrt{\frac{\mathrm{d}r_1}{\mathrm{d}l_2|l_2^*}\frac{\mathrm{d}r_2}{\mathrm{d}l_1|l_1^*}} \end{cases} \qquad (15)$$

their analytical expressions depend on the behavior of the agents and will be discussed in the relative sections.

For this kind of two-dimensional dynamical systems it is also common to consider the second iterate and, therefore, to study two decoupled one-dimensional dynamics:

$$\begin{cases} l_1(t+2) = r_1(l_2(t+1)) = r_1(r_2(l_1(t))) \\ l_2(t+2) = r_2(l_1(t+1)) = r_2(r_1(l_2(t))) \end{cases}$$

that is

$$\begin{cases} l_1(t+2) = g_1(l_1(t)) \\ l_2(t+2) = g_2(l_2(t)) \end{cases} \qquad (16)$$

where $g_1 = r_1 \circ r_2$ and $g_2 = r_2 \circ r_1$. In fact, [14] shows that for two-dimensional maps with this structure cycles and fixed points are related to one-dimensional maps similar to (16). Furthermore, in [6] this approach is extended to study the chaotic attractor and basins of two-dimensional maps. Therefore, in the following we will use these properties to study the stability of some fixed points.

The different reaction functions, and consequently the parameters values, determine the number of fixed points, which may range from 1 to 4 as illustrated in Fig. 4.

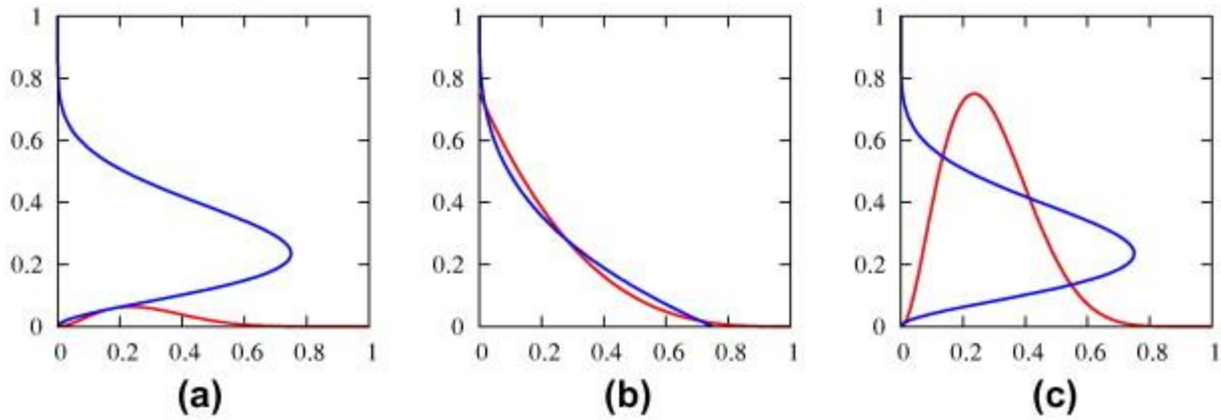

Fig. 4 Intersections of reaction functions for the subordinates. (a) 2 Intersections, $a_1 = a_2 = 3,\ b_1 = b_2 = 7.5,\ e_1 = .75,\ e_2 = .0638$; (b) 3 intersections, $a_1 = a_2 = 1,\ b_1 = b_2 = 4,\ e_1 = e_2 = .75$; (c) 4 intersections, $a_1 = a_2 = 3,\ b_1 = b_2 = 7.5,\ e_1 = e_2 = .75$.

Finally, it is easy to observe that the model is symmetric w.r.t. index *i* , as$T(l_1,l_2,a_1,b_1,e_1)=T(l_2,l_1,a_2,b_2,e_2)$). This leads to a symmetric structure of the bifurcation curves in the 2-dimensional parameter planes $(a_1, a_2),\ (b_1, b_2)$ and $(e_1,e_2)$with respect to the lines $a_1 = a_2,\ b_1 = b_2$, and $e_1=e_2$ respectively.

In the following we analyze the different configurations in the subordinate matching.

3.1. Tolerant subordinates

The case in which at least one of the subordinates is tolerant is straightforward. Assume for the sake of simplicity that the tolerant subordinate is the second one; then his reaction curve is a straight line which can be represented as a vertical line. As a consequence there exists a unique intersection between the subordinates' reaction function. As the eigenvalues are both 0, the fixed point is stable. In particular, when engagement for the tolerant agent is $e_i=\beta/(\alpha+\beta)$ and the reaction function for the other one is such that $r_{-i}(\beta/(\alpha+\beta))=\beta/(\alpha+\beta)$ the fixed point is focal. We have therefore proved the following proposition.

Proposition 3.1.

When at least one of the subordinates is tolerant there exists a unique stable fixed point. The effort allocation is focal if and only if the tolerant subordinate's allocation is focal and the best reply to focal allocation is focal.

3.2. Imitators subordinates

Proposition 3.2.

When both subordinates are imitators there can be either one or two or infinite fixed points. The origin is always a fixed point and if $a_1>a_2/(a_2-1)$*, then it is stable.*

Proof.

In this case we have $b_1=b_2=1$ and the dynamics is

$$\begin{cases} l_1(t+1) = e_1 l_2^{a_1-1}(t) \\ l_2(t+1) = e_2 l_1^{a_2-1}(t) \end{cases} \tag{17}$$

Fixed points are given by the solution of the system

$$\begin{cases} l_1 = e_1 l_2^{a_1-1} \\ l_2 = e_2 l_1^{a_2-1} \end{cases} \tag{18}$$

The origin $l_1=l_2=0$ is always a solution. Furthermore, plugging l_2 expression in the first equation we obtain

$$l_1 = e_1 e_2^{a_1-1} l_1^{(a_1-1)(a_2-1)} \tag{19}$$

and rearranging we have

$$l_1 = e_1^{\frac{1}{a_1+a_2-a_1a_2}} e_2^{\frac{a_1-1}{a_1+a_2-a_1a_2}} \tag{20}$$

This point is feasible if and only if $a_1<a_2/(a_2-1)$; in this case the coordinates are

$$l_1^* = e_1^{\frac{1}{a_1+a_2-a_1a_2}} e_2^{\frac{a_1-1}{a_1+a_2-a_1a_2}} \text{ and } l_2^* = e_1^{\frac{a_2-1}{a_1+a_2-a_1a_2}} e_2^{\frac{1}{a_1+a_2-a_1a_2}}.$$

Finally, if $e_1=e_2=1$ and $a_1=a_2/(a_2-1)$ we have coexistence of infinite fixed points and infinite period-two cycles. In the special case in which $a_1=a_2=2$ they have respectively the form $(\bar{l}, \bar{l})$, where $\bar{l} \in [0,1]$ and $((\bar{l}_1, \bar{l}_2) \rightleftarrows (\bar{l}_2, \bar{l}_1))$, with $\bar{l}_1, \bar{l}_2 \in [0,1]$.

As it concerns the stability, we consider the second iterate of the map which consists of two one-dimensional, decoupled maps:

$$\begin{aligned} l_1(t+2) &= e_1 e_2^{a_1-1} l_1(t)^{(a_1-1)(a_2-1)} := g_1(l_1(t)) \\ l_2(t+2) &= e_1^{a_2-1} e_2 l_2(t)^{(a_1-1)(a_2-1)} := g_2(l_2(t)) \end{aligned} \tag{21}$$

The first derivatives are

$$\begin{aligned} \frac{dg_1}{dl_1} &= (a_1-1)(a_2-1) e_1 e_2^{a_1-1} l_1^{(a_1-1)(a_2-1)-1} \\ \frac{dg_2}{dl_2} &= (a_1-1)(a_2-1) e_1^{a_2-1} e_2 l_2^{(a_1-1)(a_2-1)-1} \end{aligned} \tag{22}$$

As it concerns fixed point (0,0), if $a_1>a_2/(a_2-1)$ then

$$\lim_{l_1 \to 0^+} \frac{dg_1}{dl_1} = \lim_{l_2 \to 0^+} \frac{dg_2}{dl_2} = 0$$

and the origin is stable. On the other hand, if $a_1<a_2/(a_2-1)$ then

$$\lim_{l_1\to 0^+}\frac{dg_1}{dl_1}=\lim_{l_2\to 0^+}\frac{dg_2}{dl_2}=+\infty$$

and the origin is unstable.

About the stability of the other point, when considering the Jacobian we have

$$J(l_1^*,l_2^*)=\begin{pmatrix} 0 & (a_1-1)\left(e_1^{\frac{2-a_2}{a_1+a_2-a_1a_2}}e_2^{\frac{a_1-2}{a_1+a_2-a_1a_2}}\right) \\ (a_2-1)\left(e_1^{\frac{a_2-2}{a_1+a_2-a_1a_2}}e_2^{\frac{2-a_1}{a_1+a_2-a_1a_2}}\right) & 0 \end{pmatrix}$$

The eigenvalues are $\lambda_{1,2}=\pm\sqrt{(a_1-1)(a_2-1)}$, therefore when the point is feasible, i.e.,$a_1<a_2/(a_2-1)$, it is always stable.

3.3. Imitator vs compensator

Proposition 3.3.

When subordinates have respectively imitating and compensating behaviors then there is always one and only one fixed point.

Proof.

The case of one imitator and one compensator occurs when $b_1=a_2=1$. Fixed points are given by the solution of the system

$$\begin{cases} l_1=e_1 l_2^{a_1-1} \\ l_2=e_2(1-l_1)^{b_2-1} \end{cases} \tag{23}$$

If $e_1\neq 0,\ e_2\neq 0$ then, solving the first equation for l_2 and putting together, we obtain

$$\left(\frac{l_1}{e_1}\right)^{\frac{1}{a_1-1}}=e_2(1-l_1)^{b_2-1} \tag{24}$$

The left-hand side is an increasing function with respect to l_1 and passes through points (0,0) and $(1,1/e_1^{\frac{1}{a_1-1}})$; the right-hand side function is decreasing with respect tol_1 and passes through points (0,e_2) and (1,0), therefore there always exists a unique fixed point (l_1^*,l_2^*) with $l_i^*\in(0,1),\ i=1,2$.

If e_1=0, then dynamics (23) reduces to $l_1=0,\ l_2=e_2$; if e_2=0, it reduces to the origin.

3.4. Imitator vs intolerant

Proposition 3.4.

When one of the subordinates has an imitative behavior and the other one is intolerant, the origin is always a fixed point; if $a_1>a_2/(a_2-1)$*, then it is stable. A sufficient condition to have another fixed point different from the origin is* $a_1<a_2/(a_2-1)$.

Proof.

In the case of one imitator and one intolerant, that is $b_1=1$ and $\theta_1=1$, we have

$$\begin{cases} l_1 = e_1 l_2^{a_1-1} \\ l_2 = e_2 \frac{l_1^{a_2-1}(1-l_1)^{b_2-1}}{\theta_2} \end{cases} \tag{25}$$

We have always the root $l_1=0$, therefore (0,0) is always a fixed point. If $e_1 \neq 0$ and $e_2 \neq 0$, solving the first equation for l_2 and putting together we obtain

$$\left(\frac{l_1}{e_1}\right)^{\frac{1}{a_1-1}} = e_2 \frac{l_1^{a_2-1}(1-l_1)^{b_2-1}}{\theta_2} \tag{26}$$

The other possible roots are given by

$$e_1^{\frac{1}{1-a_1}} l_1^{\frac{a_1+a_2-a_1a_2}{a_1-1}} = \frac{e_2}{\theta_2}(1-l_1)^{b_2-1} \tag{27}$$

The number of the other fixed points depends on the sign of $a_1+a_2-a_1a_2$. In fact, while the right-hand side of Eq. (27) is a decreasing power function passing through$(0,e_2/\theta_2)$ and (1,0), the left-hand side changes shape depending on the sign of$a_1+a_2-a_1a_2$. When $a_1+a_2-a_1a_2>0$ the left-hand side function is an increasing power function passing through (0,0) and $\left(1, e_1^{\frac{1}{1-a_1}}\right)$; therefore, there exists always a unique intersection. In this case there are two fixed points, one of which is the origin. By contrast, when $a_1+a_2-a_1a_2\leqslant 0$ the left-hand side of Eq. (27) is either a constant or a negative power and there may be either two or one or no intersections.

As it concerns the stability, we consider only the origin which is always a fixed point. The second iterate of the map consists of two one-dimensional, decoupled maps:

$$l_1(t+2) = e_1\left(\frac{e_2 l_1(t)^{a_2-1}(1-l_1(t))^{b_2-1}}{\theta_2}\right)^{a_1-1} := g_1(l_1(t)) \tag{28}$$

$$l_2(t+2) = \frac{e_2\left(e_1 l_2(t)^{a_1-1}\right)^{a_2-1}\left(1-e_1 l_2(t)^{a_1-1}\right)^{b_2-1}}{\theta_2} := g_2(l_2(t))$$

The first derivatives are

$$\frac{dg_1}{dl_1} = e_1\left(\frac{e_2}{\theta_2}\right)^{a_1-1}(a_1-1)l_1^{(a_2-1)a_1-a_2}(1-l_1)^{(b_2-1)a_1-b_2}[(2-a_2-b_2)l_1+a_2-1] \tag{29}$$

$$\frac{dg_2}{dl_2} = \frac{e_2}{\theta_2}(a_1-1)e_1^{a_2-1}l_2^{(a_2-1)a_1-a_2}\left(1-e_1 l_2^{a_1-1}\right)^{b_2-2}\left[(2-a_2-b_2)e_1 l_2^{a_1-1}+a_2-1\right]$$

Therefore, if $a_1>a_2/(a_2-1)$ then

$$\lim_{l_1\to 0^+}\frac{dg_1}{dl_1} = \lim_{l_2\to 0^+}\frac{dg_2}{dl_2} = 0$$

and the origin is stable. On the other hand, if $a_1<a_2/(a_2-1)$ then

$$\lim_{l_1\to 0^+}\frac{dg_1}{dl_1} = \lim_{l_2\to 0^+}\frac{dg_2}{dl_2} = +\infty$$

and the origin is unstable.

3.5. Compensator subordinates

Proposition 3.5.

When both the subordinates are compensators there exists always at least a fixed point. Furthermore, if and only if the first agent's engagement is 1 then one of the fixed points is (1,0)*which is stable when* $b_2>2$*and unstable when* $b_2<2$*. Similarly, if and only if the second agent's engagement is 1 one of the fixed points is* (0,1)*which is stable when* $b_1>2$*and unstable when* $b_1<2$*.*

Proof.

The case of two compensators occurs when $a_1=a_2=1$. In this case fixed points are given by the solution of the system

$$\begin{cases} l_1 = e_1(1-l_2)^{b_1-1} \\ l_2 = e_2(1-l_1)^{b_2-1} \end{cases} \tag{30}$$

Solving the first equation for l_2 and putting together we obtain

$$1-\left(\frac{l_1}{e_1}\right)^{\frac{1}{b_1-1}} = e_2(1-l_1)^{b_2-1} \tag{31}$$

This equation can be analyzed graphically. Both the left and right hand functions are decreasing as $b_1,b_2>1$; they are concave or convex if b_1, b_2 are respectively smaller or larger than 2. The left-hand side is an affine transformation of a power function. It is decreasing with respect to l_1 and passes through (0,1) and $(e_1,0)$. The right-hand side is also a power function; it is decreasing with respect to l_1 and passes through$(0,e_2)$ and (1,0). Therefore, there always exists at least a solution.

For the second part of the claim, when the engagement of the first subordinate is maximum, we have e_1=1 and it is immediate to prove that this is necessary and sufficient in order to have the fixed point (1,0). As it concerns the stability, we consider the second iterates of the map:

$$l_1(t+2) = \left(1 - e_2(1 - l_1(t))^{b_2-1}\right)^{b_1-1} := g_1(l_1(t)) \tag{32}$$
$$l_2(t+2) = e_2\left(1 - (1 - l_2(t))^{b_1-1}\right)^{b_2-1} := g_2(l_2(t))$$

The first derivatives are

$$\frac{\mathrm{d}g_1}{\mathrm{d}l_1} = e_2(b_1-1)(b_2-1)(1-l_1)^{b_2-2}\left(1 - e_2(1-l_1)^{b_2-1}\right)^{b_1-2} \tag{33}$$
$$\frac{\mathrm{d}g_2}{\mathrm{d}l_2} = e_2(b_1-1)(b_2-1)(1-l_2)^{b_1-2}\left(1 - (1-l_2)^{b_1-1}\right)^{b_2-2}$$

Therefore, if b_2>2 then

$$\lim_{l_1 \to 1^-} \frac{\mathrm{d}g_1}{\mathrm{d}l_1} = \lim_{l_2 \to 0^+} \frac{\mathrm{d}g_2}{\mathrm{d}l_2} = 0$$

and the point (1,0) is stable. On the other hand, if b_2<2 then

$$\lim_{l_1 \to 1^-} \frac{\mathrm{d}g_1}{\mathrm{d}l_1} = \lim_{l_2 \to 0^+} \frac{\mathrm{d}g_2}{\mathrm{d}l_2} = +\infty$$

and the point (1,0) it is unstable.

The case in which the second subordinate engagement is maximum is similar to the previous one.

The analysis of the roots of Eq. (31) is not as simple as in the case of two imitators. In fact, it is possible to find examples of two and even three intersections as illustrated in Fig. 4(b). Finally, in the special case in which b_1=b_2=2 and e_1=e_2=1 there is a continuum of solutions. This case is similar to the one we examined in Section 3.2. There is coexistence of infinite fixed points $(\bar{l}, \bar{l})$, where $\bar{l} \in [0,1]$ and infinite period two-cycles $\left((\bar{l}_1, \bar{l}_2) \rightleftarrows (\bar{l}_2, \bar{l}_1)\right)$, where $\bar{l}_1, \bar{l}_2 \in [0,1]$.

3.6. Compensator vs intolerant

Proposition 3.6.

When the first subordinate has compensating behavior and the other is intolerant, there is at least a fixed point. Furthermore, (1,0)*is a fixed point if and only if the engagement of the first subordinate is maximum. The condition for the stability of* (1,0)*is that* b_2>2*.*

Proof.

In this case we have $a_1=1$ and therefore we must analyze

$$\begin{cases} l_1 = e_1(1-l_2)^{b_1-1} \\ l_2 = e_2\frac{l_1^{a_2-1}(1-l_1)^{b_2-1}}{\theta_2} \end{cases} \tag{34}$$

Solving the first equation for l_2 and putting together we obtain

$$1-\left(\frac{l_1}{e_1}\right)^{\frac{1}{b_1-1}} = e_2\frac{l_1^{a_2-1}(1-l_1)^{b_2-1}}{\theta_2}. \tag{35}$$

As the second subordinate is intolerant the right-hand side is bell shaped[7] and the left-hand side is convex and decreasing passing through points (0,1) and $(1,1-(l_1/e_1)^{1/(b_1-1)})$. Therefore there exists always at least one intersection.

Furthermore, by substituting (1,0) into Eq. (34) this effort allocation is a fixed point if and only if $e_1=1$, that is, the first agent's engagement is maximum.

As it concerns the stability of (1,0), we again consider the second iterates of the map:

$$\begin{aligned} l_1(t+2) &= \left(1-\frac{e_2}{\theta_2}l_1(t)^{a_2-1}\right)^{b_1-1}(1-l_1(t))^{b_2-1} := g_1(l_1(t)) \\ l_2(t+2) &= \frac{e_2}{\theta_2}(1-l_2(t))^{(b_1-1)(a_2-1)}\left(1-(1-l_2(t))^{b_1-1}\right)^{b_2-1} := g_2(l_2(t)) \end{aligned} \tag{36}$$

The first derivatives are

$$\begin{aligned} \frac{dg_1}{dl_1} &= (b_1-1)\frac{e_2}{\theta_2}(1-l_1)^{b_2-2}l_1^{a_2-2}((a_2+b_2-2)l_1-a_2+1)\left(1-\frac{e_2(1-l_1)^{b_2-1}l_1^{a_2-1}}{\theta_2}\right)^{b_1-2} \\ \frac{dg_2}{dl_2} &= \frac{e_2}{\theta_2}(b_1-1)(1-l_2)^{b_1-2}\left(1-(1-l_2)^{b_1-1}\right)^{b_2-2}\left((1-l_2)^{b_1-1}\right)^{a_2-2}\cdot\left[(a_2+b_2-2)(1-l_2)^{b_1-1}+1-a_2\right] \end{aligned} \tag{37}$$

Therefore, if $b_2>2$ then

$$\lim_{l_1\to 1^-}\frac{dg_1}{dl_1} = \lim_{l_2\to 0^+}\frac{dg_2}{dl_2} = 0$$

and the point (1,0) is stable. On the other hand, if $b_2<2$ then

$$\lim_{l_1\to 1^-}\frac{dg_1}{dl_1} = \lim_{l_2\to 0^+}\frac{dg_2}{dl_2} = +\infty$$

and the point is unstable.

Furthermore, from graphical analysis it is immediate to observe that there may be up to three fixed points even in the case in which the compensating agent's engagement is not maximum. For an example see Fig. 5.

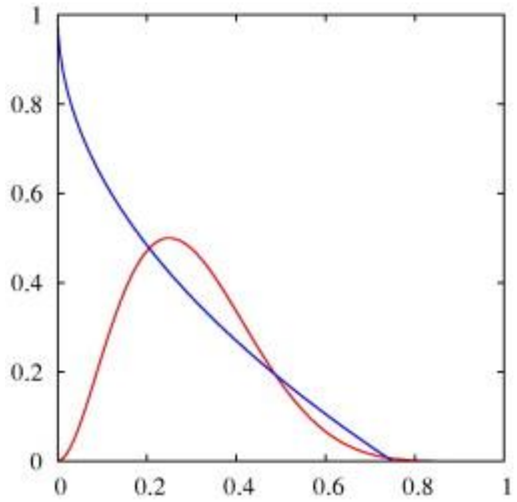

Fig. 5 Intersections of reaction function for a compensator subordinate $a_1 = 1.0,\ b_1 = 3.0,\ e_1 = .75$ and an intolerant one $a_2 = 3.0,\ b_2 = 7.0,\ e_2 = .5$; in this case we have three intersections.

3.7. Intolerant subordinates

Proposition 3.7.

When both subordinates are intolerant the origin is always a fixed point. The condition for the stability of the origin is that both a_1 *and* a_2 *are greater than two.*

Proof.

In the case of two intolerant subordinates to find the fixed point we have to solve

$$\begin{cases} l_1 = \frac{e_1}{\theta_1} l_2^{a_1-1}(1-l_2)^{b_1-1} \\ l_2 = \frac{e_2}{\theta_2} l_1^{a_2-1}(1-l_1)^{b_2-1} \end{cases} \tag{38}$$

Obviously (0,0) is a fixed point.

As it concerns the stability, we consider only the origin which is always a fixed point. The second iterate of the map again consists of two one-dimensional, decoupled maps:

$$l_1(t+2) = \frac{e_1}{\theta_1}\left(\frac{e_2(1-l_1(t))^{b_2-1}l_1(t)^{a_2-1}}{\theta_2}\right)^{a_1-1}\left(1-\frac{e_2(1-l_1(t))^{b_2-1}l_1(t)^{a_2-1}}{\theta_2}\right)^{b_1-1} := g_1(l_1(t)) \tag{39}$$
$$l_2(t+2) = \frac{e_2}{\theta_2}\left(\frac{e_1(1-l_2(t))^{b_1-1}l_2(t)^{a_1-1}}{\theta_1}\right)^{a_2-1}\left(1-\frac{e_1(1-l_2(t))^{b_1-1}l_2(t)^{a_1-1}}{\theta_1}\right)^{b_2-1} := g_2(l_2(t))$$

The first derivatives are

$$
\begin{aligned}
\frac{\mathrm{d}g_1}{\mathrm{d}l_1} = {} & \frac{e_1}{\theta_1}\left(\frac{e_2}{\theta_2}\right)^{a_1-1}\left(1-\frac{e_2(1-l_1)^{b_2-1}l_1^{a_2-1}}{\theta_2}\right)^{b_1-2}(1-l_1)^{a_1b_2-a_1-b_2}l_1^{a_1a_2-a_1-a_2} \\
& \times\bigg[(a_1-1)((a_2-1)(1-l_1)-(b_2-1)l_1)\left(1-\frac{e_2(1-l_1)^{b_2-1}l_1^{a_2-1}}{\theta_2}\right) \\
& +(b_1-1)((b_2-1)l_1-(a_2-1)(1-l_1))\left(\frac{e_2(1-l_1)^{b_2-1}l_1^{a_2-1}}{\theta_2}\right) \\
\frac{\mathrm{d}g_2}{\mathrm{d}l_2} = {} & \frac{e_2}{\theta_2}\left(\frac{e_1}{\theta_1}\right)^{a_2-1}\left(1-\frac{e_1(1-l_2)^{b_1-1}l_2^{a_1-1}}{\theta_1}\right)^{b_2-2}(1-l_2)^{a_2b_1-a_2-b_1}l_2^{a_1a_2-a_1-a_2} \\
& \times\bigg[(a_2-1)((a_1-1)(1-l_2)-(b_1-1)l_2)\left(1-\frac{e_1(1-l_2)^{b_1-1}l_2^{a_1-1}}{\theta_1}\right) \\
& +(b_2-1)((b_1-1)l_2-(a_1-1)(1-l_2))\left(\frac{e_1(1-l_2)^{b_1-1}l_2^{a_1-1}}{\theta_1}\right)\bigg]
\end{aligned}
\tag{40}
$$

Therefore, if $a_1>a_2/(a_2-1)$ then

$$\lim_{l_1\to 0^+}\frac{\mathrm{d}g_1}{\mathrm{d}l_1} = \lim_{l_2\to 0^+}\frac{\mathrm{d}g_2}{\mathrm{d}l_2} = 0$$

and the origin is stable. On the other hand, if $a_1<a_2/(a_2-1)$ then

$$\lim_{l_1\to 0^+}\frac{\mathrm{d}g_1}{\mathrm{d}l_1} = \lim_{l_2\to 0^+}\frac{\mathrm{d}g_2}{\mathrm{d}l_2} = +\infty$$

and the origin is unstable.

Since with two intolerant subordinates both reaction functions have a bell's shape, besides the origin there may be up to three more fixed point as illustrated in Fig. 4(c). In fact in this case the analysis is qualitatively similar to the one presented in [10].

The results we have found are summarized in Table 3.

	Tolerant	Imitator	Compensator	Intolerant
Tolerant	A unique fixed point	A unique fixed point	A unique fixed point	A unique fixed point
Imitator		1,2, infinite fixed points	A unique fixed point	1,2 or 3 fixed points
Compensator			1,2,3, infinite fixed points	1,2 or 3 fixed points
Intolerant				1,2,3 or 4 fixed points

Table 3 Fixed points depending of the behavior of subordinates.

4. Effort dynamics analysis

The dynamics of the work group affects its productivity as seen in [10]. There it was proven that, when subordinates are identical in terms of capacities and intolerance, there exist two equilibrium points: the null production equilibrium (0,0) and the focal equilibrium. Since in this paper we only consider same capacity subordinates, it is important to see if this result holds also with this functional form. Therefore, we analyze the dynamics when the subordinates have the same engagement and the same belief about the focal allocation.

We obtain results similar to [10] as illustrated in Fig. 6. We can see the basins of the two stable fixed points and the one of the 2-period cycle represented with different colors. Starting from any point belonging to the red region the trajectory converges to the origin. Similarly, trajectories starting in the green region converge to the other fixed point. Finally, initial conditions in the yellow regions converge to the 2-period cycle. In the picture we provide also an example for each case where the lines connecting the trajectories points are for illustrative purpose only and the arrows point to the respective attractors. For some values of intolerance these symmetric equilibria are stable. When the common intolerance is large enough, as in the case illustrated, there exists an unstable middle equilibrium (l_1^M, l_2^M) between these two equilibria.

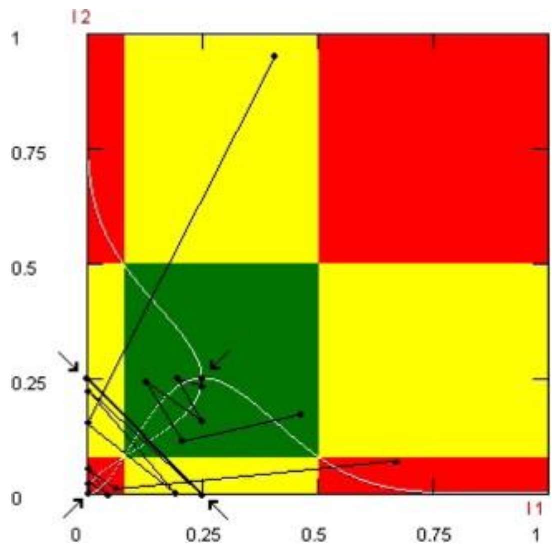

Fig. 6 Phase plane (l_1,l_2) with periodic attractors and their respective basins – represented in different colors – for $a_1 = a_2 = 19/6,\ b_1 = b_2 = 15/2$ and e_1=e_2=.25. The initial conditions of the trajectories are respectively $l_1(0) = .40,\ l_2(0) = .17$ (fixed point -green basin), $l_1(0)$=0.41, $l_2(0)$=.95 (2-period cycle -yellow basin), and $l_1(0) = .67,\ l_2(0) = 0.7$ (origin -red basin). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

When subordinates are tolerant the only fixed point is the focal equilibrium. As the common value of intolerance increases, its basin of attraction reduces. These results not only confirms the findings of [10] but allow us to conclude that they are not numerical artifacts as the functional form we consider is different from the one in the literature.

Starting from this result, can the dynamics be chaotic even when the subordinates have the same capacity? A first interesting case is to consider what happens when one of the two subordinates has different engagement in the common task. In this case also we consider identical subordinates and study their effort allocation in the common task as the engagement of one of them varies in [0,1].

The bifurcation diagram is illustrated in Fig. 7. Firstly, we can observe that the behavior of the two subordinates is linked. As a consequence, the group production will also follow this individual pattern; this phenomenon is common in work groups as illustrated in [12]. Furthermore, it is interesting to observe that when the engagement is below a certain threshold the production is null; then moving up to e_1=.25 – in this case both subordinates have the same value of engagement- the effort allocated in the common task increases for both subordinates. As the engagement increases further the effort allocation of the fixed engagement subordinate decreases. Further on a cascade of period-doubling bifurcation leads to chaos. Finally, the work group production collapses to 0 as no effort is allocated in the common tasks. The behavior we find after the first bifurcation, that is the coexistence of period cycles and, for greater values of the first subordinate's engagement value, the chaotic cycles, resembles the case of subordinates with different capacities as in [10].

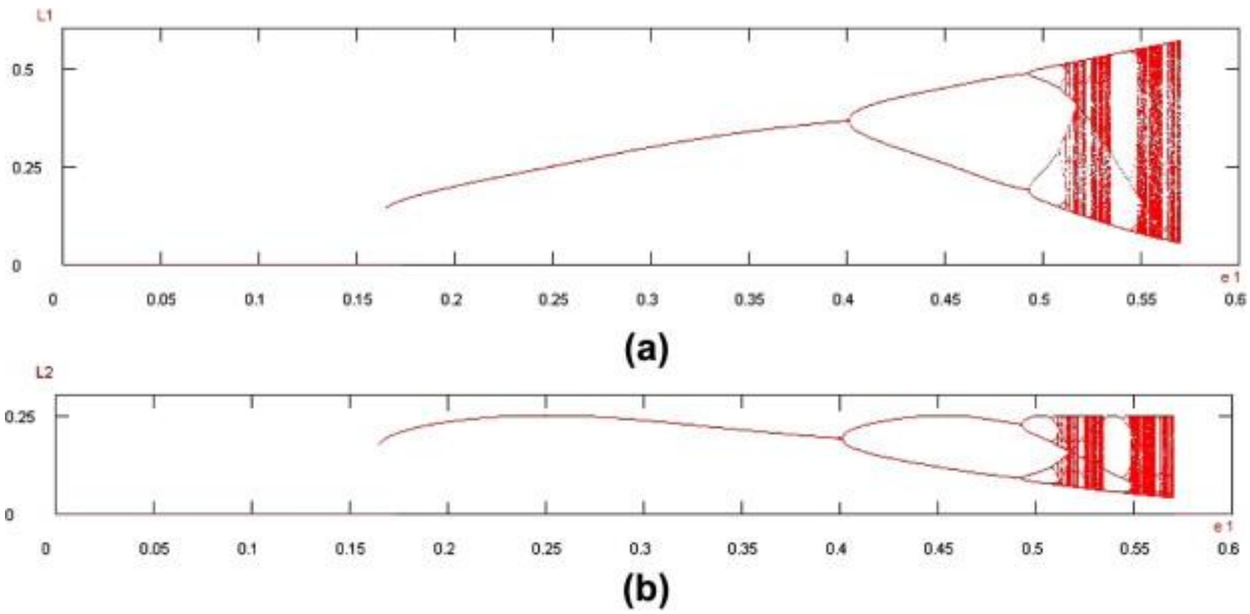

Fig. 7 Bifurcation diagrams of subordinates 1's effort (top) and 2's (bottom) as $e_1 \in [0,0.6]$ with parameters values $a_1 = a_2 = 19/6,\ b_1 = b_2 = 15/2$ and e_2=.25 and initial condition $l_1(0)=l_2(0)$=.40.

In Fig. 8 we can see some of the coexisting finite period cycles. Again, as in Fig. 6, black dots indicate the sequence of periodical points visited at each iteration, while the joining line are depicted for illustrative purpose only. Apart from the red region, which corresponds to initial conditions of trajectories converging to the origin, we can see five 16-period stable cycles. Following the terminology introduced in [6], four of them are homogeneous cycles deriving from a stable 8-period cycle of the map $g_1(x)$, one of which –

bottom left in the figure – is a Markov-Perfect-Equilibrium[8] (MPE); the last one is a mixed 16-period cycle which derives from the fixed point in the origin and shows the retaliation actions. On the top left the basins of attraction of each cycle are depicted; their colors match those at the bottom left of the respective cycle.

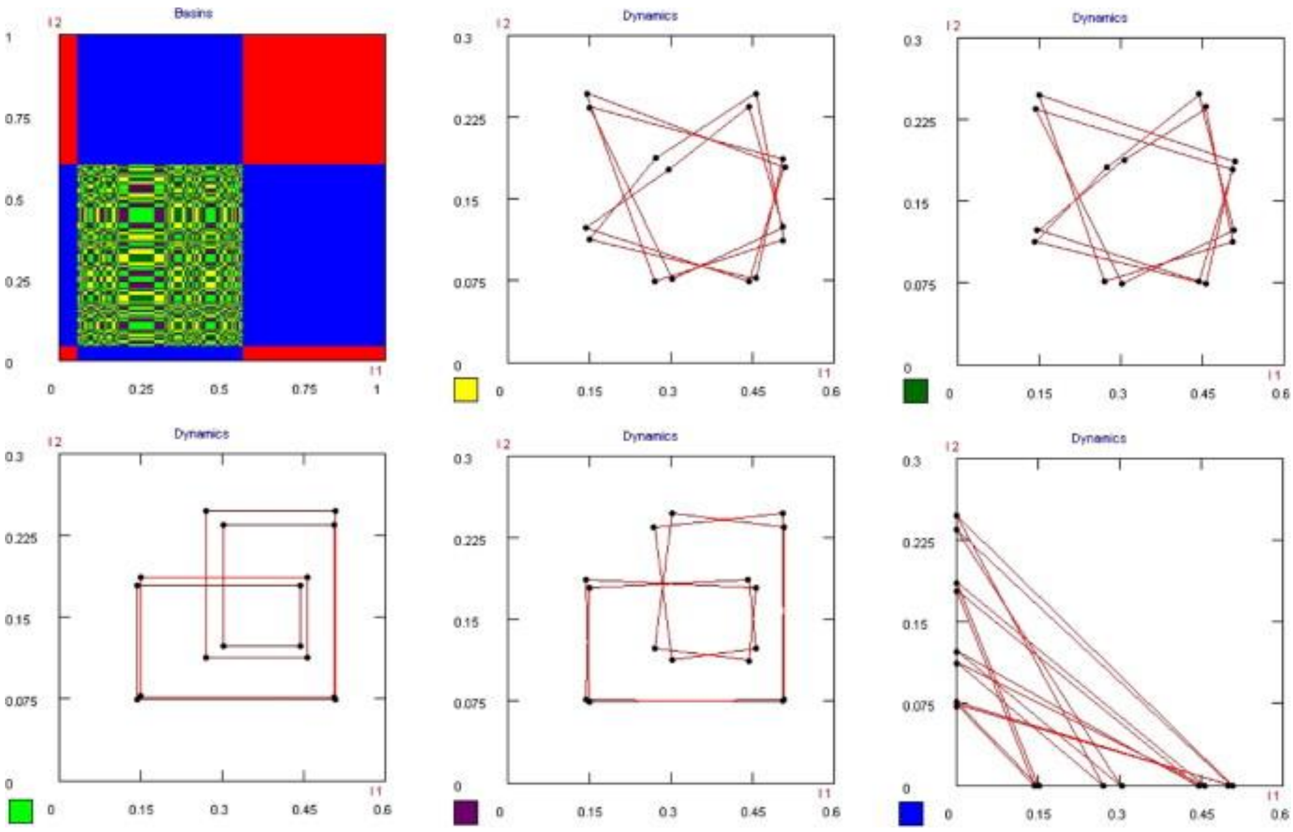

Fig. 8 Basins of attraction and respective periodic cycles with $a_1 = a_2 = 19/6,\ b_1 = b_2 = 15/2,\ e_1 = .509$ and e_2=.25 On the bottom left a MPE is illustrated, while a retaliation cycle is at the bottom right.

As in [10], the MPE is interesting in terms of group dynamics. In fact, in this case only one of the subordinates at the time changes his effort allocation. This means that each subordinate at every other turn believes that his previous allocation was optimal. In this case, the allocation dynamics is midway between a constant allocation (a fixed point) and a situation in which both subordinates adjust their allocation at each period. Finally, also the mixed 16-period cycle deriving from the fixed point in the origin is interesting in terms of group dynamics; in this case, every other turn, one of the subordinates alternately stops cooperating with his colleague. This sort of sequence retaliation may be interpreted in terms of intragroup conflict.

These kinds of cycle are quite similar to the ones found in [10] when subordinates have different capacities. Their presence confirms that with intolerant subordinates even when their capacity is the same, chaotic behaviors may occur. One of the reasons for this is that, when subordinates have the same belief about the focal allocation and a different engagement, it results in a situation similar to those in which the capacity of the subordinates is different.

So far we have examined what happens when the subordinates' engagement changes; now it is interesting to analyze what happens as one subordinate has different beliefs about the focal allocation as illustrated in Fig. 3(b). As we mentioned in Section 2, the belief a subordinate has about the focal allocation determines how he reacts to the effort of the other.

First, let us consider the case in which two intolerant subordinates have the same engagement. As the belief of one of them varies the intersection point different from the origin moves and eventually disappears as illustrated in Fig. 9(a). This phenomenon has consequences on the work group dynamics since, when the only intersection between the reaction functions is the origin, the production collapses as illustrated in the bifurcation diagram in Fig. 9(b). It should be noted that when the two subordinates' intolerance is sufficiently small this intersection never vanishes and production remains positive.

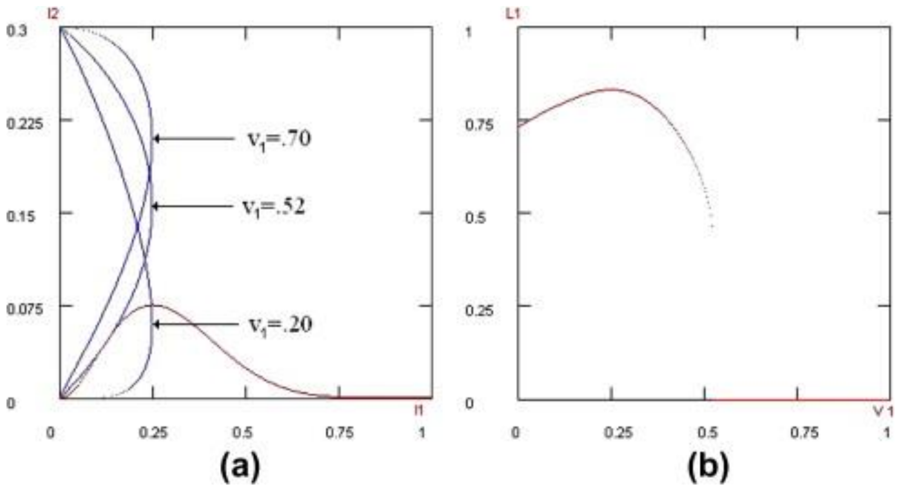

Fig. 9 (a) intersections between the reaction curves for different beliefs of subordinate 1 and constant intolerancek_1=1.4699; (b) bifurcation diagram of subordinates 1's effort as $v_1 \in [0,1]$. Other parameters are: $a_2 = 19/6,\ b_2 = 15/2,\ e_1 = e_2 = .25$. Initial condition is $l_1(0)=l_2(0)=.45$.

Finally, we observe how the effect of the belief of a subordinate depends on the engagement of the other. This is illustrated in Fig. 10(a) where, for different values of parameters v_1 and e_2, we indicate with different colors the k-period cycles which may exist. Obviously, period *k* is different in regions with different colors; the central zone shows parameter values for which the dynamics is chaotic. Fig. 10(b) and (c) show respectively the one-dimensional bifurcation diagrams of the state variable l_1 as the parameter v_1 increases along each of the two lines in Fig. 10(a); that is, for e_2=0.505and e_2=0.75. We can observe that, when the belief of subordinate 1 is larger than some threshold values depending on the engagement of subordinate 2, the effort allocation in the common task collapses and the group production drops to zero.

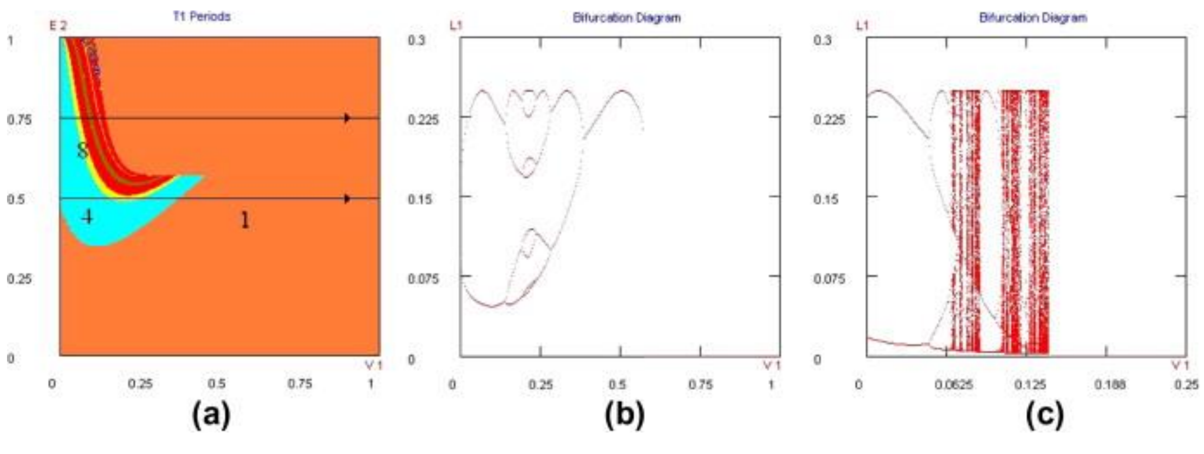

Fig. 10 Bifurcation diagrams for two intolerant agents with parameters $a_1,\ b_1$ such that $k_1(a_1, b_1) = k_1(19/6, 15/2),\ e_1 = 0.25;\ a_2 = 19/6,\ b_2 = 15/2$; (a) two-dimensional bifurcation diagram in the plane (v_1,e_2) with $v_1 \in [0,1]$ and $e_2 \in [0,1]$; (b) one-dimensional bifurcation diagram of the state variable l_1 as the parameter v_1 increases along the line shown in (a), with fixed valuee_2=0.505; (c) one-dimensional bifurcation diagram of the state variable l_1 as the parameter $v_1$increases along the line shown in (a), with fixed value e_2=0.75.

The coexistence of different attractors and chaotic dynamics can be found also when only one subordinate is intolerant. For example, in the case of one imitator and one intolerant; or when one compensator is matched with one intolerant. On the other hand, the other matchings are much less interesting and therefore not analyzed here, as it is sufficient the analysis we provided in Section 3.

5. Conclusion

In this paper we extended the classes of behavior for subordinates interacting in a supervised work group. According to [25], organizational behavior focuses on few work-related attitudes. In particular, most of the literature has looked at job satisfaction, job involvement, and organizational commitment. Nevertheless, other attitudes may be important: perceived organizational support and employee engagement. As, in the model we consider, the structure of interaction is rather fixed, we introduced engagement in the task. The new functional form allowed us to take into account the effort each subordinate expects from the colleague and his reaction – depending on the fact that the observed effort may be lower or higher than his expectation.

The introduction of this functional form allowed us to shed light on different aspects of the group dynamics. In fact, according to previous research, since subordinates with different capacity seem to be one of the principal causes for chaotic dynamics, we considered work groups in which the subordinates have the same capacities and analyzed what other aspects could affect the dynamics. Furthermore, we provided an exhaustive analysis of the equilibria which may result when subordinates with different behaviors are matched and we derived the number of possible equilibria in each case. Finally, we analyzed the stability of the origin. At this point the production collapses as both subordinates work only with the supervisor. Therefore, the conditions we found are interesting in determining what may cause this outcome which is the worst possible from the organizational perspective.

One important lesson we can derive from our analysis is that having subordinates with the same capacity is not sufficient to avoid chaotic dynamics. In fact, from the economics point of view, having subordinates with the same capacity can be related to the so-called shutdown of inefficient agents [20] and corresponds to a situation in which only efficient subordinates are employed. Nevertheless, this could not be sufficient to avoid conflicts within the group when considering also some of the important noneconomic facts suggested by [5]. In addition, since according to the literature intolerance may be the result of exposure to inequity, as long as subordinates are intolerant to deviation from what they expect from colleagues, any difference can trigger retaliation, cycles, chaotic dynamics and eventually the collapse of production. Therefore, simply matching individual according to their capacity may not solve the problem.

Furthermore, incentive schemes play an important role. Compensation schemes which are appropriate for fully rational subordinates may not be effective in practice. Rather, it is important to implement compensation systems which are fair in terms of distributive and procedural justice instead of being optimal for the fully rational agents. Finally, it is important to monitor how the employees evaluate the compensation system in order to prevent intolerance.

In order to apply this kind of analysis to a real situation it would be necessary to estimate several parameters. First, monitoring the subordinates' efforts and the resulting production could provide the production function parameters. Second, to estimate the subordinates' reaction function, we could either use Grounded Theory [17] and [30] as suggested in [3] and [13], or some of the techniques for analyzing real data in behavioral sciences as proposed in [18].

Further research will start from the findings of this paper and will address the consequences on the dynamics of strategies to reduce intolerance. Several factors such as engagement, and perception of the colleague effort seem to affect the dynamics of the work group. Therefore, it will be interesting to evaluate how interventions on these aspects may be able to avoid chaotic dynamics, steer the group from one to another of the coexisting cycles and, eventually, to increase the production.

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Notes:

1 The case Γ=0 is trivial and therefore, omitted.
2 It is immediate to observe that the two tasks are not additive; for a discussion the reader may refer to [29].
3 This notation is commonly used in the Game Theory literature, see for example [15].
4 In [11] it was proved that there exist infinitely many solutions among which one is rather natural and can be interpreted as focal in the sense of [28]. There, Schelling considers points “ which are focal for each person’s expectation of what the other expects him to expect to be expected to do” ([28, p. 57]).
5 Forcing $0^0=1$ the map expression could be simpler:

$$l_i(t+1) = r_i(l_{-i}(t)) = \frac{e_i}{\theta_i}(l_{-i}(t))^{a_i-1}(1-l_{-i}(t))^{b_i-1} \quad a_i, b_i \geqslant 1$$

With

$$\theta_i = \begin{cases} \left(\frac{a_i-1}{a_i+b_i-2}\right)^{a_i-1}\left(1-\frac{a_i-1}{a_i+b_i-2}\right)^{b_i-1} & \text{if } a_i > 1 \text{ and } b_i > 1 \\ 1 & \text{otherwise} \end{cases}$$

6 The careful reader can observe that the k_i parameter we introduce here is not exactly the same of the one used in [10]. Nevertheless, by a simple change of scale the two intolerance parameters become comparable.
7 We recall that the reaction function is a transformation of a beta probability density function.
8 This kind of equilibrium has been discussed in oligopoly dynamics, see for example[21] and [22] and also in [6] where the computation of the periodic points, as well as the structure of the basins, has been analyzed.

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