



# Threshold corrections, generalised prepotentials and Eichler integrals

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Received 5 March 2015; received in revised form 10 June 2015; accepted 11 June 2015

Available online 12 June 2015

Editor: Stephan Stieberger

## Abstract

We continue our study of one-loop integrals associated to BPS-saturated amplitudes in  $\mathcal{N} = 2$  heterotic vacua. We compute their large-volume behaviour, and express them as Fourier series in the complexified volume, with Fourier coefficients given in terms of Niebur–Poincaré series in the complex structure modulus. The closure of Niebur–Poincaré series under modular derivatives implies that such integrals derive from holomorphic prepotentials  $f_n$ , generalising the familiar prepotential of  $\mathcal{N} = 2$  supergravity. These holomorphic prepotentials transform anomalously under T-duality, in a way characteristic of Eichler integrals. We use this observation to compute their quantum monodromies under the duality group. We extend the analysis to modular integrals with respect to Hecke congruence subgroups, which naturally arise in compactifications on non-factorisable tori and freely-acting orbifolds. In this case, we derive new explicit results including closed-form expressions for integrals involving the  $\Gamma_0(N)$  Hauptmodul, a full characterisation of holomorphic prepotentials including their quantum monodromies, as well as concrete formulæ for holomorphic Yukawa couplings.

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<http://dx.doi.org/10.1016/j.nuclphysb.2015.06.009>

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## 1. Introduction

The leading radiative corrections to scattering processes in closed string theories compactified on a  $d$ -dimensional torus are computed by modular integrals of the form

$$\mathcal{I} = \int_{\mathcal{F}} d\mu \Gamma_{d+k,d} \Phi, \quad (1.1)$$

where  $\Gamma_{d+k,d}$  is the partition function of the Narain lattice with signature  $(d+k, d)$ ,  $\Phi$  is an appropriate modular form of negative weight  $(-\frac{k}{2}, 0)$  under the modular group  $\Gamma = \text{SL}(2; \mathbb{Z})$  of the worldsheet torus, and  $\mathcal{F}$  is a fundamental domain for the action of  $\Gamma$  on the Poincaré upper half plane  $\mathbb{H}$ . In order to extract quantitative predictions from string theory, it is therefore essential to develop efficient tools to compute such modular integrals.

In the traditional lattice unfolding method [1–3], one decomposes the lattice sum  $\Gamma_{d+k,d}$  into orbits under the modular group, and reduces each orbit to a single term while unfolding the integration domain. Although the resulting expression is very useful in extracting the asymptotic expansion of  $\mathcal{I}$  in the limit where the volume of the torus  $T^d$  becomes large, it depends on a choice of chamber in the Narain moduli space, and T-duality invariance is no longer manifest.

In a series of recent works [5–7], we proposed a new method, which assumes that  $\Phi$  can be written as an absolutely convergent Poincaré series under the modular group. Upon unfolding the integration domain against the sum over images, and evaluating the integral, the result is expressed as a sum over charges satisfying a BPS-type condition. This can be recognised as a particular type of Poincaré series for the automorphism group  $\text{O}(d+k, d; \mathbb{Z})$  of the Narain lattice, which is thus manifestly invariant under T-duality. The analytic structure of the integral near points of enhanced gauge symmetry is also manifest, while the large volume asymptotics is no longer apparent. In fact, the method can be applied even when the integrand does not include the Narain lattice, *e.g.* in ten-dimensional string vacua.

If  $\Phi$  is a generic non-holomorphic modular form, there is no general known way to represent it as a Poincaré series. For BPS-saturated amplitudes, however, the modular form  $\Phi$  turns out to be weakly almost holomorphic, *i.e.* a polynomial in the ring  $\mathbb{C}[\hat{E}_2, E_4, E_6, 1/\Delta]$ . In that case, it can actually be decomposed as a linear combination of so-called Niebur–Poincaré series  $\mathcal{F}(s, \kappa, w; z)$  [6,8]. These Poincaré series, introduced in the mathematics literature in [9,10], have the virtues of being eigenmodes of the Laplacian on  $\mathbb{H}$  with a pole of order<sup>1</sup>  $\kappa$  at the cusp  $z = \infty$  and of being closed under modular derivatives and Hecke operators. In addition, they reduce to harmonic Maaß forms when  $s = 1 - \frac{w}{2}$ , or to modular derivatives thereof when  $s = 1 - \frac{w}{2} + n$  with  $n$  a positive integer. For suitable linear combinations, the non-analytic parts of  $\mathcal{F}(s, \kappa, w; z)$  (encoded in the ‘shadow’) cancel, producing the desired weakly almost holomorphic form  $\Phi$ . The integral (1.1) is thus reduced to a sum of integrals of the form

$$\mathcal{I}(s, \kappa) = \int_{\mathcal{F}} d\mu \Gamma_{d+k,d} \mathcal{F}(s, \kappa, -k/2), \quad (1.2)$$

<sup>1</sup> While the integrands appearing in heterotic string theory only have a first order pole at  $z = \infty$ , it is useful to allow poles of arbitrary order  $\kappa \geq 1$  as such poles typically occur in modular integrals for congruent subgroups, relevant for orbifold compactifications. Moreover, Niebur–Poincaré series  $\mathcal{F}(s, \kappa', w)$  with  $\kappa' > 1$  occur in the Fourier expansion of modular integrals of the form (1.2), even if  $\kappa = 1$ .

each of which can be computed by unfolding the fundamental domain against the Niebur–Poincaré series [6].

In this work, we compute the large volume expansion of the modular integral (1.2), thus providing an alternative way to extract the asymptotics of (1.1) within our unfolding method. The advantage of decomposing  $\Phi$  into a sum of Niebur–Poincaré series is that the integral (1.1) is then decomposed into a sum of terms which are separately eigenmodes of the Laplacian on the Narain moduli space

$$G_{d+k,d} = O(d+k, d; \mathbb{Z}) \backslash SO(d+k, d) / SO(d+k) \times SO(d), \tag{1.3}$$

and which behave in a simple way under covariant derivatives with respect to the moduli.

For simplicity, we restrict our attention to integrals involving lattices of signature (2, 2), where the moduli space (1.3) is parametrised by the standard variables  $(T = T_1 + iT_2, U = U_1 + iU_2) \in \mathbb{H}_T \times \mathbb{H}_U$ . They correspond to the Kähler modulus and complex structure of a two-torus, and are identified under separate actions of the modular group  $SL(2; \mathbb{Z})$  on  $T$  and  $U$ , and under the involution  $\sigma : T \leftrightarrow U$ . This case is relevant when the internal space includes a  $T^2$  factor, e.g. in the context of heterotic strings compactified on  $T^2$  (with 16 supercharges), on  $K3 \times T^2$  (with 8 supercharges), or on  $\mathcal{N} = 1$  orbifolds with  $\mathcal{N} = 2$  sub-sectors, upon restriction to backgrounds with trivial or discrete Wilson lines.<sup>2</sup> We consider both the limit where the volume  $T_2$  is scaled to infinity keeping the complex structure modulus  $U$  fixed, and the limit where both  $T_2$  and  $U_2$  are scaled to infinity (in which case the asymptotic expansion depends on a choice of chamber,  $T_2 > U_2$  vs.  $T_2 < U_2$ ). In the latter case, the result is expressed as an infinite sum of a special type of single-valued polylogarithms, known in the mathematics literature as Bloch–Wigner–Ramakrishnan polylogarithms [11], which coincide with the real part of the ‘combined polylogarithms’ introduced in [13]. In the former case,  $T_2 \rightarrow \infty$  with  $U$  fixed, we show that the Fourier coefficients of the integral  $\mathcal{I}(s, \kappa)$  with respect to  $T_1$  are given by the same Niebur–Poincaré series  $\mathcal{F}(s, \kappa, 0)$  present in the integrand, now as functions of the  $U$  modulus. This is a manifestation of a phenomenon first observed in [14] which, as we shall see, can be traced to the average value property of eigenmodes of the Laplacian on the Poincaré upper half-plane. An important consequence of this observation is that, using the closure of Niebur–Poincaré series under modular derivatives, one may express the integral (1.2) as an iterated derivative of a ‘harmonic prepotential’  $h_n$ [13]

$$\mathcal{I}(n+1, \kappa) = 4 \operatorname{Re} \left[ \frac{(-D_T D_U)^n}{n!} h_n(T, U) \right] - 24 \sigma_1(\kappa) \delta_{n,0} \log T_2 U_2. \tag{1.4}$$

The prepotential  $h_n$  is by construction harmonic in  $(T, U)$  and transforms covariantly under  $SL(2; \mathbb{Z})_T \times SL(2; \mathbb{Z})_U$ , while the last term originates from the infrared divergence of the integral when  $n = 0$ . Crucially, it is also possible to construct a holomorphic prepotential  $f_n(T, U)$  which satisfies the same relation (1.4). This holomorphic prepotential is well-defined up to the addition of polynomials of degree  $2n$  in  $T$  and  $U$  with real coefficients, which lie in the kernel of the operator  $\operatorname{Re}(D_T D_U)^n$ . As a result, it need not transform homogeneously under T-duality, but may pick up an additional degree  $2n$  polynomial. This type of anomalous behaviour under modular transformations is characteristic of Eichler integrals. Indeed, we trace the modular anomaly of  $f_n(T, U)$  to the presence of a term proportional to the Eichler integral of the holomorphic Eisenstein series  $E_{2n+2}$  in the Fourier expansion. We use this fact to fully characterise the T-duality transformations of the generalised prepotential  $f_n$ .

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<sup>2</sup> We defer the analysis of asymmetric lattices with non-trivial Wilson lines to a forthcoming publication.

From a physics viewpoint, the prepotentials  $f_n$  for  $n > 1$  can be viewed as generalisations of the usual holomorphic prepotential  $F$  that governs  $\mathcal{N} = 2$  supergravity in four dimensions – or more accurately, of the one-loop contribution to it [4,15–17]. In fact, it was already recognised in [15] that the quadratic terms appearing in the behaviour of the one-loop prepotential  $f_1$  under monodromies around singularities in moduli space can be obtained from contour integrals of suitable modular forms, which are Eichler integrals in essence. Our results enable us to recover the monodromy matrices computed in [15] in a streamlined fashion, and to extend the analysis to  $n > 1$ . The case  $n = 2$  was studied in the context of  $F^4$  couplings in heterotic strings compactified on  $T^2$  [18,19], where a relation of the generalised prepotentials  $f_n$  to periods on symmetric products of  $n$  K3's was suggested. However, the analogue of special geometry for  $n > 1$  is not understood at present. As a step in this direction, we construct a class of ‘generalised Yukawa couplings’, obtained from  $f_n$  by iterated derivatives with respect to the  $T$  and  $U$  moduli, which compute modular integrals with insertions of momenta and are free of modular anomalies.

From a mathematics viewpoint, the prepotentials  $f_n$  can also be viewed as interesting generalisations of Borcherds automorphic products [20] – or rather, logarithms thereof. Indeed, in the special case where  $\mathcal{F}(1+n, 1, -2n)$  is a weakly holomorphic modular form,  $f_n$  can be written schematically as a polylogarithm sum

$$f_n(T, U) = \frac{1}{2^{2n}} \sum_{N, M} F_n(NM) \text{Li}_{2n+1}(q_T^M q_U^N) + P_{2n}(T, U), \quad (1.5)$$

where  $F_n(M)$  are the Fourier coefficients of  $\mathcal{F}(1+n, 1, -2n)$  and  $P_{2n}$  is the polynomial ambiguity mentioned above. For  $n = 0$ ,  $\text{Li}_1(z) = -\log(1-z)$ , so  $f_0$  is recognised as the logarithm of a Borcherds automorphic product. For  $n > 0$ ,  $f_n$  is no longer the logarithm of a product, but it provides an interesting example of a holomorphic ‘Mock’ modular form on  $(\mathbb{H}_T \times \mathbb{H}_U)/\sigma$  with known singular behaviour at all divisors.

Returning to physics, the integrals (1.1) do not exhaust the class of one-loop amplitudes of interest in string theory. In fact, in compactifications on non-factorisable tori [21,22] or on freely-acting orbifolds [23–27] the integrand is typically expressed as a sum of various sectors  $(h, g)$  involving shifted Narain lattices  $\Gamma_{2,2}[\frac{h}{g}]$  coupled to modular forms  $\Phi[\frac{h}{g}]$ . Although the full integrand is, by construction, invariant under the full modular group  $\text{SL}(2; \mathbb{Z})$ , each term is only invariant under a suitable congruence subgroup. These setups have not been extensively analysed so far in the literature and call for a systematic investigation. Upon partial unfolding [7], one may compute the integral by focusing on a single sector now integrated over the fundamental domain of the congruence subgroup. In the case of freely-acting  $\mathbb{Z}_N$  orbifolds of  $\mathcal{N} = 2$  compactifications, the relevant subgroup is the Hecke congruence subgroup  $\Gamma_0(N)$ . In parallel with the  $\text{SL}(2; \mathbb{Z})$  case, one may decompose  $\Phi[\frac{h}{g}]$  as a linear combination of the  $\Gamma_0(N)$  Niebur–Poincaré series  $\mathcal{F}_\alpha(1+n, \kappa, 0)$  attached to the cusp  $\alpha$  [7]. One may then evaluate the modular integral by unfolding the fundamental domain  $\mathcal{F}_N$  of  $\Gamma_0(N)$  against the Poincaré series. This procedure was suggested in [7] and applied to the study of radiative corrections to gauge couplings in classically stable non-supersymmetric heterotic vacua in [26]. The result is a representation of the modular integral as a Poincaré series under the T-duality group, in general a congruence subgroup of  $\text{O}(2, 2; \mathbb{Z})$ .

In this work, we continue this analysis and extract the large-volume expansion of one-loop modular integrals for  $\Gamma_0(N)$  with  $N$  prime. We obtain expressions which extend the results for  $\text{SL}(2; \mathbb{Z})$  and are nicely covariant with respect to the choice of cusp  $\alpha$ . The advantage of our approach, based on the special properties of  $\mathcal{F}_\alpha(1+n, \kappa, 0)$ , is that we obtain a  $T$ -Fourier series expansion with  $U$ -dependent coefficients which are manifestly invariant under  $\Gamma_0(N)_U$ , and

which actually are given in terms of the same class of Niebur–Poincaré series. This property is instrumental in allowing us to derive new explicit results. Namely, in the  $n = 0$  case we re-sum the integral in terms of the  $\Gamma_0(N)$  Hauptmodul by generalising the celebrated  $\mathrm{SL}(2; \mathbb{Z})$  Borcherds product formula to the case of Hecke congruence subgroups. For  $n \geq 1$  we identify generalised holomorphic prepotentials and fully characterise their transformation properties under T-duality. Finally, we express the generalised holomorphic Yukawa couplings in terms of the standard generators of the ring of  $\Gamma_0(N)$  modular forms.

The paper is organised as follows: in Section 2, after recalling basic properties of Niebur–Poincaré and Eisenstein series, we compute the large volume asymptotics of the modular integral (1.2) or, equivalently, the Fourier expansion with respect to  $T_1$ , starting from its T-duality invariant representation. We express the results in terms of single-valued polylogarithms, and make contact with the literature. In Section 3 we integrate this result in terms of generalised harmonic and holomorphic prepotentials, explain their relation to Eichler integrals, and determine their anomalous transformation properties under T-duality. We construct generalised Yukawa couplings by taking suitable modular derivatives of the generalised prepotentials, and connect them to modular integrals with lattice-momentum insertions. Section 4 extends the previous analysis to modular integrals involving shifted Narain lattices. We obtain new explicit results including ‘cusp-covariant’ Fourier series expansions, closed expressions for special classes of integrals involving the  $\Gamma_0(N)$  Hauptmodul, a full characterisation of the generalised holomorphic prepotentials, as well as concrete formulæ for the generalised holomorphic Yukawa couplings. Appendix A collects our conventions for the Kloosterman sums and zeta functions, and Appendix B provides a detailed derivation of the modular properties of the relevant Eichler integrals.

## 2. Modular integrals for $\mathrm{SL}(2; \mathbb{Z})$

In this section, we shall focus on modular integrals (1.1) involving the product of a signature  $(2, 2)$  Narain lattice  $\Gamma_{2,2}$  times a weakly almost holomorphic modular functions  $\Phi(\tau)$ . As anticipated, the latter can be uniquely decomposed in terms of Niebur–Poincaré series of zero weight [6,8]. Thus, we begin by collecting some useful properties of Eisenstein and Niebur–Poincaré series which are instrumental for the subsequent analysis.

### 2.1. Niebur–Poincaré series for $\mathrm{SL}(2; \mathbb{Z})$

The Niebur–Poincaré series [8–10]<sup>3</sup>

$$\begin{aligned} \mathcal{F}(s, \kappa, w; z) &= \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathcal{M}_{s,w}(-\kappa y) e^{-2\pi i \kappa x} |w| \gamma \\ &= \frac{1}{2} \sum_{(c,d)=1} (cz + d)^{-w} \mathcal{M}_{s,w} \left( -\frac{\kappa y}{|cz + d|^2} \right) \\ &\quad \times \exp \left\{ -2i\pi \kappa \left( \frac{a}{c} - \frac{cx + d}{c|cz + d|^2} \right) \right\}, \end{aligned} \quad (2.1)$$

<sup>3</sup> We shall henceforth omit the explicit dependence of the Niebur–Poincaré series on the modular parameter  $z$  whenever it is clear from the context.

defines a one-parameter family of modular forms of non-positive weight  $w$  and order  $\kappa$  pole in the nome  $q = e^{2i\pi z}$  at the cusp  $z \equiv x + iy = \infty$ . Its seed is expressed in terms of the Whittaker  $M$ -function

$$\mathcal{M}_{s,w}(t) = |4\pi t|^{-\frac{w}{2}} M_{\frac{w}{2}\text{sgn}(t),s-\frac{1}{2}}(4\pi |t|), \tag{2.2}$$

whose behaviour at  $t \rightarrow 0$  guarantees absolute convergence when  $\text{Re}(s) > 1$ . The Poincaré series can be shown to admit a meromorphic continuation to the complex  $s$  plane. Moreover, in the domain of analyticity,  $\mathcal{F}(s, \kappa, w)$  is an eigenmode of the weight- $w$  hyperbolic Laplacian  $\Delta_z$ ,

$$\left[ \Delta_z + \frac{1}{2}s(1-s) + \frac{1}{8}w(w+2) \right] \mathcal{F}(s, \kappa, w) = 0, \tag{2.3}$$

where

$$\Delta_z = 2\bar{D}_z D_z, \quad \text{with} \quad \bar{D}_z = -i\pi y^2 \partial_{\bar{z}}, \quad D_z = \frac{i}{\pi} \left( \partial_z - \frac{iw}{2y} \right). \tag{2.4}$$

Its Fourier expansion reads

$$\mathcal{F}(s, \kappa, w) = \mathcal{M}_{s,w}(-\kappa y) e^{-2i\pi\kappa x} + \sum_{m \in \mathbb{Z}} \tilde{\mathcal{F}}_m(s, \kappa, w; y) e^{2i\pi m x}, \tag{2.5}$$

where, in the mode with frequency  $m = -\kappa$ , we have separated the contribution of the seed. Explicitly,

$$\begin{aligned} \tilde{\mathcal{F}}_0(s, \kappa, w; y) &= f_0(s, \kappa, w) y^{1-s-\frac{w}{2}}, \\ \tilde{\mathcal{F}}_m(s, \kappa, w; y) &= f_m(s, \kappa, w) \mathcal{W}_{s,w}(my), \end{aligned} \tag{2.6}$$

with

$$\begin{aligned} f_0(s, \kappa, w) &= \frac{2^{2-w} i^{-w} \pi^{1+s-\frac{w}{2}} |\kappa|^{s-\frac{w}{2}} \Gamma(2s-1) \sigma_{1-2s}(\kappa)}{\Gamma(s-\frac{w}{2}) \Gamma(s+\frac{w}{2}) \zeta(2s)}, \\ f_m(s, \kappa, w) &= \frac{4\pi i^{-w} |\kappa| \Gamma(2s)}{\Gamma(s+\frac{w}{2} \text{sgn}(m))} \left| \frac{m}{\kappa} \right|^{\frac{w}{2}} \mathcal{Z}(m, -\kappa; s). \end{aligned} \tag{2.7}$$

In these expressions,  $\mathcal{W}_{s,w}(t)$  is related to the Whittaker  $W$ -function

$$\mathcal{W}_{s,w}(t) = |4\pi t|^{-\frac{w}{2}} W_{\frac{w}{2}\text{sgn}(t),s-\frac{1}{2}}(4\pi |t|), \tag{2.8}$$

$\sigma_\ell(n)$  is the divisor function, while  $\mathcal{Z}(a, b; s)$  is the associated Kloosterman–Selberg zeta function defined in (A.4).

The Niebur–Poincaré series are closed under the action of the modular derivatives,

$$\begin{aligned} D_z \mathcal{F}(s, \kappa, w) &= 2\kappa \left( s + \frac{w}{2} \right) \mathcal{F}(s, \kappa, w + 2), \\ \bar{D}_z \mathcal{F}(s, \kappa, w) &= \frac{1}{8\kappa} \left( s - \frac{w}{2} \right) \mathcal{F}(s, \kappa, w - 2), \end{aligned} \tag{2.9}$$

and of the Hecke operator<sup>4</sup>

$$\mathcal{F}(s, \kappa, w) = H_\kappa^{(z)} \mathcal{F}(s, 1, w). \tag{2.10}$$

For the special values  $s = \frac{w}{2}$  and  $s = 1 - \frac{w}{2}$ , within the domain of analyticity,  $\mathcal{F}(s, \kappa, w)$  becomes a harmonic Maaß form. For negative weight, the case of interest for most applications to string theory [6], the Niebur–Poincaré series is absolutely convergent at the latter value  $s = 1 - \frac{w}{2}$ . Moreover, since the first cusp form for  $SL(2; \mathbb{Z})$  appears at weight twelve, for  $w \in \{0, -2, -4, -6, -8, -12\}$  the shadow of  $\mathcal{F}(1 - \frac{w}{2}, \kappa, w)$  vanishes and the Niebur–Poincaré series is actually holomorphic, hence valued in the polynomial ring generated by  $E_4, E_6$  and  $\Delta^{-1}$ , where  $E_{2n}$  are the holomorphic Eisenstein series and  $\Delta$  is the modular discriminant. Other Niebur–Poincaré series of interest are  $\mathcal{F}(1 - \frac{w}{2} + n, \kappa, w)$ , with  $n$  a positive integer, which can be reached from the harmonic Maaß form  $\mathcal{F}(1 - \frac{w}{2} + n, \kappa, w - 2n)$  by acting with  $n$  modular derivatives. In fact, one can prove [6–8] that any weak almost holomorphic modular form of negative weight in the polynomial ring generated by  $\hat{E}_2, E_4, E_6$  and  $\Delta^{-1}$ , with  $\hat{E}_2$  being the almost holomorphic Eisenstein series of weight two, can be uniquely decomposed into a linear combination of Niebur–Poincaré series,

$$\Phi(z) = \sum_{0 < m \leq \kappa} \sum_{\ell=0}^n d_\ell(m) \mathcal{F}(1 - \frac{w}{2} + \ell, m, w). \tag{2.11}$$

The coefficients  $d_\ell(m)$  are determined by the principal part of the  $q$ -Laurent expansion of  $\Phi$  itself [7]. The integer  $n$  counts the maximal power of  $\hat{E}_2$  in the ring decomposition of  $\Phi$ , and is known as the *depth*. An important fact is that for each value of  $\ell$ , the shadows cancel in the linear combination (2.11).

It will also be useful to introduce the non-holomorphic weight- $w$  Eisenstein series

$$\begin{aligned} E(s, w; z) &= \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y^{s - \frac{w}{2}} |w \gamma \\ &= \frac{1}{2} \sum_{(c,d)=1} \frac{y^{s - \frac{w}{2}}}{(cz + d)^w |cz + d|^{2s-w}} \end{aligned} \tag{2.12}$$

which is absolutely convergent for  $\text{Re}(s) > 1$ , and admits the Fourier series expansion

$$\begin{aligned} E(s, w; z) &= y^{s - \frac{w}{2}} + \frac{4^{1-s} i^{-w} \pi \Gamma(2s - 1) \zeta(2s - 1)}{\Gamma(s + \frac{w}{2}) \Gamma(s - \frac{w}{2}) \zeta(2s)} y^{1-s - \frac{w}{2}} \\ &+ \sum_{m \neq 0} \frac{i^{-w} (4\pi |m|)^{\frac{w}{2}} \pi^s}{\Gamma(s + \frac{w}{2} \text{sgn}(m)) \zeta(2s)} |m|^{-s} \sigma_{2s-1}(|m|) \mathcal{W}_{s,w}(my) e^{2i\pi mx}. \end{aligned} \tag{2.13}$$

The Eisenstein series  $E(s, w)$  can be obtained formally as the limit  $\kappa \rightarrow 0$  of  $|4\pi\kappa|^{\frac{w}{2}-s} \mathcal{F}(s, \kappa, w)$ , and thus shares several of the properties of the Niebur–Poincaré series. Namely, it is an eigenmode of the weight- $w$  hyperbolic Laplacian

<sup>4</sup> The Hecke operator  $H_n^{(z)}$  acts on the Fourier modes of a weight- $w$  modular form  $\Phi = \sum_m \Phi(m, y) e^{2i\pi mx}$  as

$$H_n^{(z)} \Phi(m, y) = n^{1-w} \sum_{d|(m,n)} d^{w-1} \Phi(nm/d^2, d^2 y/n).$$

It satisfies the commutative algebra  $H_\kappa^{(z)} H_\lambda^{(z)} = \sum_{d|(\kappa,\lambda)} d^{1-w} H_{\kappa\lambda/d^2}^{(z)}$ .

$$\left[ \Delta_w + \frac{1}{2}s(1-s) + \frac{1}{8}w(w+2) \right] E(s, w; z) = 0, \tag{2.14}$$

is closed under the action of the modular derivatives

$$\begin{aligned} D_z E(s, w; z) &= \frac{1}{2\pi} \left( s + \frac{w}{2} \right) E(s, w+2; z), \\ \bar{D}_z E(s, w; z) &= \frac{\pi}{2} \left( s - \frac{w}{2} \right) E(s, w-2; z), \end{aligned} \tag{2.15}$$

and becomes harmonic in  $z$  when  $s = \frac{w}{2}$  or  $s = 1 - \frac{w}{2}$ . For  $w > 2$ ,  $E(\frac{w}{2}, w)$  coincides with the holomorphic Eisenstein series  $E_w$ .

### 2.2. Evaluation of the modular integral

We can now proceed to the study of the integral (1.1). Since any weak almost holomorphic modular form  $\Phi$  can be decomposed as in (2.11) it is sufficient to focus our attention on the basic regularised modular integral

$$\mathcal{I}(s, \kappa; T, U) = \text{R.N.} \int_{\mathcal{F}} d\mu \Gamma_{2,2}(T, U; \tau) \mathcal{F}(s, \kappa, 0; \tau), \tag{2.16}$$

where  $\Gamma_{2,2}(T, U; \tau)$  is the partition function of the even self-dual lattice of signature (2, 2), and R.N. stands for the modular invariant renormalisation prescription of [5,6,28] for treating the infra-red divergences arising at the cusp. For  $s = 1$ , the integral (2.16) develops a simple pole, which must be subtracted [6,8]. By abuse of notation we denote by  $\mathcal{I}(1, \kappa; T, U)$  the result of this subtraction. The partition function

$$\Gamma_{2,2}(T, U; \tau) = \tau_2 \sum_{m_1, m_2, n^1, n^2 \in \mathbb{Z}} q^{\frac{1}{4}|p_L|^2} \bar{q}^{\frac{1}{4}|p_R|^2}, \tag{2.17}$$

which we refer to as the Narain lattice, is a sum over integers  $m_i$  and  $n^i$  identified with the Kaluza–Klein momentum and winding numbers along a two-dimensional torus. The left-moving and right-moving momenta on the two-dimensional lattice

$$\begin{aligned} p_R &= \frac{m_2 - U m_1 + T n^1 + T U n^2}{\sqrt{T_2 U_2}}, \\ p_L &= \frac{m_2 - U m_1 + \bar{T} n^1 + \bar{T} U n^2}{\sqrt{T_2 U_2}}, \end{aligned} \tag{2.18}$$

depend on the Kähler modulus  $T = T_1 + iT_2$  and the complex structure modulus  $U = U_1 + iU_2$  of the two-torus.  $\Gamma_{2,2}(T, U; \tau)$  is modular invariant under  $\text{SL}(2; \mathbb{Z})$ , which requires the modular weight of the Niebur–Poincaré series in (2.16) to vanish.

Eq. (2.17) corresponds to the so-called Hamiltonian representation of the Narain lattice, and exhibits manifest invariance under the duality group  $\text{O}(2, 2; \mathbb{Z}) \sim \text{SL}(2; \mathbb{Z})_T \times \text{SL}(2; \mathbb{Z})_U \times \sigma_{T,U}$ , where the  $\mathbb{Z}_2$  generator  $\sigma_{T,U}$  exchanges the  $T$  and  $U$  moduli. A Poisson summation over the momenta  $m_i$  yields the so-called Lagrangian representation

$$\Gamma_{2,2}(T, U; \tau) = T_2 \sum_{A \in \text{Mat}_{2 \times 2}(\mathbb{Z})} e^{-2\pi i T \det(A)} \exp \left[ -\frac{\pi T_2}{\tau_2 U_2} \left| \begin{pmatrix} 1 & U \\ & A \end{pmatrix} \begin{pmatrix} -\tau \\ 1 \end{pmatrix} \right|^2 \right], \tag{2.19}$$



which now exhibits manifest invariance under  $SL(2; \mathbb{Z})_\tau \times SL(2; \mathbb{Z})_U \times \sigma_{\tau,U}$ . As a consequence,  $\Gamma_{2,2}(T, U; \tau)$  is invariant under the action of  $SL(2; \mathbb{Z})_\tau \times SL(2; \mathbb{Z})_T \times SL(2; \mathbb{Z})_U \times \sigma_{\tau,T,U}$ , where  $\sigma_{\tau,T,U}$  permutes the  $\tau, T$  and  $U$  moduli. This triality results in the identities

$$\Delta_T \Gamma_{2,2} \equiv \Delta_U \Gamma_{2,2} \equiv \Delta_\tau \Gamma_{2,2}, \quad H_n^{(\tau)} \Gamma_{2,2} \equiv H_n^{(T)} \Gamma_{2,2} \equiv H_n^{(U)} \Gamma_{2,2}, \tag{2.20}$$

where  $\Delta_z$  is the weight-zero hyperbolic Laplacian in the variable  $z$ . Together with the differential equation (2.3), the first equation in (2.20) implies that the modular integral (2.16) satisfy

$$\left[ \Delta_T + \frac{1}{2}s(1-s) \right] \mathcal{I}(s, \kappa) = \left[ \Delta_U + \frac{1}{2}s(1-s) \right] \mathcal{I}(s, \kappa) = 12 \sigma_1(\kappa) \delta_{s,1}, \tag{2.21}$$

where the source term is due to the subtraction of the simple pole in the integral (2.16) at  $s = 1$ .

Moreover, it suffices to consider the integral  $\mathcal{I}(s) \equiv \mathcal{I}(s, 1)$  since eqs. (2.10), (2.20) together with the self-adjointness of the Hecke operator allows one to express the integral (2.16) for general positive integer  $\kappa$  as

$$\mathcal{I}(s, \kappa) = H_\kappa^{(T)} \mathcal{I}(s) = H_\kappa^{(U)} \mathcal{I}(s). \tag{2.22}$$

2.2.1. The BPS state sum representation

A novel procedure for evaluating the integral (2.16) has been discussed in [6]. It amounts to unfolding the  $SL(2; \mathbb{Z})$  fundamental domain  $\mathcal{F}$  against the absolutely convergent Niebur–Poincaré series. One thus arrives at the Schwinger-like representation

$$\mathcal{I}(s) = \sum_{\text{BPS}} \int_0^\infty \frac{d\tau_2}{\tau_2} \mathcal{M}_{s,0}(-\tau_2) e^{-\frac{\pi\tau_2}{2}(|p_L|^2 + |p_R|^2)}, \tag{2.23}$$

where  $\sum_{\text{BPS}}$  denotes a sum over integers  $m_i, n^i$  subject to the ‘BPS constraint’<sup>5</sup>

$$m_1 n^1 + m_2 n^2 = 1. \tag{2.24}$$

Performing the  $\tau_2$ -integral then yields [6]

$$\mathcal{I}(s) = \Gamma(s) \sum_{\text{BPS}} \left( \frac{|p_L|^2}{4} \right)^{-s} {}_2F_1 \left( s, s; 2s; \frac{4}{|p_L|^2} \right). \tag{2.25}$$

The sum is absolutely convergent for  $\text{Re}(s) > 1$  and can be analytically continued to a meromorphic function on the complex  $s$  plane with a simple pole at  $s = 1$  [6,8]. The renormalisation prescription implies that this pole should be subtracted in defining the value of (2.25) at  $s = 1$ . Note that the hypergeometric function reduces to simple elementary functions for integral positive values of  $s$  [6].

Actually, eq. (2.25) provides an explicit Poincaré series representation of the automorphic function  $\mathcal{I}_{2,2}(s, 1)$  of  $O(2, 2; \mathbb{Z})$ . To illustrate this point, we note that the BPS constraint (2.24) allows one to write

$$|p_L|^2 = \frac{|\tilde{T} - \tilde{U}|^2}{\tilde{T} \tilde{U}_2}, \quad \text{with} \quad \tilde{T} = \gamma_T \cdot T, \quad \tilde{U} = \gamma_U \cdot U, \tag{2.26}$$

<sup>5</sup> BPS states exist for any charges such that  $m_1 n^1 + m_2 n^2 \leq 1$ , so (2.24) picks only a subset of the allowed BPS states associated to the  $q$ -pole, which corresponds to the ground state of the non-supersymmetric side of the heterotic string.

where  $\gamma_T$  and  $\gamma_U$  are both  $SL(2; \mathbb{Z})$  matrices. One may thus express the integral

$$\mathcal{I}(s) = \sum_{\gamma \in \Gamma_{\text{diag}} \setminus (\Gamma_T \times \Gamma_U)} \varphi \left( \frac{|\tilde{T} - U|^2}{T_2 U_2} \right) \Big| \gamma, \tag{2.27}$$

as the  $O(2, 2; \mathbb{Z})$  Poincaré series, with seed

$$\varphi(z) = \Gamma(s) \left( \frac{z}{4} \right)^{-s} {}_2F_1 \left( s, s; 2s; \frac{4}{z} \right), \tag{2.28}$$

invariant under the diagonal subgroup  $\Gamma_{\text{diag}}$  of  $\Gamma_T \times \Gamma_U$ . It is straightforward to see that the action of the Hecke operator  $H_\kappa \mathcal{I}(s, 1) = \mathcal{I}(s, \kappa)$  amounts to rescaling  $|p_L|^2 \rightarrow |p_L|^2/\kappa$  and modifying the BPS constraint to  $m_1 n^1 + m_2 n^2 = \kappa$ .

The BPS-state-sum representation of the modular integral is thus manifestly invariant under the action of the duality group and is valid throughout the Narain moduli space. In particular, it makes manifest the singularities of the integral (2.16) in the vicinity of points of symmetry enhancement where  $|p_R|^2 \rightarrow 0$  for some choice of the integers  $m_i, n^i$  [6]. If, however, one is interested in the asymptotic behaviour of (2.16) at the boundary of the moduli space, one should instead resort to a Fourier-series representation of the integral. This is best accomplished at the level of the Schwinger-like integral (2.23) as we now show.

### 2.2.2. The Fourier series representation

The BPS constraint (2.24) admits an infinite number of solutions which are in one-to-one correspondence with elements of  $SL(2; \mathbb{Z})$ . For any given co-prime pair  $(n^1, n^2)$  the Euclidean algorithm provides a pair of co-prime integers  $(m_1^*, m_2^*)$  solving (2.24). The most general solution can thus be obtained as

$$m_1 = m_1^* + \tilde{M} n^2, \quad m_2 = m_2^* - \tilde{M} n^1, \tag{2.29}$$

for any  $\tilde{M} \in \mathbb{Z}$ . Plugging this solution in (2.23) and Poisson summing over  $\tilde{M}$ , provides the desired Fourier expansion of the integral,

$$\begin{aligned} \mathcal{I}(s) &= \sum_{M \in \mathbb{Z}} \sum_{(n^1, n^2)=1} \sqrt{T_2 \tilde{U}_2} e^{2i\pi M(T_1 - \tilde{U}_1)} \\ &\times \int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}} \mathcal{M}_{s,0}(-\tau_2) \exp \left[ -\pi \tau_2 \left( \frac{T_2}{\tilde{U}_2} + \frac{\tilde{U}_2}{T_2} \right) - \frac{\pi M^2 T_2 \tilde{U}_2}{\tau_2} \right] \end{aligned} \tag{2.30}$$

where

$$\tilde{U} \equiv \tilde{U}_1 + i\tilde{U}_2 = \frac{m_1^* U - m_2^*}{n^2 U + n^1} \equiv \gamma \cdot U, \quad \text{with} \quad \gamma = \begin{pmatrix} m_1^* & -m_2^* \\ n^2 & n^1 \end{pmatrix} \in SL(2; \mathbb{Z}). \tag{2.31}$$

The integral in (2.30) can be readily evaluated for any  $T_2$  and  $\tilde{U}_2$ , using the explicit expression

$$\mathcal{M}_{s,0}(t) = 2^{2s-1} \Gamma(s + \frac{1}{2}) (4\pi|t|)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi|t|), \tag{2.32}$$

and eq. (16) of Section 8.6 in [12],<sup>6</sup> and reads

$$\mathcal{I}(s) = \mathcal{I}^{(0)}(s) + \mathcal{I}^{(+)}(s) + \mathcal{I}^{(-)}(s), \tag{2.33}$$

with the zero-frequency mode given by

$$\mathcal{I}^{(0)}(s) = 2^{4s-2} \sqrt{4\pi} \Gamma(s - \frac{1}{2}) \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} (T_2 \tilde{U}_2)^s \left( T_2 + \tilde{U}_2 + |T_2 - \tilde{U}_2| \right)^{1-2s}, \tag{2.34}$$

the positive-frequency modes given by

$$\begin{aligned} \mathcal{I}^{(+)}(s) = & \sum_{M>0} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \frac{e^{2i\pi M(T_1 - \tilde{U}_1)}}{M} \mathcal{M}_{s,0} \left( \frac{1}{2} M \left[ T_2 + \tilde{U}_2 - |T_2 - \tilde{U}_2| \right] \right) \\ & \times \mathcal{W}_{s,0} \left( \frac{1}{2} M \left[ T_2 + \tilde{U}_2 + |T_2 - \tilde{U}_2| \right] \right), \end{aligned} \tag{2.35}$$

and the negative-frequency modes simply obtained via complex conjugation,  $\mathcal{I}^{(-)}(s) = [\mathcal{I}^{(+)}(s^*)]^*$ . Notice that we have recognised the sum over the co-prime integers  $(n^1, n^2)$  as the sum over cosets  $\Gamma_\infty \setminus \Gamma$  where  $\Gamma = \Gamma_U$ .

Actually, the Fourier series expansion has different coefficients in the chambers  $T_2 > U_2$  and  $T_2 < U_2$ . Without loss of generality we can always assume that both  $T$  and  $U$  lie in the fundamental domain  $\mathcal{F}$ . Then, whenever  $U$  satisfies  $U_2 < T_2$ , so do all its images  $\tilde{U} = \gamma U$ , for any  $\gamma \in \Gamma_\infty \setminus \Gamma$ . On the contrary, when  $T_2 > 1$  all points  $U$  with  $U_2 > T_2$  are mapped to points  $\tilde{U}$  with  $\tilde{U}_2 < T_2$ , except when  $\gamma$  is in the coset of the identity. If  $T_2 < 1$ , then both  $U$  and  $\tilde{U} = -1/U$  may satisfy the inequality  $\tilde{U}_2 > T_2$ .

As a result, the Fourier coefficients for  $\mathcal{I}(s)$  read

$$\begin{aligned} \mathcal{I}^{(0)}(s) = & 2^{2s} \sqrt{4\pi} \Gamma(s - \frac{1}{2}) \left[ T_2^{1-s} E(s, 0; U) \right. \\ & \left. + \frac{1}{2} \sum_{\gamma=1, S} \Theta(\tilde{U}_2 - T_2) \left( T_2^s \tilde{U}_2^{1-s} - T_2^{1-s} \tilde{U}_2^s \right) \right], \end{aligned} \tag{2.36}$$

and

$$\begin{aligned} \mathcal{I}^{(+)}(s) = & 2 \sum_{M>0} \left[ \frac{e^{2i\pi MT_1}}{M} \mathcal{W}_{s,0}(MT_2) \mathcal{F}(s, M, 0; U) + \sum_{\gamma=1, S} \Theta(\tilde{U}_2 - T_2) \right. \\ & \left. \times \frac{e^{2i\pi M(T_1 - \tilde{U}_1)}}{M} \left( \mathcal{M}_{s,0}(MT_2) \mathcal{W}_{s,0}(M\tilde{U}_2) - \mathcal{W}_{s,0}(MT_2) \mathcal{M}_{s,0}(M\tilde{U}_2) \right) \right], \end{aligned} \tag{2.37}$$

where  $\Theta(x)$  is the Heaviside function. Note that, upon Fourier expanding the Niebur–Poincaré series, the expressions in the two chambers are related by  $T \leftrightarrow U$  exchange. Indeed, one could

<sup>6</sup> We reproduce this formula for convenience:

$$\int_0^\infty \frac{dt}{t} e^{-\frac{1}{2t} - \beta t} I_\nu(\gamma t) = 2 K_\nu \left( \sqrt{\beta + \sqrt{\beta^2 - \gamma^2}} \right) I_\nu \left( \sqrt{\beta - \sqrt{\beta^2 - \gamma^2}} \right) \quad (\beta > \gamma > 0).$$

have obtained the very same results by alternatively solving the BPS constraint in terms of pairs of co-prime momenta  $(m_1, m_2)$ , which would result in a Fourier expansion with respect to  $U$ , convergent in the chamber  $U_2 > T_2$ , with coefficients manifestly invariant under  $SL(2; \mathbb{Z})_T$ .

For convenience, we shall henceforth focus our attention to the fundamental chamber  $T_2 > U_2$ , where the Heaviside function in (2.36) and (2.37) vanishes, and the above expressions simplify to

$$\begin{aligned} \mathcal{I}^{(0)}(s) &= 2^{2s} \sqrt{4\pi} \Gamma(s - \frac{1}{2}) T_2^{1-s} E(s, 0; U), \\ \mathcal{I}^{(+)}(s) &= 2 \sum_{M>0} \frac{e^{2i\pi MT_1}}{M} \mathcal{W}_{s,0}(MT_2) \mathcal{F}(s, M, 0; U). \end{aligned} \tag{2.38}$$

Remarkably, using (2.10), the coefficient of the  $M$ th Fourier mode can be seen to involve the Hecke transform of the same function  $\mathcal{F}(s, 1, 0)$  as the one appearing in the integrand of (2.16), except that the former depends on  $U$  while the latter depends on  $\tau$ . This is a manifestation of a fact first noticed in [14], whose rationale will be explained in Section 2.2.3.

While the most relevant case for heterotic string theory is  $\kappa = 1$ , on which we have concentrated so far, the results above may be extended to higher values of  $\kappa$  by acting with the Hecke operator. Specifically, one obtains

$$\begin{aligned} \mathcal{I}^{(0)}(s, \kappa) &= 2^{2s} \sqrt{4\pi} \Gamma(s - \frac{1}{2}) (\kappa T_2)^{1-s} \sigma_{2s-1}(\kappa) E(s, 0; U), \\ \mathcal{I}^{(+)}(s, \kappa) &= 2 \sum_{d|\kappa} \sum_{M>0} \frac{e^{2\pi idMT_1}}{M} \mathcal{W}_{s,0}(dMT_2) \mathcal{F}\left(s, \frac{\kappa M}{d}, 0; U\right). \end{aligned} \tag{2.39}$$

### 2.2.3. Unfolding against the Narain lattice

An alternative approach for obtaining the Fourier series representation of the modular integral, commonly used in the past string theory literature, is to unfold the fundamental domain against the Narain lattice [1–3], rather than against the Niebur–Poincaré series. To this end, one resorts to the Lagrangian representation of the  $(2, 2)$  lattice and decomposes the sum of winding numbers  $A$  into orbits of the  $SL(2; \mathbb{Z})_\tau$  action. The vanishing orbit contribution, corresponding to the trivial matrix  $A = 0$ , is absent in this case since the average of any Niebur–Poincaré series over the fundamental domain vanishes. The degenerate orbits, corresponding to non-vanishing matrices  $A$  with  $\det A = 0$ , precisely reproduce  $\mathcal{I}^{(0)}$  in (2.38). The non-degenerate orbits, associated to the set of matrices

$$A = \begin{pmatrix} k & j \\ 0 & p \end{pmatrix}, \quad 0 \leq j < k, \quad p \neq 0, \tag{2.40}$$

can all be reconstructed from the contribution of the unit matrix  $A = \mathbb{1}$ , via the action of the Hecke operator  $H_{\det(A)}$  on the complex structure  $U$ . In fact, one can write

$$\mathcal{I}_{\text{non-deg}}(s) = 4 T_2 \operatorname{Re} \sum_{M>0} e^{2i\pi MT} H_M^{(U)} \int_{\mathbb{C}_+} d\mu e^{-\frac{\pi MT_2}{\tau_2 U_2} |U-\tau|^2} \mathcal{F}(s, 1, 0; \tau), \tag{2.41}$$

and upon evaluating the integral over the upper-complex plane one reproduces the non-vanishing frequency contributions to the Fourier expansion (2.38).

In the framework of the old unfolding method, the fact that the Fourier coefficients in (2.38) involve the Hecke transform of the same function  $\mathcal{F}(s, 1, 0)$  appearing in the integrand is a con-

sequence of Fay’s theorem [29], which states that any eigenfunction  $G(z)$  of the hyperbolic Laplacian with eigenvalue  $s(1 - s)$ , not necessarily automorphic, has the mean value property

$$\frac{1}{m(\rho; s)} \int_{\mathbb{C}_+} d\mu \rho(r) G(z) = G(z_0). \tag{2.42}$$

Here  $r$  is the geodesic distance between the points  $z$  and  $z_0$ , and  $\rho(r)$  is an arbitrary ‘measure’ on the complex upper plane  $\mathbb{C}_+$  that depends on  $r$  but not on the polar angle between  $z$  and  $z_0$ , and the weight factor  $m(\rho; s)$  is given by

$$m(\rho; s) = 2\pi \int_0^\infty P_s(\cosh r) \rho(r) \sinh r \, dr, \tag{2.43}$$

with  $P_s(t)$  the Legendre function. The integral in (2.41) is seen to be a special case of (2.42) with  $z = \tau, z_0 = U$  and

$$\rho(r) = e^{-\frac{\pi MT_2}{\tau_2 U_2} |\tau - U|^2} = e^{-4\pi MT_2 \sinh^2 \frac{r}{2}}. \tag{2.44}$$

The integral (2.43) can be computed with the help of 7.141.5 in [30],

$$m(\rho; s) = \frac{e^{-2\pi MT_2}}{MT_2} \mathcal{W}_{s,0}(MT_2), \tag{2.45}$$

which thus yields the result (2.38).

### 2.3. The special cases of integral $s$

Let us now focus on the cases  $s = 1 + n$ , with  $n$  a non-negative integer. For low enough values of  $n$  the Niebur–Poincaré series  $\mathcal{F}(1 + n, 1, 0)$  reduce to weak almost holomorphic modular forms, and their modular integral (2.16) admits alternative representations. The case  $s = 1$  is special in that  $\mathcal{F}(1, 1, 0) = j(z) + 24$  is holomorphic,<sup>7</sup> and  $\mathcal{I}(1)$  is related to Borcherds’ automorphic products. For  $n$  large the Niebur–Poincaré series are genuinely non-holomorphic. However, as discussed in Section 2.1, the linear combinations (2.11) still reduce to weak almost holomorphic modular forms, and are thus amenable to the same type of treatment.

#### 2.3.1. The case $s = 1$

As already noted, for  $s = 1$  the integral (2.16) has a pole which must be subtracted [6,8]. This pole originates from the non-holomorphic Eisenstein series  $E(s, 0; U)$  appearing in (2.38). Using the first Kronecker limit formula one thus obtains

$$\mathcal{I}^{(0)}(1) = -24 \log \left( T_2 U_2 |\eta(U)|^4 \right) + \text{const}, \tag{2.46}$$

where the additive constant is computable with the renormalisation scheme of [5,28], but inconsequential for our purposes.

The positive frequency modes are regular at  $s = 1$  and, using  $\mathcal{W}_{1,0}(t) = e^{-2\pi|t|}$ , they simplify to

<sup>7</sup> Our convention for the Klein  $j$ -function does not involve a constant term in its Fourier expansion.

$$\begin{aligned} \mathcal{I}^{(+)}(1) &= 2 \sum_{M>0} \frac{q_T^M}{M} \mathcal{F}(1, M, 0; U) \\ &= 2 \sum_{M>0} \frac{q_T^M}{M} \left[ \mathcal{F}(1, M, 0; U) - \tilde{\mathcal{F}}_0(1, M, 0; U) \right] - 48 \log \left( q_T^{-1/24} \eta(T) \right), \end{aligned} \tag{2.47}$$

where we have introduced the nome  $q_T$  associated to the Kähler modulus  $T$ , and  $\tilde{\mathcal{F}}_0(1, M, 0) = 24 \sigma_1(M)$  is the constant zero-mode of Niebur–Poincaré series, see (2.6).

Using holomorphy and modularity in the  $T$  and  $U$  variables, together with the behaviour  $\mathcal{F}(1, M, 0) - \tilde{\mathcal{F}}_0(1, M, 0) \sim q_U^{-M}$  at the cusp  $\infty$ , one may show

$$\sum_{M>0} \frac{q_T^M}{M} \left[ \mathcal{F}(1, M, 0; U) - \tilde{\mathcal{F}}_0(1, M, 0; U) \right] = -\log q_T (j(T) - j(U)), \tag{2.48}$$

which is simply a reformulation of the Borcherds product formula [20]

$$q_T^{-1} \prod_{M>0, N \in \mathbb{Z}} \left( 1 - q_T^M q_U^N \right)^{c(NM)} = j(T) - j(U), \tag{2.49}$$

where  $c(M)$  are the Fourier coefficients of the Klein  $j$ -function.

Putting things together, the modular integral reads

$$\mathcal{I}(1) = -24 \log(T_2 U_2 |\eta(T) \eta(U)|^4) - \log |j(T) - j(U)|^4 + \text{const}. \tag{2.50}$$

The first term on the r.h.s. originates from the constant in the decomposition  $\mathcal{F}(1, 1, 0) = j(z) + 24$  and is the celebrated result of [3], whereas the second term originates from the Klein function and can be uniquely determined by harmonicity, automorphy and by the singularities of the modular integral [4,31]. These originate from additional massless states appearing at the special points  $T = U$  and their  $\text{SL}(2; \mathbb{Z})$  images. Similar automorphic products arise for  $\kappa > 1$ , and can be obtained by acting with the Hecke operators  $H_\kappa$  in either  $T$  or  $U$  variables, but have more complicated singular divisors.

### 2.3.2. The case $s = 1 + n$ with $n > 0$

For integer  $s = 1 + n > 1$  the Niebur–Poincaré series  $\mathcal{F}(1 + n, 1, 0)$  becomes weakly almost holomorphic. Using the fact

$$\mathcal{W}_{1+n,0}(t) = (-1)^n n! (4\pi|t|)^{-n} e^{-2\pi|t|} L_n^{(-1-2n)}(4\pi|t|), \tag{2.51}$$

with  $L_n^{(\alpha)}$  being the associated Laguerre polynomials, we can express the integral in terms of the ‘combined polylogarithms’ introduced in [13]

$$\mathcal{L}_{(k)}(z) = \sum_{\ell=0}^k \frac{(k + \ell)!}{\ell!(k - \ell)!(4\pi)^\ell} y^{k-\ell} \text{Li}_{k+\ell+1}(e^{2i\pi z}). \tag{2.52}$$

To this end, notice that the non-holomorphic Eisenstein series may be expressed as

$$\begin{aligned} E(1 + n, 0; z) &= y^{1+n} + \frac{\sqrt{\pi} \Gamma(n + \frac{1}{2}) \zeta(2n + 1)}{n! \zeta(2n + 2)} y^{-n} \\ &\quad + 2 \text{Re} \left[ \frac{\pi^{1+n} y^{-n}}{n! \zeta(2n + 2)} \sum_{N>0} \mathcal{L}_{(n)}(Nz) \right], \end{aligned} \tag{2.53}$$

so that  $\mathcal{I}^{(0)}$  can be easily written in terms of the single-valued polylogarithms. As for the non-vanishing modes, we recall the Fourier series expansion of the Niebur–Poincaré series at the special value  $s = 1 + n$ ,

$$\mathcal{F}(1 + n, 1, 0) = \sum_{N \geq -1} F_n(N) (4\pi y)^{-n} L_n^{(-1-2n)}(4\pi Ny) q^N, \tag{2.54}$$

where

$$\begin{aligned} F_n(-1) &= \Gamma(2n + 2), \\ F_n(0) &= \frac{(2\pi)^{2+2n} (-1)^n}{\zeta(2n + 2)}, \\ F_n(N > 0) &= 4\pi (-1)^n \Gamma(2n + 2) \frac{\mathcal{Z}(N, -1; 1 + n)}{N^n}, \end{aligned} \tag{2.55}$$

as can be derived from eq. (2.7) and Appendix A. Using (2.7), one may check that the  $F_n(N)$ ’s are in fact the Fourier coefficients of  $\mathcal{F}(1 + n, 1, -2n) = \sum_{N \geq -1} F_n(N) q^N$ .

Using the following expression for the product of two associated Laguerre polynomials

$$(n!)^2 L_n^{(-1-2n)}(u) L_n^{(-1-2n)}(v) = \sum_{k=0}^n \sum_{\ell=0}^k \frac{(n+k)! (k+\ell)!}{k! (n-k)! \ell! (k-\ell)!} (uv)^{n-k} (u+v)^{k-\ell}, \tag{2.56}$$

one finds

$$\mathcal{I}^{(+)} = \frac{2(-1)^n}{n!} \sum_{M>0} \sum_{N \in \mathbb{Z}} F_n(MN) \sum_{k=0}^n \frac{(n+k)!}{k! (n-k)!} \frac{(MN)^n}{(4\pi MNT_2U_2)^k} L_{(k)}(MT + NU), \tag{2.57}$$

in agreement with [13]. Notice that the combination (2.52) of the polylogarithms has the property that its real part is a single-valued function in the complex plane [11]. This ensures that the modular integral be a well-defined function in the fundamental Weyl chamber.

### 3. Generalised prepotentials for $SL(2; \mathbb{Z})$

Using the behaviour (2.9) of Niebur–Poincaré series under the action of modular derivatives, we show in this section that for integer  $s = 1 + n > 1$ , the modular integral (2.16) can be expressed in terms of a holomorphic function  $f_n(T, U)$ , which we refer to as the generalised prepotential,<sup>8</sup>

$$\mathcal{I}(1 + n) = 4 \operatorname{Re} \frac{(-D_T D_U)^n}{n!} f_n(T, U). \tag{3.1}$$

The case  $n = 1$  arises in computations of one-loop threshold corrections to gauge couplings in heterotic string theory compactified on  $K3 \times T^2$ , where  $f_1(T, U)$  is identified as the one-loop correction to the usual holomorphic prepotential  $F(X)$  encoding the vector-multiplet self-couplings

<sup>8</sup> This equation holds for  $n < 5$  and  $n = 6$ , such that  $\mathcal{F}(1 + n, 1, -2n)$  is a weak holomorphic modular form. For  $n = 5$  and  $n \geq 7$ ,  $\mathcal{I}(1 + n)$  can always be integrated to a harmonic prepotential. For suitable linear combinations such that the shadows of  $\mathcal{F}(1 + n, \kappa, -2n)$  cancel, there is no obstruction in taking the prepotential to be holomorphic.

in  $\mathcal{N} = 2$  supergravity theories in four dimensions [15–17]. The case  $n = 2$  arose in studies of  $F^4$  and  $R^4$  couplings in heterotic strings on  $T^2$  [18], although the physical significance of the prepotential  $f_2$  in the context of eight-dimensional supergravity remains to be elucidated. The generalised prepotentials  $f_n$  were first introduced in [13], where integrals of the type (1.1) were considered with  $\Phi$  being an arbitrary weakly holomorphic modular form. Since  $\Phi$  is in general a mixture of Niebur–Poincaré series  $\mathcal{F}(1 + n, 1, 0)$  with different values of  $n$ , the integral (1.1) becomes a sum of iterated derivatives of various prepotentials  $f_n$ . One advantage of decomposing  $\Phi$  into a sum of Niebur–Poincaré series  $\mathcal{F}(1 + n, 1, 0)$  is that it allows one to disentangle these contributions.

While it is interesting and useful that the integral (2.16) can be integrated to a holomorphic function, the drawback is that this function is ambiguous up to polynomials of degree  $2n$  in  $T$  and  $U$  with real coefficients, as these polynomials lie in the kernel of  $\text{Re}(D_T D_U)^n$ . This implies that under T-duality, the prepotential  $f_n$  need not be a standard modular form of weight  $-2n$ , but rather can transform inhomogeneously, picking up extra polynomials under the action of the modular group. This behaviour is characteristic of Eichler integrals. Indeed, we shall see that  $f_n$  contains a  $T$ -independent term proportional to the Eichler integral of the holomorphic Eisenstein series  $E_{2n+2}(U)$ , which is responsible for this modular anomaly. Alternatively, one may express the integral as an iterated derivative of a harmonic function  $h_n(T, U)$ , which is modular covariant with weight  $-2n$  but not holomorphic. This is in fact our first step in establishing (3.1).

### 3.1. Generalised harmonic prepotentials

To derive the harmonic prepotentials associated to the integral  $\mathcal{I}(1 + n)$ , we recall the action (2.9) and (2.15) of the modular derivatives on the Niebur and Eisenstein Poincaré series, and observe that

$$\begin{aligned}
 D^n q^N &= (-2N)^n \mathcal{W}_{n+1,0}(Ny) e^{2\pi i N x}, \\
 D^n 1 &= \frac{2^{2n}}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right) (-2\pi y)^{-n},
 \end{aligned}
 \tag{3.2}$$

provided  $q^M$  is treated as a generic mode in the Fourier-series expansion of a modular form of weight  $-2n$  in  $z$ . Using (2.9) and (2.15) we may then cast our result (2.38) for the integral as the action of the iterated derivatives

$$\mathcal{I}(1 + n) = 4 \text{Re} \frac{(-D_T D_U)^n}{n!} h_n(T, U),
 \tag{3.3}$$

with

$$\begin{aligned}
 h_n(T, U) &= (2\pi)^{2n+1} E(1 + n, -2n; U) \\
 &\quad + \sum_{N>0} \frac{2}{(2N)^{2n+1}} q_T^N \mathcal{F}(1 + n, N, -2n; U).
 \end{aligned}
 \tag{3.4}$$

Since the Eisenstein and Niebur–Poincaré series appearing in (3.4) have  $s = 1 - \frac{w}{2}$ , the function  $h_n(T, U)$  is harmonic in  $U$ , and holomorphic in  $T$ , and we shall refer to it as the generalised harmonic prepotential. For low values of  $n$ , the non-vanishing frequency modes are in fact also holomorphic in  $U$ , but the zero-mode part never is. Importantly,  $h_n$  is manifestly modular under the action of  $\text{SL}(2; \mathbb{Z})_U$ , with weight  $-2n$ . Yet, it is not modular under the full T-duality group,



in particular it is not invariant under  $T \leftrightarrow U$ . Note that  $h_n$  is a priori defined up to terms in the kernel of  $\text{Re}(D_T D_U)^n$ . Our choice ensures that  $h_n$  is modular under  $\text{SL}(2; \mathbb{Z})_U$ .

As before, this result generalises to the case of  $\kappa > 1$  by the action of the Hecke operator on either  $T$  or  $U$ . Specifically,

$$h_n(T, U; \kappa) = \kappa^{-n} H_\kappa \cdot h_n(T, U). \tag{3.5}$$

The overall factor is a consequence of the identity

$$\kappa^n H_\kappa D^n \mathcal{F}(s, 1, w) = D^n \mathcal{F}(s, \kappa, w). \tag{3.6}$$

Notice, that generalised *harmonic* prepotentials can always be introduced independently of the value of  $n$ , even when the Niebur–Poincaré series in the integrand is no longer weak almost holomorphic but is rather genuinely non-holomorphic, which occurs when  $n = 5$  and  $n > 6$ .

### 3.2. Generalised holomorphic prepotentials and Eichler integrals

As we have seen, the harmonic prepotential  $h_n$  emerges rather naturally and has the advantage of being modular covariant, at the cost of sacrificing holomorphy. Although this is legitimate from a mathematical viewpoint, it is however difficult to reconcile with the holomorphic structure of  $\mathcal{N} = 2$  supergravity, since  $h_1$  should compute the radiative correction to the holomorphic prepotential  $F(X)$ . As we shall see, this is not a problem and holomorphy may always be restored, whenever the Niebur–Poincaré series in the integrand is weakly almost holomorphic.

For  $n = 1, 2, 3, 4, 6$  the only source of non-holomorphy in the harmonic prepotential is due to the zero-frequency mode  $\mathcal{I}^{(0)}$ . As a result, we shall first restrict our analysis to this contribution. Using (2.15) one has

$$E(1+n, 0) = \frac{(2\pi)^n}{n!} D^n E(1+n, -2n) = \frac{(2\pi)^n}{n!} \text{Re} D^n E(1+n, -2n), \tag{3.7}$$

where  $E(1+n; -2n)$  is a harmonic Maaß form. As a result, it can be decomposed

$$E(1+n; -2n) = \Phi_h + \Phi_{\text{nh}}, \tag{3.8}$$

into its holomorphic (Mock modular) part  $\Phi_h$  and the non-holomorphic complement  $\Phi_{\text{nh}}$ , with

$$\Phi_h(z) = \left(\frac{z}{2i}\right)^{2n+1} + \frac{(-1)^n \pi \zeta(2n+1)}{2^{2n+1} \zeta(2n+2)} + \frac{(-1)^n \pi}{2^{2n} \zeta(2n+2)} \sum_{N>0} \sigma_{-2n-1}(N) q^N, \tag{3.9}$$

and

$$\begin{aligned} \Phi_{\text{nh}}(x, y) = & y^{2n+1} - \left(\frac{z}{2i}\right)^{2n+1} + \frac{(-1)^n \pi \zeta(2n+1)}{2^{2n+1} \zeta(2n+2)} \\ & + \frac{(-1)^n \pi}{2^{2n} \zeta(2n+2)} \sum_{N>0} \sigma_{-2n-1}(N) \frac{\Gamma(2n+1; 4\pi N y)}{\Gamma(2n+1)} q^{-N}. \end{aligned} \tag{3.10}$$

The remarkable identities

$$\begin{aligned} \text{Re} \left[ D^n \left(\frac{z}{2i}\right)^{2n+1} \right] &= D^n y^{2n+1}, \\ D^n \left[ \Gamma(2n+1; 4\pi N y) q^{-N} \right] &= \frac{(2n)!}{(4\pi N)^n} \left[ D^n q^N \right]^*, \end{aligned} \tag{3.11}$$

together with equations (3.2) yield the non-trivial result

$$D^n \Phi_h(z) = [D^n \Phi_{nh}(x, y)]^* , \tag{3.12}$$

which originates from the fact that both the shadow and the ghost [6] of the harmonic Maaß form  $E(n + 1, -2n)$  are proportional to the holomorphic Eisenstein series  $E_{2n+2}(z)$ . In fact, one can prove that

$$D^{2n+1} \Phi_h = [(D^*)^{n+1} D^n \Phi_{nh}]^* = y^{-2-2n} [\bar{D} \Phi_{nh}]^* , \tag{3.13}$$

where the left-hand side defines the ghost and the right-hand side defines the shadow.

The above considerations allow one to write

$$\mathcal{I}^{(0)}(n + 1) = \frac{4}{n!} \text{Re}(-D_T D_U)^n f_n^{(0)} , \tag{3.14}$$

where

$$f_n^{(0)}(U) = \alpha \tilde{E}_{-2n}(U) , \tag{3.15}$$

is the zero-frequency mode of the desired generalised holomorphic prepotential, written in terms of

$$\tilde{E}_{-2n}(z) = \frac{\zeta(2n + 2)}{2\pi i} z^{2n+1} + \frac{\zeta(2n + 1)}{2} + \sum_{N>0} \sigma_{-1-2n}(N) q^N , \tag{3.16}$$

and

$$\alpha = \frac{(-1)^n (2\pi)^{2n+2}}{2^{2n} \zeta(2n + 2)} . \tag{3.17}$$

One may recognise  $\tilde{E}_{-2n}(z)$  as the Eichler integral of the holomorphic Eisenstein series  $E_{2n+2}(z)$  [32], defined by

$$E_{2n+2}(z) = \frac{2\pi i}{(2n + 1)! \zeta(2n + 2)} \partial_z^{2n+1} \tilde{E}_{-2n}(z) . \tag{3.18}$$

As anticipated, this holomorphic contribution does not transform covariantly under the action of the modular group, but rather involves an inhomogeneous term, as we shall discuss in the following section. Altogether, the generalised *holomorphic* prepotential reads

$$f_n(T, U) = \alpha \tilde{E}_{-2n}(U) + \sum_{N>0} \frac{2}{(2N)^{2n+1}} q_T^N \mathcal{F}(1 + n, N, -2n; U) . \tag{3.19}$$

It is important to stress that this choice of holomorphic prepotential is not unique and one has the freedom of adding a polynomial of degree  $2n$  in  $T$  and  $U$  with real coefficients. In (3.19) we have made a convenient choice for this polynomial.

Upon Fourier-expanding  $\mathcal{F}(1 + n, 1, -2n) = \sum_{N \geq -1} F_n(N) q^N$ , the generalised holomorphic prepotential admits the alternative representation

$$f_n(T, U) = \alpha \tilde{E}_{-2n}(U) + 2^{-2n} \sum_{\substack{N>0 \\ M \in \mathbb{Z}}} F_n(NM) \text{Li}_{2n+1}(q_T^N q_U^M) \tag{3.20}$$

in terms of polylogarithms. By further Fourier expanding the Eichler integral and rearranging terms, one arrives at

$$\begin{aligned}
 f_n(T, U) = & \alpha \zeta(2n + 2) \left[ \frac{U^{2n+1}}{2\pi i} - \frac{1}{2} \mathcal{Z}(0, 0; 1 + n) \right] + \frac{F_n(-1)}{2^{2n}} \text{Li}_{2n+1} \left( \frac{q_T}{q_U} \right) \\
 & + \frac{1}{2^{2n}} \sum_{N, M \geq 0} F_n(NM) \text{Li}_{2n+1}(q_T^N q_U^M), \tag{3.21}
 \end{aligned}$$

which exposes the asymmetry of the generalised holomorphic prepotentials under  $T \leftrightarrow U$  exchange, originating from the singularity at  $T = U$  in the term proportional to  $F_n(-1)$ .

Although the zero-frequency part  $\mathcal{I}^{(0)}$  may always be expressed in terms of a holomorphic function  $f_n^{(0)}$  for any  $n$ , this is no longer true for the non-vanishing frequency modes when  $n = 5$  and  $n > 6$ , since the Niebur–Poincaré series  $\mathcal{F}(1 + n, N, -2n)$  are genuine harmonic Maaß forms [6]. This is related to the fact that, for these values of  $n$ ,  $\mathcal{F}(1 + n, 1, 0)$  is a genuine non-holomorphic function. However, one can always associate a generalised holomorphic prepotential to the linear combination (2.11), representing weak almost holomorphic modular forms, since the shadows will cancel also in the prepotential.

Similarly to the harmonic case, the generalised holomorphic prepotential  $f_n(T, U; \kappa)$  may be straightforwardly obtained via the action of the Hecke operator  $H_\kappa$  as in eq. (3.5).

### 3.3. Modular properties of generalised holomorphic prepotentials

As anticipated, the generalised holomorphic prepotential of eq. (3.19) does not transform homogeneously with weight  $-2n$  under the action of the T-duality group. Rather, owing to the inherent polynomial ambiguity in  $f_n$ , one has

$$\begin{aligned}
 (cT + d)^{2n} (c'U + d')^{2n} f_n \left( \frac{aT + b}{cT + d}, \frac{a'U + b'}{c'U + d'} \right) = & f_n(T, U) + P_{\gamma, \gamma'}(T, U), \\
 f_n(T, U) = & f_n(U, T) + P_\sigma(T, U), \tag{3.22}
 \end{aligned}$$

where  $P_{\gamma, \gamma'}(T, U)$  and  $P_\sigma(T, U)$  are polynomials of degree  $2n$  in  $(T, U)$  with real coefficients. These polynomials themselves depend on the choice of polynomial ambiguity in  $f_n$ , which we have fixed in the definition (3.19). It is important to note that there is no choice of this ambiguity that can trivialise all polynomials  $P$ 's.

As far as the action of  $\text{SL}(2; \mathbb{Z})_U$  is concerned, the anomaly can be traced to the presence of the Eichler integral  $\tilde{E}_{-2n}(U)$  in the zero-th Fourier mode (3.15) with respect to  $T$ . Indeed, under a generic element  $\gamma \in \text{SL}(2; \mathbb{Z})$ , the Eichler integral transforms as

$$(cz + d)^{2n} \tilde{E}_{-2n} \left( \frac{az + b}{cz + d} \right) = \tilde{E}_{-2n}(z) + P_\gamma(z), \tag{3.23}$$

with  $P_\gamma(z)$  being a degree  $2n$  polynomial in the  $z$  variable, that depends on the choice of  $\gamma$ . The polynomials  $P_\gamma(z)$  are uniquely determined by  $P_T(z)$  and  $P_S(z)$ , which are associated to the standard generators  $T$  and  $S$  of the modular group given in (B.4). It is straightforward to derive  $P_T$  since the nome  $q$  is trivially invariant and the only source of inhomogeneity is the monomial  $z^{2n+1}$ . As a result,

$$P_T(z) = \frac{\zeta(2n + 2)}{2\pi i} \sum_{k=0}^{2n} \binom{2n + 1}{k} z^k. \tag{3.24}$$

The transformation under  $S$  is more involved and can be determined by the analytic properties of the  $L$ -series

$$L^*(s) = \int_0^\infty dy y^{s-1} \sum_{N>0} \sigma_{-1-2n}(N) e^{-2\pi Ny} = (2\pi)^{-s} \Gamma(s) \zeta(s) \zeta(s + 2n + 1), \tag{3.25}$$

associated to  $\tilde{E}_{-2n}$ , as we derive in [Appendix B](#). Notice that although the  $L$ -series satisfies the functional equation  $s \rightarrow w - s$ , with  $w = -2n$ , and has simple poles at  $s = 0$  and  $s = -2n$  as in the case for modular forms of weight  $-2n$ , the additional poles at  $s = 1 - 2p$  with  $p = 0, 1, \dots, n + 1$  are actually responsible for the inhomogeneous transformation [\(3.23\)](#), and the associated residues uniquely fix

$$P_S(z) = -\frac{(2\pi i)^{2n+1}}{2} \sum_{k=1}^n \frac{B_{2k} B_{2n-2k+2}}{(2k)! (2n - 2k + 2)!} z^{2k-1}, \tag{3.26}$$

thus generalising the result of [\[32\]](#) to generic  $n$ .

Clearly,  $P_T$  and  $P_S$  determine the behaviour of the generalised holomorphic prepotential  $f_n(T, U)$  under the action of  $SL(2; \mathbb{Z})_U$ . Moreover, using the property of the polylogarithms

$$Li_k(e^{2i\pi z}) + (-1)^k Li_k(e^{-2i\pi z}) = -\frac{(2\pi i)^k}{k!} B_k(z), \quad 0 < \text{Re}(z) < 1, \tag{3.27}$$

with  $B_k(z)$  the Bernoulli polynomials, one obtains

$$P_\sigma(T, U) = -\frac{(2\pi i)^{2n+1}}{2^{2n}} \left[ B_{2n+1}(T - U) - T^{2n+1} + U^{2n+1} \right], \tag{3.28}$$

when  $T$  and  $U$  lie in the fundamental chamber, with real parts in the interval  $(0, 1)$ . This expression generalises the result of [\[4\]](#) to  $n > 1$ .

The knowledge of the polynomials  $P_S$ ,  $P_T$  and  $P_\sigma$  fully determines the transformation of the prepotential under the duality group, because any transformation  $g' \in SL(2; \mathbb{Z})_T$  can be obtained by conjugating the associated transformation  $g \in SL(2; \mathbb{Z})_U$  by  $\sigma$ , *i.e.*  $g' = \sigma^{-1} g \sigma$ . However, it is important to note that, since  $P_{\sigma^2} \neq 0$ , the generator  $\sigma$  is no longer an involution at the quantum level, and as a result, modular transformations  $g$  and  $g'$  no longer commute. The group generated by these transformations, which we call the monodromy group, is an extension of the classical T-duality group  $SL(2; \mathbb{Z})_T \times SL(2; \mathbb{Z})_U \times \sigma$  given by the braid group of  $(X \times X) \setminus \Delta$ , where  $X$  is the Riemann sphere with three punctures located at images of  $e^{i\pi/2}$ ,  $e^{i\pi/3}$  and  $i\infty$  under the action of the Klein  $j$  function, and  $\Delta$  is the diagonal divisor in  $X \times X$ . The braid group is generated by the elements  $g_1 : U \rightarrow -1/U$ ,  $g_2 : U \rightarrow -1/(U + 1)$  and  $\sigma$ , obeying the relations

$$g_1^2 = g_2^3 = \mathbb{1}, \quad g_i \sigma g_j \sigma^{-1} = \sigma g_j \sigma^{-1} g_i. \tag{3.29}$$

Using the results above, we find that the generators act on the generalised holomorphic prepotential as

$$\begin{aligned} f_n\left(T, -\frac{1}{U}\right) &= U^{-2n} \left[ f_n(T, U) + \alpha P_S(U) \right], \\ f_n\left(T, -\frac{1}{U+1}\right) &= (U+1)^{-2n} \left[ f_n(T, U) + \alpha (P_T(U) + P_S(U+1)) \right], \\ f_n(T, U) &= f_n(U, T) + P_\sigma(T, U), \\ f_n\left(-\frac{1}{T}, U\right) &= T^{-2n} \left[ f_n(T, U) + \alpha P_S(T) - P_\sigma(T, U) + T^{2n} P_\sigma\left(-\frac{1}{T}, U\right) \right], \end{aligned}$$

$$f_n\left(-\frac{1}{T+1}, U\right) = (T+1)^{-2n} \left[ f_n(T, U) + \alpha (P_S(T+1) + P_T(T)) - P_\sigma(T, U) + (T+1)^{2n} P_\sigma\left(-\frac{1}{T+1}, U\right) \right], \tag{3.30}$$

with  $P_T$ ,  $P_S$  and  $P_\sigma$  the polynomials given in eqs. (3.24), (3.26) and (3.28).

For  $n = 1$ , our analysis is in full agreement with the results of [4,15] and extends it to higher values of  $n$ . The precise matching with [15] at the level of the holomorphic prepotential is obtained by adding to  $f_1(T, U)$  the polynomial  $3 + 5U + 3U^2$  that lies in the kernel of  $\text{Re}(-D_T D_U)$ .

### 3.4. Generalised Yukawa couplings

In  $\mathcal{N} = 2$  supergravity in four dimensions, the Yukawa couplings  $Y_{ijk}$  are obtained from the third derivatives of the prepotential  $F$ . They are modular covariant, and, in the case of  $i = j = k = T$  or  $U$ , holomorphic in  $(T, U)$ . Similarly, for heterotic strings compactified on  $T^2$ , the one-loop contributions to  $F_{\mu\nu}^4$  couplings are fifth derivatives of the generalised prepotential  $f_2$ . In this subsection, we construct a similar set of functions  $Y_{T^p U^q}$  with  $p + q = 2n + 1$ , which we call generalised Yukawa couplings, which are proportional to  $(2n + 1)$ th derivatives of the generalised prepotential  $f_n$ , independent of the polynomial ambiguity, holomorphic when  $p = 0$  or  $q = 0$ , modular covariant for any  $p, q$ , and related to one-loop modular integrals with momentum insertions. The physical significance of these functions is not clear at present, but we hope that they may provide hints towards a generalisation of the notion of special geometry to  $n > 1$ .

We shall begin by studying the simplest instance of holomorphic generalised Yukawa couplings

$$Y_{T^{2n+1}}(T, U) = -2^{-2n-1} D_T^{2n+1} f_n(T, U) = \left(\frac{\partial_T}{2\pi i}\right)^{2n+1} f_n(T, U), \tag{3.31}$$

where in the last equality we used Bol’s identity [33], which equates the  $(1 - w)$ -th modular derivative of a modular function of negative integer weight  $w < 0$  to the ordinary  $(1 - w)$ -th derivative. Notice that, since the degree  $2n$  polynomial in the inhomogeneous transformation of  $f_n$  is annihilated by the derivatives,  $Y_{T^{2n+1}}$  indeed transforms as a weight  $(2n + 2, -2n)$  holomorphic modular form with respect to  $\text{SL}(2; \mathbb{Z})_T \times \text{SL}(2; \mathbb{Z})_U$ , for  $n$  sufficiently small. Inserting the expressions (3.19) and (3.20), one obtains

$$\begin{aligned} Y_{T^{2n+1}}(T, U) &= 2^{-2n} \sum_{N>0} q_T^N \mathcal{F}(1 + n, N, -2n; U) \\ &= 2^{-2n} \sum_{\substack{N>0 \\ M \in \mathbb{Z}}} N^{2n+1} F_n(MN) \frac{q_T^N q_U^M}{1 - q_T^N q_U^M}, \end{aligned} \tag{3.32}$$

thus generalising the ‘multi-cover’ formula for the standard Yukawa coupling  $Y_{T^3}$  [34]. This expression clearly exhibits a simple pole divergence at  $q_T = q_U$ ,

$$Y_{T^{2n+1}} \sim \frac{(2n + 1)!}{2^{2n}} \frac{q_T}{q_U - q_T}. \tag{3.33}$$

This behaviour, together with holomorphy and modularity, uniquely fixes the expression of the Yukawa couplings,

$$\begin{aligned}
 Y_{T^3} &= \frac{3}{2} \frac{E_4(T) E_4(U) E_6(U)}{\Delta(U) [j(T) - j(U)]}, & Y_{T^5} &= \frac{15}{2} \frac{E_6(T) E_4^2(U)}{\Delta(U) [j(T) - j(U)]}, \\
 Y_{T^7} &= \frac{315}{4} \frac{E_4^2(T) E_6(U)}{\Delta(U) [j(T) - j(U)]}, & Y_{T^9} &= \frac{2835}{2} \frac{E_4(T) E_6(T) E_4(U)}{\Delta(U) [j(T) - j(U)]},
 \end{aligned}
 \tag{3.34}$$

which may be written compactly as

$$Y_{T^{2n+1}}(T, U) = 2^{-2n} \frac{E_{2n+2}(T) \mathcal{F}(1+n, 1, -2n; U)}{j(T) - j(U)}, \tag{3.35}$$

in terms of the holomorphic Eisenstein and Niebur–Poincaré series with  $s = 1 - \frac{w}{2}$ . Note that the  $Y_{T^{2n+1}}$  have already been discussed in the literature in the context of heterotic string compactifications on  $K3 \times T^2$  and on  $T^2$  [18,19,35], respectively. Similar results can be obtained for  $Y_{U^{2n+1}}$  upon exchanging  $T$  and  $U$  in the above expressions.

The holomorphic generalised Yukawa couplings can be related to modular integrals with lattice-momentum insertions. In fact,

$$\begin{aligned}
 D_T^{n+1} \bar{D}_U^n \mathcal{I}(1+n) &= (-1)^{n+1} \frac{4(2n)!}{n!} Y_{T^{2n+1}}(T, U) \\
 &\quad + \frac{2}{n!} D_T^{n+1} \bar{D}_U^n [(-D_T D_U)^n f_n(T, U)]^*,
 \end{aligned}
 \tag{3.36}$$

where we have used again Bol’s identity and the fact that  $\bar{D}^n D^n$  is proportional to the identity operator when acting on Niebur–Poincaré series. The second term on the r.h.s. actually vanishes due to holomorphy of  $f_n$  and of non-trivial combinatorial identities. Moreover, one can trade the covariant derivatives for insertions of lattice momenta to obtain

$$Y_{T^{2n+1}} = \frac{(-1)^{n+1}}{2^{4n+3}} \frac{U_2^n}{T_2^{n+1}} \int_{\mathcal{F}} d\mu \tau_2 \langle p_L^{2n+1} \bar{p}_R \rangle \mathcal{F}(1+n, 1, -2n). \tag{3.37}$$

Here we have employed the notation

$$\langle f(p) \rangle = \tau_2 \sum_{m_i, n^i} f(p) q^{\frac{1}{4}|p_L|^2} \bar{q}^{\frac{1}{4}|p_R|^2} \tag{3.38}$$

for the Narain lattice with generic momentum insertion  $f(p)$ .

Also in this case, the generalisation to higher values of  $\kappa$  is straightforward and can be obtained via the action of the Hecke operator

$$Y_{T^{2n+1}}(T, U; \kappa) = \kappa^{-n} H_\kappa \cdot Y_{T^{2n+1}}(T, U). \tag{3.39}$$

We can now move to discuss more general Yukawa couplings, which can be obtained as mixed derivatives of the generalised holomorphic prepotential. Aside from sporadic cases [18,19,36], these have not been extensively studied in the literature, and indeed there are ambiguities in their proper definition. In these cases Bol’s identity is no longer applicable and modularity dictates a non-holomorphic completion. For instance, in the combination

$$\begin{aligned}
 Y_{T^p U^q}(T, U; \kappa) &= -\frac{D_U^q}{2^{2n+1}} \left[ D_T^p f_n(T, U; \kappa) - \frac{(2\pi)^{2q-2} p!}{(2n)!(q-1)!} \frac{U_2^{2n}}{T_2^{2n-2q+2}} \right. \\
 &\quad \left. \times \left( D_T^{q-1} D_U^{2n} f_n(T, U; \kappa) \right)^* \right],
 \end{aligned}
 \tag{3.40}$$

with  $p + q = 2n + 1$  and  $1 \leq q \leq n < p$ , the degree  $2n$  polynomial in the inhomogeneous transformation of  $f_n$  is exactly cancelled. After some tedious algebra, one may cast them as modular integrals with lattice momentum insertions

$$Y_{T^p U^q} = (-1)^{n+1} \frac{\kappa^q q!}{2^{4n-2p+1}} \frac{U_2^{n-q}}{T_2^{p-n}} \int_{\mathcal{F}} d\mu \tau_2 \langle p_L^{p-q} \bar{p}_R \rangle \mathcal{F}(1 + n, \kappa, 2q - 2n). \tag{3.41}$$

These mixed Yukawa couplings are clearly non-holomorphic functions, but have holomorphic weights  $2p - 2n$  and  $2q - 2n$  with respect to  $SL(2; \mathbb{Z})_T$  and  $SL(2; \mathbb{Z})_U$ , respectively.

Furthermore, observe that the quantity inside the brackets in eq. (3.40) is itself modular covariant and may be expressed as the modular integral

$$(-1)^n \frac{(2\pi)^{2q} \kappa^{q-n-1} (2p)!}{2^{4n-q+4} n!} \frac{U_2^n}{T_2^{p-n}} \int_{\mathcal{F}} d\mu \tau_2^{q-1} \langle p_L^p p_R^{q-1} \rangle \mathcal{F}(1 + n, \kappa, q - p - 1). \tag{3.42}$$

This is a non-trivial result, and allows one to straightforwardly compute the Fourier series expansions of such integrals by expressing them as derivative of the known generalised prepotential.

#### 4. Modular integrals for $\Gamma_0(N)$

In perturbative computations in many string theory models, one encounters one-loop integrals over the fundamental domain of various congruence subgroups rather than of the full modular group. This happens *e.g.* in orbifold compactifications involving non-factorisable tori [21,22] or when supersymmetry is spontaneously broken, either partially or completely [23–27]. In these cases, the one-loop amplitudes can be expressed as

$$\mathcal{I} = \int_{\mathcal{F}} d\mu \sum_{h,g} \mathcal{A} \left[ \begin{matrix} h \\ g \end{matrix} \right], \tag{4.1}$$

where the sum over  $h$  runs over the untwisted and twisted orbifold sectors and the sum over  $g$  imposes the orbifold projection. Clearly, the integrand is invariant under the full modular group, but each independent term in the sum is typically invariant only under some congruence subgroup of  $SL(2; \mathbb{Z})$ , except for the untwisted unprojected sector  $(h, g) = (0, 0)$  which is invariant under the full modular group. This suggests that an efficient way of evaluating (4.1) is to partially unfold the modular integral [7], *i.e.* collect the  $(h, g)$  sectors into orbits of  $SL(2; \mathbb{Z})$  and keep one element in each orbit, now integrated over the fundamental domain of the corresponding congruence subgroup. Schematically,

$$\sum_{h,g} \mathcal{A} \left[ \begin{matrix} h \\ g \end{matrix} \right] = \sum_a \sum_{\gamma \in \Gamma_a \backslash \Gamma} \mathcal{A}_a | \gamma \tag{4.2}$$

where the amplitudes  $\mathcal{A}_a$  are now invariant under the congruence subgroup  $\Gamma_a \subset \Gamma$ , and the sum over  $a$  runs over the possible subgroups. As a result,

$$\mathcal{I} = \sum_a \mathcal{I}_a = \sum_a \int_{\mathcal{F}_a} d\mu \mathcal{A}_a, \tag{4.3}$$

$\mathcal{F}_a$  being the fundamental domain of  $\Gamma_a$ . The integrals  $\mathcal{I}_a$  can then be computed by employing similar methods as in the case of the full modular group [7].

For simplicity, we shall restrict our analysis to the case of abelian  $\mathbb{Z}_2$  and  $\mathbb{Z}_3$  orbifolds, which play an important role in string vacuum constructions. These cases are special in that they involve a single Hecke congruence subgroup,  $\Gamma_0(2)$  and  $\Gamma_0(3)$ , respectively. Other  $\mathbb{Z}_M$  and non-abelian orbifolds can be dealt with similar methods, though they typically involve several orbits.

We shall further focus on BPS-saturated amplitudes, where the orbifold blocks are of the form

$$\mathcal{A} \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \right] = \frac{1}{N} \Gamma_{2,2} \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \right] (T, U) \Phi \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \right], \tag{4.4}$$

where  $\Phi \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \right]$  is a weakly almost holomorphic modular form under  $\Gamma_0(N)$  and

$$\Gamma_{2,2} \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \right] (T, U, \tau) = \tau_2 \sum_{m_i, n^i \in \mathbb{Z}} e^{\frac{2\pi i g}{N} (\lambda_i n^i + \mu^i m_i)} q^{\frac{1}{4} |p_L(h)|^2} \bar{q}^{\frac{1}{4} |p_R(h)|^2}, \tag{4.5}$$

is the shifted Narain lattice partition function with momenta

$$\begin{aligned} p_L(h) &= \frac{1}{\sqrt{T_2 U_2}} \left[ m_2(h) - U m_1(h) + \bar{T} (n^1(h) + U n^2(h)) \right], \\ p_R(h) &= \frac{1}{\sqrt{T_2 U_2}} \left[ m_2(h) - U m_1(h) + T (n^1(h) + U n^2(h)) \right], \end{aligned} \tag{4.6}$$

where  $m_i(h) = m_i + \frac{h}{N} \lambda_i$ ,  $n^i(h) = n^i + \frac{h}{N} \mu^i$ . The integral shift vector  $(\lambda_i, \mu^i)$  is constrained by modular invariance to satisfy  $\lambda_i \mu^i = 0 \pmod{N}$ .

For  $\mathbb{Z}_N$  orbifolds with  $N = 2, 3$ , partial unfolding leads to

$$\mathcal{I} = \frac{1}{N} \int_{\mathcal{F}} d\mu \Gamma_{2,2} \left[ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] (T, U) \Phi \left[ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] + \frac{1}{N} \sum_{g=1}^{N-1} \int_{\mathcal{F}_N} d\mu \Gamma_{2,2} \left[ \begin{smallmatrix} 0 \\ g \end{smallmatrix} \right] (T, U) \Phi \left[ \begin{smallmatrix} 0 \\ g \end{smallmatrix} \right], \tag{4.7}$$

where  $\mathcal{F}_N$  is the fundamental domain of  $\Gamma_0(N)$ . The relation  $\Phi \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \right] = \Phi \left[ \begin{smallmatrix} N-h \\ N-g \end{smallmatrix} \right]$ , valid for conjugate orbifold sectors, allows one to write

$$\begin{aligned} \mathcal{I} &= \frac{1}{N} \int_{\mathcal{F}} d\mu \Gamma_{2,2} \left[ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] \left[ \Phi \left[ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] - \sum_{\gamma \in \Gamma_0(N) \setminus \Gamma} \Phi \left[ \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] \Big|_{\gamma} \right] \\ &+ \int_{\mathcal{F}_N} d\mu \Phi \left[ \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] \frac{1}{N} \sum_{g=0}^{N-1} \Gamma_{2,2} \left[ \begin{smallmatrix} 0 \\ g \end{smallmatrix} \right]. \end{aligned} \tag{4.8}$$

Although this way of writing may appear contrived, it is actually instrumental in the subsequent analysis, since the sum over  $g$  enforces a  $\mathbb{Z}_N$  projection on the momenta and/or windings.

The term in the square bracket is  $SL(2; \mathbb{Z})$  invariant, and therefore the first integral can be straightforwardly evaluated using the results of [5,6] and of Sections 2, 3. The second integral instead can be treated using the techniques in [7]. Since any weak almost holomorphic modular form of  $\Gamma_0(N)$  can be uniquely decomposed in terms of a linear combination of Niebur–Poincaré series [7], we shall restrict our analysis to the prototype integral involving the shifted lattice times the Niebur–Poincaré series of  $\Gamma_0(N)$ .



### 4.1. Symmetries of the shifted Narain lattice

To study the symmetries of the shifted Narain lattice, it is convenient to introduce the matrices

$$M = \begin{pmatrix} n^1 & m_2 \\ n^2 & -m_1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \mu_1 & \mu_2 \\ \lambda^2 & -\lambda^1 \end{pmatrix}, \tag{4.9}$$

that encode the momentum and winding numbers, and the shift vectors. The shifted Narain lattice then reads

$$\begin{aligned} \Gamma_{2,2}(T, U, \tau; \Lambda) &\equiv \frac{1}{N} \sum_{g \in \mathbb{Z}_N} \Gamma_{2,2} \begin{bmatrix} 0 \\ g \end{bmatrix} (T, U, \tau) \\ &= \sum_{g \in \mathbb{Z}_N} \sum_{M \in \text{Mat}_{2 \times 2}(\mathbb{Z})} \frac{e^{\frac{2i\pi g}{N} \text{tr}(M \Lambda)}}{N} e^{-2i\pi \tau \det(M)} \\ &\quad \times \exp \left[ -\frac{\pi \tau_2}{T_2 U_2} \left| (1 \quad U) M \begin{pmatrix} T \\ 1 \end{pmatrix} \right|^2 \right]. \end{aligned} \tag{4.10}$$

Its duality symmetry is clearly a subgroup of  $SL(2; \mathbb{Z})_T \times SL(2; \mathbb{Z})_U \times \sigma$ , which is the duality group for  $\Lambda = 0$ , and can be easily determined as follows. A generic transformation in  $SL(2; \mathbb{Z})_T \times SL(2; \mathbb{Z})_U$  can be reabsorbed by the matrix redefinition

$$M \rightarrow \tilde{M} = \sigma_1 \gamma_U^t \sigma_1 M \gamma_T, \tag{4.11}$$

where  $\sigma_1$  is the Pauli matrix, and  $\gamma_\alpha \in SL(2; \mathbb{Z})_\alpha$ , at the cost of changing the shift matrix

$$\Lambda \rightarrow \tilde{\Lambda} = \gamma_T^{-1} \Lambda \sigma_1 \gamma_U^{-t} \sigma_1. \tag{4.12}$$

This transformation acts on the lattice as

$$\Gamma_{2,2}(\gamma_T T, \gamma_U U; \Lambda) = \Gamma_{2,2}(T, U; \tilde{\Lambda}) \tag{4.13}$$

and is a symmetry if and only if

$$\Lambda = \gamma_T^{-1} \Lambda \sigma_1 \gamma_U^{-t} \sigma_1 \pmod{N}. \tag{4.14}$$

Similarly, one can show that the involution  $\sigma : T \leftrightarrow U$  is a symmetry if and only if

$$\Lambda = \sigma_1 \Lambda^t \sigma_1 \pmod{N}. \tag{4.15}$$

Actually, one has the option of combining  $\sigma$  with any other involution which is in  $SL(2; \mathbb{Z})_T \times SL(2; \mathbb{Z})_U$  but not in the perturbative duality group of  $\Gamma_{2,2}(T, U; \Lambda)$ .

As an example, let us consider the case of a momentum shift along the first direction,  $\mu_i = 0 = \lambda^2, \lambda^1 = 1$ . Solving eqs. (4.14) and (4.15) yield the duality group  $\Gamma^1(N)_T \times \Gamma_1(N)_U \ltimes \sigma_{T \leftrightarrow -1/U}$ , where the involution is now given by  $\sigma S$ , where  $S : z \rightarrow -1/z$  is an order-two element which is in  $SL(2; \mathbb{Z})$  but not in  $\Gamma_1(N)$  or  $\Gamma^1(N)$ . Notice, however, that any even-weight modular form of the congruence subgroup  $\Gamma_1(N)$  is automatically modular under the larger Hecke subgroup  $\Gamma_0(N)$ . As a result, since the lattice has zero-weight with respect to the  $T$  and  $U$  variables, the effective duality group is always enhanced and in this case reads  $\Gamma^0(N)_T \times \Gamma_0(N)_U \ltimes \sigma_{T \leftrightarrow -1/U}$ . Similarly, for the case of a winding shift along the second cycle of the  $T^2$ , with  $\mu_2 = 1$  and  $\mu_1 = 0 = \lambda^i$ , the duality group is  $\Gamma_0(N)_T \times \Gamma_0(N)_U \ltimes \sigma_{T \leftrightarrow U}$ . Actually, the lattice partition function is now invariant under the permutations of  $T, U, \tau$

$$\Gamma_{2,2}(T, U, \tau; \Lambda) = \Gamma_{2,2}(U, T, \tau; \Lambda) = \Gamma_{2,2}(T, \tau, U; \Lambda), \tag{4.16}$$

just like the unshifted lattice  $\Gamma_{2,2}(T, U, \tau)$ . An important by-product is the relation between the Atkin–Lehner [37] involution  $z' = -1/(Nz)$  in the  $T, U$  and  $\tau$  variables

$$\Gamma_{2,2}\left(T, -\frac{1}{NU}, \tau; \Lambda\right) = \Gamma_{2,2}\left(-\frac{1}{NT}, U, \tau; \Lambda\right) = \Gamma_{2,2}\left(T, U, -\frac{1}{N\tau}; \Lambda\right). \tag{4.17}$$

Before we conclude, we observe that different shift vectors can actually be mapped into each other by a suitable redefinition of the  $T$  and  $U$  moduli [38].

#### 4.2. Niebur–Poincaré series for $\Gamma_0(N)$

In this subsection we recall basic facts about modular forms and Poincaré series with respect to the Hecke congruence subgroups  $\Gamma_0(N)$  with  $N$  prime, referring the reader to the existing literature for more details [7,39–41]. The fundamental domain  $\mathcal{F}_N = \Gamma_0(N) \backslash \mathbb{H}$  possesses two cusps, at  $z = 0$  and  $z = \infty$ . These special points are characterised by their *width*  $m_\alpha$ , which essentially counts the number of copies of the  $SL(2; \mathbb{Z})$  fundamental domain in  $\mathcal{F}_N$  that end at the cusp  $z_\alpha$ . As a result, one finds  $m_\infty = 1$  while  $m_0 = N$ . The two cusps are mapped to each other via the Atkin–Lehner involution  $z \rightarrow -1/(Nz)$ . For a given seed  $f(z)$ , invariant under  $z \rightarrow z + 1$ , one defines the Poincaré series attached to the cusp  $\mathfrak{a}$  via

$$F_\alpha(z) = \sum_{\gamma \in \Gamma_\alpha \backslash \Gamma_0(N)} f(z) \Big|_w \sigma_\alpha^{-1} \gamma, \tag{4.18}$$

where  $\Big|_w$  denotes the Petersson slash operator,  $\Gamma_\alpha$  is the stabiliser of the cusp  $z_\alpha$ , and  $\sigma_\alpha$  are  $SL(2; \mathbb{R})$  matrices which relate  $\Gamma_\alpha$  to  $\Gamma_\infty$ , with

$$\sigma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 0 & 1/\sqrt{N} \\ -\sqrt{N} & 0 \end{pmatrix}. \tag{4.19}$$

A convenient choice of Poincaré series of  $\Gamma_0(N)$  consists in choosing the same seed  $f(z) = \mathcal{M}_{s,w}(-\kappa y) e^{-2i\pi\kappa x}$  as for  $SL(2; \mathbb{Z})$  Niebur–Poincaré series. We shall denote them by  $\mathcal{F}_\alpha(s, \kappa, w)$ , and we shall refer to them as the  $\Gamma_0(N)$  Niebur–Poincaré series attached to the cusp  $\mathfrak{a}$ . They are absolutely convergent for  $\text{Re}(s) > 1$ , and can thus be used for unfolding the fundamental domain. The functions  $\mathcal{F}_\infty(s, \kappa, w)$  and  $\mathcal{F}_0(s, \kappa, w)$  are actually related via the action of the Atkin–Lehner involution, namely

$$\mathcal{F}_0(s, \kappa, w; z) = N^{-w/2} z^{-w} \mathcal{F}_\infty(s, \kappa, w; -1/Nz). \tag{4.20}$$

By construction, they are eigenmodes of the weight- $w$  hyperbolic Laplacian with the same eigenvalue as in (2.3) and satisfy the same closure properties under modular differentiations as in (2.9). As a result, they are harmonic Maaß forms for  $s = 1 - \frac{w}{2}$ , and turn out to be weakly holomorphic for low enough weight, when the shadow vanishes identically. The Hecke operators  $H_n^{(N)}$  of  $\Gamma_0(N)$ <sup>9</sup> acts on the Niebur–Poincaré series as

<sup>9</sup> The Hecke operator  $H_n^{(N)}$  of  $\Gamma_0(N)$  acts on the Fourier modes of a weight- $w$  modular form  $\Phi = \sum_m \phi(m, y) e^{2i\pi m x}$  as

$$H_n^{(N)} \phi(m, y) = n^{1-w} \sum_{d|(m,n)} d^{w-1} \chi(d) \phi(nm/d^2, d^2 y/n),$$

$$H_\kappa^{(N)} \mathcal{F}_a(s, 1, w; z) = \mathcal{F}_a(s, \kappa, w; z) - \delta_{a\infty} \mathcal{F}(s, \kappa/N, w; Nz). \tag{4.21}$$

The second term in the r.h.s. is understood to be present only when  $a = \infty$  and  $N|\kappa$ . It is important to note that  $H_\kappa^{(N)}$  is self-adjoint with respect to the Petersson inner product only when  $(N, \kappa) = 1$  [40,41].

The Fourier expansion of the Niebur–Poincaré series  $\mathcal{F}_a(s, \kappa, w)$  around the two cusps can be found in [7]. For small enough (negative) weight, in the holomorphic case  $s = 1 - \frac{w}{2}$ , one has

$$\begin{aligned} \mathcal{F}_a\left(1 - \frac{w}{2}, \kappa, w\right) |_{w \sigma_b} = & \delta_{ab} \frac{\Gamma(2-w)}{q^\kappa} + \frac{4\pi^2 \kappa}{(2\pi i \kappa)^w} Z_{ab}\left(0, \kappa; 1 - \frac{w}{2}\right) \\ & + 4\pi \kappa \Gamma(2-w) \sum_{m \geq 1} \left(\frac{-m}{\kappa}\right)^{\frac{w}{2}} Z_{ab}\left(m, -\kappa; 1 - \frac{w}{2}\right) q^m, \end{aligned} \tag{4.22}$$

where  $Z_{ab}(a, b; s)$  and  $\mathcal{Z}_{ab}(a, b; s)$  are the Kloosterman–Selberg zeta functions relating the two cusps  $a$  and  $b$  defined in Appendix A.<sup>10</sup>

As for the case of the modular group, the Niebur–Poincaré series  $\mathcal{F}_a(1 - \frac{w}{2} + n, \kappa, w)$  can be used to decompose any generic  $\Gamma_0(N)$  modular form  $\Phi$  of negative weight as a linear combination

$$\Phi(z) = \sum_{a=0, \infty} \sum_{-\kappa_a \leq m < 0} \sum_{p=0}^n d_{a,p}(m) \mathcal{F}_a\left(1 - \frac{w}{2} + p, m, w\right). \tag{4.23}$$

The coefficients  $d_{a,p}(m)$  are uniquely determined by matching the singular behaviour of both sides of the equation around the two cusps. Their explicit expression can be found in [7].

We shall also need the non-holomorphic weight- $w$  Eisenstein series  $E_a(s, w)$  obtained by choosing the seed  $f(z) = y^s$  in (4.18). As in the  $SL(2; \mathbb{Z})$  case, they are eigenmodes of the hyperbolic Laplacian with the same eigenvalue as the Niebur–Poincaré series, absolutely convergent for  $\text{Re}(s) > 1$ , and satisfy the same closure properties under modular differentiation as in (2.15). For  $s = 1 - \frac{w}{2}$  and  $s = \frac{w}{2}$  the Eisenstein series become harmonic Maaß forms. For even weight  $w > 2$ ,  $E_{a,w} \equiv E_a(\frac{w}{2}, w)$  reduces to the conventional holomorphic Eisenstein series  $E_w$  of  $\Gamma_0(N)$  attached to the cusp  $a$ . The Eisenstein series  $E_a(s, w)$  can actually be expressed as linear combinations of the  $SL(2; \mathbb{Z})$  ones with rescaled arguments,

$$\begin{aligned} E_\infty(s, w; z) &= \frac{N^{s+\frac{w}{2}} E(s, w; Nz) - E(s, w; z)}{N^{2s} - 1}, \\ E_0(s, w; z) &= \frac{N^s E(s, w; z) - N^{\frac{w}{2}} E(s, w; Nz)}{N^{2s} - 1}. \end{aligned} \tag{4.24}$$

From these relations, the Fourier series expansions around the cusp  $b$  of the  $\Gamma_0(N)$  Eisenstein series attached to the cusp  $a$  can be straightforwardly extracted and one obtains

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with  $\chi(d)$  the trivial Dirichlet character mod  $N$ , such that  $\chi(d) = 1$  if  $(N, d) = 1$  and  $\chi(d) = 0$  otherwise. It satisfies the commutative algebra  $H_\kappa^{(N)} H_\lambda^{(N)} = \sum_{d|(\kappa, \lambda)} d^{1-w} \chi(d) H_{\kappa\lambda/d^2}^{(N)}$ .

<sup>10</sup> Note that in the present work we employ a different notation for the Kloosterman–Selberg zeta function than the one of [7].

$$\begin{aligned}
 E_{\mathbf{a}}(s, w; z)|_{\mathfrak{b}} &= \delta_{\mathbf{ab}} y^{s-\frac{w}{2}} + i^{-w} \sqrt{\pi} \frac{\Gamma(s) \Gamma(s-\frac{1}{2})}{\Gamma(s+\frac{w}{2}) \Gamma(s-\frac{w}{2})} Z_{\mathbf{ab}}(0, 0; s) y^{1-s-\frac{w}{2}} \\
 &+ i^{-w} \pi^s (4\pi)^{\frac{w}{2}} \sum_{m \neq 0} \frac{1}{\Gamma(s+\frac{w}{2} \operatorname{sgn}(m))} |m|^{s+\frac{w}{2}-1} Z_{\mathbf{ab}}(0, m; s) \\
 &\times \mathcal{W}_{s,w}(my) e^{2\pi imx}.
 \end{aligned}
 \tag{4.25}$$

### 4.3. Properties of the $\Gamma_0(N)$ modular integrals

We can now proceed to the computation of the prototype integral

$$\mathcal{I}_N(s, \kappa; \mathbf{a}) = \text{R.N.} \int_{\mathcal{F}_N} d\mu \mathcal{F}_{\mathbf{a}}(s, \kappa, 0) \Gamma_{2,2}(T, U; \Lambda)
 \tag{4.26}$$

by unfolding  $\mathcal{F}_N$  against the Niebur–Poincaré series. Any  $\Gamma_0(N)$  integral involving the shifted Narain lattice times a generic weakly almost holomorphic modular form can be straightforwardly derived from (4.26) by virtue of the decomposition (4.23). The symbol R.N. stands for the  $\Gamma_0(N)$  invariant renormalisation prescription of [7] for treating the infra-red divergences that may arise at the two cusps. Special care is needed for  $s = 1$  where the integral develops a simple pole, in which case, the renormalisation prescription amounts to its proper subtraction.

From the differential properties of the Niebur–Poincaré series and from the identities

$$\Delta_T \Gamma_{2,2}(T, U; \Lambda) = \Delta_U \Gamma_{2,2}(T, U; \Lambda) = \Delta_{\tau} \Gamma_{2,2}(T, U; \Lambda),
 \tag{4.27}$$

obeyed by the shifted Narain lattice, for any choice of the shift  $\Lambda$ , it turns out that the  $\Gamma_0(N)$  modular integrals  $\mathcal{I}_N(s, \kappa; \mathbf{a})$  are themselves eigenmodes of both the (weight-zero) hyperbolic Laplacians  $\Delta_T$  and  $\Delta_U$  with eigenvalue  $\frac{1}{2}s(s-1)$ . For  $s = 1$  a constant source-term appears, similarly to the  $\text{SL}(2; \mathbb{Z})$  case (2.21).

Whenever the shifted lattice  $\Gamma_{2,2}(T, U; \Lambda)$  is symmetric under the exchange of the modular parameter  $\tau$  with either the  $T$  and  $U$  moduli, and provided that  $(\kappa, N) = 1$  so that the  $\Gamma_0(N)$  Hecke operator is self-adjoint, one can restrict the analysis to the case  $\kappa = 1$  and  $\mathbf{a} = \infty$ , since all other cases can be recovered via the action of the Hecke and Atkin–Lehner operators.

Indeed, following the steps outlined in Section 2 one finds

$$\mathcal{I}_N(s, \kappa; \mathbf{a}) = H_{\kappa}^{(N)} \cdot \mathcal{I}_N(s; \mathbf{a}),
 \tag{4.28}$$

where the Hecke operator acts on either  $T$  or  $U$ , or on both, depending on the available symmetries and  $\mathcal{I}_N(s; \mathbf{a}) \equiv \mathcal{I}_N(s, 1; \mathbf{a})$ .

Moreover, the integrals associated to the two cusps are not independent, but are related by the action of the Atkin–Lehner involution  $\sigma_0$ . In fact, since

$$\mathcal{F}_0(s, \kappa, w) = \mathcal{F}_{\infty}(s, \kappa, w)|\sigma_0,
 \tag{4.29}$$

and since  $\sigma_0$  is self-adjoint on the fundamental domain of  $\Gamma_0(N)$ , one finds

$$\mathcal{I}_N(s; \mathbf{a}) = \mathcal{I}_N(s; \infty)|\sigma_{\mathbf{a}},
 \tag{4.30}$$

where the Petersson slash operator can act on either variable, depending on the available symmetries. For  $s = 1 + n$ , this relation can be extended to the level of both the harmonic and holomorphic prepotential

$$f_n(T, U; \mathfrak{a}) = f_n(T, U; \infty)|\sigma_{\mathfrak{a}} + \text{polynomial}, \tag{4.31}$$

due to the commutation property of the involution  $\sigma_{\mathfrak{a}}$  with the modular derivative. Similarly, for  $(\kappa, N) = 1$ , one may extend the relation (3.5) to both the harmonic and holomorphic prepotentials of  $\Gamma_0(N)$

$$f_n(T, U, \kappa; \mathfrak{a}) = \kappa^{-n} H_{\kappa}^{(N)} \cdot f_n(T, U; \mathfrak{a}). \tag{4.32}$$

#### 4.4. Evaluation of the modular integral $\mathcal{I}_N(s; \mathfrak{a})$

For concreteness, we shall evaluate the integral (4.26) for  $\kappa = 1$ , and we shall consider explicitly only the case of a winding shift along the second cycle of the two-torus, with  $\mu_2 = 1$  and  $\mu_1 = \lambda^i = 0$ , since the other cases can be treated similarly. In addition to the duality symmetry  $\Gamma_0(N)_T \times \Gamma_0(N)_U \times \sigma_{T \leftrightarrow U}$ , inherited by the integral, the shifted lattice is also invariant under the triality symmetry  $T \leftrightarrow U \leftrightarrow \tau$ , which implies that the results for the cusp at zero, or for higher values of  $\kappa$ , can be obtained by acting with the Atkin–Lehner involution or Hecke operator on  $\mathcal{I}(s; \infty)$ .

For the cusp at  $\infty$  the shifted lattice  $\Gamma_{2,2}(T, U; \Lambda)$  reduces to the conventional unshifted Narain lattice sum merely subjected to the constraint  $n^2 = 0 \pmod N$ . One may thus readily obtain the result of the integral  $\mathcal{I}(s; \infty)$  from its  $\text{SL}(2; \mathbb{Z})$  cousin  $\mathcal{I}(s)$ , both in the BPS-sum representation and in the Fourier-series one.

Unfolding the fundamental domain against the Niebur–Poincaré series yields the BPS-state sum representation [7]

$$\begin{aligned} \mathcal{I}_N(s; \mathfrak{a}) &= \frac{m_{\mathfrak{a}}}{N} \Gamma(s) \sum_{g=0}^{N-1} \sum_{\text{BPS}(v_{\mathfrak{a}}g)} e^{2i\pi u_{\mathfrak{a}}gn_2/N} {}_2F_1\left(s, s; 2s; \frac{4}{m_{\mathfrak{a}} |p_L(v_{\mathfrak{a}}g)|^2}\right) \\ &\quad \times \left(\frac{m_{\mathfrak{a}} |p_L(v_{\mathfrak{a}}g)|^2}{4}\right)^{-s}, \end{aligned} \tag{4.33}$$

where the BPS-sum extends over integers  $m_i, n^i$  subject to the ‘BPS-constraint’

$$m_{\mathfrak{a}} \left( |p_L(v_{\mathfrak{a}}g)|^2 - |p_R(v_{\mathfrak{a}}g)|^2 \right) = 4, \tag{4.34}$$

with the shifted left-moving and right-moving momenta given in eq. (4.6). The ratio  $u_{\mathfrak{a}}/v_{\mathfrak{a}}$  identifies the coordinate of the cusp  $\mathfrak{a}$  on the complex upper plane, i.e.  $(u_0, v_0) = (0, 1)$  and  $(u_{\infty}, v_{\infty}) = (1, 0)$ .

Following the same arguments as in Section 2.2.1, this expression can be cast as the Poincaré series

$$\mathcal{I}_N(s; \mathfrak{a}) = \sum_{\gamma \in \Gamma_{\text{diag}} \setminus (\Gamma_T \times \Gamma_U)} \varphi\left(\frac{|\bar{T} - U|^2}{T_2 U_2}\right) \Big|_{\sigma_{\mathfrak{a},i}^{-1}} \gamma, \tag{4.35}$$

where the seed  $\varphi(z)$  is the same as in eq. (2.28), and the involution  $\sigma_{\mathfrak{a},i}$  acts on either of the  $T$  and  $U$  variables. For generic  $N$  including  $N = 1$ ,  $\Gamma_T \times \Gamma_U = \Gamma_0(N)_T \times \Gamma_0(N)_U$  and  $\Gamma_{\text{diag}} = \text{diag}(\Gamma_0(N)_T \times \Gamma_0(N)_U)$ .

This representation is valid at any point in the Narain moduli space. It is tailored to exploit the analytic structure of the integral in the vicinity of points of gauge symmetry enhancement and provides a Poincaré series representation, thus making the duality symmetry manifest. As in

the  $SL(2; \mathbb{Z})$  case, points of gauge symmetry enhancement arise at  $p_R(v_\alpha g) = 0$ , i.e.  $T = \sigma_\alpha U$  together with their  $\Gamma_0(N)$  images [6]. Note that the singular locus depends on a choice of cusp.

The Fourier series representation depends on a choice of Weyl chamber. Although the structure of Weyl chambers depends on the value of  $N$ , in the chamber  $T_2 > U_2$  one finds

$$\mathcal{I}_N(s; \mathbf{a}) = \mathcal{I}_N^{(0)}(s; \mathbf{a}) + \mathcal{I}_N^{(+)}(s; \mathbf{a}) + \mathcal{I}_N^{(-)}(s; \mathbf{a}), \tag{4.36}$$

with

$$\begin{aligned} \mathcal{I}_N^{(0)}(s; \mathbf{a}) &= 2^{2s} \sqrt{4\pi} \Gamma(s - \frac{1}{2}) T_2^{1-s} E_\mathbf{a}(s, 0; U), \\ \mathcal{I}_N^{(+)}(s; \mathbf{a}) &= 2 \sum_{M>0} \frac{e^{2i\pi MT_1}}{M} \mathcal{W}_{s,0}(MT_2) \mathcal{F}_\mathbf{a}(s, M, 0; U), \end{aligned} \tag{4.37}$$

and  $\mathcal{I}_N^{(-)}(s; \mathbf{a}) = [\mathcal{I}_N^{(+)}(s; \mathbf{a})]^*$ .

Notice that although these expressions are valid for  $N$  prime, the choice  $N = 1$  reduces to eq. (2.16), if we assume that, in this case,  $\mathbf{a}$  labels the unique cusp  $\infty$  of the modular group. In fact, in this case  $\mathcal{F}_\infty^{(1)}(s, 1, 0) \equiv \mathcal{F}(s, 1, 0)$ ,  $\mathcal{F}_1 \equiv \mathcal{F}$ , while for the Narain lattice, any shift vector  $\Lambda$  acts trivially by simply relabelling the integral momenta and windings, and thus  $\Gamma_{2,2}^{(1)}(T, U; \Lambda) \equiv \Gamma_{2,2}(T, U)$ .

Of particular interest in string theory are the cases where  $s = 1 + n$  is a positive integer. The Fourier series expansion then admits alternative representations in terms of polylogarithms or, for  $n = 0$ , can be summed in terms of combinations of the  $\Gamma_0(N)$  Hauptmodul, thus generalising the Borcherds product formulæ valid for  $SL(2; \mathbb{Z})$ .

#### 4.4.1. The case $s = 1$

The case  $s = 1$  is special since the R.N. prescription requires the subtraction of a simple pole from the non-holomorphic Eisenstein series entering the Fourier mode with vanishing frequency. Using the first Kronecker limit formula one obtains

$$\mathcal{I}_N^{(0)}(1; \mathbf{a}) = -\frac{24}{(N-1)v_N} \log \left( T_2^{N-1} U_2^{N-1} \left| \left[ \frac{\eta^N(NU)}{\eta(U)} \Big| \sigma_\mathbf{a} \right] \right|^4 \right) + \text{const}, \tag{4.38}$$

where  $v_N = \text{vol}(\mathcal{F}_N)/\text{vol}(\mathcal{F})$  is the index of  $\Gamma_0(N)$  inside  $SL(2; \mathbb{Z})$ . Notice that this expression is valid also for  $N = 1$ , if we allow the formal limit  $N \rightarrow 1$ . In fact,  $\eta(NU) \sim \eta(U) + \mathcal{O}(N - 1)$ , and thus

$$\lim_{N \rightarrow 1} -\frac{24}{(N-1)v_N} \log \left( T_2^{N-1} U_2^{N-1} \left| \frac{\eta^N(NU)}{\eta(U)} \right|^4 \right) = -24 \log \left( T_2 U_2 |\eta(U)|^4 \right), \tag{4.39}$$

in agreement with eq. (2.46). The positive-frequency modes are regular at  $s = 1$  and read

$$\begin{aligned} \mathcal{I}_N^{(+)}(1; \mathbf{a}) &= 2 \sum_{M>0} \frac{q_T^M}{M} \mathcal{F}_\mathbf{a}(1, M, 0) \\ &= 2 \sum_{M>0} \frac{q_T^M}{M} \left[ \mathcal{F}_\mathbf{a}(1, M, 0) - \tilde{\mathcal{F}}_\mathbf{a}^{(0)}(1, M, 0) \right] \\ &\quad + 4\pi i T \delta_{\mathbf{a}\infty} - \frac{24}{(N-1)v_N} \log \left( \left[ \frac{\eta^N(NT)}{\eta(T)} \Big| \sigma_\mathbf{a} \right] \right)^2, \end{aligned} \tag{4.40}$$

where  $\tilde{\mathcal{F}}_a^{(0)}(1, M, 0) = 4\pi^2 M Z_{a\infty}(0, M; 1)$  is the constant zero-mode of the Niebur–Poincaré series of  $\Gamma_0(N)$  attached to the cusp  $a$ .

Using the properties of holomorphy and modularity in the  $T$  and  $U$  variables, together with the singularity structure of  $\mathcal{F}_a(1, M, 0) - \tilde{\mathcal{F}}_a^{(0)}(1, M, 0)$  around the two cusps, one obtains the identity

$$\sum_{M>0} \frac{q_T^M}{M} \left[ \mathcal{F}_a(1, M, 0) - \tilde{\mathcal{F}}_a^{(0)}(1, M, 0) \right] + 2\pi i T \delta_{a\infty} = -\log \frac{(j_\infty(T) - j_a(U))}{\left(j_\infty(T) - \frac{24}{N-1}\right)^{\delta_{a0}}}, \tag{4.41}$$

where  $j_a(z) = \sum_M c_a(M) q^M$ . In fact,  $j_\infty(z)$  is the Hauptmodul for  $\Gamma_0(N)$ , while  $j_0(z) = j_\infty(-1/Nz)$  is its image under the Atkin–Lehner involution. Eq. (4.41) generalises the Borcherds product formula [20] to the case of  $\Gamma_0(N)$  congruence subgroups

$$\prod_{K>0, L\in\mathbb{Z}} \left( \frac{(1 - q_T^K q_U^L)^N}{1 - q_T^{NK} q_U^{NL}} \right)^{c_a(KL)} = \frac{(j_\infty(T) - j_a(U))^N}{(j(NT) - j(NU))^{\delta_{a\infty}} \left(j_\infty(T) - \frac{24}{N-1}\right)^{N\delta_{a0}}}. \tag{4.42}$$

Putting things together, the modular integral reads

$$\mathcal{I}_N(1; a) = -\frac{24}{(N-1)v_N} \log \left( T_2^{N-1} U_2^{N-1} \left| \left[ \frac{\eta^N(NU)}{\eta(U)} \middle| \sigma_a \right] \left[ \frac{\eta^N(NT)}{\eta(T)} \middle| \sigma_a \right] \right|^4 \right) - \log \frac{|j_\infty(T) - j_a(U)|^4}{\left| j_\infty(T) - \frac{24}{N-1} \right|^{4\delta_{a0}}} + \text{const}. \tag{4.43}$$

The first term on the r.h.s. originates from the constant term in the decomposition  $\mathcal{F}_a(1, 1, 0) = j_a(z) + \tilde{\mathcal{F}}_a^{(0)}(1, 1, 0)$ , whereas the second term originates from the  $\Gamma_0(N)$  Hauptmodul. By employing the identity

$$(j_0(T) - j_\infty(U)) \left( j_\infty(T) - \frac{24}{N-1} \right) = (j_0(U) - j_\infty(T)) \left( j_\infty(U) - \frac{24}{N-1} \right), \tag{4.44}$$

it is straightforward to verify that the expression (4.43) is indeed invariant under the duality symmetry  $\Gamma_0(N)_T \times \Gamma_0(N)_U \ltimes \sigma_{T \leftrightarrow U}$ . The modular integral (4.43) develops logarithmic singularities at the point  $T = U$  and its  $\Gamma_0(N)$  images, ascribed to the presence of extra massless states that would arise at symmetry enhancement points in the Narain moduli space.

Before closing this section, we shall also give the expressions for the harmonic integrals involving the  $\Gamma_0(N)$  Hauptmodul for  $N = 2, 3$ . Using the result

$$\text{R.N.} \int_{\mathcal{F}_N} d\mu \Gamma_{2,2} \left[ \begin{matrix} 0 \\ 1 \end{matrix} \right] = -\frac{1}{N-1} \log(T_2 U_2)^{N-1} \left| \frac{\eta^N(NT) \eta^N(NU)}{\eta(T) \eta(U)} \right|^4 + \text{const}, \tag{4.45}$$

for integrals of the shifted Narain lattice established in [7], together with eq. (4.8), we obtain

$$\begin{aligned} \text{R.N.} \int_{\mathcal{F}_N} d\mu \Gamma_{2,2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} j_\infty = & -\frac{N}{N-1} \log \left| \frac{j_\infty(T) - j_\infty(U)}{\left(j_\infty(T) - \frac{24}{N-1}\right) \left(j_\infty(U) - \frac{24}{N-1}\right)} \right|^4 \\ & + \frac{1}{N-1} \log |j(T) - j(U)|^4 + \text{const}, \end{aligned} \tag{4.46}$$

and

$$\text{R.N.} \int_{\mathcal{F}_N} d\mu \Gamma_{2,2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(j_0 - \frac{24}{N-1}\right) = -\frac{N}{N-1} \log |j_0(T) - j_\infty(U)|^4 + \text{const}, \tag{4.47}$$

which generalise the  $\text{SL}(2; \mathbb{Z})$  result (2.50) first obtained in [4] to the  $\Gamma_0(N)$  integrals. These results have been used in [26] to derive the universal behaviour of gauge thresholds in stable non-supersymmetric heterotic vacua.

4.4.2. The case  $s = 1 + n$  with  $n > 0$

For integer  $s = 1 + n > 1$ , and for small enough  $n$ , the Niebur–Poincaré series  $\mathcal{F}_\alpha^{(N)}(1 + n, 1, 0)$  becomes almost holomorphic and admits the Fourier series expansion at the cusp  $\infty$

$$\mathcal{F}_\alpha^{(N)}(1 + n, 1, 0) = \sum_m F_{\alpha,n}(m) (4\pi y)^{-n} L_n^{(-2n-1)}(4\pi m y) q^m, \tag{4.48}$$

in terms of the associated Laguerre polynomials, where  $F_{\alpha,n}$  are the Fourier coefficients of  $\mathcal{F}_\alpha(1 + n, 1, -2n)$  at the cusp at infinity,

$$\begin{aligned} F_{\alpha,n}(-1) &= \Gamma(2n + 2) \delta_{\alpha\infty}, \\ F_{\alpha,n}(0) &= (2\pi)^{2n+2} (-1)^n Z_{\alpha\infty}(0, 1; n + 1), \\ F_{\alpha,n}(L > 0) &= 4\pi (-1)^n \Gamma(2n + 2) \frac{Z_{\alpha\infty}(L, -1; n + 1)}{L^n}. \end{aligned} \tag{4.49}$$

Similarly to the  $\text{SL}(2; \mathbb{Z})$  case, the integral may be expressed in terms of the single-valued polylogarithms of eq. (2.52). One finds

$$\begin{aligned} \mathcal{I}_N^{(+)}(1 + n; \mathfrak{a}) &= \frac{\delta_{\alpha\infty}}{N} \mathcal{I}^{(+)}(NT, NU) + \frac{2(-1)^n}{n!} \sum_{\substack{M>0 \\ L \in \mathbb{Z}}} F_{\alpha,n}(LM) \\ &\times \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!} \frac{(ML)^n}{(4\pi MLT_2U_2)^k} \hat{\mathcal{L}}_{(k)}^{(N)}(MT + LU), \end{aligned} \tag{4.50}$$

where we have defined the ‘combined  $\Gamma_0(N)$  polylogarithms’

$$\hat{\mathcal{L}}_{(k)}^{(N)}(z) \equiv \mathcal{L}_{(k)}(z) - \frac{1}{N^{2k+1}} \mathcal{L}_{(k)}(Nz). \tag{4.51}$$

The zero mode  $\mathcal{I}_N^{(0)}$  similarly admits a representation in terms of single-valued polylogarithms, by using (2.53) together with the functional relations (4.24) in order to express the non-holomorphic Eisenstein series  $E_\alpha(1 + n, 0; z)$  of  $\Gamma_0(N)$  in terms of the  $\text{SL}(2; \mathbb{Z})$  ones,  $E(1 + n, 0; z)$  and  $E(1 + n, 0; Nz)$ .



#### 4.5. Generalised holomorphic prepotentials for $\Gamma_0(N)$

As in the case of the full modular group, for integer  $s = 1 + n > 1$  one may define generalised harmonic and holomorphic prepotentials also for the  $\Gamma_0(N)$  integral  $\mathcal{I}_N$  via

$$\begin{aligned} \mathcal{I}_N(1 + n; \mathfrak{a}) &= 4 \operatorname{Re} \frac{(-D_T D_U)^n}{n!} h_n(T, U; \mathfrak{a}) \\ &= 4 \operatorname{Re} \frac{(-D_T D_U)^n}{n!} f_n(T, U; \mathfrak{a}). \end{aligned} \tag{4.52}$$

The expression of the integral in terms of iterated derivatives of the prepotential is again possible due to the closure property of Niebur–Poincaré series under modular differentiation.

The prepotentials for  $\Gamma_0(N)$  can be obtained by an analysis parallel to the  $\operatorname{SL}(2; \mathbb{Z})$  case and one obtains

$$\begin{aligned} h_n(T, U; \mathfrak{a}) &= (2\pi)^{2n+1} E_{\mathfrak{a}}(n + 1, -2n; U) \\ &\quad + \sum_{M>0} \frac{2}{(2M)^{2n+1}} q_T^M \mathcal{F}_{\mathfrak{a}}(1 + n, M, -2n; U), \end{aligned} \tag{4.53}$$

for the generalised harmonic and

$$\begin{aligned} f_n(T, U; \mathfrak{a}) &= \frac{(-1)^n (2\pi)^{2n+2}}{2^{2n} \zeta(2n + 2)} \tilde{E}_{-2n, \mathfrak{a}}(U) \\ &\quad + \sum_{M>0} \frac{2}{(2M)^{2n+1}} q_T^M \mathcal{F}_{\mathfrak{a}}(1 + n, M, -2n; U), \end{aligned} \tag{4.54}$$

for the generalised holomorphic prepotentials. Here,  $\tilde{E}_{-2n, \mathfrak{a}}(U)$  is the  $\Gamma_0(N)$  Eichler integral of weight  $-2n$  associated to the holomorphic Eisenstein series  $E_{2n+2, \mathfrak{a}}$ , attached to the cusp  $\mathfrak{a}$ . Its Fourier expansion about a cusp  $\mathfrak{b}$  reads

$$\begin{aligned} \tilde{E}_{-2n, \mathfrak{a}}(z) |_{\sigma_{\mathfrak{b}}} &= \zeta(2n + 2) \left[ \frac{z^{2n+1}}{2\pi i} \delta_{\mathfrak{a}\mathfrak{b}} + \frac{1}{2} Z_{\mathfrak{a}\mathfrak{b}}(0, 0; n + 1) \right. \\ &\quad \left. + \sum_{M>0} Z_{\mathfrak{a}\mathfrak{b}}(0, M; n + 1) q^M \right]. \end{aligned} \tag{4.55}$$

The  $\Gamma_0(N)$  Eichler integral  $\tilde{E}_{-2n, \mathfrak{a}}$  inherits the functional relation properties of the Eisenstein series  $E_{\mathfrak{a}}(n + 1, -2n; z)$  and may be expressed in terms of  $\operatorname{SL}(2; \mathbb{Z})$  Eichler integrals via

$$\begin{aligned} \tilde{E}_{-2n, \infty}(z) &= \frac{N \tilde{E}_{-2n}(Nz) - \tilde{E}_{-2n}(z)}{N^{2n+2} - 1}, \\ \tilde{E}_{-2n, 0}(z) &= \frac{N^{2n+1} \tilde{E}_{-2n}(z) - \tilde{E}_{-2n}(Nz)}{N^n (N^{2n+2} - 1)}. \end{aligned} \tag{4.56}$$

Of course, since holomorphic prepotentials are only defined up to terms in the kernel of  $\operatorname{Re}(-D_T D_U)^n$ , the expression (4.54) for  $f_n(T, U; \mathfrak{a})$  corresponds to a specific choice for the polynomial ambiguity. Notice that, due to the inhomogeneous transformation of the Eichler integral, the action of  $\sigma_0$  now relates the holomorphic prepotential  $f_n(T, U; 0)$  attached to the cusp  $\mathfrak{a} = 0$  to the prepotential  $f_n(T, U; \infty)$  attached to the cusp  $\mathfrak{a} = \infty$  with an additional polynomial dependence in the  $U$  variable, so that eq. (4.31) becomes

$$f_n(T, U; 0) = f_n(T, U; \infty)|\sigma_0 - \alpha \frac{N^{2n+1} P_S(U) - P_S(NU)}{N^n (N^{2n+2} - 1)}, \tag{4.57}$$

where  $P_S(z)$  is the polynomial transformation of the  $SL(2; \mathbb{Z})$  Eichler integral given in (3.26) and  $\alpha$  is the constant defined in (3.17).

In terms of the Fourier coefficients  $F_{\alpha,n}(M)$  of the Niebur–Poincaré series  $\mathcal{F}_\alpha(1+n, 1, -2n)$ , we may cast the generalised holomorphic prepotential in the alternative representation

$$\begin{aligned} f_n(T, U; \mathfrak{a}) &= \frac{\delta_{\mathfrak{a}\infty}}{N^{2n+1}} f_n(NT, NU) + \frac{F_{\alpha,n}(-1)}{2^{2n}} \hat{\text{Li}}_{2n+1}^{(N)}\left(\frac{q_T}{q_U}\right) \\ &+ \frac{1}{2^{2n}} \sum_{M,L \geq 0} F_{\alpha,n}(ML) \hat{\text{Li}}_{2n+1}^{(N)}(q_T^M q_U^L) \\ &+ \frac{1}{2} \alpha \zeta(2n+2) \left[ Z_{\mathfrak{a}\infty}(0, 0; n+1) - \delta_{\mathfrak{a}\infty} Z(0, 0; n+1) \right. \\ &\left. - 2(1 - N^{-2n-1}) \zeta(2n+1) Z_{\mathfrak{a}\infty}(0, 1; n+1) \right], \end{aligned} \tag{4.58}$$

where  $f_n(T, U)$  is the  $SL(2; \mathbb{Z})$  prepotential (3.21) and, for convenience, we have introduced the ‘ $\Gamma_0(N)$  polylogarithms’

$$\hat{\text{Li}}_k^{(N)}(z) \equiv \text{Li}_k(z) - \frac{1}{N^k} \text{Li}_k(z^N). \tag{4.59}$$

Note that eq. (4.58) is cusp-covariant and reduces to the  $SL(2; \mathbb{Z})$  prepotential by setting  $N = 1$  and  $\mathfrak{a} = \infty$ , whereby the Kloosterman–Selberg zeta function  $Z_{\infty\infty}(a, b; s)$  simply reduces to its  $SL(2; \mathbb{Z})$  counterpart,  $Z(a, b; s)$ .

Although the positive frequency modes in  $T$  of the generalised holomorphic prepotential do transform covariantly as weight  $-2n$  modular forms in the  $U$  variable, the zero-frequency mode does not. Clearly, the inhomogeneous polynomial term is completely determined by the transformation properties of the Eichler integral  $\tilde{E}_{-2n,\mathfrak{a}}(z)$  under the generators of  $\Gamma_0(N)$ . Since it is always possible to decompose the latter into repeated applications of the generators  $S$  and  $T$  of the full modular group, one may uniquely fix the polynomial transformation of  $\tilde{E}_{-2n,\mathfrak{a}}(z)$  under any element of  $\Gamma_0(N)$  by using eq. (4.56) and the known transformation properties of the Eichler integral of  $SL(2; \mathbb{Z})$ .

Moreover, the behaviour of the generalised holomorphic prepotentials under  $\sigma_{T \leftrightarrow U}$  can be straightforwardly determined from eq. (4.58) and from the property (3.27) of polylogarithms to be

$$f_n(T, U; \mathfrak{a}) - f_n(U, T; \mathfrak{a}) = \delta_{\mathfrak{a}\infty} P_\sigma(T, U), \tag{4.60}$$

with  $T$  and  $U$  lying in the fundamental chamber, with real parts in the interval  $(0, 1)$  and with  $P_\sigma(T, U)$  being the polynomial (3.28).

#### 4.6. Generalised Yukawa couplings for $\Gamma_0(N)$

In complete analogy to the  $SL(2; \mathbb{Z})$  case, one may compute interactions in the low energy  $\mathcal{N} = 2$  supergravity by acting with suitable derivatives on the  $\Gamma_0(N)$  generalised holomorphic prepotential. Here we shall restrict our attention to the simplest case of holomorphic generalised Yukawa couplings attached to the cusp  $\mathfrak{a}$

$$Y_{T^{2n+1}}(T, U; \mathbf{a}) = -2^{-2n-1} D_T^{2n+1} f_n(T, U; \mathbf{a}) = \left(\frac{\partial_T}{2\pi i}\right)^{2n+1} f_n(T, U; \mathbf{a}). \tag{4.61}$$

The degree  $2n$  polynomial in the inhomogeneous transformation of  $f_n$  is again annihilated by the derivatives and the Yukawa coupling  $Y_{T^{2n+1}}$  transforms as a weight  $2n + 2$  and  $-2n$  holomorphic modular form under  $\Gamma_0(N)_T$  and  $\Gamma_0(N)_U$ , respectively. In terms of the Fourier coefficients  $F_{\mathbf{a},n}(M)$  of the Niebur–Poincaré series  $\mathcal{F}_{\mathbf{a}}(1 + n, 1, -2n)$  expanded around the cusp  $\infty$ , one finds explicitly

$$\begin{aligned} Y_{T^{2n+1}}(T, U; \mathbf{a}) &= 2^{-2n} \sum_{M>0} q_T^M \mathcal{F}_{\mathbf{a}}(1 + n, M, -2n; U) \\ &= \delta_{\mathbf{a}\infty} Y_{T^{2n+1}}(NT, NU) \\ &\quad + 2^{-2n} \sum_{\substack{M>0 \\ L\in\mathbb{Z}}} M^{2n+1} F_{\mathbf{a},n}(ML) \mathcal{R}_N(q_T^M q_U^L), \end{aligned} \tag{4.62}$$

which generalises the “multi-cover” formula for the standard Yukawa couplings  $Y_{T^3}$  to the case of freely-acting  $\mathbb{Z}_N$  orbifolds. To lighten the notation, we have introduced the rational function

$$\mathcal{R}_N(z) \equiv \frac{z}{1-z} - \frac{z^N}{1-z^N}. \tag{4.63}$$

From the above expression, one may readily extract the simple pole divergence at  $q_T = q_U$ ,

$$Y_{T^{2n+1}}(T, U; \mathbf{a}) \sim \delta_{\mathbf{a}\infty} \frac{(2n + 1)!}{2^{2n}} \frac{q_T}{q_U - q_T}. \tag{4.64}$$

This singularity, together with holomorphy, modularity and eq. (4.62) uniquely fixes the expression of the Yukawa couplings

$$Y_{T^{2n+1}}(T, U; \mathbf{a}) = 2^{-2n} \frac{E_{2n+2,\infty}(T) \mathcal{F}_{\mathbf{a}}(1 + n, 1, -2n; U)}{j_{\infty}(T) - j_{\mathbf{a}}(U)}, \tag{4.65}$$

where it is understood that the holomorphic Eisenstein series, the Niebur–Poincaré series and the Hauptmodul pertain to the Hecke congruence subgroup  $\Gamma_0(N)$ . For  $N = 1$  one recovers the  $SL(2; \mathbb{Z})$  result (3.35). Moreover, using the results of [7], it is straightforward to express the generalised Yukawa couplings  $Y_{T^{2n+1}}(T, U; \mathbf{a})$  in terms of the holomorphic modular forms of  $\Gamma_0(N)$ . For instance, one gets

$$\begin{aligned} Y_{T^3}(T, U; \infty) &= \frac{1}{6} \frac{[4X^2(T) - E_4(T)][E_6(U) - 2X(U)E_4(U)]}{\Delta_8(U) [j_{\infty}(T) - j_{\infty}(U)]}, \\ Y_{T^3}(T, U; 0) &= \frac{1}{24} \frac{[4X^2(T) - E_4(T)][E_6(U) + X(U)E_4(U)]}{\Delta_8(U) [j_{\infty}(T) - j_0(U)]}, \end{aligned} \tag{4.66}$$

for  $N = 2$ , and

$$\begin{aligned} Y_{T^3}(T, U; \infty) &= \frac{3}{2048} \frac{[9X^2(T) - 4E_4(T)][9X^2(U) - 4E_4(U)]}{\Delta_6(U) [j_{\infty}(T) - j_{\infty}(U)]}, \\ Y_{T^3}(T, U; 0) &= \frac{3}{2048} \frac{[9X^2(T) - 4E_4(T)][X^2(U) - 4E_4(U)]}{\Delta_6(U) [j_{\infty}(T) - j_0(U)]}, \end{aligned} \tag{4.67}$$

for  $N = 3$ . In these expressions  $E_w(z)$  are the holomorphic Eisenstein series of  $SL(2; \mathbb{Z})$ ,  $\Delta_w$  is the weight- $w$  cusp form of  $\Gamma_0(N)$ , and  $X(z) = E_2(z) - N E_2(Nz)$  is the weight-two holomorphic modular form of  $\Gamma_0(N)$  [7].

The generalisation to higher values of  $\kappa$ , with  $(\kappa, N) = 1$ , is also straightforward and may be obtained by acting with the Hecke operator on the  $\kappa = 1$  result,

$$Y_{T^{2n+1}}(T, U, \kappa; \mathfrak{a}) = \kappa^{-n} H_\kappa^{(N)} \cdot Y_{T^{2n+1}}(T, U; \mathfrak{a}). \tag{4.68}$$

**Acknowledgements**

We are grateful to Per Berglund and Kathrin Bringmann for discussions. C.A. would like to thank the TH Unit at CERN and the Arnold Sommerfeld Centre at the Ludwig-Maximilians-Universität München and the Max-Planck-Institut für Physik in München for hospitality during different stages of this project. I.F. would like to thank the Physics Department of Torino University for hospitality. This work was partially supported by the European ERC Advanced Grant No. 226455 ‘‘Supersymmetry, Quantum Gravity and Gauge Fields’’ (SUPERFIELDS) and by the Compagnia di San Paolo contract ‘‘Modern Application in String Theory’’ (MAST) TO-Call3-2012-0088.

**Appendix A. Kloosterman–Selberg zeta function**

The classical Kloosterman sums for the modular group  $SL(2; \mathbb{Z})$  are defined as

$$S(m, n; c) \equiv \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^*} \exp \left[ \frac{2\pi i}{c} (m d + n d^{-1}) \right], \tag{A.1}$$

where  $m, n$  and  $c$  are integers and  $d^{-1}$  denotes the arithmetic inverse of  $d \pmod c$ . Clearly, it is symmetric under the exchange of  $m$  and  $n$ . One defines the Kloosterman–Selberg zeta function as in [39]

$$Z(m, n; s) \equiv \sum_{c>0} \frac{S(m, n; c)}{c^{2s}}. \tag{A.2}$$

In the special case when  $mn = 0$ , the Kloosterman–Selberg zeta function reduces to

$$Z(0, 0; s) = \frac{\zeta(2s - 1)}{\zeta(2s)}, \quad Z(0, m; s) = \frac{\sigma_{1-2s}(m)}{\zeta(2s)}, \tag{A.3}$$

where  $\sigma_t(m) = \sum_{d|m} d^t$  is the divisor function.

For  $mn \neq 0$ , it is convenient to introduce the *associated* Kloosterman–Selberg zeta function following [6,39]

$$\mathcal{Z}(m, n; s) \equiv \frac{1}{2\sqrt{|mn|}} \sum_{c>0} \frac{S(m, n; c)}{c} \times \begin{cases} J_{2s-1} \left( \frac{4\pi}{c} \sqrt{mn} \right) & \text{if } mn > 0 \\ I_{2s-1} \left( \frac{4\pi}{c} \sqrt{-mn} \right) & \text{if } mn < 0 \end{cases}, \tag{A.4}$$

where  $I_s(z)$  and  $J_s(z)$  are the Bessel  $I$  and  $J$  functions. Note that  $\mathcal{Z}(m, n; s)$  is related to  $Z(m, n; s)$  via [39]

$$\mathcal{Z}(m, n; s) = \pi (4\pi^2 |mn|)^{s-1} \sum_{k=0}^{\infty} \frac{(-4\pi^2 mn)^k}{k! \Gamma(2s + k)} Z(m, n; s + k). \tag{A.5}$$

The Kloosterman zeta functions  $Z$  and  $\mathcal{Z}$  can also be defined in the case of Hecke congruence subgroups of  $SL(2; \mathbb{Z})$ . The Kloosterman–Selberg zeta function associated to a pair of cusps  $\mathfrak{ab}$  of  $\Gamma_0(N)$  is defined for  $\text{Re}(s) > 1$  by the absolutely convergent sum

$$Z_{\mathfrak{ab}}(m, n; s) \equiv \sum_{\begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \sigma_a^{-1} \Gamma_0(N) \sigma_b / \Gamma_\infty} \frac{e^{2i\pi(m\frac{d}{c} + n\frac{a}{c})}}{c^{2s}}, \tag{A.6}$$

of  $2 \times 2$  real matrices  $\begin{pmatrix} a & * \\ c & d \end{pmatrix}$  in the double coset  $\Gamma_\infty \backslash \sigma_a^{-1} \Gamma_0(N) \sigma_b / \Gamma_\infty$ . Here,  $\sigma_a$  is the scaling matrix [39] associated to the cusp  $a$  and  $\Gamma_\infty$  is the stabiliser of the cusp at  $\infty$ .

Similarly to the  $SL(2; \mathbb{Z})$  case, for  $mn = 0$ , the Kloosterman–Selberg zeta function may be evaluated in terms of the Riemann zeta function. For  $N$  prime, one finds

$$\begin{aligned} Z_{\infty\infty}(0, 0; s) &= \frac{N-1}{N^{2s}-1} \frac{\zeta(2s-1)}{\zeta(2s)}, \\ Z_{\infty 0}(0, 0; s) &= \frac{N^{2s-1}-1}{N^{s-1}(N^{2s}-1)} \frac{\zeta(2s-1)}{\zeta(2s)}, \end{aligned} \tag{A.7}$$

and

$$\begin{aligned} Z_{\infty\infty}(0, m; s) &= \frac{N \sigma_{1-2s}(m/N) - \sigma_{1-2s}(m)}{(N^{2s}-1) \zeta(2s)}, \\ Z_{\infty 0}(0, m; s) &= \frac{N^{2s-1} \sigma_{1-2s}(m) - \sigma_{1-2s}(m/N)}{N^{s-1}(N^{2s}-1) \zeta(2s)}, \end{aligned} \tag{A.8}$$

where it is understood that  $\sigma_{1-2s}(m/N)$  vanishes unless  $N$  divides  $m$ .

Similarly, for  $mn \neq 0$  one defines the *associated* Kloosterman–Selberg zeta function attached to the pair of cusps  $\mathfrak{ab}$  via

$$\begin{aligned} \mathcal{Z}_{\mathfrak{ab}}(m, n; s) &\equiv \frac{1}{2\sqrt{|mn|}} \sum_{\begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \sigma_a^{-1} \Gamma_0(N) \sigma_b / \Gamma_\infty} \frac{e^{2i\pi(m\frac{d}{c} + n\frac{a}{c})}}{c} \\ &\times \begin{cases} J_{2s-1}\left(\frac{4\pi}{c}\sqrt{mn}\right) & \text{if } mn > 0 \\ I_{2s-1}\left(\frac{4\pi}{c}\sqrt{-mn}\right) & \text{if } mn < 0 \end{cases}. \end{aligned} \tag{A.9}$$

Notice that the definitions (A.6) and (A.9) valid for any positive integer  $N$ , include the case of the full modular group for  $N = 1$ , where one recovers the  $SL(2; \mathbb{Z})$  Kloosterman zeta functions

$$Z_{\infty\infty}(m, n; s)|_{N=1} = Z(m, n; s), \quad \mathcal{Z}_{\infty\infty}(m, n; s)|_{N=1} = \mathcal{Z}(m, n; s). \tag{A.10}$$

### Appendix B. Modular properties of the Eichler integral

In this appendix, following [32], we explicitly derive the transformation properties of the Eichler integral  $\tilde{E}_{-2n}(z)$ , with even negative weight  $w = -2n$ , with respect to the action of the modular group  $SL(2; \mathbb{Z})$ . We recall the definition

$$\tilde{E}_{-2n}(z) = \frac{\zeta(2n+2)}{2\pi i} z^{2n+1} + \frac{\zeta(2n+1)}{2} + \sum_{N>0} \sigma_{-1-2n}(N) q^N, \tag{B.1}$$

and its general transformation property

$$(cz + d)^{2n} \tilde{E}_{-2n} \left( \frac{az + b}{cz + d} \right) = \tilde{E}_{-2n}(z) + P_\gamma(z),$$

with  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2; \mathbb{Z}),$  (B.2)

and  $P_\gamma(z)$  is a degree  $2n$  polynomial depending on the transformation  $\gamma$  and satisfies the cocycle relation

$$P_{\gamma\gamma'} = P_\gamma|_{-2n} \gamma' + P_{\gamma'}. \tag{B.3}$$

It is uniquely specified by  $P_T(z)$  and  $P_S(z)$ , with  $T$  and  $S$  the two generators of the modular group,

$$T : z \rightarrow z + 1, \quad S : z \rightarrow -1/z. \tag{B.4}$$

The polynomial  $P_T(z)$  can be straightforwardly derived, and reads

$$P_T(z) = \frac{\zeta(2n + 2)}{2\pi i} \left[ (z + 1)^{2n+1} - z^{2n+1} \right] = \frac{\zeta(2n + 2)}{2\pi i} \sum_{k=0}^{2n} \binom{2n + 1}{k} z^k. \tag{B.5}$$

The identification of  $P_S(z)$  is a bit more involved. We have by definition

$$z^{2n} \tilde{E}_{-2n}(-1/z) - \tilde{E}_{-2n}(z) = \sum_{k=0}^{2n} C_k z^k, \tag{B.6}$$

for some coefficients  $C_k$ . Note that the simple change of variable  $z \rightarrow -1/z$  implies the relations

$$C_k = (-1)^{k+1} C_{2n-k} \quad \text{for } k = 0, 1, \dots, 2n, \tag{B.7}$$

and, a result, only half of the coefficients are independent. To determine the independent coefficients let us consider the case where  $z = iy$ , and define the function

$$\begin{aligned} \tilde{e}_{-2n}(y) &= \sum_{N>0} \sigma_{-1-2n}(N) e^{-2\pi Ny} \\ &= \tilde{E}_{-2n}(iy) - \frac{\zeta(2n + 1)}{2} - \frac{\zeta(2n + 2)}{2\pi} (-1)^n y^{2n+1}. \end{aligned} \tag{B.8}$$

From eq. (B.6) one finds

$$\tilde{e}_{-2n}(y) - (-1)^n y^{2n} \tilde{e}_{-2n}(1/y) = \sum_{k=-1}^{2n+1} A_k y^k, \tag{B.9}$$

where

$$\begin{aligned} A_{-1} &= \frac{\zeta(2n + 2)}{2\pi} & A_0 &= -C_0 - \frac{\zeta(2n + 1)}{2}, \\ A_{2n+1} &= (-1)^{n+1} \frac{\zeta(2n + 2)}{2\pi}, & A_{2n} &= (-1)^n \frac{\zeta(2n + 1)}{2} - (-1)^n C_{2n}, \end{aligned} \tag{B.10}$$

and  $A_m = -i^m C_m$ , for  $m = 1, \dots, 2n - 1$ .

The coefficients  $A_k$ , and thus the  $C_k$ , can then be determined by computing the Mellin transform

$$L^*(s) = \int_0^\infty dy y^{s-1} \tilde{e}_{-2n}(y). \tag{B.11}$$

A straightforward evaluation of the integral yields

$$L^*(s) = (2\pi)^{-s} \Gamma(s) \zeta(s) \zeta(s + 2n + 1). \quad (\text{B.12})$$

Alternatively, using the relation (B.9), we can write

$$L^*(s) = \int_1^\infty dy y^{s-1} \tilde{e}_{-2n}(y) + (-1)^n \int_0^1 dy y^{2n+s-1} \tilde{e}_{-2n}(1/y) + \sum_{k=-1}^{2n+1} \frac{A_k}{s+k}. \quad (\text{B.13})$$

By matching the singularities of the two expressions (B.12) and (B.13) one thus finds the following expression for the inhomogeneous contribution

$$P_S(z) = -\frac{(2\pi i)^{2n+1}}{2} \sum_{k=1}^n \frac{B_{2k} B_{2n-2k+2}}{(2k)! (2n-2k+2)!} z^{2k-1}, \quad (\text{B.14})$$

that generalises [32] to the cases  $n > 1$ .

Alternatively, the polynomials  $P_Y$  could be computed using the method of [42].

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