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# WEIGHTED ENERGY ESTIMATES FOR $p$ -EVOLUTION EQUATIONS IN $SG$ -CLASSES

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ABSTRACT. We prove energy estimates for linear  $p$ -evolution equations in weighted Sobolev spaces under suitable assumptions on the behavior at infinity of the coefficients with respect to the space variables. As a consequence we obtain well posedness for the related Cauchy problem in the Schwartz spaces  $\mathcal{S}(\mathbb{R})$  and  $\mathcal{S}'(\mathbb{R})$ .

## 1. Introduction

Let us start by considering the Cauchy problem

$$(1.1) \quad \begin{cases} P(t, x, D_t, D_x)u(t, x) = f(t, x) & (t, x) \in [0, T] \times \mathbb{R}, \\ u(0, x) = g(x) & x \in \mathbb{R}, \end{cases}$$

$D = -i\partial$ , where  $P(t, x, D_t, D_x)$  is a differential evolution operator of the form

$$(1.2) \quad P(t, x, D_t, D_x) = D_t + a_p(t)D_x^p + \sum_{j=0}^{p-1} a_j(t, x)D_x^j,$$

with  $p \in \mathbb{N}, p \geq 2$ ,  $a_p \in C([0, T]; \mathbb{R})$  and  $a_j \in C([0, T]; \mathcal{B}^\infty)$ , where  $\mathcal{B}^\infty$  stands for the class of complex valued  $C^\infty(\mathbb{R}_x)$  functions with uniformly bounded derivatives.

Operators of the form above are usually referred to as “ $p$ -evolution operators”; the condition that  $a_p(t)$  is real valued means that the principal symbol of  $P$  (in the sense of Petrowski) has the real characteristic  $\tau = -a_p(t)\xi^p$ ; by the Lax-Mizohata theorem, this is a necessary condition to have a unique solution in Sobolev spaces of the Cauchy problem (1.1) in a neighborhood of  $t = 0$ , for any  $p \geq 1$ . Notice that in the case  $p = 1$  operator (1.2) is strictly hyperbolic; in the case  $p = 2$ , operators of the form (1.2) with real characteristics are usually called “Schrödinger-type evolution operators”, being the Schrödinger operator the most relevant model in the class.

A wide literature concerning the well-posedness of problem (1.1) in Sobolev spaces exists for  $p = 1, 2$ . For general  $p \geq 2$ , many results are known when the coefficients  $a_j(t, x)$  are real valued, see for instance [1, 2, 3, 11, 13, 15]. When the coefficients  $a_j(t, x)$  are complex valued for some  $1 \leq j \leq p-1$ , then we know from [20, 8] that some decay conditions for  $|x| \rightarrow \infty$  must be required on the imaginary part of the coefficients in order to obtain  $H^\infty$ -well posedness. In the papers [20, 21], Ichinose has given necessary and sufficient conditions for the case  $p = 2$ ,  $x \in \mathbb{R}$ . Kajitani and Baba [22] then proved that, for  $p = 2$  and  $a_2(t)$  constant,  $x \in \mathbb{R}^n$ , the Cauchy problem (1.1) is  $H^\infty$ -well posed if

$$(1.3) \quad \text{Im } a_1(t, x) = \mathcal{O}(|x|^{-\sigma}), \quad \sigma \geq 1, \quad \text{as } |x| \rightarrow \infty,$$

uniformly with respect to  $t \in [0, T]$ . Second order equations with  $p = 2$  and decay conditions as  $|x| \rightarrow \infty$  have been considered, for example, in [12, 16]. Cicognani and Colombini [14] treated

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the case  $p = 3$  proving  $H^\infty$ -well posedness under the conditions

$$(1.4) \quad \begin{aligned} |\operatorname{Im} a_2| &\leq C a_3(t) \langle x \rangle^{-1}, \\ |\operatorname{Im} a_1| + |\operatorname{Re} \partial_x a_2| &\leq C a_3(t) \langle x \rangle^{-1/2}. \end{aligned}$$

Recently, the first author et al. [6] extended the results of [14] and [22] to the case  $p \geq 4$ , giving sufficient conditions for  $H^\infty$  well posedness of the Cauchy problem for the operator (1.2); results in [6] have then been generalized to pseudo-differential systems in [5], and to higher order equations in [4]; semi-linear 3-evolution equations have been then studied in [7]. Recently, in [8] a necessary condition of decay at infinity for the coefficients of (1.2) with arbitrary  $p \geq 2$  has been given.

In this paper we want to consider the Cauchy problem (1.1) when  $P(t, x, D_t, D_x)$  is an evolution operator of the form

$$(1.5) \quad P(t, x, D_t, D_x) = D_t + a_p(t, D_x) + \sum_{j=0}^{p-1} a_j(t, x, D_x),$$

$p \in \mathbb{N}$ ,  $p \geq 2$ , where  $a_j$  are pseudo-differential operators with symbols  $a_j$  of order  $j$  for  $0 \leq j \leq p$ , and for every  $t \in [0, T]$ ,  $x, \xi \in \mathbb{R}$  we have:  $a_p(t, \xi) \in \mathbb{R}$ ,  $a_j(t, x, \xi) \in \mathbb{C}$  for  $0 \leq j \leq p-1$ .

In [5] it has been proved the following:

**Theorem 1.1.** *The Cauchy problem (1.1) for the operator (1.5) is  $H^\infty$  well posed under the assumptions:*

$$(1.6) \quad |\partial_\xi a_p(t, \xi)| \geq C_p |\xi|^{p-1} \quad \forall t \in [0, T], \quad |\xi| \gg 1,$$

for some  $C_p > 0$  and

$$(1.7) \quad |\operatorname{Im} \partial_\xi^\alpha a_j(t, x, \xi)| \leq C_\alpha \langle x \rangle^{-\frac{j}{p-1}} \langle \xi \rangle_h^{j-\alpha}, \quad 1 \leq j \leq p-1$$

$$(1.8) \quad |\operatorname{Im} \partial_\xi^\alpha D_x a_j(t, x, \xi)| \leq C_\alpha \langle x \rangle^{-\frac{j-1}{p-1}} \langle \xi \rangle_h^{j-\alpha}, \quad 2 \leq j \leq p-1$$

$$(1.9) \quad |\operatorname{Im} \partial_\xi^\alpha D_x^\beta a_j(t, x, \xi)| \leq C_\alpha \langle x \rangle^{-\frac{j-[\beta/2]}{p-1}} \langle \xi \rangle_h^{j-\alpha}, \quad 1 \leq \left\lfloor \frac{\beta}{2} \right\rfloor \leq j-1, \quad 3 \leq j \leq p-1$$

for all  $(t, x, \xi) \in [0, T] \times \mathbb{R}^2$  and for some  $C_\alpha > 0$ , where  $\langle \cdot \rangle_h = \sqrt{h^2 + |\cdot|^2}$ ,  $h \geq 1$ . More precisely, there exists  $\sigma > 0$  such that for all  $f \in C([0, T]; H^s)$  and  $g \in H^s$  there is a unique solution  $u \in C([0, T]; H^{s-\sigma})$  of (1.1), (1.5) which satisfies the following energy estimate:

$$(1.10) \quad \|u(t, \cdot)\|_{s-\sigma}^2 \leq C_s \left( \|g\|_s^2 + \int_0^t \|f(\tau, \cdot)\|_s^2 d\tau \right) \quad \forall t \in [0, T],$$

for some  $C_s > 0$ .

Formula (1.10) shows that the Cauchy problem (1.1), (1.5) is  $H^\infty$  well posed with loss of  $\sigma$  derivatives, in the sense that the solution is less regular than the Cauchy data. This phenomenon, which is usual in the theory of degenerate hyperbolic equations, appears so also in the theory of non-degenerate  $p$ -evolution equations for  $p \geq 2$ , and has been yet observed in [13, 21, 22]. Notice that assumptions (1.6)-(1.9) are consistent with the conditions in (1.3), (1.4). The loss of derivatives appearing in (1.10) for the solution of (1.1) is explicitly computed in [5], and it can be avoided by slightly strengthening the sole assumption (1.7) for  $j = p-1$ . Formula (1.10) gives so an accurate information about the regularity of the solution, but, in spite of the very precise decay conditions on the coefficients, it does not say anything about the behavior of the solution as  $|x| \rightarrow \infty$ .

This suggests us to change the setting of the Cauchy problem (1.1) to gain the possibility of giving similar precise information on the behavior of the solution for  $|x| \rightarrow \infty$ ; namely, one

could try to obtain energy estimates in suitable weighted Sobolev spaces and well-posedness in the Schwartz spaces  $\mathcal{S}(\mathbb{R})$ ,  $\mathcal{S}'(\mathbb{R})$ .

Results of the above type have been proved for strictly hyperbolic equations ( $p = 1$ ) by Cordes [17]; we also recall similar results when the coefficients are not Lipschitz continuous in  $t$ , see [9, 10]. The natural framework consists in dealing with pseudo differential operators with symbols in the classes  $SG^{m_1, m_2} = SG^{m_1, m_2}(\mathbb{R}^2)$ , with  $m_j \in \mathbb{R}$ ,  $j = 1, 2$ , defined as the class of all functions  $p(x, \xi) \in C^\infty(\mathbb{R}^2)$  satisfying the following estimates:

$$(1.11) \quad \|p\|_{\alpha, \beta} := \sup_{(x, \xi) \in \mathbb{R}^2} \langle \xi \rangle^{-m_1 + \alpha} \langle x \rangle^{-m_2 + \beta} |\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| < \infty$$

for every  $\alpha, \beta \in \mathbb{N}$ . We refer to [17, 18, 24, 25] for a detailed calculus for this class. In the following we shall prove energy estimates in the weighted Sobolev spaces  $H_{s_1, s_2}(\mathbb{R})$ ,  $s_j \in \mathbb{R}$ ,  $j = 1, 2$ , defined as the space of all  $u \in \mathcal{S}'(\mathbb{R})$  satisfying the following condition:

$$(1.12) \quad \|u\|_{s_1, s_2} = \|\langle x \rangle^{s_2} \langle D \rangle^{s_1} u\|_{L^2} < \infty,$$

where we denote by  $\langle D \rangle^{s_1}$  the Fourier multiplier with symbol  $\langle \xi \rangle^{s_1}$ . It is worth to recall that for  $s_2 = 0$  we recapture the standard Sobolev spaces and that the following identities hold:

$$(1.13) \quad \bigcap_{s_1, s_2 \in \mathbb{R}} H_{s_1, s_2}(\mathbb{R}) = \mathcal{S}(\mathbb{R}), \quad \bigcup_{s_1, s_2 \in \mathbb{R}} H_{s_1, s_2}(\mathbb{R}) = \mathcal{S}'(\mathbb{R}).$$

Moreover we recall that  $\mathcal{S}(\mathbb{R})$  is dense in  $H_{s_1, s_2}(\mathbb{R})$  for any  $s_1, s_2 \in \mathbb{R}$ .

The main result of the paper is the following.

**Theorem 1.2.** *Let  $P(t, x, D_t, D_x)$  be an operator of the form (1.5) and assume that the following conditions hold:*

$$(1.14) \quad a_p \in C([0, T]; SG^{p, 0}),$$

$$(1.15) \quad |\partial_\xi a_p(t, \xi)| \geq C_p |\xi|^{p-1} \quad \forall t \in [0, T], \quad |\xi| \gg 1, \quad \text{with } C_p > 0,$$

$$(1.16) \quad a_j \in C([0, T]; SG^{j, -j/(p-1)}), \quad j = 0, \dots, p-1.$$

*Then, the Cauchy problem (1.1) is well-posed in  $\mathcal{S}(\mathbb{R})$ ,  $\mathcal{S}'(\mathbb{R})$ . More precisely, there exists  $\sigma > 0$  such that for all  $s_1, s_2 \in \mathbb{R}$ ,  $f \in C([0, T]; H_{s_1, s_2}(\mathbb{R}))$  and  $g \in H_{s_1, s_2}(\mathbb{R})$  there is a unique solution  $u \in C([0, T]; H_{s_1, s_2 - \sigma}(\mathbb{R}))$  which satisfies the following energy estimate:*

$$(1.17) \quad \|u(t, \cdot)\|_{s_1, s_2 - \sigma}^2 \leq C_s \left( \|g\|_{s_1, s_2}^2 + \int_0^t \|f(\tau, \cdot)\|_{s_1, s_2}^2 d\tau \right) \quad \forall t \in [0, T],$$

for some  $C = C(s_1, s_2) > 0$ .

**Remark 1.3.** The energy estimate (1.17) shows that under our assumptions we can obtain well-posedness in  $\mathcal{S}(\mathbb{R})$  and  $\mathcal{S}'(\mathbb{R})$  *without any loss of derivatives* for the solution  $u$  of (1.1), paying this with a modification of the rate of decay/growth at infinity of the solution with respect to the Cauchy data. The solution of (1.1) has so the same regularity as the Cauchy data; but, if we start from data with a prescribed decay at infinity, then a loss of decay appears in the solution; similarly, if the data have a fixed polynomial growth at infinity, then the solution presents a stronger growth. Moreover the solution exists uniquely and the precise value of  $\sigma$  is computed, see formulas (3.4) and (3.13).

**Remark 1.4.** The proof of Theorem 1.2 is in part inspired by [6], but it takes advantage of the fact that in the new framework we are considering we can admit initial data with polynomial growth with respect to the space variable. On the other hand, the assumptions on the coefficients given in our paper are stronger than the ones used in [5, 6]. Hence, one can modify the approach and define the functions  $\lambda_{p-k}$  in the following by (2.14) for  $1 \leq k \leq p-1$

(hence also for  $k = 1$ ) as in [6] and repeat readily the argument of the proof using the estimates of Lemma 2.1 in [6] instead of Lemma 2.5 in the case  $k = 1$ . In this way we are able to prove that there exists  $\sigma' > 0$  such that for all  $s_1, s_2 \in \mathbb{R}$ ,  $f \in C([0, T]; H_{s_1, s_2}(\mathbb{R}))$  and  $g \in H_{s_1, s_2}(\mathbb{R})$  there is a unique solution  $u \in C([0, T]; H_{s_1 - \sigma', s_2}(\mathbb{R}))$  which satisfies the following energy estimate:

$$(1.18) \quad \|u(t, \cdot)\|_{s_1 - \sigma', s_2}^2 \leq C_s \left( \|g\|_{s_1, s_2}^2 + \int_0^t \|f(\tau, \cdot)\|_{s_1, s_2}^2 d\tau \right) \quad \forall t \in [0, T],$$

for some  $C = C(s_1, s_2) > 0$ . We do not prove here this alternative result, the proof being a repetition of the one of Theorem 1.1 in [6] in our functional setting. Finally Theorem 1.2 opens two important issues. First of all it seems to suggest that the loss of derivatives proved in [6] may not really appear, although we have to remark that our assumptions are slightly stronger than the ones used in [6]. Moreover, it would be interesting to clarify if the loss of regularity in the second Sobolev index may really appear. We shall treat these two problems in a future paper.

**Remark 1.5.** If the condition (1.16) with  $j = p - 1$  is strengthened into

$$a_{p-1} \in C([0, T]; SG^{p-1, -(1+\epsilon)}),$$

for any  $\epsilon > 0$ , then the Cauchy problem (1.1) is well posed in  $\mathcal{S}(\mathbb{R})$ ,  $\mathcal{S}'(\mathbb{R})$  without loss of derivatives and without modification of the behavior at infinity.

**Remark 1.6.** We observe that the assumption (1.16) in Theorem 1.2 can be slightly weakened without changing the argument of the proof. Namely we can replace the condition (1.16) with the following

$$(1.19) \quad \operatorname{Re} a_j \in C([0, T]; SG^{j, 0}), \quad \operatorname{Im} a_j \in C([0, T]; SG^{j, -j/(p-1)}), \quad 0 \leq j \leq p - 1.$$

The argument of the proof remains essentially the same but it involves more complicate notation. For this reason we prefer to present our main result using the more simple assumption (1.16). We refer to Remark 3.7 at the end of the paper for some comments on the more refined result. Finally we observe that if  $a_j(t, x, D_x)$  are differential operators, assumptions (1.14), (1.15), (1.19) are consistent with the ones given in [6, 14, 22] for the corresponding case  $a_p(t) > C_p > 0 \forall t \in [0, T]$ .

## 2. Preliminaries

In this section we collect some basic notions on SG classes of pseudo-differential operators and prove some preliminary results which will be used in the proof of Theorem 1.2 in the next section.

**2.1. SG-pseudo-differential operators.** We first recall that SG classes can be regarded as a particular case of general Hörmander classes, see [19, Chapter XVIII]. A specific calculus in different functional settings can be found in [17], [18], [24], [25]. Here we recall only some basic facts which will be used in the proof of our result. In general, fixed  $d \in \mathbb{N} \setminus \{0\}$ , the space  $SG^{m_1, m_2}(\mathbb{R}^{2d})$  is the space of all functions  $p(x, \xi) \in C^\infty(\mathbb{R}^{2d})$  satisfying the following estimates:

$$(2.1) \quad \|p\|_{\alpha, \beta} := \sup_{(x, \xi) \in \mathbb{R}^{2d}} \langle \xi \rangle^{-m_1 + |\alpha|} \langle x \rangle^{-m_2 + |\beta|} |\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| < \infty$$

for every  $\alpha, \beta \in \mathbb{N}$ . We can associate to every  $p \in SG^{m_1, m_2}(\mathbb{R}^{2d})$  a pseudo-differential operator defined by

$$(2.2) \quad Pu(x) = p(x, D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\langle x, \xi \rangle} p(x, \xi) \hat{u}(\xi) d\xi.$$

The operator  $p(x, D)$  is a linear continuous map  $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$  which extends to a continuous map  $\mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ . Concerning the action of these operators on weighted Sobolev spaces, we have that if  $p \in SG^{m_1, m_2}(\mathbb{R}^{2d})$ , then the map

$$p(x, D) : H_{s_1, s_2}(\mathbb{R}^d) \rightarrow H_{s_1 - m_1, s_2 - m_2}(\mathbb{R}^d)$$

is continuous for every  $s_1, s_2 \in \mathbb{R}$ , where the space  $H_{s_1, s_2}(\mathbb{R}^d)$  is obviously defined in arbitrary dimension as the space of all  $u \in \mathcal{S}'(\mathbb{R}^d)$  satisfying (1.12). We also recall the following result concerning the composition and the adjoint of SG operators.

**Proposition 2.1.** *Let  $p \in SG^{m_1, m_2}(\mathbb{R}^{2d})$  and  $q \in SG^{m'_1, m'_2}(\mathbb{R}^{2d})$ . Then there exists a symbol  $s \in SG^{m_1 + m'_1, m_2 + m'_2}(\mathbb{R}^{2d})$  such that  $p(x, D)q(x, D) = s(x, D) + R$  where  $R$  is a smoothing operator  $\mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ . Moreover,  $s$  has the following asymptotic expansion*

$$s(x, \xi) \sim \sum_{\alpha} \alpha!^{-1} \partial_{\xi}^{\alpha} p(x, \xi) D_x^{\alpha} q(x, \xi)$$

i.e. for every  $N \geq 1$ , we have

$$s(x, \xi) - \sum_{|\alpha| < N} \alpha!^{-1} \partial_{\xi}^{\alpha} p(x, \xi) D_x^{\alpha} q(x, \xi) \in SG^{m_1 + m'_1 - N, m_2 + m'_2 - N}(\mathbb{R}^{2d}).$$

**Proposition 2.2.** *Let  $p \in SG^{m_1, m_2}(\mathbb{R}^{2d})$  and let  $P^*$  be the  $L^2$ -adjoint of  $p(x, D)$ . Then there exists a symbol  $p^* \in SG^{m_1, m_2}(\mathbb{R}^{2d})$  such that  $P^* = p^*(x, D) + R'$ , where  $R'$  is a smoothing operator  $\mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ . Moreover,  $p^*$  has the following asymptotic expansion*

$$p^*(x, \xi) \sim \sum_{\alpha} \alpha!^{-1} \partial_{\xi}^{\alpha} \overline{D_x^{\alpha} p(x, \xi)}$$

i.e. for every  $N \geq 1$ , we have

$$p^*(x, \xi) - \sum_{|\alpha| < N} \alpha!^{-1} \partial_{\xi}^{\alpha} \overline{D_x^{\alpha} p(x, \xi)} \in SG^{m_1 - N, m_2 - N}(\mathbb{R}^{2d}).$$

We also recall the definition of the class  $S^m(\mathbb{R}^{2d})$ ,  $m \in \mathbb{R}$ , defined as the space of all symbols  $p(x, \xi) \in C^{\infty}(\mathbb{R}^{2d})$  satisfying

$$|\partial_{\xi}^{\alpha} \partial_x^{\beta} p(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m - |\alpha|}, \quad (x, \xi) \in \mathbb{R}^{2d}$$

for every  $\alpha, \beta \in \mathbb{N}^d$ . It is important for the sequel to notice that

$$(2.3) \quad SG^{m_1, m_2}(\mathbb{R}^{2d}) \subset S^{m_1}(\mathbb{R}^{2d})$$

for any  $m_1, m_2 \in \mathbb{R}$  with  $m_2 \leq 0$  and that the operators with symbols in  $S^0(\mathbb{R}^{2d})$  map continuously  $H_{s_1, s_2}(\mathbb{R}^d)$  to itself for every  $s_1, s_2 \in \mathbb{R}$ .

In the proof of Theorem 1.2 we shall also use the sharp Gårding inequality applied to SG operators. This result is known as a particular case of [19, Thm. 18.6.14]. However, for our purposes we also need a precise estimate of the order of the remainder with respect to  $\xi$ , which has been proved only for symbols in the Hörmander classes  $S^m(\mathbb{R}^{2d})$ , see [23, Thm. 4.2]. Nevertheless, we have to observe that the operators we shall consider have negative order with respect to  $x$ . Hence, in view of the inclusion (2.3), we can base the proof of this result on the classical sharp Gårding inequality for standard Hörmander symbols and estimate the order of the remainder term with respect to  $\xi$  by looking at its classical asymptotic expansion. Namely we have the following result.

**Theorem 2.3.** *Let  $m_1 \geq 0, m_2 \leq 0$ ,  $a \in SG^{m_1, m_2}(\mathbb{R}^{2d})$  with  $\operatorname{Re} a(x, \xi) \geq 0$ . Then there exist pseudo-differential operators  $Q = q(x, D)$ ,  $\tilde{R} = \tilde{r}(x, D)$  and  $R_0 = r_0(x, D)$  with symbols, respectively,  $q \in SG^{m_1, m_2}(\mathbb{R}^{2d})$ ,  $\tilde{r} \in SG^{m_1-1, m_2}(\mathbb{R}^{2d})$  and  $r_0 \in S^0(\mathbb{R}^{2d})$  such that*

$$(2.4) \quad a(x, D) = q(x, D) + \tilde{r}(x, D) + r_0(x, D)$$

$$(2.5) \quad \operatorname{Re}\langle q(x, D)u, u \rangle \geq 0 \quad \forall u \in \mathcal{S}(\mathbb{R}^d).$$

*Proof.* Since  $m_2 \leq 0$ , then  $SG^{m_1, m_2}(\mathbb{R}^{2d}) \subset S^{m_1}(\mathbb{R}^{2d})$ . Hence, the classical Gårding inequality gives the existence of two symbols  $q$  and  $r$  such that  $a(x, D) = q(x, D) + r(x, D)$  and  $q(x, D)$  satisfies (2.5). Let us now consider the asymptotic expansion of the remainder term  $r(x, D)$ . By Theorem 4.2 in [23], we have that

$$(2.6) \quad r(x, \xi) \sim \psi_1(\xi)D_x a(x, \xi) + \sum_{|\alpha+\beta| \geq 2} \psi_{\alpha, \beta}(\xi) \partial_\xi^\alpha D_x^\beta a(x, \xi),$$

for some real valued functions  $\psi_1, \psi_{\alpha, \beta}$  with  $\psi_1 \in SG^{-1, 0}(\mathbb{R}^{2d})$  and  $\psi_{\alpha, \beta} \in SG^{(|\alpha|-|\beta|)/2, 0}(\mathbb{R}^{2d})$ . In particular, we have that

$$r(x, \xi) = \psi_1(\xi)D_x a(x, \xi) + \sum_{2 \leq |\alpha+\beta| \leq 2m_1-1} \psi_{\alpha, \beta}(\xi) \partial_\xi^\alpha D_x^\beta a(x, \xi) + r_0(x, \xi),$$

for a symbol  $r_0 \in S^0(\mathbb{R}^{2d})$ . Moreover, it is easy to verify that  $\psi(\xi)D_x a(x, \xi) \in SG^{m_1-1, m_2-1}(\mathbb{R}^{2d})$  and that

$$\sum_{2 \leq |\alpha+\beta| \leq 2m_1-1} \psi_{\alpha, \beta}(\xi) \partial_\xi^\alpha D_x^\beta a(x, \xi) \in SG^{m_1-1, m_2}(\mathbb{R}^{2d}).$$

Then we have that the symbol

$$\tilde{r}(x, \xi) = \psi_1(\xi)D_x a(x, \xi) + \sum_{2 \leq |\alpha+\beta| \leq 2m_1-1} \psi_{\alpha, \beta}(\xi) \partial_\xi^\alpha D_x^\beta a(x, \xi) \in SG^{m_1-1, m_2}(\mathbb{R}^{2d}).$$

This concludes the proof.  $\square$

**Remark 2.4.** In the sequel of the paper we will often replace the weight function  $\langle \xi \rangle$  with  $\langle \xi \rangle_h = (h^2 + |\xi|^2)^{1/2}$  for some  $h \geq 1$  to prove our results. It is clear that this modification does not change the definition of the class  $SG^{m_1, m_2}(\mathbb{R}^{2d})$  and of the spaces  $H^{s_1, s_2}(\mathbb{R}^d)$ , and their properties.

**2.2. Changes of variables and conjugations.** The idea of the proof of Theorem 1.2 is to prove an energy estimate in  $L^2(\mathbb{R})$  for the operator

$$(2.7) \quad iP = \partial_t + ia_p(t, D_x) + \sum_{j=0}^{p-1} ia_j(t, x, D_x) = \partial_t + A(t, x, D_x).$$

We have

$$(2.8) \quad \begin{aligned} \frac{d}{dt} \|u\|_0^2 &= 2 \operatorname{Re}\langle \partial_t u, u \rangle = 2 \operatorname{Re}\langle iP u, u \rangle - 2 \operatorname{Re}\langle A u, u \rangle \\ &\leq \|f\|_0^2 + \|u\|_0^2 - 2 \operatorname{Re}\langle A u, u \rangle. \end{aligned}$$

Notice that  $2 \operatorname{Re}\langle A u, u \rangle = \langle (A + A^*)u, u \rangle$ , with  $A^*$  the formal adjoint of  $A$ , and  $A + A^*$  is an operator with symbol in  $SG^{p-1, -1}$ , hence with positive order with respect to  $\xi$ . This implies that the desired energy estimate is not straightforward and in order to obtain it, we need to transform the Cauchy problem (1.1) into an equivalent one of the form

$$(2.9) \quad \begin{cases} P_\lambda u_\lambda = f_\lambda \\ u_\lambda(0, x) = g_\lambda, \end{cases}$$

where  $P_\lambda = D_t - iA_\lambda$  and  $\operatorname{Re} A_\lambda(t, x, \xi) \geq 0$ ; then we apply Theorem 2.3 to obtain the estimate from below

$$\operatorname{Re}\langle A_\lambda v, v \rangle \geq -c\|v\|_0^2$$

for  $v \in \mathcal{S}(\mathbb{R})$  and for some positive constant  $c$ . This, computing as in (2.8), will give an  $L^2$  energy estimate for the solution  $u_\lambda$  of the Cauchy problem (2.9). The operator  $P_\lambda$  will be the result of  $p-1$  conjugations of  $P$  with operators of the form  $e^{\lambda_{p-k}(x, D_x)}$ ,  $k = 1, \dots, p-1$ , namely:

$$(iP)_\lambda := (e^{\lambda_1(x, D_x)})^{-1} \dots (e^{\lambda_{p-2}(x, D_x)})^{-1} (e^{\lambda_{p-1}(x, D_x)})^{-1} (iP) e^{\lambda_{p-1}(x, D_x)} e^{\lambda_{p-2}(x, D_x)} \dots e^{\lambda_1(x, D_x)}.$$

Here and in the following we shall denote by  $e^{\pm\lambda_{p-k}(x, D_x)}$ ,  $k = 1, \dots, p-1$ , the operators with symbols  $e^{\pm\lambda_{p-k}(x, \xi)}$  and the functions  $\lambda_{p-k}$  will be chosen such that:

- $\lambda_{p-k}(x, \xi)$  are real valued,  $1 \leq k \leq p-1$ ;
- $e^{\lambda_{p-1}(x, \xi)} \in SG^{0, M_{p-1}}$  for some  $M_{p-1} > 0$  and  $e^{\lambda_{p-k}(x, \xi)} \in SG^{0, 0}$  for  $2 \leq k \leq p-1$ ;
- the operator  $e^{\lambda_{p-k}(x, D_x)}$  is invertible for every  $1 \leq k \leq p-1$  and the principal part of  $(e^{\lambda_{p-k}(x, D_x)})^{-1}$  is  $e^{-\lambda_{p-k}(x, D_x)}$ ;
- the operator

$$A_\lambda := (e^{\lambda_1(x, D_x)})^{-1} \dots (e^{\lambda_{p-2}(x, D_x)})^{-1} (e^{\lambda_{p-1}(x, D_x)})^{-1} (iA) e^{\lambda_{p-1}(x, D_x)} e^{\lambda_{p-2}(x, D_x)} \dots e^{\lambda_1(x, D_x)}$$

is such that  $\operatorname{Re}\langle A_\lambda v, v \rangle \geq -c\|v\|_0^2 \forall v(t, \cdot) \in \mathcal{S}(\mathbb{R})$ .

After the transformation of the problem (1.1) into (2.9) with  $P_\lambda$  defined by (2.10) and  $f_\lambda$  and  $g_\lambda$  given by

$$(2.11) \quad f_\lambda = (e^{\lambda_1(x, D_x)})^{-1} \dots (e^{\lambda_{p-1}(x, D_x)})^{-1} f, \quad g_\lambda = (e^{\lambda_1(x, D_x)})^{-1} \dots (e^{\lambda_{p-1}(x, D_x)})^{-1} g,$$

we will obtain an energy estimate in  $L^2(\mathbb{R})$  for the new variable

$$(2.12) \quad u_\lambda(t, x) = (e^{\lambda_1(x, D_x)})^{-1} \dots (e^{\lambda_{p-2}(x, D_x)})^{-1} (e^{\lambda_{p-1}(x, D_x)})^{-1} u(t, x)$$

which will yield to an estimate of the form (1.17) for the solution  $u$  of (1.1).

Let us now define the functions  $\lambda_j$ . We set

$$(2.13) \quad \lambda_{p-1}(x, \xi) := M_{p-1} \omega \left( \frac{\xi}{h} \right) \int_0^x \frac{1}{\langle y \rangle} dy,$$

and for  $2 \leq k \leq p-1$

$$(2.14) \quad \lambda_{p-k}(x, \xi) := M_{p-k} \omega \left( \frac{\xi}{h} \right) \langle \xi \rangle_h^{-k+1} \int_0^x \langle y \rangle^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle y \rangle}{\langle \xi \rangle_h^{p-1}} \right) dy,$$

where  $M_{p-1}, M_{p-2}, \dots, M_1$  are positive constants to be chosen later on,  $\omega \in C^\infty(\mathbb{R})$  is such that

$$(2.15) \quad \omega(\xi) = \begin{cases} 0 & |\xi| \leq 1 \\ \operatorname{sgn}(\partial_\xi a_p(t, \xi)) & |\xi| \geq R \end{cases}$$

for some  $R > 1$ , and  $\psi \in C_0^\infty(\mathbb{R})$  is such that  $0 \leq \psi(y) \leq 1 \forall y \in \mathbb{R}$ ,  $\psi(y) = 1$  for  $|y| \leq \frac{1}{2}$ ,  $\psi(y) = 0$  for  $|y| \geq 1$ . Notice that assumption (1.15) ensures the existence of  $R > 0$  such that for every fixed  $\xi$  with  $|\xi| > R$  the sign of the function  $\partial_\xi a_p(t, \xi)$  remains constant for every  $t \in [0, T]$ , then  $\omega$  is well defined and does not depend on  $t$ .

Definition (2.14) is in part inspired by [5, 6]; more precisely, the symbols  $\lambda_{p-k}$  in (2.14) are exactly the same as in [5], while the symbol  $\lambda_{p-1}$  in (2.13) is new: it can be considered only in the framework of the  $SG$  calculus, where symbols with polynomial growth in  $x$  can be handled. The setting we are using allows to construct a transformation with a "stronger"  $\lambda_{p-1}$  with respect to [5] still remaining in (weighted) Sobolev spaces.



**Lemma 2.5.** *The function  $\lambda_{p-1}$  defined by (2.13) satisfies the following estimates:*

$$(2.16) \quad |\lambda_{p-1}(x, \xi)| \leq M_{p-1}(1 + \ln\langle x \rangle)$$

$$(2.17) \quad |\partial_\xi^\alpha \partial_x^\beta \lambda_{p-1}(x, \xi)| \leq M_{p-1} C_{\alpha, \beta} \langle x \rangle^{-\beta} \langle \xi \rangle_h^{-\alpha} \quad \alpha \geq 0, \beta \geq 1,$$

$$(2.18) \quad |\partial_\xi^\alpha \lambda_{p-1}(x, \xi)| \leq M_{p-1} C'_{\alpha, R} \langle \xi \rangle_h^{-\alpha} (1 + \ln\langle x \rangle) \chi_{E_{h, R}}(\xi), \quad \alpha \geq 1,$$

with positive constants  $C, C_{\alpha, \beta}, C'_{\alpha, R}$ , and where  $\chi_{E_{h, R}}$  is the characteristic function of the set  $E_{h, R} = \{\xi \in \mathbb{R} \mid h \leq |\xi| \leq hR\}$ .

*Proof.* A simple explicit computation of the integral in (2.13) gives

$$(2.19) \quad |\lambda_{p-1}(x, \xi)| \leq M_{p-1} \log(2\langle x \rangle) < M_{p-1}(1 + \ln\langle x \rangle).$$

For  $\beta \geq 1$  we have

$$(2.20) \quad |\partial_x^\beta \lambda_{p-1}(x, \xi)| = M_{p-1} \left| \omega \left( \frac{\xi}{h} \right) \partial_x^{\beta-1} \langle x \rangle^{-1} \right| \leq M_{p-1} C_\beta \langle x \rangle^{-\beta},$$

and for  $\alpha, \beta \geq 1$ :

$$(2.21) \quad \begin{aligned} |\partial_\xi^\alpha \partial_x^\beta \lambda_{p-1}(x, \xi)| &= M_{p-1} \left| \omega^{(\alpha)} \left( \frac{\xi}{h} \right) h^{-\alpha} \partial_x^{\beta-1} \langle x \rangle^{-1} \right| \leq M_{p-1} C_{\alpha, \beta} h^{-\alpha} \langle x \rangle^{-\beta} \\ &\leq M_{p-1} C_{\alpha, \beta} \langle \xi \rangle_h^{-\alpha} \langle x \rangle^{-\beta}. \end{aligned}$$

Finally, for  $\alpha \geq 1$ :

$$(2.22) \quad \begin{aligned} |\partial_\xi^\alpha \lambda_{p-1}(x, \xi)| &= M_{p-1} \left| \omega^{(\alpha)} \left( \frac{\xi}{h} \right) h^{-\alpha} \int_0^x \frac{1}{\langle y \rangle} dy \right| \\ &\leq M_{p-1} C_\alpha h^{-\alpha} \ln\langle x \rangle \chi_{E_{h, R}}(\xi) \\ &\leq M_{p-1} C_{\alpha, R} \langle \xi \rangle_h^{-\alpha} \ln\langle x \rangle \chi_{E_{h, R}}(\xi), \end{aligned}$$

since  $h^{-1} \leq \langle R \rangle \langle \xi \rangle_h^{-1}$  on  $E_{h, R}$ . □

**Lemma 2.6.** *Let  $\lambda_{p-k}, k = 2, \dots, p-1$  be defined by (2.14). Then for every  $\alpha, \beta \in \mathbb{N}$ , there exists a constant  $C_{k, \alpha, \beta} > 0$  such that*

$$(2.23) \quad \begin{aligned} |\partial_\xi^\alpha \partial_x^\beta \lambda_{p-k}(x, \xi)| &\leq C_{k, \alpha, \beta} M_{p-k} \langle x \rangle^{\frac{k-1}{p-1} - \beta} \langle \xi \rangle_h^{-\alpha - k + 1} \chi_\xi(x) \\ &\leq C_{k, \alpha, \beta} M_{p-k} \langle x \rangle^{-\beta} \langle \xi \rangle_h^{-\alpha}, \end{aligned}$$

where  $\chi_\xi(x)$  denotes the characteristic function of the set  $\{x \in \mathbb{R} \mid \langle x \rangle \leq \langle \xi \rangle_h^{p-1}\}$ . In particular, we have that  $\lambda_{p-k} \in SG^{0,0}$  for  $2 \leq k \leq p-1$ .

*Proof.* See [6, Lemma 2.1]. □

From the estimates proved in Lemma 2.5 and 2.6 we obtain, by simply applying the Faà di Bruno formula, the following estimates for the symbols  $e^{\pm \lambda_{p-k}(x, \xi)}$ . We leave the details of the proof to the reader.

**Lemma 2.7.** *Let  $\lambda_{p-k}, k = 1, \dots, p-1$  be defined by (2.13) and (2.14). Then*

$$(2.24) \quad |e^{\pm \lambda_{p-1}(x, \xi)}| \leq K \langle x \rangle^{M_{p-1}},$$

$$(2.25) \quad |\partial_\xi^\alpha e^{\pm \lambda_{p-1}(x, \xi)}| \leq C_\alpha [1 + \ln\langle x \rangle \chi_{E_{h, R}}(\xi)]^\alpha \langle \xi \rangle_h^{-\alpha} e^{\pm \lambda_{p-1}(x, \xi)}, \quad \alpha \geq 1,$$

$$(2.26) \quad |\partial_x^\beta e^{\pm \lambda_{p-1}(x, \xi)}| \leq C_\beta \langle x \rangle^{-\beta} e^{\pm \lambda_{p-1}(x, \xi)}, \quad \beta \geq 1,$$

$$(2.27) \quad |\partial_\xi^\alpha \partial_x^\beta e^{\pm \lambda_{p-1}(x, \xi)}| \leq C_{\alpha, \beta} [1 + \ln\langle x \rangle \chi_{E_{h, R}}(\xi)]^\alpha \langle x \rangle^{-\beta} \langle \xi \rangle_h^{-\alpha} e^{\pm \lambda_{p-1}(x, \xi)}, \quad \alpha, \beta \geq 1,$$

$$(2.28) \quad |\partial_\xi^\alpha \partial_x^\beta e^{\pm \lambda_{p-k}(x, \xi)}| \leq C_{\alpha, \beta, k} \langle x \rangle^{-\beta} \langle \xi \rangle_h^{-\alpha} e^{\pm \lambda_{p-k}(x, \xi)}, \quad 2 \leq k \leq p-1, \alpha, \beta \in \mathbb{N},$$

for some positive constants  $K, C_\alpha, C_\beta, C_{\alpha,\beta}, C_{\alpha,\beta,k}$ .

The next two results state the invertibility of the operators  $e^{\lambda_{p-k}(x,D_x)}$  for  $k = 1, \dots, p-1$ .

**Lemma 2.8.** *Let  $\lambda_{p-1}(x, \xi)$  be defined by (2.13). Then there exists  $h_1 \geq 1$  such that for  $h \geq h_1$  the operator  $e^{\lambda_{p-1}(x,D)}$  is invertible and*

$$(2.29) \quad (e^{\lambda_{p-1}(x,D_x)})^{-1} = e^{-\lambda_{p-1}(x,D_x)}(I + R_{p-1})$$

where  $I$  is the identity operator and  $R_{p-1}$  has principal symbol given by

$$r_{p-1,-1}(x, \xi) \in SG^{-1,0}.$$

*Proof.* By Proposition 2.1 it follows that

$$e^{\lambda_{p-1}(x,D_x)} e^{-\lambda_{p-1}(x,D_x)} = I - r_{p-1,-1}(x, D) + r_{p-1,-2}(x, D),$$

where

$$r_{p-1,-1}(x, \xi) = \partial_\xi \lambda_{p-1}(x, \xi) D_x \lambda_{p-1}(x, \xi)$$

and

$$r_{p-1,-2}(x, \xi) \sim \sum_{m \geq 2} \frac{1}{m!} \partial_\xi^m e^{\lambda_{p-1}(x,\xi)} D_x^m e^{-\lambda_{p-1}(x,\xi)}.$$

By (2.17) and (2.18)

$$r_{p-1,-1}(x, \xi) \in SG^{-1,-1+\varepsilon} \quad \text{for every } \varepsilon > 0$$

and

$$r_{p-1,-2}(x, \xi) \in SG^{-2,-2+\varepsilon} \quad \text{for every } \varepsilon > 0.$$

More precisely,

$$(2.30) \quad \begin{aligned} |\partial_\xi^\alpha D_x^\beta r_{p-1,-1}(x, \xi)| &\leq \sum_{\alpha_1 + \alpha_2 = \alpha} (c_{\alpha_1, \alpha_2, \beta} |\partial_\xi^{\alpha_1 + 1} \lambda_{p-1}| \cdot |\partial_\xi^{\alpha_2} D_x^{\beta+1} \lambda_{p-1}| + \\ &\quad + \sum_{\beta_1 + \beta_2 = \beta, \beta_1 \neq 0} c_{\alpha_1, \alpha_2, \beta_1, \beta_2} |\partial_\xi^{\alpha_1 + 1} D_x^{\beta_1} \lambda_{p-1}| \cdot |\partial_\xi^{\alpha_2} D_x^{\beta_2 + 1} \lambda_{p-1}|) \\ &\leq C_{\alpha, \beta, R} \langle \xi \rangle_h^{-\alpha} \langle x \rangle^{-\beta} \langle \xi \rangle_h^{-1} (\ln \langle x \rangle + 1) \langle x \rangle^{-1} \\ &\leq 2C_{\alpha, \beta, R} \cdot h^{-1} \langle \xi \rangle_h^{-\alpha} \langle x \rangle^{-\beta}, \quad \forall \alpha, \beta \geq 0. \end{aligned}$$

Setting  $r_{p-1}(x, \xi) := r_{p-1,-1}(x, \xi) - r_{p-1,-2}(x, \xi)$ , we have also

$$(2.31) \quad |\partial_\xi^\alpha D_x^\beta r_{p-1}(x, \xi)| \leq C'_{\alpha, \beta, R} \cdot h^{-1} \langle \xi \rangle_h^{-\alpha} \langle x \rangle^{-\beta}, \quad \forall \alpha, \beta \geq 0$$

and for some  $C'_{\alpha, \beta, R} > 0$ ; this means that for  $h$  large enough, operator  $I - R_{p-1}$  is invertible by Neumann series and  $\sum_{n=0}^{+\infty} R_{p-1}^n$  is the inverse operator. Similar considerations hold for  $e^{-\lambda_{p-1}(x,D_x)} e^{\lambda_{p-1}(x,D_x)}$ ; thus  $e^{-\lambda_{p-1}} \sum_{n=0}^{+\infty} R_{p-1}^n$  is a left and right inverse for  $e^{\lambda_{p-1}(x,D_x)}$ . The lemma is then proved.  $\square$

**Lemma 2.9.** *Let  $\lambda_{p-k}(x, \xi), k = 2, \dots, p-1$  be defined by (2.14). Then for every  $k$  there exists  $h_k \geq 1$  such that for  $h \geq h_k$  the operator  $e^{\lambda_{p-k}(x,D)}$  is invertible and*

$$(2.32) \quad (e^{\lambda_{p-k}(x,D_x)})^{-1} = e^{-\lambda_{p-k}(x,D_x)}(I + R_{p-k})$$

where  $I$  is the identity operator and  $R_{p-k}$  has principal symbol

$$r_{p-k,-k}(x, \xi) = \partial_\xi \lambda_{p-k}(x, \xi) D_x \lambda_{p-k}(x, \xi) \in SG^{-k, -\frac{p-k}{p-1}}.$$

*Proof.* The construction of the inverse is completely analogous to the one of Lemma 2.8. Moreover, by (2.23) we have that

$$\begin{aligned}
(2.33) \quad |r_{p-k,-k}(x, \xi)| &= |\partial_\xi \lambda_{p-k}(x, \xi) D_x \lambda_{p-k}(x, \xi)| \leq C_k M_{p-k}^2 \langle x \rangle^{\frac{2k-1}{p-1}-1} \langle \xi \rangle_h^{-2k+1} \chi_\xi(x) \\
&\leq C_k M_{p-k}^2 \langle x \rangle^{-\frac{p-k}{p-1}} \langle \xi \rangle_h^{-k} \\
&\leq C_k M_{p-k}^2 h^{-1}
\end{aligned}$$

since on the support of  $\chi_\xi(x)$  we have  $\langle x \rangle \leq \langle \xi \rangle_h^{p-1}$ . The derivatives of  $r_{p-k,-k}$  can be estimated similarly. Thus, for  $h$  large enough we obtain (2.32), and from (2.33) we have  $r_{p-k,-k} \in SG^{-k, -\frac{p-k}{p-1}}$ .  $\square$

### 3. The proof of Theorem 1.2

The proof of Theorem 1.2 needs some preparation. As announced in Section 2, we shall reduce the Cauchy problem (1.1) to the problem (2.9), where the operator  $P_\lambda$  is defined by (2.10) and the functions  $f_\lambda, g_\lambda$  and  $u_\lambda$  are given respectively by (2.11) and (2.12). We first prove the following result.

**Proposition 3.1.** *Let  $P_\lambda$  be defined by (2.10). Then we have:*

$$(iP)_\lambda = \partial_t + ia_p(t, D_x) + \sum_{\ell=1}^{p-1} Q_{p-\ell}(t, x, D_x) + r_0(t, x, D_x),$$

for some operators  $Q_{p-\ell}$  with symbols  $Q_{p-\ell}(t, x, \xi) \in SG^{p-\ell, 0}$ , satisfying

$$\operatorname{Re} \langle Q_{p-\ell} v, v \rangle \geq 0 \quad v \in \mathcal{S}(\mathbb{R}), \quad 1 \leq \ell \leq p-1$$

and for some operator  $r_0$  with symbol in  $S^0$ .

Proposition 3.1 will be proved in  $p-1$  steps each of them corresponding to a conjugation with an operator of the form  $e^{\lambda_{p-k}}, k = 1, \dots, p-1$ . We know by (2.23) and (2.24) that the operators  $e^{\pm \lambda_{p-1}(x, D_x)}$  have order  $(0, M_{p-1})$ , while the other operators  $e^{\pm \lambda_{p-k}(x, D_x)}$  have order  $(0, 0)$  for  $k = 2, \dots, p-1$ . In the proof of the energy estimate (1.17) the first conjugation will play an essential role since it determines the loss of  $H_{s_1, s_2}$  regularity; the others all work similarly to each other. This is the reason why we shall organize the proof of Proposition 3.1 as follows. We present in detail the first two transformations and then we argue by induction. Each of these steps corresponds to a different lemma. Before this we give a preliminary result which states for an operator with symbol  $a(x, \xi) \in SG^{m_1, 0}$  the form of the composed operator  $e^{-\lambda_i(x, D_x)} a(x, D_x) e^{\lambda_i(x, D_x)}, i = 1, \dots, p-1$ , with  $\lambda_i$  defined as in the previous section.

**Lemma 3.2.** *Let  $a \in SG^{m_1, 0}$  and let  $\lambda_i, i = 1, \dots, p-1$  be defined by (2.13), (2.14). Then the symbol of  $e^{-\lambda_i(x, D_x)} a(x, D_x) e^{\lambda_i(x, D_x)}, i = 1, \dots, p-1$  is given by:*

$$\begin{aligned}
(3.1) \quad (e^{-\lambda_i} a e^{\lambda_i})(x, \xi) &= a + \sum_{\alpha=1}^{m_1-1} \frac{1}{\alpha!} \partial_\xi^\alpha a \cdot e^{-\lambda_i} \cdot D_x^\alpha e^{\lambda_i} \\
&+ \sum_{\gamma=1}^{m_1-1} \frac{1}{\gamma!} \partial_\xi^\gamma e^{-\lambda_i} D_x^\gamma (a e^{\lambda_i}) + \sum_{\gamma=1}^{m_1-2} \sum_{\alpha=1}^{m_1-1} \frac{1}{\alpha! \gamma!} \partial_\xi^\gamma e^{-\lambda_i} D_x^\gamma (\partial_\xi^\alpha a \cdot D_x^\alpha e^{\lambda_i}) + r_0
\end{aligned}$$

with  $r_0 \in SG^{0,0}$ . Moreover the symbol

$$r(x, \xi) = \sum_{\gamma=1}^{m_1-1} \frac{1}{\gamma!} \partial_\xi^\gamma e^{-\lambda_i} D_x^\gamma (a e^{\lambda_i}) + \sum_{\gamma=1}^{m_1-2} \sum_{\alpha=1}^{m_1-1} \frac{1}{\alpha! \gamma!} \partial_\xi^\gamma e^{-\lambda_i} D_x^\gamma (\partial_\xi^\alpha a \cdot D_x^\alpha e^{\lambda_i}) \in SG^{m_1-1,0}.$$

*Proof.* The proof easily follows by Proposition 2.1.  $\square$

**Remark 3.3.** If we assume that  $a \in S^{m_1}$  instead of  $a \in SG^{m_1,0}$ , then Lemma 3.2 still holds with  $r \in S^{m_1-1}$ .

**Lemma 3.4.** *Let  $h$  be as in Lemma 2.8 and consider, for  $h \geq h_1$ , the operator  $(iP)_1 = (e^{\lambda_{p-1}(x, D_x)})^{-1} (iP) e^{\lambda_{p-1}(x, D_x)}$ . There exist operators  $Q_{p-1}(t, x, D_x)$ ,  $a_{j,1}(t, x, D_x)$  and  $r_1(t, x, D_x)$  with symbols  $Q_{p-1}(t, x, \xi) \in SG^{p-1,-1}$ ,  $a_{j,1}(t, x, \xi) \in SG^{j,-j/(p-1)}$ ,  $1 \leq j \leq p-2$  and  $r_1(t, x, \xi) \in S^0$  such that*

$$(iP)_1 = \partial_t + ia_p(t, D_x) + Q_{p-1}(t, x, D_x) + \sum_{j=1}^{p-2} ia_{j,1}(t, x, D_x) + r_1(t, x, D_x),$$

and

$$\operatorname{Re}\langle Q_{p-1}(t, x, D_x)v, v \rangle \geq 0, \quad \forall v \in \mathcal{S}(\mathbb{R}).$$

*Proof.* We first notice that, by Lemma 2.8 we have

$$\begin{aligned} (iP)_1(t, x, D_x) &= \partial_t + \sum_{j=0}^p (e^{\lambda_{p-1}(x, D_x)})^{-1} ia_j e^{\lambda_{p-1}(x, D_x)} = \\ &= \partial_t + \sum_{j=0}^p e^{-\lambda_{p-1}(x, D_x)} ia_j e^{\lambda_{p-1}(x, D_x)} + \sum_{j=0}^p e^{-\lambda_{p-1}(x, D_x)} iR_{p-1} a_j e^{\lambda_{p-1}(x, D_x)} \\ &= \partial_t + e^{-\lambda_{p-1}(x, D_x)} ia_p e^{\lambda_{p-1}(x, D_x)} + \sum_{j=1}^{p-1} e^{-\lambda_{p-1}(x, D_x)} i(a_j + R_{p-1} a_{j+1}) e^{\lambda_{p-1}(x, D_x)} + s_0 \\ (3.2) \quad &= \partial_t + e^{-\lambda_{p-1}(x, D_x)} ia_p e^{\lambda_{p-1}(x, D_x)} + \sum_{j=1}^{p-1} e^{-\lambda_{p-1}(x, D_x)} i\tilde{a}_j e^{\lambda_{p-1}(x, D_x)} + s_0 \end{aligned}$$

for some  $s_0(t, x, D_x)$  of order  $(0, 0)$ , and with new operators  $\tilde{a}_j = a_j + R_{p-1} a_{j+1}$  having symbol  $\tilde{a}_j(t, x, \xi) \in SG^{j,-j/(p-1)}$ . We now apply formula (3.1), observing that in the case  $i = p-1$ , the term  $r$  vanishes for  $|\xi| \geq hR$  since it is a sum of products with at least one  $\xi$ -derivative of  $\lambda_{p-1}$  appearing in each factor, see (2.18). Then in particular, the term  $r$  has order  $(0, 0)$  since it is compactly supported in  $\xi$  and with order 0 in  $x$ . Hence we obtain for the related operators:

$$\begin{aligned} (3.3) \quad (iP)_1 &= \partial_t + ia_p + \sum_{\alpha=1}^{p-1} \frac{1}{\alpha!} \partial_\xi^\alpha ia_p \cdot e^{-\lambda_{p-1}} \cdot D_x^\alpha e^{\lambda_{p-1}} \\ &\quad + \sum_{j=1}^{p-1} \left( i\tilde{a}_j + \sum_{\alpha=1}^{j-1} \frac{1}{\alpha!} \partial_\xi^\alpha i\tilde{a}_j \cdot e^{-\lambda_{p-1}} \cdot D_x^\alpha e^{\lambda_{p-1}} \right) + s_0 \\ &= \partial_t + ia_p + \partial_\xi a_p \partial_x \lambda_{p-1} + i\tilde{a}_{p-1} + \sum_{\alpha=2}^{p-1} \frac{1}{\alpha!} \partial_\xi^\alpha ia_p \cdot e^{-\lambda_{p-1}} \cdot D_x^\alpha e^{\lambda_{p-1}} + \sum_{j=1}^{p-2} i\tilde{a}_j \\ &\quad + \sum_{j=1}^{p-1} \sum_{\alpha=1}^{j-1} \frac{1}{\alpha!} \partial_\xi^\alpha i\tilde{a}_j \cdot e^{-\lambda_{p-1}} \cdot D_x^\alpha e^{\lambda_{p-1}} + \tilde{s}_0 \end{aligned}$$

for a term  $\tilde{s}_0$  with symbol in  $SG^{0,0}$ . Now we consider the real part of the terms of order  $p-1$  with respect to  $\xi$  in (3.3), and use (1.15), (1.16), (2.15), (2.20) for  $|\xi| \geq hR$  to get:

$$\begin{aligned} \operatorname{Re}(i\tilde{a}_{p-1} + \partial_\xi a_p \partial_x \lambda_{p-1}) &= -\operatorname{Im} \tilde{a}_{p-1} + \partial_\xi a_p \partial_x \lambda_{p-1} \\ &= -\operatorname{Im} a_{p-1} + M_{p-1} |\partial_\xi a_p(t, \xi)| \langle x \rangle^{-1} \\ &\geq (2^{-(p-1)/2} C_p M_{p-1} - C) \langle \xi \rangle_h^{p-1} \langle x \rangle^{-1}, \end{aligned}$$

where we used that  $\tilde{a}_j = a_j$  for  $|\xi| > hR$ , and we used also the inequality:  $\langle a \rangle_h \leq \sqrt{2}|a|$  for  $a \in \mathbb{R}$ ,  $|a| \geq h$ . We thus obtain

$$\operatorname{Re}(i\tilde{a}_{p-1} + \partial_\xi a_p \partial_x \lambda_1) \geq 0$$

if we choose the constant  $M_{p-1}$  large enough:

$$(3.4) \quad M_{p-1} \geq 2^{(p-1)/2} C / C_p.$$

We can then apply Theorem 2.3 to the operator  $i\tilde{a}_{p-1} + \partial_\xi a_p \partial_x \lambda_{p-1}$  and obtain that there exist pseudo-differential operators  $Q_{p-1}(t, x, D_x)$ ,  $\tilde{R}_{p-2}(t, x, D_x)$  and  $R_0$  with symbols  $Q_{p-1}(t, x, \xi) \in SG^{p-1, -1}$ ,  $\tilde{R}_{p-2}(t, x, \xi) \in SG^{p-2, -1}$ ,  $R_0(t, x, \xi) \in S^0$  such that

$$\operatorname{Re} \langle Q_{p-1}(t, x, D_x) v, v \rangle \geq 0, \quad \forall v \in \mathcal{S}(\mathbb{R})$$

and

$$i\tilde{a}_{p-1} + \partial_\xi a_p \partial_x \lambda_{p-1} = Q_{p-1} + \tilde{R}_{p-2} + R_0.$$

Finally we estimate the last three terms in the right-hand side of (3.3). We observe that for  $2 \leq \alpha \leq p-1$  we have

$$(3.5) \quad |\partial_\xi^\alpha a_p \cdot e^{-\lambda_{p-1}} \cdot D_x^\alpha e^{\lambda_{p-1}}| \leq C'_p \langle \xi \rangle_h^{p-\alpha} \langle x \rangle^{-1-\alpha} \leq C'_p \langle \xi \rangle_h^{p-2} \langle x \rangle^{-\frac{p-2}{p-1}},$$

since  $\alpha + 1 \geq 3 > (p-2)/(p-1)$ . Similarly we can estimate the derivatives of the above symbol observing that the worst case occurs when  $\xi$ -derivatives of order  $\gamma$  fall on the term  $e^{-\lambda_{p-1}}$ . By (2.27) this produces a term of type  $(\ln \langle x \rangle)^\gamma$  in the estimates but since

$$(\ln \langle x \rangle)^\gamma \langle x \rangle^{-1-\alpha} \leq (\ln \langle x \rangle)^\gamma \langle x \rangle^{-3} \leq C \langle x \rangle^{-\frac{p-2}{p-1}}$$

for every  $\gamma > 0$ , we conclude that  $\partial_\xi^\alpha a_p \cdot e^{-\lambda_{p-1}} \cdot D_x^\alpha e^{\lambda_{p-1}} \in SG^{p-2, -(p-2)/(p-1)}$ . By similar arguments we obtain for  $1 \leq \alpha \leq j-1$  that  $\partial_\xi^\alpha \tilde{a}_j \cdot e^{-\lambda_{p-1}} \cdot D_x^\alpha e^{\lambda_{p-1}} \in SG^{j-1, -(j-1)/(p-1)}$ . Hence, we can gather the last three terms in the right-hand side of (3.3) and the remainder term  $\tilde{R}_{p-2}$  obtained from the application of Theorem 2.3 and conclude that

$$(iP)_1 = \partial_t + ia_p(t, D_x) + Q_{p-1}(t, x, D_x) + \sum_{j=1}^{p-2} ia_{j,1}(t, x, D_x) + r_1(t, x, D)$$

for some  $a_{j,1} \in SG^{j, -j/(p-1)}$  and  $r_1 \in S^0$ . Lemma 3.4 is proved.  $\square$

To perform the second transformation, we need to compute the operator

$$(iP)_2 = (e^{\lambda_{p-2}(x, D_x)})^{-1} (iP)_1 e^{\lambda_{p-2}(x, D_x)}.$$

By Lemma 2.9, for  $h \geq h_2$  there exists an operator  $R_{p-2}$  with principal symbol

$$r_{p-2, -2}(x, \xi) = \partial_\xi \lambda_{p-2}(x, \xi) D_x \lambda_{p-2}(x, \xi) \in SG^{-2, -(p-2)/(p-1)}$$

such that:

$$(iP)_2 = e^{-\lambda_{p-2}(x, D_x)} (I + R_{p-2}) (iP)_1 e^{\lambda_{p-2}(x, D_x)}.$$

We have the following result.

**Lemma 3.5.** *Let  $h_1, h_2$  be as in Lemmas 2.8 and 2.9 and let  $h \geq \max\{h_1, h_2\}$ . Then there exist pseudo-differential operators  $Q_{p-2}(t, x, D_x)$ ,  $a_{j,2}(t, x, D_x)$  and  $r_2(t, x, D_x)$  with symbols  $Q_{p-2}(t, x, \xi) \in SG^{p-2,0}$ ,  $a_{j,2}(t, x, \xi) \in SG^{j-j/(p-1)}$  for  $1 \leq j \leq p-3$ ,  $r_2(t, x, \xi) \in S^0$  such that:*

$$(iP)_2 = \partial_t + ia_p(t, D_x) + Q_{p-1}(t, x, D_x) + Q_{p-2}(t, x, D_x) + \sum_{j=1}^{p-3} ia_{j,2}(t, x, D_x) + r_2(t, x, D_x),$$

and

$$\operatorname{Re}\langle Q_{p-2}(t, x, D_x)v, v \rangle \geq 0, \quad \forall v \in \mathcal{S}(\mathbb{R}).$$

*Proof.* By Lemma 3.4 we have (omitting the notation  $(t, x, D_x)$ ):

$$(3.6) \quad (iP)_2 = \partial_t + e^{-\lambda_{p-2}} \left( ia_p + Q_{p-1} + \sum_{j=1}^{p-2} ia_{j,1} + r_1 \right) e^{\lambda_{p-2}} \\ + e^{-\lambda_{p-2}} \left( iR_{p-2}a_p + R_{p-2}Q_{p-1} + \sum_{j=1}^{p-2} iR_{p-2}a_{j,1} + R_{p-2}r_1 \right) e^{\lambda_{p-2}},$$

where  $a_{j,1}(t, x, D)$  have symbols in  $SG^{j-j/(p-1)}$ , and  $r_1$  has a symbol in  $S^0$ . Now we observe that  $iR_{p-2}a_p \in SG^{p-2, -(p-2)/(p-1)}$ ,  $R_{p-2}Q_{p-1} \in SG^{p-3, -(p-3)/(p-1)}$  and  $R_{p-2}ia_{j,1} \in SG^{j-2, -(j-2)/(p-1)}$  for every  $j = 1, \dots, p-2$ . Then we can write (3.6) as follows:

$$(iP)_2 = \partial_t + e^{-\lambda_{p-2}(x, D_x)} \left( ia_p + Q_{p-1} + \sum_{j=1}^{p-2} i\tilde{a}_{j,1} \right) e^{\lambda_{p-2}(x, D_x)} + s_0$$

for some  $s_0$  with symbol in  $S^0$  and with new symbols  $\tilde{a}_{j,1} \in SG^{j-j/(p-1)}$ . Here we have considered the composition  $e^{\lambda_{p-2}}(R_{p-2}r_1)e^{\lambda_{p-2}}$  using Remark 3.3. It is important to underline that the operator  $i\tilde{a}_{p-2,1}$  has the symbol:

$$(3.7) \quad i\tilde{a}_{p-2,1}(t, x, \xi) = ia_{p-2,1}(t, x, \xi) + ia_p(t, \xi)\partial_\xi \lambda_{p-2}(t, x, \xi) D_x \lambda_{p-2}(t, x, \xi),$$

where the first term in the right-hand side depends only on the constant  $M_{p-1}$  and the second only on  $M_{p-2}$  which will be chosen later on in this second transformation. But now, by formula

(3.1), we get:

$$\begin{aligned}
(iP)_2 &= \partial_t + ia_p + Q_{p-1} + i\tilde{a}_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2} + \sum_{j=1}^{p-3} i\tilde{a}_{j,1} \\
&+ \sum_{\alpha=2}^{p-1} \frac{1}{\alpha!} \partial_\xi^\alpha ia_p \cdot e^{-\lambda_{p-2}} \cdot D_x^\alpha e^{\lambda_{p-2}} - ia_p \partial_\xi \lambda_{p-2} D_x \lambda_{p-2} + \sum_{\gamma=2}^{p-1} \frac{1}{\gamma!} \partial_\xi^\gamma e^{-\lambda_{p-2}} \cdot (ia_p D_x^\gamma e^{\lambda_{p-2}}) \\
&+ \sum_{\gamma=1}^{p-2} \sum_{\alpha=1}^{p-1} \frac{1}{\alpha! \gamma!} \partial_\xi^\gamma e^{-\lambda_{p-2}} D_x^\gamma (\partial_\xi^\alpha a_p D_x^\alpha e^{\lambda_{p-2}}) \\
&+ \sum_{\alpha=1}^{p-2} \frac{1}{\alpha!} \partial_\xi^\alpha Q_{p-1} \cdot e^{-\lambda_{p-2}} \cdot D_x^\alpha e^{\lambda_{p-2}} + \sum_{j=1}^{p-2} \sum_{\alpha=1}^{j-1} \frac{1}{\alpha!} \partial_\xi^\alpha i\tilde{a}_{j,1} \cdot e^{-\lambda_{p-2}} \cdot D_x^\alpha e^{\lambda_{p-2}} \\
&+ \sum_{\gamma=1}^{p-2} \frac{1}{\gamma!} \partial_\xi^\gamma e^{-\lambda_{p-2}} D_x^\gamma (Q_{p-1} e^{\lambda_{p-2}}) + \sum_{\alpha=1}^{p-2} \sum_{\gamma=1}^{p-3} \frac{1}{\alpha! \gamma!} \partial_\xi^\gamma e^{-\lambda_{p-2}} D_x^\gamma (\partial_\xi^\alpha Q_{p-1} D_x^\alpha e^{\lambda_{p-2}}) \\
&+ \sum_{j=1}^{p-2} \sum_{\gamma=1}^{j-1} \frac{1}{\gamma!} \partial_\xi^\gamma e^{-\lambda_{p-2}} D_x^\gamma (i\tilde{a}_{j,1} e^{\lambda_{p-2}}) + \sum_{j=1}^{p-2} \sum_{\alpha=1}^{j-1} \sum_{\gamma=1}^{j-2} \frac{1}{\alpha! \gamma!} \partial_\xi^\gamma e^{-\lambda_{p-2}} D_x^\gamma (\partial_\xi^\alpha (i\tilde{a}_{j,1}) D_x^\alpha e^{\lambda_{p-2}}) \\
&+ s_0.
\end{aligned}$$

By estimating as before the orders of the terms appearing in the above formula, using (2.23) and by (3.7) we obtain that

$$(3.8) \quad (iP)_2 = \partial_t + ia_p + Q_{p-1} + \partial_\xi a_p \partial_x \lambda_{p-2} + ia_{p-2,1} + \sum_{j=1}^{p-3} i\tilde{a}_{j,1} + \tilde{s}_0,$$

for some operators  $\tilde{a}_{j,1}(t, x, D)$  with symbols  $\tilde{a}_{j,1}(t, x, \xi) \in SG^{j, -j/(p-1)}$ , and with a term  $\tilde{s}_0$  with symbol in  $S^0$ . Now we want to apply Theorem 2.3 to the operator  $ia_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2}$ , namely to the term of order  $p-2$  with respect to  $\xi$ . We observe that for  $|\xi| \geq hR$  we have:

$$\begin{aligned}
\operatorname{Re}(ia_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2}) &= -\operatorname{Im}a_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2} \\
&= -\operatorname{Im}a_{p-2,1} + M_{p-2} \partial_\xi a_p \omega \left( \frac{\xi}{h} \right) \langle x \rangle^{-\frac{p-2}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) \langle \xi \rangle_h^{-1} \\
&= -\operatorname{Im}a_{p-2,1} + M_{p-2} |\partial_\xi a_p(t, \xi)| \langle \xi \rangle_h^{-1} \langle x \rangle^{-\frac{p-2}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) \\
&\geq -C \langle \xi \rangle_h^{p-2} \langle x \rangle^{-\frac{p-2}{p-1}} + M_{p-2} C_p |\xi|^{p-1} \langle \xi \rangle_h^{-1} \langle x \rangle^{-\frac{p-2}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) \\
&\geq -C \langle \xi \rangle_h^{p-2} \langle x \rangle^{-\frac{p-2}{p-1}} + M_{p-2} C_p 2^{-(p-1)/2} \langle \xi \rangle_h^{p-2} \langle x \rangle^{-\frac{p-2}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) \\
&= (-C + M_{p-2} C_p 2^{-(p-1)/2}) \langle \xi \rangle_h^{p-2} \langle x \rangle^{-\frac{p-2}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) \\
&\quad - C \langle \xi \rangle_h^{p-2} \langle x \rangle^{-\frac{p-2}{p-1}} \left( 1 - \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) \right) \\
&\geq (-C + M_{p-2} C_p 2^{-(p-1)/2}) \langle \xi \rangle_h^{p-2} \langle x \rangle^{-\frac{p-2}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) - C',
\end{aligned}$$

where  $C = C(M_{p-1})$  is a constant depending on the already chosen  $M_{p-1} > 0$  and, in the last inequality we used the fact that on the support of  $1 - \psi\left(\frac{\langle x \rangle}{\langle \xi \rangle^{p-1}}\right)$  we have  $\langle \xi \rangle_h^{p-2} \langle x \rangle^{-\frac{p-2}{p-1}} \leq C'$ . Choosing the constant  $M_{p-2} \geq C(M_{p-1})2^{(p-1)/2}/C_p$ , we obtain that

$$\operatorname{Re}(ia_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2}) \geq -C' \quad \text{for } |\xi| \geq hR.$$

Then we can apply Theorem 2.3 to the symbol  $ia_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2} + C' \in SG^{p-2,0}$ . There exist operators  $Q_{p-2}(t, x, D)$ ,  $\tilde{R}_{p-3}$ ,  $R_0$  such that

$$\begin{aligned} ia_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2} + C' &= Q_{p-2} + \tilde{R}_{p-3} + R_0, \\ \operatorname{Re}\langle Q_{p-2}v, v \rangle &\geq 0 \quad \forall v \in \mathcal{S}(\mathbb{R}). \end{aligned}$$

It is now crucial to observe that  $Q_{p-2}(t, x, \xi) \in SG^{p-3,0}$ ,  $R_0 \in S^0$  whereas by (2.6) we have  $\tilde{R}_{p-3} \in SG^{p-3, -(p-3)/(p-1)}$  since the constant  $C'$  does not appear in the expression of the symbol. As a matter of fact we have

$$\tilde{r}_{p-3}(x, \xi) = \psi_1(\xi)D_x(ia_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2}) + \sum_{|\alpha+\beta| \geq 2} \psi_{\alpha,\beta}(\xi) \partial_\xi^\alpha D_x^\beta (ia_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2}),$$

which is in  $SG^{p-3, -(p-3)/(p-1)}$ . The proposition is proved.  $\square$

We now prove by induction on  $n \geq 2$  the following:

**Lemma 3.6.** *Given  $n \in \mathbb{N}$ ,  $n \geq 2$ , we can find a constant  $h_n \geq 1$  such that for  $h \geq h_n$  there exist pseudo-differential operators  $Q_{p-\ell}(t, x, D_x)$ ,  $1 \leq \ell \leq n$ ,  $a_{j,n}(t, x, D_x)$ ,  $1 \leq j \leq p-n-1$ , and  $r_n(t, x, D_x)$  with symbols  $Q_{p-\ell}(t, x, \xi) \in SG^{p-\ell,0}$ ,  $a_{j,n}(t, x, \xi) \in SG^{j,-j/(p-1)}$ ,  $r_n(t, x, \xi) \in S^0$  such that the operator*

$$(iP)_n =: (e^{\lambda_{p-n}(x, D_x)})^{-1} \dots (e^{\lambda_{p-2}(x, D_x)})^{-1} (e^{\lambda_{p-1}(x, D_x)})^{-1} (iP) e^{\lambda_{p-1}(x, D_x)} e^{\lambda_{p-2}(x, D_x)} \dots e^{\lambda_{p-n}(x, D_x)}$$

can be written in the form:

$$(iP)_n = \partial_t + ia_p(t, D_x) + \sum_{\ell=1}^n Q_{p-\ell}(t, x, D_x) + \sum_{j=1}^{p-n-1} ia_{j,n}(t, x, D_x) + r_n(t, x, D_x),$$

and

$$(3.9) \quad \operatorname{Re}\langle Q_{p-\ell}(t, x, D_x)v, v \rangle \geq 0, \quad \forall v \in \mathcal{S}(\mathbb{R}), \quad 1 \leq \ell \leq n.$$

*Proof.* For  $n = 2$  this is exactly the statement of Lemma 3.5. Let us suppose that for  $h \geq h_{n-1}$  it holds

$$(3.10) \quad (iP)_{n-1} = \partial_t + ia_p(t, D_x) + \sum_{\ell=1}^{n-1} Q_{p-\ell}(t, x, D_x) + \sum_{j=1}^{p-n} ia_{j,n-1}(t, x, D_x) + r_{n-1}(t, x, D_x),$$

for some pseudo-differential operators  $Q_{p-\ell}(t, x, D_x)$ ,  $1 \leq \ell \leq n-1$ ,  $a_{j,n-1}(t, x, D_x)$ ,  $1 \leq j \leq p-n$ , and  $r_{n-1}(t, x, D_x)$  with symbols  $Q_{p-\ell}(t, x, \xi) \in SG^{p-\ell,0}$ ,  $a_{j,n-1}(t, x, \xi) \in SG^{j,-j/(p-1)}$ ,  $r_{n-1}(t, x, \xi) \in S^0$  such that

$$\operatorname{Re}\langle Q_{p-\ell}(t, x, D_x)v, v \rangle \geq 0, \quad \forall v \in \mathcal{S}(\mathbb{R}), \quad 1 \leq \ell \leq n-1,$$

and consider the operator

$$(iP)_n = (e^{\lambda_{p-n}(x, D_x)})^{-1} (iP)_{n-1} e^{\lambda_{p-n}(x, D_x)}.$$

Lemma 2.9 gives, for  $h$  large enough, say  $h \geq \tilde{h}_n$ , that

$$(iP)_n = e^{-\lambda_{p-n}(x, D_x)} (I + R_{p-n}) (iP)_{n-1} e^{\lambda_{p-n}(x, D_x)},$$



with a pseudo-differential operator  $R_{p-n}$  having the principal symbol

$$r_{p-n,-n}(x, \xi) = \partial_\xi \lambda_{p-n}(x, \xi) D_x \lambda_{p-n}(x, \xi) \in SG^{-n, -(p-n)/(p-1)}.$$

Thus, for  $h \geq h_n = \max\{\tilde{h}_n, h_{n-1}\}$  and by (3.10) we have (omitting  $(t, x, D_x)$  in the notation):

$$\begin{aligned} (iP)_n &= \partial_t + e^{-\lambda_{p-n}} \left( ia_p + \sum_{\ell=1}^{n-1} Q_{p-\ell} + \sum_{j=1}^{p-n} ia_{j,n-1} + r_{n-1} \right) e^{\lambda_{p-n}} \\ &\quad + e^{-\lambda_{p-n}} \left( iR_{p-n}a_p + \sum_{\ell=1}^{n-1} R_{p-n}Q_{p-\ell} + \sum_{j=1}^{p-n} iR_{p-n}a_{j,n-1} + R_{p-n}r_{n-1} \right) e^{\lambda_{p-n}}. \end{aligned}$$

Now we notice that  $R_{p-n}Q_{p-\ell} \in SG^{p-n-\ell, -(p-n)/(p-1)} \subset SG^{p-n, -(p-n)/(p-1)}$  since  $1 \leq \ell \leq n-1$ ,  $iR_{p-n}a_p \in SG^{p-n, -(p-n)/(p-1)}$ , and  $R_{p-n}ia_{j,n-1} \in SG^{j-n, -(p-n)/(p-1)} \subset SG^{j-n, -(j-n)/(p-1)}$ ,  $j = 1, \dots, p-n$ . So we can write  $(iP)_n$  as follows:

$$(iP)_n = \partial_t + e^{-\lambda_{p-n}(x, D_x)} \left( ia_p + \sum_{\ell=1}^{n-1} Q_{p-\ell} + \sum_{j=1}^{p-n} i\tilde{a}_{j,n-1} \right) e^{\lambda_{p-n}(x, D_x)} + s_0$$

for some  $s_0$  with symbol in  $S^0$ , and with new symbols  $\tilde{a}_{j,n-1} \in SG^{j, -j/(p-1)}$ . Again, we need to notice that at level  $p-n$  we get:

$$i\tilde{a}_{p-n,n-1}(t, x, \xi) = ia_{p-n,n-1}(t, x, \xi) + ia_p(t, \xi) \partial_\xi \lambda_{p-n}(t, x, \xi) D_x \lambda_{p-n}(t, x, \xi)$$

where the first term in the right-hand side depends on the constants  $M_{p-1}, \dots, M_{p-n+1}$  chosen before and the second one depends only on the constant  $M_{p-n}$  which will be chosen later on; this does not cause any trouble in the choice of the constant  $M_{p-n}$  because, again by (3.1), we get:

$$\begin{aligned} (iP)_n &= \partial_t + ia_p + \sum_{\ell=1}^{n-1} Q_{p-\ell} + i\tilde{a}_{p-n,n-1} + \partial_\xi a_p \partial_x \lambda_{p-n} + \sum_{j=1}^{p-n-1} i\tilde{a}_{j,n-1} \\ &\quad + \sum_{\alpha=2}^{p-n} \frac{1}{\alpha!} \partial_\xi^\alpha ia_p \cdot e^{-\lambda_{p-n}} \cdot D_x^\alpha e^{\lambda_{p-n}} - ia_p \partial_\xi \lambda_{p-n} D_x \lambda_{p-n} \\ &\quad + \sum_{\gamma=2}^{p-n} \frac{1}{\gamma!} \partial_\xi^\gamma e^{-\lambda_{p-n}} \cdot (ia_p D_x^\gamma e^{\lambda_{p-n}}) + \sum_{\gamma=1}^{p-n} \sum_{\alpha=1}^{p-n+1} c_{\alpha,\gamma} \partial_\xi^\alpha e^{-\lambda_{p-n}} \partial_\xi^\gamma ia_p D_x^{\alpha+\gamma} e^{\lambda_{p-n}} \\ &\quad + \sum_{\ell=1}^{n-1} \sum_{\alpha=1}^{p-\ell-n} \frac{1}{\alpha!} \partial_\xi^\alpha Q_{p-\ell} \cdot e^{-\lambda_{p-n}} \cdot D_x^\alpha e^{\lambda_{p-n}} + \sum_{j=1}^{p-n} \sum_{\alpha=1}^{j-1} \frac{1}{\alpha!} \partial_\xi^\alpha i\tilde{a}_{j,n-1} \cdot e^{-\lambda_{p-n}} \cdot D_x^\alpha e^{\lambda_{p-n}} \\ &\quad + \sum_{\ell=1}^{n-1} \sum_{\gamma=1}^{p-\ell} \frac{1}{\gamma!} \partial_\xi^\gamma e^{-\lambda_{p-n}} D_x^\gamma (Q_{p-\ell} e^{\lambda_{p-n}}) + \sum_{\ell=1}^{n-1} \sum_{\alpha=1}^{p-n} \sum_{\gamma=1}^{p-\ell-1} \frac{1}{\alpha! \gamma!} \partial_\xi^\alpha e^{-\lambda_{p-n}} D_x^\gamma (\partial_\xi^\alpha Q_{p-\ell} D_x^\alpha e^{\lambda_{p-n}}) \\ &\quad + \sum_{j=1}^{p-n} \sum_{\gamma=1}^{j-1} \frac{1}{\gamma!} \partial_\xi^\gamma e^{-\lambda_{p-n}} D_x^\gamma (i\tilde{a}_{j,n-1} e^{\lambda_{p-n}}) + \sum_{j=1}^{p-n} \sum_{\alpha=1}^{j-1} \sum_{\gamma=1}^{j-2} \frac{1}{\gamma!} \partial_\xi^\alpha e^{-\lambda_{p-n}} D_x^\gamma (\partial_\xi^\alpha (i\tilde{a}_{j,n-1}) D_x^\alpha e^{\lambda_{p-n}}) \\ &\quad + \tilde{t}_0 \\ (3.11) &= \partial_t + ia_p + \sum_{\ell=1}^{n-1} Q_{p-\ell} + (\partial_\xi a_p \partial_x \lambda_{p-n} + ia_{p-n,n-1}) + \sum_{j=1}^{p-n-1} i\tilde{a}_{j,n-1} + \tilde{s}_0, \end{aligned}$$

for some operators  $\tilde{a}_{j,n-1}(t, x, D_x)$  with symbols  $\tilde{a}_{j,n-1}(t, x, \xi) \in SG^{j,-j/(p-1)}$ ,  $1 \leq j \leq p-n-1$ , depending on  $M_{p-1}, \dots, M_{p-n+1}$  and where  $\tilde{s}_0$  is a term containing operators with symbol in  $S^0$ .

As done in the second transformation, we now look at the real part of the terms of order  $p-n$  with respect to  $\xi$  in (3.11); for  $|\xi| \geq hR$  we have by (1.16),(2.14) and for a positive constant  $C = C(M_{p-1}, \dots, M_{p-n+1})$ :

$$\begin{aligned} \operatorname{Re}(ia_{p-n,n-1} + \partial_\xi a_p \partial_x \lambda_{p-n}) &= -\operatorname{Im}a_{p-n,n-1} + M_{p-n} \partial_\xi a_p \omega \left( \frac{\xi}{h} \right) \langle x \rangle^{-\frac{p-n}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) \langle \xi \rangle_h^{-n+1} \\ &\geq -C \langle \xi \rangle_h^{p-n} \langle x \rangle^{-\frac{p-n}{p-1}} + M_{p-n} C_p |\xi|^{p-1} \langle \xi \rangle_h^{-n+1} \langle x \rangle^{-\frac{p-n}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) \\ &\geq (-C + M_{p-n} C_p 2^{-(p-1)/2}) \langle \xi \rangle_h^{p-n} \langle x \rangle^{-\frac{p-n}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) - C' \\ &\geq -C' \end{aligned}$$

if we choose  $M_{p-n} \geq C(M_{p-1}, \dots, M_{p-n+1}) 2^{(p-1)/2} / C_p$ . An application of Theorem 2.3 to the symbol  $ia_{p-n,n-1} + \partial_\xi a_p \partial_x \lambda_{p-n} + C' \in SG^{p-n,0}$  gives then the existence of operators  $Q_{p-n}(t, x, D)$ ,  $\tilde{R}_{p-n-1}(t, x, D)$ ,  $R_0(t, x, D)$  with symbols respectively  $Q_{p-n}(t, x, \xi) \in SG^{p-n,0}$ ,  $\tilde{R}_{p-n-1} \in SG^{p-n-1, -(p-n)/(p-1)}$ ,  $R_0 \in S^0$  such that

$$\begin{aligned} ia_{p-n,n-1} + \partial_\xi a_p \partial_x \lambda_{p-n} + C' &= Q_{p-n} + \tilde{R}_{p-n-1} + R_0, \\ \operatorname{Re}\langle Q_{p-n} v, v \rangle &\geq 0 \quad \forall v \in \mathcal{S}(\mathbb{R}). \end{aligned}$$

Then, from (3.11) we finally obtain

$$\begin{aligned} (iP)_n &= \partial_t + ia_p + \sum_{\ell=1}^{n-1} Q_{p-\ell} + Q_{p-n} + \tilde{R}_{p-n-1} + \sum_{j=1}^{p-n-1} i\tilde{a}_{j,n-1} + s_0 + R_0 \\ &= \partial_t + ia_p + \sum_{\ell=1}^n Q_{p-\ell} + \sum_{j=1}^{p-n-1} ia_{j,n} + r_n \end{aligned}$$

for some operators  $a_{j,n}$ ,  $r_n$  with symbols  $a_{j,n} \in SG^{j,-j/(p-1)}$ ,  $r_n \in S^0$  and where  $Q_{p-\ell}$  satisfy (3.9) for  $1 \leq \ell \leq n$ . The lemma is proved.  $\square$

Proposition 3.1 follows directly from Lemmas 3.4, 3.5 and 3.6.

*Proof of Theorem 1.2.* By the change of variable (2.12) the Cauchy problem (1.1) in the unknown  $u(t, x)$  is reduced to the Cauchy problem (2.9) for the unknown  $u_\lambda(t, x)$ , where  $P_\lambda$  is defined by (2.10) and  $f_\lambda$  and  $g_\lambda$  are defined by (2.11). Now we apply Proposition 3.1 to derive an energy estimate for the solution to the Cauchy problem (2.9). For every  $v \in C^1([0, T], \mathcal{S}(\mathbb{R}))$  we have

$$\begin{aligned} \frac{d}{dt} \|v\|_0^2 &= 2 \operatorname{Re}\langle \partial_t v, v \rangle \\ &= 2 \operatorname{Re}\langle (iP_\lambda)v, v \rangle - 2 \sum_{\ell=1}^{p-1} \operatorname{Re}\langle Q_{p-\ell} v, v \rangle - 2 \operatorname{Re}\langle r_0 v, v \rangle \\ &\leq C(\|P_\lambda v\|_{L^2}^2 + \|v\|_{L^2}^2). \end{aligned}$$

By standard arguments from the energy method we deduce that, for all  $s = (s_1, s_2) \in \mathbb{R}^2$  and every  $v \in C^1([0, T], \mathcal{S}(\mathbb{R}))$  the following estimate holds

$$(3.12) \quad \|v(t, \cdot)\|_{s_1, s_2}^2 \leq c' \left( \|v(0, \cdot)\|_{s_1, s_2}^2 + \int_0^t \|P_\lambda v(\tau, \cdot)\|_{s_1, s_2}^2 d\tau \right) \quad \forall t \in [0, T],$$

for some  $c' > 0$ . The energy estimate (3.12) can be extended by a density argument to a function  $v \in C^1([0, T], H_{s_1, s_2}(\mathbb{R}))$  for every  $s_1, s_2 \in \mathbb{R}$ . This implies that if  $f_\lambda \in C([0, T], H_{s_1, s_2}(\mathbb{R}))$ ,  $g_\lambda \in H_{s_1, s_2}(\mathbb{R})$ , then the Cauchy problem (2.9) has a unique solution  $u_\lambda(t, x) \in C^1([0, T], H_{s_1, s_2}(\mathbb{R})) \cap C([0, T], H_{s_1+p, s_2}(\mathbb{R}))$  satisfying (3.12). By the relation (2.12) between  $u$  and  $u_\lambda$  and by Lemma 2.7, we obtain existence and uniqueness of a solution  $u$  of (1.1). Moreover, from (2.11), (2.12) and (3.12) we get

$$(3.13) \quad \|u\|_{s_1, s_2 - 2M_{p-1}}^2 \leq c_1 \|u_\lambda\|_{s_1, s_2 - M_{p-1}}^2 \leq c_2 \left( \|g_\lambda\|_{s_1, s_2 - M_{p-1}}^2 + \int_0^t \|f_\lambda\|_{s_1, s_2 - M_{p-1}}^2 d\tau \right) \\ \leq c_3 \left( \|g\|_{s_1, s_2}^2 + \int_0^t \|f\|_{s_1, s_2}^2 d\tau \right)$$

for some  $c_1, c_2, c_3 > 0$ . This gives the energy estimate (1.17), and proves well posedness in  $\mathcal{S}, \mathcal{S}'$  of the Cauchy problem (1.1).  $\square$

**Remark 3.7.** As outlined in the Introduction, Theorem 1.2 can be proved replacing the assumption (1.16) by the weaker condition (1.19) and repeating readily the argument of the proof above. For the sake of brevity we leave the details to the reader. We limit ourselves to observe that the assumption (1.19) is sufficient for the application of Theorem 2.3 in the proofs of Lemmas 3.4, 3.5, 3.6 where only the imaginary parts of the symbols  $a_j(t, x, \xi)$  are involved and that every step can be performed identically with the only difference that the symbols  $a_{j, \ell}, \ell = 1, \dots, p-1$  appearing in the above Lemmas are now such that  $\text{Re } a_{j, \ell} \in SG^{j, 0}$  and  $\text{Im } a_{j, \ell} \in SG^{j, -j/(p-1)}$ . Nevertheless this is sufficient to obtain the assertion of Proposition 3.1.

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