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This is the author's manuscript

Original Citation:

Availability:

This version is available http://hdl.handle.net/2318/148821 since 2016-01-18T19:13:11Z

Published version:

DOI:10.1017/S0960129513000339

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12 October 2024

arXiv:1204.5338v2 [math.LO] 11 Oct 2012

Under consideration for publication in Math. Struct. in Comp. Science

Wadge-like reducibilities on arbitrary quasi-Polish spaces

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Received 12 October 2012

The structure of the Wadge degrees on zero-dimensional spaces is very simple (almost well-ordered), but for many other natural non-zero-dimensional spaces (including the space of reals) this structure is much more complicated. We consider weaker notions of reducibility, including the so-called $\mathbf{\Delta}^0_\alpha$-reductions, and try to find for various natural topological spaces X the least ordinal α_X such that for every $\alpha_X \leq \beta < \omega_1$ the degree-structure induced on X by the $\mathbf{\Delta}^0_\beta$-reductions is simple (i.e. similar to the Wadge hierarchy on the Baire space). We show that $\alpha_X \leq \omega$ for every quasi-Polish space X, that $\alpha_X \leq 3$ for quasi-Polish spaces of dimension $\neq \infty$, and that this last bound is in fact optimal for many (quasi-)Polish spaces, including the real line and its powers.

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† Victor Selivanov has been supported by a Marie Curie International Research Staff Exchange Scheme Fellowship within the 7th European Community Framework Programme and by DFG-RFBR Grant 436 RUS 113/1002/01.

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1. Introduction

A subset A of the Baire space $\mathcal{N} = \omega^\omega$ is *Wadge reducible* to a subset B if and only if $A = f^{-1}(B)$ for some continuous function $f\colon \mathcal{N} \to \mathcal{N}$. The structure of Wadge degrees (i.e. the quotient-structure of $(\mathscr{P}(\mathcal{N}), \leq_{\mathsf{W}})$) is fairly well understood and turns our to be rather simple. In particular, the structure $(\mathbf{B}(\mathcal{N}), \leq_{\mathsf{W}})$ of the Wadge degrees of Borel sets is semi-well-ordered by (Wadge 1984), i.e. it has no infinite descending chain and for every $A, B \in \mathbf{B}(\mathcal{N})$ we have $A \leq_{\mathsf{W}} B$ or $\mathcal{N} \setminus B \leq_{\mathsf{W}} A$, which implies that the antichains have size at most two. More generally: if all the Boolean combinations of sets in a pointclass $\mathbf{\Gamma} \subseteq \mathscr{P}(\mathcal{N})$ (closed under continuous preimages) are determined, then the Wadge structure restricted to $\mathbf{\Gamma}$ is semi-well-ordered. For example, under the Axiom of Projective Determinacy PD the structure of the Wadge degrees of projective sets is semi-well-ordered, and under the Axiom of Determinacy AD, the whole Wadge degree-structure remains semi-well-ordered.

The Wadge degree-structure refines the structure of levels (more precisely, of the Wadge complete sets in those levels) of several important hierarchies, like the stratification of the Borel sets in $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ sets, or the Hausdorff-Kuratowski difference hierarchies, and serves as a nice tool to measure the topological complexity of many problems of interest in descriptive set theory (DST) (Kechris 1995), automata theory (Perrin and Pin 2004; Selivanov 2008b), and computable analysis (CA) (Weihrauch 2000).

There are several reasons and several ways to generalize the Wadge reducibility $\leq_{\mathsf{W}}$ on the Baire space. For example, one can consider

(1) other natural classes of reducing functions in place of the continuous functions;
(2) more complicated topological spaces instead of $\mathcal{N}$ (the notion of Wadge reducibility makes sense for arbitrary topological spaces);
(3) reducibility between functions rather than reducibility between sets (the sets may be identified with their characteristic functions);
(4) more complicated reductions than the many-one reductions by continuous functions.

In any of the mentioned directions a certain progress has been achieved, although in many cases the situation typically becomes more complicated than in the classical case.

For what concerns the possibility of using other sets of functions as reducibilities between subsets of $\mathcal{N}$, in a series of papers, A. Andretta, D. A. Martin, and L. Motto Ros considered the degree-structures obtained by replacing continuous functions with one of the following classes:

(a) the class of Borel functions, i.e. of those $f\colon \mathcal{N} \to \mathcal{N}$ such that $f^{-1}(U)$ is Borel for every open (equivalently, Borel) set U (see (Andretta and Martin 2003));
(b) the class D_α of $\mathbf{\Delta}^0_\alpha$-functions (for $\alpha < \omega_1$), i.e. of those $f\colon \mathcal{N} \to \mathcal{N}$ such that $f^{-1}(D) \in$

$\mathbf{\Delta}^0_\alpha$ for every $D \in \mathbf{\Delta}^0_\alpha$ (see (Andretta 2006) for the case $\alpha = 2$, and (Motto Ros 2009) for the general case);

(c) for $\gamma < \omega_1$ an additively closed ordinal, the collection $\mathcal{B}_\gamma$ of all functions of Baire class $< \gamma$, i.e. of those $f\colon \mathcal{N} \to \mathcal{N}$ for which there is $\alpha < \gamma$ such that $f^{-1}(U) \in \mathbf{\Sigma}^0_\alpha$ for every open set U (see (Motto Ros 2010a))[†];

(d) the class of $\mathbf{\Sigma}^1_n$ functions (for $n \in \omega$), i.e. the class of those $f\colon \mathcal{N} \to \mathcal{N}$ such that $f^{-1}(U) \in \mathbf{\Sigma}^1_n$ for every open (equivalently, $\mathbf{\Sigma}^1_n$) set U (see (Motto Ros 2010b)).

It turns out that the degree-structures resulting from (a)–(c), as for the Wadge degrees case, are all semi-well-ordered when restricted to the class of Borel sets or, provided that all Boolean combinations of sets in $\mathbf{\Gamma}$ are determined, to any pointclass $\mathbf{\Gamma} \subseteq \mathscr{P}(\mathcal{N})$ closed under continuous preimages (hence, in particular, to the entire $\mathscr{P}(\mathcal{N})$ when AD is assumed),[‡] and that under the full AD also the degree-structures resulting from (d) are semi-well-ordered (on the entire $\mathscr{P}(\mathcal{N})$): thus, we obtain a series of natural classifications of subsets of the Baire space which are weaker than the Wadge one.

Concerning Polish spaces different from the Baire space, using the methods developed in (Wadge 1984) it is immediate to check that the structure of Wadge degrees on any zero-dimensional Polish space remains semi-well-ordered (this follows also from Proposition 5.4). On the other hand, P. Hertling showed in (Hertling 1996) that the Wadge hierarchy on the real line $\mathbb{R}$ is much more complicated than the structure of Wadge degrees on the Baire space. In particular, there are infinite antichains and infinite descending chains in the structure of Wadge degrees of $\mathbf{\Delta}^0_2$ sets. Recently, this result has been considerably strengthened in (Ikegami *et al.* 2012). Moreover, P. Schlicht also showed in (Schlicht 2012) that the structure of Wadge degrees on *any* non zero-dimensional metric space must contain infinite antichains, and V. Selivanov showed in (Selivanov 2005) that the Wadge hierarchy is more complicated also when considering other natural topological spaces (e.g. the so-called ω-algebraic domains).

As already noted, if one passes from continuous reductions between sets to continuous reductions between functions, the situation becomes much more intricate. Even when considering the simplest possible generalization, namely continuous reductions between partitions of the Baire space into $3 \leq k \in \omega$ subsets, the degree-structure obtained is rather complicated, e.g. there are antichains of arbitrarily large finite size. On the other hand, it is still a well-quasi-order (briefly, a wqo), i.e. it has neither infinite descending chains nor infinite antichains: hence it can still serve as a scale to measure the topological complexity of k-partitions of the Baire space — see the end of Subsection 2.6 and the references contained therein.

In the fourth direction (more complicated reductions), the so called Weihrauch reducibility became recently rather popular: it turns out to be very useful in characterizing the topological complexity of some important computational problems, and also in understanding the computational content of some important classical mathematical theorems

[†] Notice that we cannot take a single level of the Baire stratification because in general it is not closed under composition, and hence does not give a preorder when used as reducibility between sets of reals.

[‡] In fact, for the cases of Borel functions and $\mathbf{\Delta}^0_\alpha$-functions, the corresponding degree-structures are even isomorphic to the Wadge one.

— see e.g. (Hertling 1996; Brattka and Gherardi 2011a; Brattka and Gherardi 2011b; Kudinov *et al.* 2010).

In this paper we aim to make the first three kinds of generalizations interact with each other, namely we will consider some weaker versions of the Wadge reducibility (including the ones mentioned above), and study the degree structures induced by them on arbitrary *quasi-Polish* spaces, a collection of spaces recently identified in (de Brecht 2011) by M. de Brecht as a natural class of spaces for DST and CA. Each of these degree-structures should be intended as a tool for measuring the complexity of subsets (or partitions) of the space under consideration: a structure like the Wadge one is nearly optimal for this goal, but, as already noticed, we get a reasonable notion of complexity also if the structure is just a wqo. If instead the degree structure contains infinite antichains but is well-founded, then we can at least assign a rank to the degrees (even if this rank could be not completely meaningful), while if it is also ill-founded it becomes completely useless as a notion of classification. These considerations justify the following terminology: a degree-structure obtained by considering a notion of reducibility (between sets or partitions) on a topological space will be called

- ***very good*** if it is semi-well-ordered;
- ***good*** if it is a wqo;
- ***bad*** if it contains infinite antichains;
- ***very bad*** if it contains both infinite descending chains and infinite antichains.

By the results mentioned above, the Wadge hierarchy on any non zero-dimensional Polish space is always bad, but we will show that for many other natural reducibilities, the corresponding hierarchy is very good on a great number of spaces. This is obtained by computing the minimal complexity of an isomorphism between such spaces and the Baire space. In particular, after recalling some preliminaries in Section 2 and introducing various reducibility notions in Section 3, we will show in Section 4 that all uncountable quasi-Polish spaces are pairwise D_ω-isomorphic, and that any quasi-Polish space of topological dimension $\neq \infty$ is even D_3-isomorphic to $\mathcal{N}$ (and that, in general, the indices ω and 3 cannot be lowered). This fact, together with the results from (Motto Ros 2009; Motto Ros 2010a), implies that the degree-structures induced by the classes of functions D_α and $\mathcal{B}_\gamma$ (where $\gamma < \omega_1$ is additively closed) on *any* uncountable quasi-Polish space are very good (when restricted to the degrees of Borel sets, or even, under corresponding determinacy assumptions, to the degrees of sets in any larger pointclass $\mathbf{\Gamma} \subseteq \mathscr{P}(\mathcal{N})$) whenever $\alpha \geq \omega$, and that the same is true also for $\alpha \geq 3$ when considering quasi-Polish spaces of dimension $\neq \infty$. In Section 5 we will show that these results are nearly optimal by showing that the degree-structure induced by the class of functions D_2 is (very) bad on many natural Polish spaces (like the real line $\mathbb{R}$ and its powers), and that the Wadge hierarchy can fail to be very good also on extremely simple countable quasi-Polish spaces.

2. Notation and preliminaries

In this section we introduce a great deal of notation that will be used throughout the paper. The notation for pointclasses and for isomorphisms between topological spaces will be introduced at the beginning of Sections 3 and 4, respectively.

2.1. *Notation*

Unless otherwise specified, we will always work in $\mathsf{ZF} + \mathsf{DC}$, i.e. in the usual Zermelo-Frænkel set theory together with the Axiom of Dependent Choice.

We freely use the standard set-theoretic notation like $|X|$ for the cardinality of X, $X \times Y$ for the Cartesian product of X and Y, $X \sqcup Y$ for the disjoint union of X and Y, Y^X for the set of all functions $f\colon X \to Y$, and $\mathscr{P}(X)$ for the set of all subsets of X. Given a product $X \times Y$ we denote by π_0 (respectively, π_1) the projection on the first (respectively, the second) coordinate. For $A \subseteq X$, $\overline{A}$ denotes the complement $X \setminus A$ of A in X. For $\mathcal{A} \subseteq \mathscr{P}(X)$, $BC(\mathcal{A})$ denotes the Boolean closure of $\mathcal{A}$, i.e. the set of finite Boolean combinations of sets in $\mathcal{A}$. We identify each nonzero natural number n with the set of its predecessors $\{0, \dots, n-1\}$, and the set of natural numbers, which will be denoted by ω, with its order type under $<$. The first uncountable ordinal is denoted by ω_1, while the class of all ordinal numbers is denoted by On. Given a set X and a natural number $i \in \omega$, we let $[X]^i = \{Y \subseteq X \mid |Y| = i\}$. Given an arbitrary partially ordered set $(X, \leq)$ (briefly, a *poset*), we denote by $<$ its strict part, i.e. the relation on X defined by $x < y \iff x \leq y \wedge x \neq y$.

2.2. *Spaces and pointclasses*

We assume the reader be familiar with the basic notions of topology. The collection of all open subsets of a space X (i.e. the topology of X) is denoted by τ_X or, when the space is clear from the context, simply by τ. We abbreviate "topological space" to "space" and denote by $\mathscr{X}$ the collection of all (topological) spaces. Let $\mathscr{Y} \subseteq \mathscr{X}$: we say that $X \in \mathscr{X}$ is *universal for* $\mathscr{Y}$ if every $Y \in \mathscr{Y}$ can be *topologically embedded* into X, i.e. there is a subspace $X' \subseteq X$ such that Y is homeomorphic to X' (where X' is endowed with the relative topology inherited from X). A space X is *connected* if there are no nonempty clopen proper subsets of X, and *totally disconnected* if every connected subset contains at most one point. A space X is called *locally connected* if every element has arbitrarily small connected open neighborhoods. A space X is called *σ-compact* if it can be written as a countable union of compact sets. For any space X, define the transfinite descending sequence $\langle X^{(\alpha)} \mid \alpha \in \mathsf{On}\rangle$ of closed subsets of X as follows: $X^{(0)} = X$, $X^{(\alpha+1)} =$ the set of non-isolated points of $X^{(\alpha)}$ (where x is an isolated point of a space X if $\{x\}$ is open in X), and $X^{(\alpha)} = \bigcap\{X^{(\beta)} \mid \beta < \alpha\}$ if α is a limit ordinal. The space X is called *scattered* if and only if $\bigcap_{\alpha \in \mathsf{On}} X^{(\alpha)} = \emptyset$.

Let $\mathcal{N} = \omega^\omega$ be the set of all infinite sequences of natural numbers (i.e. of all functions $\xi\colon \omega \to \omega$). Let ω^* be the set of finite sequences of elements of ω, including the empty sequence. For $\sigma \in \omega^*$ and $\xi \in \mathcal{N}$, we write $\sigma \sqsubseteq \xi$ to denote that σ is an initial segment of ξ. We denote the concatenation of σ and ξ by $\sigma\xi = \sigma \cdot \xi$, and the set of all extensions of σ in $\mathcal{N}$ by $\sigma \cdot \mathcal{N}$. For $\xi \in \mathcal{N}$, we can write $\xi = \xi(0)\xi(1)\cdots$ where $\xi(i) \in \omega$ for each $i < \omega$. Notations in the style of regular expressions like 0^ω, 0^m1^n or 0^*1 have the obvious standard meaning: for example, 0^ω is the ω-sequence constantly equal to 0, 0^m1^n is the sequence formed by m-many 0's followed by n-many 1's, $0^*1 = \{0^m1 \mid m \in \omega\}$ is the set

of all sequences constituted by a finite (possibly empty) sequence of 0's followed by (a unique) 1, and so on.

When we endow $\mathcal{N}$ with the product of the discrete topologies on ω we obtain the so-called *Baire space*. This topology coincides with the topology generated by the collection of sets of the form $\sigma \cdot \mathcal{N}$ for $\sigma \in \omega^*$. The Baire space is of primary interest for DST and CA: its importance stems from the fact that many countable objects are coded straightforwardly by elements of $\mathcal{N}$, and it has very specific topological properties. In particular, it is a perfect zero-dimensional space and the spaces $\mathcal{N}^2$, $\mathcal{N}^\omega$, $\omega \times \mathcal{N} = \mathcal{N} \sqcup \mathcal{N} \sqcup \dots$ (endowed with the product topology) are all homeomorphic to $\mathcal{N}$.

The subspace $\mathcal{C} = 2^\omega$ of $\mathcal{N}$ formed by the infinite binary strings (endowed with the relative topology inherited from $\mathcal{N}$) is known as the *Cantor space*. In this paper, we will also consider the space ω (with the discrete topology), the space $\mathbb{R}$ of reals (with the standard topology), and the space of irrationals number (with the relative topology inherited from $\mathbb{R}$), which is homeomorphic to $\mathcal{N}$.

A *pointclass* on the space X is a collection $\mathbf{\Gamma}(X)$ of subsets of X. A *family of pointclasses* is a family $\mathbf{\Gamma} = \{\mathbf{\Gamma}(X) \mid X \in \mathscr{X}\}$ indexed by arbitrary topological spaces such that each $\mathbf{\Gamma}(X)$ is a pointclass on X and $\mathbf{\Gamma}$ is closed under continuous preimages, i.e. $f^{-1}(A) \in \mathbf{\Gamma}(X)$ for every $A \in \mathbf{\Gamma}(Y)$ and every continuous function $f\colon X \to Y$ (families of pointclasses are sometimes called *boldface pointclasses* by other authors). In particular, any pointclass $\mathbf{\Gamma}(X)$ in such a family is downward closed under the Wadge reducibility on X.

Trivial examples of families of pointclasses are $\mathcal{E}, \mathcal{F}$, where $\mathcal{E}(X) = \{\emptyset\}$ and $\mathcal{F}(X) = \{X\}$ for any space $X \in \mathscr{X}$. Another basic example is given by the collection $\{\tau_X \mid X \in \mathscr{X}\}$ of the topologies of all the spaces.

Finally, we define some operations on families of pointclasses which are relevant to hierarchy theory. The usual set-theoretic operations will be applied to the families of pointclasses pointwise: for example, the union $\bigcup_i \mathbf{\Gamma}_i$ of the families of pointclasses $\mathbf{\Gamma}_0, \mathbf{\Gamma}_1, \dots$ is defined by $(\bigcup_i \mathbf{\Gamma}_i)(X) = \bigcup_i \mathbf{\Gamma}_i(X)$. A large class of such operations is induced by the set-theoretic operations of L. V. Kantorovich and E. M. Livenson which are now better known under the name "ω-Boolean operations" (see (Selivanov 2011) for the general definition). Among them are the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_\sigma$, where $\mathbf{\Gamma}(X)_\sigma$ is the set of all countable unions of sets in $\mathbf{\Gamma}(X)$, the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_c$, where $\mathbf{\Gamma}(X)_c$ is the set of all complements of sets in $\mathbf{\Gamma}(X)$, and the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_d$, where $\mathbf{\Gamma}(X)_d$ is the set of all differences of sets in $\mathbf{\Gamma}(X)$.

2.3. *Classical hierarchies in arbitrary spaces*

First we recall from (Selivanov 2004) the definition of the Borel hierarchy in arbitrary spaces.

Definition 2.1. For $\alpha < \omega_1$, define the family of poinclasses $\mathbf{\Sigma}^0_\alpha = \{\mathbf{\Sigma}^0_\alpha(X) \mid X \in \mathscr{X}\}$ by induction on α as follows: $\mathbf{\Sigma}^0_0(X) = \{\emptyset\}$, $\mathbf{\Sigma}^0_1(X) = \tau_X$, and $\mathbf{\Sigma}^0_2(X) = ((\mathbf{\Sigma}^0_1(X))_d)_\sigma$ is the collection of all countable unions of differences of open sets. For $\alpha > 2$, $\mathbf{\Sigma}^0_\alpha(X) = (\bigcup_{\beta<\alpha}(\mathbf{\Sigma}^0_\beta(X))_c)_\sigma$ is the class of countable unions of sets in $\bigcup_{\beta<\alpha}(\mathbf{\Sigma}^0_\beta(X))_c)$.

We also let $\mathbf{\Pi}^0_\beta(X) = (\mathbf{\Sigma}^0_\beta(X))_c$ and $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$. We call a set *proper* $\mathbf{\Sigma}^0_\beta(X)$ if it is in $\mathbf{\Sigma}^0_\beta(X) \setminus \mathbf{\Pi}^0_\beta(X)$.

Notice that by definition $\mathbf{\Sigma}^0_0 = \mathcal{E}$ and $\mathbf{\Pi}^0_0 = \mathcal{F}$. When $\lambda < \omega_1$ is a limit ordinal, we let $\mathbf{\Sigma}^0_{<\lambda} = \bigcup_{\alpha<\lambda} \mathbf{\Sigma}^0_\alpha$, and similarly $\mathbf{\Pi}^0_{<\lambda} = \bigcup_{\alpha<\lambda} \mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_{<\lambda} = \bigcup_{\alpha<\lambda} \mathbf{\Delta}^0_\alpha$. Notice that all of $\mathbf{\Sigma}^0_{<\lambda}$, $\mathbf{\Pi}^0_{<\lambda}$ and $\mathbf{\Delta}^0_{<\lambda}$ are families of pointclasses as well.

The sequence $\langle \mathbf{\Sigma}^0_\alpha(X), \mathbf{\Pi}^0_\alpha(X), \mathbf{\Delta}^0_\alpha(X) \mid \alpha < \omega_1 \rangle$ is called the *Borel hierarchy* of X. The pointclasses $\mathbf{\Sigma}^0_\alpha(X)$, $\mathbf{\Pi}^0_\alpha(X)$ are the *non-selfdual levels* of the hierarchy (i.e. they are the levels which are not closed under complementation), while the pointclasses $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$ are the *self-dual levels* (as is usual in DST, we will apply this terminology also to the levels of the other hierarchies considered below). The pointclass $\mathbf{B}(X)$ of *Borel sets* of X is the union of all levels of the Borel hierarchy, and $\mathbf{B} = \{\mathbf{B}(X) \mid X \in \mathscr{X}\}$ is the family of pointclasses of Borel sets. It is straightforward to check by induction on $\alpha, \beta < \omega_1$ that using Definition 2.1 one has the following result.

Proposition 2.2. For every $X \in \mathscr{X}$ and for all $\alpha < \beta < \omega_1$, $\mathbf{\Sigma}^0_\alpha(X), \mathbf{\Pi}^0_\alpha(X) \subseteq \mathbf{\Delta}^0_\beta(X) \subseteq \mathbf{\Sigma}^0_\beta(X), \mathbf{\Pi}^0_\beta(X)$.

Thus if $\lambda < \omega_1$ is a limit ordinal we have $\mathbf{\Sigma}^0_{<\lambda} = \mathbf{\Pi}^0_{<\lambda} = \mathbf{\Delta}^0_{<\lambda}$.

Remark 2.3. Definition 2.1 applies to all the spaces $X \in \mathscr{X}$, and Proposition 2.2 holds true in the full generality. Note that Definition 2.1 differs from the classical definition for Polish spaces (see e.g. (Kechris 1995, Section 11.B)) only for the level 2, and that for the case of Polish spaces our definition of Borel hierarchy is equivalent to the classical one. The classical definition cannot be applied in general to non metrizable spaces X (like e.g. the non discrete ω-algebraic domains) precisely because with that definition the inclusion $\mathbf{\Sigma}^0_1(X) \subseteq \mathbf{\Sigma}^0_2(X)$ may fail.

The Borel hierarchy is refined by the difference hierarchies (over the family of pointclasses $\mathbf{\Sigma}^0_\alpha$, $\alpha < \omega_1$) introduced by Hausdorff and Kuratowski. Recall that an ordinal α is called *even* (respectively, *odd*) if $\alpha = \lambda + n$ where λ is either zero or a limit ordinal, $n < \omega$, and the number n is even (respectively, odd). For an ordinal α, let $r(\alpha) = 0$ if α is even and $r(\alpha) = 1$, otherwise. For any ordinal $1 \leq \alpha < \omega_1$, consider the operation D_α sending any sequence $\langle A_\beta \mid \beta < \alpha \rangle$ of subsets of a space X to the subset of X

$$D_\alpha(\langle A_\beta \mid \beta < \alpha \rangle) = \bigcup \Big\{ A_\beta \setminus \bigcup\nolimits_{\gamma<\beta} A_\gamma \mid \beta < \alpha,\, r(\beta) \neq r(\alpha) \Big\}.$$

Definition 2.4. For any ordinal $1 \leq \alpha < \omega_1$ and any family of pointclasses $\mathbf{\Gamma}$, let $D_\alpha(\mathbf{\Gamma})(X)$ be the class of all sets of the form $D_\alpha(\langle A_\beta \mid \beta < \alpha \rangle)$, where the A_β's form an increasing (with respect to inclusion) sequence of sets in $\mathbf{\Gamma}(X)$, and then set $D_\alpha(\mathbf{\Gamma}) = \{D_\alpha(\mathbf{\Gamma})(X) \mid X \in \mathscr{X}\}$.

To simplify the notation, when $\mathbf{\Gamma} = \mathbf{\Sigma}^0_1$ we set $\mathbf{\Sigma}^{-1}_\alpha(X) = D_\alpha(\mathbf{\Sigma}^0_1)(X)$, $\mathbf{\Pi}^{-1}_\alpha(X) = \{X \setminus A \mid A \in \mathbf{\Sigma}^{-1}_\alpha(X)\}$, and $\mathbf{\Delta}^{-1}_\alpha(X) = \mathbf{\Sigma}^{-1}_\alpha(X) \cap \mathbf{\Pi}^{-1}_\alpha(X)$ for every $1 \leq \alpha < \omega_1$. Finally, we further set $\mathbf{\Sigma}^{-1}_0(X) = \{\emptyset\}$ and $\mathbf{\Pi}^{-1}_0(X) = \{X\}$.

For example, $\mathbf{\Sigma}^{-1}_4(X)$ and $\mathbf{\Sigma}^{-1}_\omega(X)$ consist of the the sets of the form, respectively, $(A_1 \setminus A_0) \cup (A_3 \setminus A_2)$ and $\bigcup_{i<\omega}(A_{2i+1} \setminus A_{2i})$, where the A_i's form an increasing sequence of

open subsets of X. Notice that the requirement that the sequence of the A_β's be increasing can be dropped (yielding to an equivalent definition of the pointclass $D_\alpha(\mathbf{\Gamma})(X)$) when $\mathbf{\Gamma}(X)$ is closed under countable unions. This in particular applies to the case when $\mathbf{\Gamma}$ is the pointclass of open sets or one of the pointclasses $\mathbf{\Sigma}^0_\alpha$. Moreover, it is easy to see that any level of the difference hierarchy over $\mathbf{\Sigma}^0_\alpha$, $\alpha < \omega_1$, is again a family of pointclasses.

2.4. *ω-continuous domains*

In this section we will briefly review the notation and some (basic) facts concerning ω-continuous domains which will be used in the following sections. For all undefined notions and for a more detailed presentation of this topic (as well as for all omitted proofs) we refer the reader to the standard monograph (Giertz *et al.* 2003).

Let $(X, \leq)$ be an arbitrary poset. The *Alexandrov topology* on $(X, \leq)$ is formed by taking the upward closed subsets of X as the open sets. The continuous functions between two spaces endowed with the Alexandrov topology coincide with the monotone (with respect to their partial orders) functions.

Let $(X, \leq)$ be a poset. A set $D \subseteq X$ is *directed* if any two elements of D have an upper bound in D. The poset $(X, \leq)$ is called a *directed-complete partial order* (briefly, *dcpo*) if any non-empty directed subset of X has a supremum in X. The *Scott topology* on a dcpo $(X, \leq)$ is formed by taking as open sets all the upward closed sets $U \subseteq X$ such that $D \cap U \neq \emptyset$ whenever D is a non-empty directed subset of X whose supremum is in U. As it is well-known, every dcpo endowed with the Scott topology is automatically a T_0 space (it is enough to observe that if $x \not\leq y$ then $x \in U$ and $y \notin U$ for $U = \{z \in X \mid z \not\leq y\}$, which is clearly Scott open). Note that the order $\leq$ may be recovered from the Scott topology because it coincides with its specialization order: $x \leq y$ if and only if x belongs to the closure of $\{y\}$ with respect to the Scott topology. An element $c \in X$ is *compact* if the set $\uparrow c = \{x \mid c \leq x\}$ is open, and the set of all compact elements of X is denoted by X_0. Note that for every $c \in X_0$, $\uparrow c$ is the smallest open neighborhood of c, and that if $(X, \leq)$ has a top element, then the closure of every non-empty open set is the entire space.

A dcpo $(X, \leq)$ is an *algebraic domain* if $\{\uparrow c \mid c \in X_0\}$ is a basis for the Scott topology of X. An *ω-algebraic domain* is an algebraic domain X such that X_0 is countable. An important example of an ω-algebraic domain is the space $P\omega$ of subsets of ω with the Scott topology on the directed-complete lattice $(\mathscr{P}(\omega), \subseteq)$ ($P\omega$ is sometimes called the *Scott domain*): in this space, the compact elements are precisely the finite subsets of ω. Another natural example which will be frequently considered in this paper is $(\omega^{\leq\omega}, \sqsubseteq)$, where $\omega^{\leq\omega} = \omega^* \cup \omega^\omega$: in this case, the compact elements are exactly the sequences in ω^*.

For any dcpo $(X, \leq)$ and $x, y \in X$, let $x \ll y$ mean that $y \in \mathrm{int}(\uparrow x)$ where int is the interior operator. This relation is transitive and $x \ll y$ implies $x \leq y$. A dcpo $(X, \leq)$ is a *continuous domain* if for any Scott-open set U and any $x \in U$ there is $b \in U$ with $b \ll x$. $(X, \leq)$ is an *ω-continuous domain* if there is a countable set $B \subseteq X$ such that for any Scott-open set U and any $x \in U$ there is $b \in U \cap B$ with $b \ll x$. Note that any ω-algebraic domain is an ω-continuous domain because we can take $B = X_0$.

In the next proposition we characterize scattered dcpo's with the Scott topology.

Proposition 2.5. Let $(X, \leq)$ be a dcpo with the Scott topology. Then X is scattered if and only if there is no infinite $\leq$-ascending chain $x_0 < x_1 < \dots$ in X.

Proof. Let X have no infinite ascending chain, i.e. $(X, \geq)$ is well-founded. Consider the (unique) rank function $\mathrm{rk}\colon X \to \mathsf{On}$ defined by $\mathrm{rk}(x) = \sup\{\mathrm{rk}(y) + 1 \mid x < y\}$ for each $x \in X$. By induction, $X^{(\alpha)} = \{x \in X \mid \mathrm{rk}(x) \geq \alpha\}$ for every ordinal $\alpha \in \mathsf{On}$, hence $\bigcap_{\alpha \in \mathsf{On}} X^{(\alpha)} = \emptyset$ and X is scattered.

It remains to show that if X has an infinite ascending chain $x_0 < x_1 < \dots$ then X is not scattered. It suffices to check that the supremum x of this chain is in $\bigcap_{\alpha \in \mathsf{On}} X^{(\alpha)}$. First notice that for every $X' \subseteq X$ containing all the x_n's, each x_n is not isolated in X' because $x_m \in \uparrow x_n \cap X'$ (the smallest open set of X' containing x_n) for every $m \geq n$. In particular, by induction on $\alpha \in \mathsf{On}$ one can easily show that $x_n \in X^{(\alpha)}$ for every $n \in \omega$. We now check by induction that $x \in X^{(\alpha)}$ for each $\alpha \in \mathsf{On}$. This is obvious when $\alpha = 0$ or α is a limit ordinal, so assume that $\alpha = \beta + 1$. Suppose, for a contradiction, that $x \notin X^{(\alpha)}$. Then by the inductive hypothesis $x \in X^{(\beta)} \setminus X^{(\beta+1)}$, so $\{x\}$ is Scott-open in $X^{(\beta)}$. Since $x_n \in X^{(\beta)}$ for all $n < \omega$, then $x_n \in \{x\}$ for some $n < \omega$, which is a contradiction because necessarily $x_n \neq x$ for every $n \in \omega$. $\square$

Corollary 2.6. In any scattered dcpo, the Scott topology coincides with the Alexandrov topology. The continuous functions between scattered dcpo's coincide with the monotone functions.

For future reference, we recall a characterization of the levels of the difference hierarchy over open sets in ω-algebraic domains obtained in (Selivanov 2004) (in (Selivanov 2008a) this was extended to the context of ω-continuous domains). Let $(X, \leq)$ be an ω-algebraic domain. A set $A \subseteq X$ is called *approximable* if for any $x \in A$ there is a compact element $c \leq x$ with $[c, x] \subseteq A$, where $[c, x] = \{y \in X \mid c \leq y \leq x\}$.

Let $(X, \leq)$ be a dcpo endowed with the Scott topology. Given $A \subseteq X$ and $n \in \omega$, a nondecreasing sequence $a_0 \leq \dots \leq a_n$ of compact elements of X is said to be *alternating for* A if $a_i \in A \iff a_{i+1} \notin A$ for every $i < n$. Notice that in this case we necessarily have $a_0 < \dots < a_n$. For this reason, a sequence as above will be also called *alternating chain for* A. An *alternating tree for* $A \subseteq X$ is a monotone function $f\colon (T, \sqsubseteq) \to (X_0, \leq)$ such that:

(1) $T \subseteq \omega^*$ is a well-founded tree (i.e. the partial order $(T, \sqsupseteq)$ is well-founded), and
(2) $f(\sigma) \in A \iff f(\sigma n) \notin A$, for each $\sigma n \in T$ (i.e. the image under f of any branch of T is an alternating chain for A).

The *rank* of f is the rank of $(T, \sqsupseteq)$. An alternating tree f is called 1-alternating (respectively, 0-alternating) if $f(\emptyset) \in A$ (respectively, $f(\emptyset) \notin A$).

Theorem 2.7. ((Selivanov 2004, Theorem 2.9) and (Selivanov 2005, Proposition 4.13)) Let X be an ω-algebraic domain, $\alpha < \omega_1$, and $A \subseteq X$. Then $A \in \mathbf{\Sigma}^{-1}_\alpha$ if and only if A and $X \setminus A$ are approximable and there is no 1-alternating tree of rank α for A. Moreover, if $\alpha < \omega$ then any set in $\mathbf{\Sigma}^{-1}_\alpha \setminus \mathbf{\Pi}^{-1}_\alpha$ (resp. in $\mathbf{\Delta}^{-1}_{\alpha+1} \setminus (\mathbf{\Sigma}^{-1}_\alpha \cup \mathbf{\Pi}^{-1}_\alpha)$) is Wadge complete in $\mathbf{\Sigma}^{-1}_\alpha$ (resp. in $\mathbf{\Delta}^{-1}_{\alpha+1}$).

2.5. *Polish and quasi-Polish spaces*

Recall that a space X is *Polish* if it is countably based and admits a metric d compatible with its topology such that (X, d) is a complete metric space. Examples of Polish spaces are the Baire space, the Cantor space, the space of reals $\mathbb{R}$ and its Cartesian powers $\mathbb{R}^n$ ($n \in \omega$), the closed unit interval $[0, 1]$, the Hilbert cube $[0, 1]^\omega$, and the space $\mathbb{R}^\omega$. It is well-known that both the Hilbert cube and $\mathbb{R}^\omega$ are universal for Polish spaces (see e.g. (Kechris 1995, Theorem 4.14)).

A natural variant of Polish spaces has recently emerged, the so-called *quasi-Polish* spaces. This class includes all Polish spaces and all ω-continuous domains (the main objects under consideration in DST and domain theory, respectively), and provides a unitary approach to their topological analysis. Moreover, it has shown to be a relevant class of spaces for CA. In the rest of this section we will provide the definition of these spaces and recall some of their properties that will be used later.

Given a set X, call a function d from $X \times X$ to the nonnegative reals *quasi-metric* whenever $x = y$ if and only if $d(x, y) = d(y, x) = 0$, and $d(x, y) \leq d(x, z) + d(z, y)$ (but we don't require d to be symmetric). In particular, every metric is a quasi-metric. Every quasi-metric on X canonically induce the topology τ_d on X, where τ_d is the topology generated by the open balls $B_d(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$ for $x \in X$ and $0 \neq \varepsilon \in \mathbb{R}^+$. A (topological) space X is called *quasi-metrizable* if there is a quasi-metric on X which generates its topology. If d is a quasi-metric on X, let $\hat{d}$ be the metric on X defined by $\hat{d}(x, y) = \max\{d(x, y), d(y, x)\}$. A sequence $\langle x_n \mid n \in \omega\rangle$ is called *d-Cauchy sequence* if for every $0 \neq \varepsilon \in \mathbb{R}^+$ there is $N \in \omega$ such that $d(x_n, x_m) < \varepsilon$ for all $N \leq n \leq m$. We say that the quasi-metric d on X is *complete* if every d-Cauchy sequence converges with respect to $\hat{d}$ (notice that this definition is coherent with the notion of completeness for a metric d, as in this case $\hat{d} = d$).

Definition 2.8. A T_0 space X is called *quasi-Polish* if it is countably based and there is a complete quasi-metric which generates its topology. When we fix a particular compatible complete quasi-metric d on X, we say that (X, d) is a *quasi-Polish metric space*.

Notice that every Polish space is automatically quasi-Polish, but, as recalled above, also every ω-continuous domain is quasi-Polish by (de Brecht 2011, Corollary 45). For example, a complete quasi-metric which is compatible with the topology of the Scott domain $P\omega$ is given by $d(x, y) = 0$ if $x \subseteq y$ and $d(x, y) = 2^{-(n+1)}$ if n is the smallest element in $x \setminus y$ (for every, $x, y \subseteq \omega$). De Brecht's paper (de Brecht 2011) shows that there is a reasonable descriptive set theory for the class of quasi-Polish spaces which extends the classical theory for Polish spaces in many directions, for example:

Proposition 2.9. (de Brecht 2011, Corollary 23) A subspace of a quasi-Polish space X is quasi-Polish if and only if it is $\mathbf{\Pi}^0_2(X)$.

It is not difficult to see that if (X, d) is a quasi-Polish metric space then $(X, \hat{d})$ is a Polish metric space, and that the following holds.

Proposition 2.10. (de Brecht 2011, Corollary 14) For every quasi-Polish metric space (X, d), $\tau_d \subseteq \tau_{\hat{d}} \subseteq \mathbf{\Sigma}^0_2(X, \tau_d)$. Hence, in particular, $\mathbf{\Sigma}^0_{<\omega}(X, \tau_d) = \mathbf{\Sigma}^0_{<\omega}(X, \tau_{\hat{d}})$, and the

identity function id_X is continuous from $(X, \tau_{\hat{d}})$ to (X, τ_d) and is $\mathbf{\Sigma}^0_2$-measurable from (X, τ_d) to $(X, \tau_{\hat{d}})$.

This implies that each quasi-Polish space is in fact a standard Borel space, and hence that the Souslin's separation theorem holds in the context of quasi-Polish spaces (de Brecht 2011, Theorem 58): if X is quasi-Polish then $\mathbf{B}(X) = \mathbf{\Delta}^1_1(X)$. Moreover, the Borel hierarchy on uncountable quasi-Polish spaces does not collapse.

Proposition 2.11. (de Brecht 2011, Theorem 18) If X is a quasi-Polish space, then for all $\alpha < \beta < \omega_1$, $\mathbf{\Sigma}^0_\alpha(X), \mathbf{\Pi}^0_\alpha(X) \subsetneq \mathbf{\Delta}^0_\beta(X) \subsetneq \mathbf{\Sigma}^0_\beta(X), \mathbf{\Pi}^0_\beta(X)$.

As for universal quasi-Polish spaces, we have the following result:

Proposition 2.12. (de Brecht 2011, Corollary 24) A space is quasi-Polish if and only if it is homeomorphic to a $\mathbf{\Pi}^0_2$-subset of $P\omega$ (with the relative topology inherited from $P\omega$). In particular, $P\omega$ is a universal quasi-Polish space.

A quasi-Polish space need not to be T_1. However one can still prove that the complexity of the singletons is not too high. Recall from e.g. (de Brecht 2011) that a space X *satisfies the T_D-axiom* if $\{x\}$ is the intersection of an open and a closed set for every $x \in X$.

Proposition 2.13.

(1) (de Brecht 2011, Proposition 8) If X is a countably based T_0-space then $\{x\} \in \mathbf{\Pi}^0_2(X)$ for any $x \in X$.
(2) (de Brecht 2011, Theorem 65) A countably based space is scattered if and only if it is a countable quasi-Polish space satisfying the T_D-axiom.
(3) (Folklore) If $(X, \leq)$ is a dcpo endowed with the Scott topology, then $\{c\}$ is the intersection of an open set and a closed set for every compact element $c \in X_0$.

Of course, a quasi-Polish space need not to have any other special topological property: for example, all nondiscrete ω-continuous domain are not Hausdorff and not regular. As for metrizability, we have the following result:

Proposition 2.14. (de Brecht 2011, Corollary 42) A metrizable space is quasi-Polish if and only if it is Polish.

Finally, among the various characterizations of the class of quasi-Polish spaces presented in (de Brecht 2011), the following one will be of interest for the results of this paper.

Proposition 2.15. (de Brecht 2011, Theorem 53) A topological space X is a quasi-Polish space if and only if it is homeomorphic to the set of non-compact elements of some ω-algebraic domain.

2.6. *Reducibilities*

In this subsection we introduce and briefly discuss some notions of reducibility which serve as tools for measuring the topological complexity of problems (e.g. sets, partitions, and so on) in DST and CA.

Definition 2.16. Let $X \in \mathscr{X}$ be a topological space. A collection of functions $\mathcal{F}$ from X to itself is called *reducibility (on X)* if it contains the identity function id_X and is closed under composition.

Given a reducibility $\mathcal{F}$ on X, one can consider the preorder $\leq^X_{\mathcal{F}}$ on $\mathscr{P}(X)$ associated to $\mathcal{F}$ obtained by setting for $A, B \subseteq X$

$$A \leq^X_{\mathcal{F}} B \iff A = f^{-1}(B) \text{ for some } f \in \mathcal{F}.$$

The preorder $\leq^X_{\mathcal{F}}$ canonically induces the equivalence relation

$$A \equiv^X_{\mathcal{F}} B \iff A \leq^X_{\mathcal{F}} B \wedge B \leq^X_{\mathcal{F}} A.$$

Given $A \subseteq X$, the set $[A]^X_{\mathcal{F}} = \{B \subseteq X \mid A \equiv^X_{\mathcal{F}} B\}$ is called the *$\mathcal{F}$-degree* of A (in X). We denote the set of $\mathcal{F}$-degrees by $\mathcal{D}^X_{\mathcal{F}}$. Notice that $\leq^X_{\mathcal{F}}$ canonically induces on $\mathcal{D}^X_{\mathcal{F}}$ the partial order $[A]^X_{\mathcal{F}} \leq [B]^X_{\mathcal{F}} \iff A \leq^X_{\mathcal{F}} B$. The structures $(\mathscr{P}(X), \leq^X_{\mathcal{F}})$ or its $\equiv^X_{\mathcal{F}}$-quotient $(\mathcal{D}^X_{\mathcal{F}}, \leq)$ are both called *$\mathcal{F}$-hierarchy (on X)* or *hierarchy of the $\mathcal{F}$-degrees*. When the space X is clear from the context, we drop the superscript referring to X in all the notation above. Notice that if $\mathcal{F}$ is the collection of all continuous functions, then $\leq_{\mathcal{F}}$ coincides with the Wadge reducibility $\leq_{\mathsf{W}}$. We will also sometimes consider the restriction of the $\mathcal{F}$-hierarchy to some suitable pointclass $\mathbf{\Gamma}(X)$: in this case, the structure $(\mathbf{\Gamma}(X), \leq_{\mathcal{F}})$ and its $\equiv_{\mathcal{F}}$-quotient (whenever it is well-defined) will be called *$(\mathbf{\Gamma}, \mathcal{F})$-hierarchy*.

An interesting variant of the reducibility between subsets of X considered above is obtained by considering X-namings instead of subsets of X.

Definition 2.17. Let X be a topological space. An *X-naming* is a function ν with domain X.

There are several natural reducibility notions for namings, the most basic of which is the following generalization of $\leq_{\mathsf{W}}$.

Definition 2.18. (Selivanov 2005; Selivanov 2011) An X-naming μ is *Wadge reducible* to an X-naming ν (in symbols $\mu \leq_{\mathsf{W}} \nu$) if $\mu = \nu \circ f$ for some continuous function $f \colon X \to X$.

An X-naming μ is *Wadge equivalent* to ν (in symbols $\mu \equiv_{\mathsf{W}} \nu$), if $\mu \leq_{\mathsf{W}} \nu$ and $\nu \leq_{\mathsf{W}} \mu$.

For any set S, one can then consider the preorder $(S^X, \leq_{\mathsf{W}})$ (or its $\equiv_{\mathsf{W}}$-quotient structure). This gives a generalization of the preorder formed by the classical Wadge reducibility on subsets of X, because if $S = \{0, 1\}$ then the structures $(\mathscr{P}(X), \leq_{\mathsf{W}})$ and $(S^X, \leq_{\mathsf{W}})$ are isomorphic: $A \leq_{\mathsf{W}} B$ if and only if $c_A \leq_{\mathsf{W}} c_B$, where $c_A \colon X \to 2$ is the characteristic function of the set $A \subseteq X$. Passing to an arbitrary set S, the Wadge reducibility between X-namings on S corresponds to the continuous reducibility between partitions of X in (at most) $|S|$-many pieces. For this reason, when $S = \kappa$ is a cardinal number the elements of κ^X are also called *κ-partitions of X*. Moreover, as for the Wadge reducibility between subsets of X, we can consider the restriction of $(S^X, \leq_{\mathsf{W}})$ to a pointclass $\mathbf{\Gamma}(X)$. In particular, when $S = k$ (for some $k \in \omega$) we denote by $(\mathbf{\Gamma}(X))_k$ the set of k-partitions $\nu \in k^X$ such that $\nu^{-1}(i) \in \mathbf{\Gamma}(X)$ for every $i < k$.

Already for $3 \leq k \in \omega$, the structure $(k^{\mathcal{N}}, \leq_{\mathsf{W}})$ becomes much more complicated than the structure of Wadge degrees on $\mathcal{N}$, but when restricted to suitable pointclasses it is quite well understood. In (van Engelen *et al.* 1987) it is shown that the structure $((\mathbf{\Delta}^1_1(\mathcal{N}))_k, \leq_{\mathsf{W}})$ is a wqo, i.e. it has neither infinite descending chain nor infinite antichain. In (Hertling 1993; Selivanov 2007) the $\equiv_{\mathsf{W}}$-quotient structures of, respectively, $((BC(\mathbf{\Sigma}^0_1)(\mathcal{N}))_k, \leq_{\mathsf{W}})$ and $((\mathbf{\Delta}^0_2(\mathcal{N}))_k, \leq_{\mathsf{W}})$ were characterized in terms of the relation of homomorphism between finite and, respectively, countable well-founded k-labeled forests. The mentioned characterizations considerably clarify the corresponding structures $((BC(\mathbf{\Sigma}^0_1)(\mathcal{N}))_k, \leq_{\mathsf{W}})$ and $((\mathbf{\Delta}^0_2(\mathcal{N}))_k, \leq_{\mathsf{W}})$, and led to deep definability theories for them developed in (Kudinov and Selivanov 2007; Kudinov and Selivanov 2009; Kudinov *et al.* 2009). In particular, both structures have undecidable first-order theories, and their automorphism groups are isomorphic to the symmetic group on k. Similar results are also known for k-partitions of $\omega^{\leq\omega}$, see (Selivanov 2010).

Of course one can consider other variants on the notion of continuous reducibility between X-namings. For example, given a reducibility $\mathcal{F}$ on X and a set S, one can consider the $\mathcal{F}$-reducibility $\leq^{S,X}_{\mathcal{F}}$ between X-namings defined in the obvious way (as usual, when S and/or X are clear from the context we will drop any reference to them in the notation): in this paper we will also provide some results related to this more general notion of reducibility.

3. Some examples of reducibilities

There are various notions of reducibility $\mathcal{F}$ that have been considered in the literature (see e.g. (Andretta 2006; Andretta and Martin 2003; Motto Ros 2009; Motto Ros 2010a; Motto Ros 2010b)). In this section we will provide several examples which are relevant for this paper.

Let $\mathbf{\Gamma}, \mathbf{\Delta}$ be two families of pointclasses and X, Y be arbitrary topological spaces. We denote by $\mathbf{\Gamma\Delta}(X, Y)$ (respectively, $\mathbf{\Gamma\Delta}[X, Y]$) the collection of functions $f\colon X \to Y$ such that $f^{-1}(A) \in \mathbf{\Gamma}(X)$ for all $A \in \mathbf{\Delta}(Y)$ (respectively, $f(A) \in \mathbf{\Delta}(Y)$ for all $A \in \mathbf{\Gamma}(X)$). Notice that if $f\colon X \to Y$ is an injection then $f \in \mathbf{\Gamma\Delta}(X, Y) \iff f^{-1} \in \mathbf{\Delta\Gamma}[f(X), X]$. If moreover $f\colon X \to Y$ is such that $f(X) \in \mathbf{\Delta}$ and $\mathbf{\Delta}$ is closed under finite intersections, then $f \in \mathbf{\Gamma\Delta}[X, Y] \iff f^{-1} \in \mathbf{\Delta\Gamma}(f(X), X)$. We abbreviate $\mathbf{\Gamma\Gamma}(X, Y)$ with $\mathbf{\Gamma}(X, Y)$ and $\mathbf{\Gamma\Gamma}[X, Y]$ with $\mathbf{\Gamma}[X, Y]$. When writing $\mathbf{\Gamma} \subseteq \mathbf{\Gamma}'$ we mean that $\mathbf{\Gamma}(X) \subseteq \mathbf{\Gamma}'(X)$ for all topological spaces X.

Remark 3.1. Let $\mathbf{\Gamma}, \mathbf{\Gamma}', \mathbf{\Delta}, \mathbf{\Delta}', \mathbf{\Lambda}$ be families of pointclasses and X, Y, Z be arbitrary topological spaces.

(1) If $\mathbf{\Gamma} \subseteq \mathbf{\Gamma}'$ and $\mathbf{\Delta} \subseteq \mathbf{\Delta}'$ then $\mathbf{\Gamma\Delta}'(X, Y) \subseteq \mathbf{\Gamma}'\mathbf{\Delta}(X, Y)$ and $\mathbf{\Gamma}'\mathbf{\Delta}[X, Y] \subseteq \mathbf{\Gamma\Delta}'[X, Y]$.
(2) If $f \in \mathbf{\Gamma\Lambda}(X, Y)$ and $g \in \mathbf{\Lambda\Delta}(Y, Z)$ then $g \circ f \in \mathbf{\Gamma\Delta}(X, Z)$.
(3) If $f \in \mathbf{\Gamma\Lambda}[X, Y]$ and $g \in \mathbf{\Lambda\Delta}[Y, Z]$ then $g \circ f \in \mathbf{\Gamma\Delta}[X, Z]$.
(4) $\mathbf{\Gamma}(X, X)$ is closed under composition and contains the identity function (hence is a reducibility on X).

In this paper, we will often consider the sets of functions given by using the levels of the Borel hierarchy in the above definitions. For ease of notation, when $\mathbf{\Gamma} = \mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Delta} = \mathbf{\Sigma}^0_\beta$

(for $\alpha, \beta < \omega_1$) we will write $\mathbf{\Sigma}^0_{\alpha,\beta}(X,Y)$ and $\mathbf{\Sigma}^0_{\alpha,\beta}[X,Y]$ instead of $\mathbf{\Sigma}^0_\alpha\mathbf{\Sigma}^0_\beta(X,Y)$ and $\mathbf{\Sigma}^0_\alpha\mathbf{\Sigma}^0_\beta[X,Y]$, respectively. Similarly, we will write $\mathbf{\Sigma}^0_{<\alpha,\beta}(X,Y)$ instead of $\mathbf{\Sigma}^0_{<\alpha}\mathbf{\Sigma}^0_\beta(X,Y)$. Moreover, we will often denote the class of continuous functions $\mathbf{\Sigma}^0_1(X,Y)$ by $\mathsf{W}(X,Y)$, and write $\mathsf{W}(X)$ (or even just W, if X is clear from the context) instead of $\mathsf{W}(X,X)$. (The symbol W stands for W. Wadge, who was the first to initiate in (Wadge 1984) a systematic study of the quasi-order $\leq_{\mathsf{W}}$ on $\mathcal{N}$.)

Some of these classes of functions are well-known in DST. For example, $\mathbf{\Sigma}^0_\alpha(X,Y)$ coincides (for $\alpha \geq 1$ and $X = Y = \mathcal{N}$) with the class of functions D_α considered in (Motto Ros 2009): for this reason, the class $\mathbf{\Sigma}^0_\alpha(X,Y)$ will be often denoted by $\mathsf{D}_\alpha(X,Y)$. When $X = Y$ we will simplify a little bit the notation by setting $\mathsf{D}_\alpha(X) = \mathsf{D}_\alpha(X,X)$, and even drop the reference to X when such space is clear from the context. Notice that the classes $\mathsf{D}_\alpha(X)$ are always reducibilities by Remark 3.1(4). It follows immediately from the above definitions that $\mathsf{D}_\alpha(X,Y) = \mathbf{\Pi}^0_\alpha(X,Y)$ for $1 \leq \alpha < \omega_1$, and that if $\alpha \geq 2$ then $\mathsf{D}_\alpha(X,Y) = \mathbf{\Delta}^0_\alpha(X,Y)$; when Y is zero-dimensional, then we also have that $\mathsf{D}_1(X,Y) = \mathbf{\Delta}^0_1(X,Y)$.

Another well-known class is $\mathbf{\Sigma}^0_{\alpha,1}(X,Y)$, the collection of all $\mathbf{\Sigma}^0_\alpha$-measurable functions from X to Y. Recall that by (Kechris 1995, Theorem 24.3) if $\alpha = \beta + 1$ and X, Y are metrizable with Y separable, the class $\mathbf{\Sigma}^0_{\alpha,1}$ coincides with the class of Baire class β functions (as defined e.g. in (Kechris 1995, Definition 24.1)). The classes $\mathbf{\Sigma}^0_{\alpha,1}(X,X)$ are not closed under composition if $\alpha > 1$: as computed in (Motto Ros 2010a, Theorem 6.4), the closure under composition of $\mathbf{\Sigma}^0_{\alpha,1}(X,X)$ is given by $\mathcal{B}_\gamma(X) = \bigcup_{\beta<\gamma}\mathbf{\Sigma}^0_{\beta,1}(X,X)$, where $\gamma = \alpha \cdot \omega$ is the first additively closed ordinal above α (as usual, we will drop the reference to X whenever such space will be clear from the context). Hence, when γ is additively closed the set $\mathcal{B}_\gamma(X)$ is a reducibility on X. The reducibilities $\mathcal{B}_\gamma(X)$ and their induced degree-structures have been studied in (Motto Ros 2010a). Notice also that in general $\bigcup_{\alpha<\gamma}\mathsf{D}_\alpha(X) \subsetneq \mathcal{B}_\gamma(X) \subsetneq \mathbf{\Sigma}^0_{<\gamma,1}(X,X)$

We now state some properties of these classes of functions.

Proposition 3.2.

(1) If $1 \leq \alpha < \beta < \omega_1$ then $\mathsf{D}_\alpha(X,Y) \subseteq \mathsf{D}_\beta(X,Y)$, and if β is limit $\mathsf{D}_\alpha(X,Y) \subseteq \mathbf{\Sigma}^0_{<\beta}(X,Y) \subseteq \mathsf{D}_\beta(X,Y)$.

(2) Let $1 \leq \alpha, \beta < \omega_1$ and $\delta = \max\{\alpha,\beta\} \cdot \omega$ (i.e. δ is the first additively closed ordinal strictly above α and β). Then $\mathbf{\Sigma}^0_{\alpha,\beta}(X,Y) \subseteq \mathbf{\Sigma}^0_{<\delta}(X,Y)$. In particular, $\bigcup_{\alpha<\gamma}\mathbf{\Sigma}^0_{\alpha,1}(X,Y) \subseteq \mathbf{\Sigma}^0_{<\gamma}(X,Y) \subseteq \mathsf{D}_\gamma(X,Y)$ for every additively closed $\gamma < \omega_1$.

Proof. The proof of (1) is straightforward, so we just consider (2). If $\alpha \leq \beta$ then $\mathbf{\Sigma}^0_{\alpha,\beta}(X,Y) \subseteq \mathbf{\Sigma}^0_\beta(X,Y) \subseteq \mathbf{\Sigma}^0_{<\delta}(X,Y)$ by (1), hence we can assume $\beta < \alpha$. Arguing by induction on $\gamma < \omega_1$, one easily obtains $\mathbf{\Sigma}^0_{\alpha,\beta}(X,Y) \subseteq \mathbf{\Sigma}^0_{\alpha+\gamma,\beta+\gamma}(X,Y)$. Let $\alpha' \leq \alpha$ be such that $\beta + \alpha' = \alpha$, and let $\gamma = \alpha + (\alpha' \cdot \omega) = \beta + (\alpha' \cdot \omega) \leq \alpha \cdot \omega = \delta$. We claim that $\mathbf{\Sigma}^0_{\alpha,\beta} \subseteq \mathbf{\Sigma}^0_{<\gamma}(X,Y)$, which obviously implies the desired result. Let $f \in \mathbf{\Sigma}^0_{\alpha,\beta}(X,Y)$ and $A \in \mathbf{\Sigma}^0_{<\gamma}(Y)$. Then for some $k < \omega$ we have $A \in \mathbf{\Sigma}^0_{\beta+(\alpha'\cdot k)}(Y)$, hence $f^{-1}(A) \in \mathbf{\Sigma}^0_{\alpha+(\alpha'\cdot k)}(X) \subseteq \mathbf{\Sigma}^0_{<\gamma}(X)$, as required. □

In particular, by Proposition 3.2(2) all Baire class 1 functions (i.e. the functions in $\mathbf{\Sigma}^0_{2,1}(X,Y)$) are in $\mathbf{\Sigma}^0_{<\omega}(X,Y)$, and hence also in $\mathsf{D}_\omega(X,Y)$.

Other kind of reducibilities which will be considered in this paper are given by classes of piecewise defined functions. Given a family of pointclasses $\mathbf{\Gamma}$, a $\mathbf{\Gamma}$-*partition* of a space X is a sequence $\langle D_n \mid n \in \omega \rangle$ of pairwise disjoint sets from $\mathbf{\Gamma}(X)$ such that $X = \bigcup_{n\in\omega} D_n$. Notice that, in particular, every $\mathbf{\Sigma}^0_\alpha$-partition of X is automatically a $\mathbf{\Delta}^0_\alpha$-partition of X (for every $\alpha < \omega_1$).

Definition 3.3. (Motto Ros 2011) Given two spaces $X, Y \in \mathscr{X}$, a collection of functions $\mathcal{F}$ from (subsets of) X to Y, and an ordinal $\alpha < \omega_1$, we will denote by $\mathsf{D}^{\mathcal{F}}_\alpha(X,Y)$ [§] the collection of those $f\colon X \to Y$ for which there is a $\mathbf{\Sigma}^0_\alpha$-partition (equivalently, a $\mathbf{\Delta}^0_\alpha$-partition) $\langle D_n \mid n \in \omega \rangle$ of X and a sequence $\langle f_n\colon D_n \to Y \mid n \in \omega \rangle$ of functions from $\mathcal{F}$ such that $f = \bigcup_{n\in\omega} f_n$.

In particular, we will be interested in the classes $\mathsf{D}^{\mathsf{W}}_\alpha(X,Y) = \mathsf{D}^{\mathsf{W}(\subseteq X,Y)}_\alpha(X,Y)$, where $\mathsf{W}(\subseteq X,Y) = \bigcup_{X'\subseteq X} \mathsf{W}(X',Y)$, for various $\alpha < \omega_1$. Notice also that $\mathbf{\Sigma}^0_\alpha(X,Y) = \mathsf{D}^{\mathbf{\Sigma}^0_\alpha(\subseteq X,Y)}_\alpha(X,Y)$ with $\mathbf{\Sigma}^0_\alpha(\subseteq X,Y) = \bigcup_{X'\subseteq X} \mathbf{\Sigma}^0_\alpha(X',Y)$, and that $\mathsf{D}^{\mathsf{W}}_\alpha(X,Y) \subseteq \mathsf{D}_\alpha(X,Y)$. As for the other classes of functions, we will write $\mathsf{D}^{\mathsf{W}}_\alpha(X)$ instead of $\mathsf{D}^{\mathsf{W}}_\alpha(X,X)$, and even drop the reference to X when there is no danger of confusion. It is not hard to check that each of the $\mathsf{D}^{\mathsf{W}}_\alpha(X)$ is a reducibility on X.

It is a remarkable theorem of Jayne and Rogers (Jayne and Rogers 1982, Theorem 5) (but see also (Motto Ros and Semmes 2010; Kačena *et al.* 2012) for a shorter and simpler proof) that if X, Y are Polish spaces[¶] then $\mathsf{D}_2(X,Y) = \mathsf{D}^{\mathsf{W}}_2(X,Y)$. This result has been recently extended to the level 3 (for the special case $X = Y = \mathcal{N}$) by B. Semmes.

Theorem 3.4. (Semmes 2009) $\mathsf{D}_3(\mathcal{N}) = \mathsf{D}^{\mathsf{W}}_3(\mathcal{N})$.

Whether this result can be extended to all $3 < n < \omega$ is a major open problem, but notice however that, as observed e.g. in (Andretta 2007; Motto Ros 2009), it is not possible to generalize the Jayne-Rogers theorem to levels $\alpha \geq \omega$. To see this we need to recall the definition of containment between functions introduced in (Solecki 1998), and the definition of a very special function, called the Pawlikowski function. We call embedding any function between two topological spaces which is an homeomorphism on its range.

Definition 3.5. (Solecki 1998) Let $X_0, X_1, Y_0, Y_1 \in \mathscr{X}$ be topological spaces and consider $f\colon X_0 \to Y_0$ and $g\colon X_1 \to Y_1$. We say that f is *contained* in g, and we write $f \sqsubseteq g$, just in case there are two embeddings $\varphi\colon X_0 \to X_1$ and $\psi\colon f(X_0) \to Y_1$ such that $\psi \circ f = g \circ \varphi$.

It is not hard to check that if f and g are as in Definition 3.5 and $f \sqsubseteq g$, then $g \in \mathbf{\Sigma}^0_{\alpha,\beta}(X_1,Y_1)$ implies $f \in \mathbf{\Sigma}^0_{\alpha,\beta}(X_0,Y_0)$ for every $1 \leq \alpha, \beta < \omega_1$.

[§] Notice that for $X = Y = \mathcal{N}$, this class of functions was denoted by $\tilde{\mathsf{D}}^{\mathcal{F}}_\alpha$ in (Motto Ros 2011). However, here we will not use the other class of piecewise defined functions considered in that paper, so we can safely simplify the notation dropping the decoration on the symbol D.

[¶] In fact the Jayne-Rogers result is even more general, in that its conclusion holds also when X, Y are arbitrary metric spaces with X an absolute Souslin-$\mathscr{F}$ set.

Let us endow the space $(\omega+1)^\omega$ with the product of the order topology on $\omega+1$. The *Pawlikowski function* is the function $P\colon (\omega+1)^\omega \to \mathcal{N}$ defined by

$$P(x)(n) = \begin{cases} x(n)+1 & \text{if } x(n) \in \omega \\ 0 & \text{if } x(n) = \omega. \end{cases}$$

Since $(\omega+1)^\omega$ is a perfect nonempty compact metrizable zero-dimensional space, it is homeomorphic to the Cantor space $\mathcal{C}$ by Brouwer's theorem (Kechris 1995, Theorem 7.4): therefore, using the fact that $\mathcal{N}$ is homeomorphic to a G_δ subset of $\mathcal{C}$, P can actually be regarded as a function from $\mathcal{C}$ to $\mathcal{C}$.

Definition 3.6. Let X, Y be arbitrary Polish spaces. We say that a function $f\colon X \to Y$ *can be decomposed* into countably many continuous functions (briefly, it is *decomposable*) if there is some partition $\langle D_n \mid n \in \omega\rangle$ of X into countably many pieces (of arbitrary complexity) such that $f \restriction D_n$ is continuous for every $n < \omega$.

Notice in particular that all the functions in $\bigcup_{\alpha<\omega_1} \mathsf{D}^{\mathsf{W}}_\alpha(X,Y)$ are decomposable by definition, and that if $f \sqsubseteq g$ and g is decomposable, then f is decomposable as well. The function P was introduced as an example of a Baire class 1 function (hence $P \in \mathbf{\Sigma}^0_{2,1}(\mathcal{C},\mathcal{C})$) which is not decomposable — see e.g. (Solecki 1998). Using this fact, we can now prove that the Jayne-Rogers theorem cannot be generalized to the infinite levels of the Borel hierarchy for several uncountable quasi-Polish spaces.

Proposition 3.7. Assume that X, Y are uncountable quasi-Polish spaces and $\omega \le \alpha < \omega_1$. If $\mathcal{C}$ can be embedded into Y, then $\mathsf{D}^{\mathsf{W}}_\alpha(X,Y) \subsetneq \mathsf{D}_\alpha(X,Y)$. In particular, $\mathsf{D}^{\mathsf{W}}_\alpha(X,Y) \subsetneq \mathsf{D}_\alpha(X,Y)$ for Y a Polish space, $Y = \omega^{\le\omega}$, or $Y = P\omega$.

Proof. First observe that $P \in \mathbf{\Sigma}^0_{<\omega}(\mathcal{C},\mathcal{C}) \subseteq \mathsf{D}_\alpha(\mathcal{C})$ by Proposition 3.2, but $P \notin \bigcup_{\alpha<\omega_1} \mathsf{D}^{\mathsf{W}}_\alpha(\mathcal{C})$ because P is not even decomposable. Suppose now that X, Y are arbitrary uncountable quasi-Polish spaces, and further assume that $\mathcal{C}$ embeds into Y.

Claim 3.7.1. Let Z be an uncountable quasi-Polish space. Then there is a continuous injection of $\mathcal{C}$ onto a $\mathbf{\Pi}^0_2$ subset of Z.

Proof of the Claim. Let d be a complete quasi-metric on Z compatible with its topology. By the Cantor-Bendixon theorem (Kechris 1995, Theorem 6.4), there is an embedding $\varphi\colon \mathcal{C} \to (Z, \tau_{\hat{d}})$. Notice that the range of φ is automatically $\tau_{\hat{d}}$-closed. Therefore by Proposition 2.10 we have that $\varphi\colon \mathcal{C} \to (Z, \tau_d)$ is continuous as well, and that its range is a $\mathbf{\Pi}^0_2$ set with respect to τ_d. □

Let $\varphi\colon \mathcal{C} \to X$ be a continuous injection with $\varphi(\mathcal{C}) \in \mathbf{\Pi}^0_2(X)$, and let $\psi\colon \mathcal{C} \to Y$ be an embedding. Pick an arbitrary $y_0 \in Y$ and define $P'\colon X \to Y$ by setting $P'(x) = \psi(P(\varphi^{-1}(x)))$ if $x \in \varphi(\mathcal{C})$ and $P'(x) = y_0$ otherwise. It is straightforward to check that $P' \in \mathbf{\Sigma}^0_{3,1}(X,Y) \subseteq \mathbf{\Sigma}^0_{<\omega}(X,Y) \subseteq \mathsf{D}_\alpha(X,Y)$ by Proposition 3.2. We will now show that $P'\colon X \to Y$ is not decomposable, which clearly implies that $P' \notin \mathsf{D}^{\mathsf{W}}_\alpha(X,Y)$, as required. Assume towards a contradiction that P' is decomposable, and let $\langle X_n \mid n \in \omega\rangle$ be a countable partition of X such that $P'_n = P' \restriction X_n$ is continuous for every $n \in \omega$. Then $\langle \varphi^{-1}(X_n) \mid n \in \omega\rangle$ is a countable partition of $\mathcal{C}$: we will show that for every $n \in \omega$

the function $P_n = P \restriction \varphi^{-1}(X_n)$ is continuous, contradicting the fact that P is not decomposable. Let $U \subseteq \mathcal{C}$ be open. Since ψ is an embedding, there is an open $V \subseteq Y$ such that $\psi(U) = V \cap \psi(\mathcal{C})$. By definition of P', $P_n^{-1}(U) = \varphi^{-1}((P'_n)^{-1}(V))$, which is open in $\varphi^{-1}(X_n)$ because φ and P'_n are both continuous: therefore P_n is continuous. □

Corollary 3.8. Assume that X, Y are uncountable quasi-Polish spaces and $\omega \leq \alpha < \omega_1$. If Y is Hausdorff, then $\mathsf{D}^{\mathsf{W}}_\alpha(X, Y) \subsetneq \mathsf{D}_\alpha(X, Y)$.

Proof. It is enough to show that if Y is an uncountable Hausdorff quasi-Polish space then there is an embedding of $\mathcal{C}$ into Y. Let $\varphi\colon \mathcal{C} \to Y$ be a continuous injection (which exists by Claim 3.7.1). We want to show that for every open $U \subseteq \mathcal{C}$, $\varphi(U)$ is open in $\varphi(\mathcal{C})$. Since $\mathcal{C}$ is compact, $\mathcal{C} \setminus U$ is compact as well. Since φ is continuous, then $D = \varphi(\mathcal{C} \setminus U)$ is compact as well, and hence also closed in Y. But then $\varphi(U) = (Y \setminus D) \cap \varphi(\mathcal{C})$ is an open set with respect to the relative topology of $\varphi(\mathcal{C})$, as required. □

The Pawlikowski function P can in fact be used to characterize decomposable functions within certain Borel classes. In (Solecki 1998), Solecki proved that if $f \in \mathbf{\Sigma}^0_{2,1}(X, Y)$ with X, Y Polish spaces[‖], then f is decomposable if and only if $P \not\sqsubseteq f$. Using the technique of changes of topologies and arguing by induction on $n < \omega$, this characterization can easily be extended[††] to the wider context of functions in $\bigcup_{1 \leq n < \omega} \mathbf{\Sigma}^0_{n,1}(X, Y)$.

Theorem 3.9. (Motto Ros 2012, Lemma 5.7) Let X, Y be Polish spaces and let f be in $\mathbf{\Sigma}^0_{n,1}(X, Y)$ for some $1 \leq n < \omega$. Then f is decomposable if and only if $P \not\sqsubseteq f$.

Finally, we recall from (Motto Ros 2010a, Proposition 6.6) the following result on the topological complexity of P.

Proposition 3.10. For every $n \in \omega$, $P \notin \mathsf{D}_n(\mathcal{C})$.

Combining all above results together, we have the following proposition (see also Lemma 5.8 in (Motto Ros 2012)).

Proposition 3.11. Let X, Y be Polish spaces. For every $n < \omega$ and $f \in \mathsf{D}_n(X, Y)$, f is decomposable.

Proof. By Proposition 3.10 and the observation following Definition 3.5, we have that $P \not\sqsubseteq f$, hence f is decomposable by Theorem 3.9. □

4. Isomorphisms of minimal complexity between quasi-Polish spaces

The following definition extends in various directions the topological notion of homeomorphism.

‖ Solecki's theorem applies to a slightly wider context, i.e. to the case when X is an analytic space and Y is separable metric.

†† In (Pawlikowski and Sabok 2012, Theorem 1.1), Solecki's characterization of decomposable functions is further extended (using different and more involved methods) to the even wider context of all Borel functions from an analytic space X to a separable metrizable space Y, but here we will not need the above characterization in such generality.

Definition 4.1.

(1) Let $\mathcal{F}$ be a collection of functions between topological spaces, and $X, Y \in \mathscr{X}$. We say that X and Y are *$\mathcal{F}$-isomorphic* ($X \simeq_{\mathcal{F}} Y$ in symbols) if there is a bijection $f\colon X \to Y$ such that both f and f^{-1} belong to $\mathcal{F}$.

(2) If $\mathbf{\Gamma}$ is a family of pointclasses, we say that two topological spaces X, Y are *$\mathbf{\Gamma}$-isomorphic* ($X \simeq_{\mathbf{\Gamma}} Y$ in symbols) if $X \simeq_{\mathcal{F}} Y$ where $\mathcal{F} = \bigcup\{\mathbf{\Gamma}(X, Y) \mid X, Y \in \mathscr{X}\}$. This is obviously equivalent to requiring that $X \simeq_{\mathcal{G}} Y$ where $\mathcal{G} = \bigcup\{\mathbf{\Gamma}[X, Y] \mid X, Y \in \mathscr{X}\}$.

(3) We say that $X, Y \in \mathscr{X}$ are *piecewise homeomorphic* ($X \simeq_{\mathsf{pw}} Y$ in symbols) if there are countable partitions $\langle X_n \mid n \in \omega\rangle, \langle Y_n \mid n \in \omega\rangle$ of, respectively, X and Y such that X_n and Y_n are homeomorphic for every $n \in \omega$.

(4) Given a family of pointclasses $\mathbf{\Gamma}$, we say that X and Y are *$\mathbf{\Gamma}$-piecewise homeomorphic* ($X \simeq_{\mathsf{pw}(\mathbf{\Gamma})} Y$ in symbols) if and only if there are partitions $\langle X_n \mid n \in \omega\rangle, \langle Y_n \mid n \in \omega\rangle$ of, respectively, X and Y consisting of sets in $\mathbf{\Gamma}$ such that X_n and Y_n are homeomorphic for every $n \in \omega$.

It is obvious that if $\mathcal{F} \subseteq \mathcal{G}$ are two sets of functions between topological spaces and $X, Y \in \mathscr{X}$ are such that $X \simeq_{\mathcal{F}} Y$ then $X \simeq_{\mathcal{G}} Y$. Since the notion of $\mathbf{\Sigma}^0_\alpha$-isomorphism, $\mathbf{\Pi}^0_\alpha$-isomorphism, and $\mathbf{\Delta}^0_\alpha$-isomorphism all coincide for $2 \leq \alpha < \omega_1$, for simplicity of notation we will write $X \simeq_\alpha Y$ instead of $X \simeq_{\mathbf{\Sigma}^0_\alpha} Y$. Similarly, when $\mathcal{F} = \bigcup\{\mathsf{D}^{\mathsf{W}}_\alpha(X, Y) \mid X, Y \in \mathscr{X}\}$ we will simply write $X \simeq^{\mathsf{W}}_\alpha Y$ instead of $X \simeq_{\mathcal{F}} Y$.

Lemma 4.2. Let $1 \leq \alpha < \omega_1$ and $X, Y \in \mathscr{X}$. Then $X \simeq^{\mathsf{W}}_\alpha Y$ if and only if $X \simeq_{\mathsf{pw}(\mathbf{\Sigma}^0_\alpha)} Y$ (equivalently, if and only if $X \simeq_{\mathsf{pw}(\mathbf{\Delta}^0_\alpha)} Y$).

Similarly, let $\mathcal{F}$ be the class of all decomposable functions. Then $X \simeq_{\mathcal{F}} Y$ if and only if $X \simeq_{\mathsf{pw}} Y$.

Proof. We just consider the first part of the lemma, as the second one can be proved in a similar way. The direction from right to left directly follows from the definition of $\mathsf{D}^{\mathsf{W}}_\alpha(X, Y)$, so let us assume that $f\colon X \to Y$ is a bijection such that $f \in \mathsf{D}^{\mathsf{W}}_\alpha(X, Y)$ and $f^{-1} \in \mathsf{D}^{\mathsf{W}}_\alpha(Y, X)$. By definition, there are partitions $\langle X'_n \mid n \in \omega\rangle$ and $\langle Y'_m \mid m \in \omega\rangle$ of, respectively, X and Y in $\mathbf{\Sigma}^0_\alpha$ pieces such that $f \restriction X'_n$ and $f^{-1} \restriction Y'_m$ are continuous for every $n, m \leq \omega$. This implies that for every $n, m < \omega$ the sets $X_{\langle n,m\rangle} = X'_n \cap f^{-1}(Y'_m)$ and $Y_{\langle m,n\rangle} = Y'_m \cap f(X'_n)$ (where $\langle \cdot, \cdot \rangle$ denotes a bijection between $\omega \times \omega$ and ω) are in $\mathbf{\Sigma}^0_\alpha$ as well and form two countable partitions of, respectively X and Y. Then it is easy to see that $f \restriction X_{\langle n,m\rangle}\colon X_{\langle n,m\rangle} \to Y_{\langle m,n\rangle}$ is a bijection witnessing that $X_{\langle n,m\rangle}$ and $Y_{\langle m,n\rangle}$ are homeomorphic, hence we are done. □

It is a classical result of DST that every two uncountable Polish spaces X, Y are $\mathbf{B}$-isomorphic (see e.g. (Kechris 1995, Theorem 15.6)). The next proposition extends this result to the context of uncountable quasi-Polish spaces and computes an upper bound for the complexity of the Borel-isomorphism according to Definition 4.1.

Proposition 4.3. Let $\mathcal{F} = \bigcup\{\mathbf{\Sigma}^0_{3,1}(X, Y) \mid X, Y \in \mathscr{X}\}$ and let X, Y be two uncountable quasi-Polish spaces. Then $X \cong_{\mathcal{F}} Y$. In particular, $X \cong_{\mathbf{\Delta}^0_{<\omega}} Y$ and hence also $X \cong_\omega Y$.

Proof. Let d_X and d_Y be complete quasi-metrics compatible with the topologies of, respectively, X and Y, and let $\hat{d}_X, \hat{d}_Y$ be the metrics induced by d_X and d_Y. Then by Proposition 2.10 $\mathrm{id}_X \colon (X\tau_{\hat{d}_X}) \to (X, \tau_{d_X})$ is a continuous function with $\mathbf{\Sigma}^0_2$-measurable inverse, and similarly for $\mathrm{id}_Y \colon (Y, \tau_{\hat{d}_Y}) \to (Y, \tau_{d_Y})$. Since $(X, \tau_{\hat{d}_X})$ and $(Y, \tau_{\hat{d}_Y})$ are uncountable Polish spaces, by e.g. (Kuratowski 1934, p. 212) there is a bijection $g \colon (X, \tau_{\hat{d}_X}) \to (Y, \tau_{\hat{d}_Y})$ such that both g and g^{-1} are $\mathbf{\Sigma}^0_2$-measurable. Hence $f = \mathrm{id}_Y \circ g \circ \mathrm{id}_X^{-1}$ is a bijection between (X, τ_{d_X}) and (Y, τ_{d_Y}) such that both f and f^{-1} are $\mathbf{\Sigma}^0_3$-measurable.

The second part of the Proposition follows from Proposition 3.2(2). □

Proposition 4.4. Every quasi-Polish space is $\mathsf{D}^{\mathsf{W}}_4$-isomorphic to an ω-algebraic domain.

Proof. Let Y be a quasi-Polish space. We can clearly assume that Y is infinite (otherwise Y itself is an ω-algebraic domain). By Proposition 2.15, there is an ω-algebraic domain X and a function $f \colon Y \to X$ such that f is an homeomorphism between Y and $X \setminus X_0$, where X_0 is the (countable) set of compact elements of X (see Subsection 2.4). By Lemma 4.2, it is enough to show that $Y \cong_{\mathsf{pw}(\mathbf{\Delta}^0_4)} X$. Let $\langle x_n \mid n \in \omega \rangle$ be an enumeration without repetitions of X_0 and $\langle y_n \mid n \in \omega \rangle$ be an enumeration without repetitions of an infinite countable subset Y_0 of Y such that $Y \setminus Y_0$ is non-empty. Then $\langle Y \setminus Y_0, \{y_n\} \mid n \in \omega \rangle$ and $\langle X \setminus (X_0 \cup f(Y_0)), \{f(y_n)\}, \{x_n\} \mid n \in \omega \rangle$ are countable partitions of, respectively, Y and X into $\mathbf{\Pi}^0_3$ pieces by Proposition 2.13(1). The function $f \restriction (Y \setminus Y_0)$ is an homeomorphism between $Y \setminus Y_0$ and $X \setminus (X_0 \cup f(Y_0))$. For every $n \in \omega$, the function sending y_{2n} to $f(y_n)$ is an homeomorphism between $\{y_{2n}\}$ and $\{f(y_n)\}$, while the function sending y_{2n+1} to x_n is an homeomorphism between $\{y_{2n+1}\}$ and $\{x_n\}$. Hence $Y \cong_{\mathsf{pw}(\mathbf{\Delta}^0_4)} X$, as required. □

Proposition 4.5.

(1) Let X, Y be countable countably based T_0-spaces. Then $X \simeq^{\mathsf{W}}_3 Y$ if and only if $|X| = |Y|$.
(2) Let X, Y be countable T_1 spaces. Then $X \simeq^{\mathsf{W}}_2 Y$ if and only if $|X| = |Y|$.
(3) Let X, Y be scattered countably based spaces. Then $X \simeq^{\mathsf{W}}_2 Y$ if and only if $|X| = |Y|$.

In particular, $X \simeq^{\mathsf{W}}_3 Y$ (respectively, $X \simeq^{\mathsf{W}}_2 Y$) for X, Y countable quasi-Polish (respectively, Polish) spaces of the same cardinality.

Proof. (1) For the nontrivial direction, notice that by Proposition 2.13(1) any bijection $f \colon X \to Y$ is a witness of $X \simeq^{\mathsf{W}}_3 Y$.

(2) It is a classical fact that a space is T_1 if and only if its singletons are closed: hence any bijection between X and Y witnesses $X \simeq^{\mathsf{W}}_2 Y$.

(3) By Proposition 2.13(2), $\{x\}$ is the intersection of an open set and a closed set for every $x \in X$, and similarly for every $y \in Y$: hence any bijection between X and Y witnesses $X \simeq^{\mathsf{W}}_2 Y$. □

Of course the general results above (Propositions 4.3,4.4 and 4.5) do not give in general an optimal bound (in the sense of Definition 4.1) on the minimal complexity of an isomorphism between two specific quasi-Polish spaces X and Y. In the next proposition we collect some easy observations concerning the possible complexity of isomorphism between concrete examples of quasi-Polish spaces, including the following:

(1) ω endowed with the discrete topology;
(2) the space $\mathbb{R}^n$ $(n \in \omega)$ endowed with the product of the order topology on $\mathbb{R}$;
(3) the ω-algebraic domain $(\omega^{\leq\omega}, \sqsubseteq)$ endowed with the Scott topology.

Remark 4.6. It is straightforward to check that if $f \in \mathsf{D}_2^{\mathsf{W}}(X, Y)$ with X σ-compact and Y an Hausdorff space, then the range of f is σ-compact as well. In particular, if X, Y are Polish spaces with X σ-compact and Y non σ-compact, then there is no onto $f \in \mathsf{D}_2^{\mathsf{W}}(X, Y)$.

Proposition 4.7.
(1) $\mathcal{N} \simeq_2^{\mathsf{W}} \omega \sqcup \mathcal{N}$;
(2) if X is a σ-compact quasi-Polish space then $\mathcal{N} \not\simeq_2^{\mathsf{W}} X$. In particular, $\mathcal{N} \not\simeq_2^{\mathsf{W}} \mathcal{C}$, $\mathcal{N} \not\simeq_2^{\mathsf{W}} \mathbb{R}^n$ for every $n < \omega$, and $\mathcal{N} \not\simeq_2^{\mathsf{W}} \omega^{\leq\omega}$;
(3) $\mathcal{N} \simeq_3^{\mathsf{W}} \mathcal{C}$. More precisely, there is a bijection $f\colon \mathcal{N} \to \mathcal{C}$ such that $f \in \mathsf{D}_2^{\mathsf{W}}(\mathcal{N}, \mathcal{C})$ and $f^{-1} \in \mathsf{D}_3^{\mathsf{W}}(\mathcal{C}, \mathcal{N})$;
(4) $\mathcal{N} \simeq_3^{\mathsf{W}} \mathbb{R}^n$ for every $1 \leq n < \omega$. More precisely, there is a bijection $f\colon \mathcal{N} \to \mathbb{R}^n$ such that $f \in \mathsf{D}_2^{\mathsf{W}}(\mathcal{N}, \mathbb{R}^n)$ and $f^{-1} \in \mathsf{D}_3^{\mathsf{W}}(\mathbb{R}^n, \mathcal{N})$;
(5) $\mathcal{N} \simeq_3^{\mathsf{W}} \omega^{\leq\omega}$. More precisely, there is a bijection $f\colon \mathcal{N} \to \omega^{\leq\omega}$ such that $f \in \mathsf{D}_2^{\mathsf{W}}(\mathcal{N}, \omega^{\leq\omega})$ and $f^{-1} \in \mathsf{D}_3^{\mathsf{W}}(\omega^{\leq\omega}, \mathcal{N})$.

Proof. (1) The space $X = \mathcal{N} \setminus \{n0^\omega \mid n \in \omega\}$ is a nonempty perfect zero-dimensional Polish space whose compact subsets all have empty interior, hence it is homeomorphic to $\mathcal{N}$ by the Alexandrov-Urysohn theorem (Kechris 1995, Theorem 7.7). Let $\hat{f}\colon X \to \mathcal{N}$ be a witness of this fact, and extend $\hat{f}$ to a bijection $f\colon \mathcal{N} \to \omega \sqcup \mathcal{N}$ by setting $f(n0^\omega) = n$ for every $n \in \omega$. Since X is open in $\mathcal{N}$ and each $\{n0^\omega\}$ is closed in $\mathcal{N}$, the partition $\langle X, \{n0^\omega\} \mid n \in \omega\rangle$ of $\mathcal{N}$ witnesses that $f \in \mathsf{D}_2^{\mathsf{W}}(\mathcal{N}, \omega \sqcup \mathcal{N})$. Conversely, the partition $\langle \mathcal{N}, \{n\} \mid n \in \omega\rangle$ is a clopen partition of $\omega \sqcup \mathcal{N}$ witnessing $f^{-1} \in \mathsf{D}_2^{\mathsf{W}}(\omega \sqcup \mathcal{N}, \mathcal{N})$.

(2) Since $\mathcal{N}$ is not σ-compact, the claim follows from Remark 4.6.

(3) By part (1), it is enough to prove the claim with $\mathcal{N}$ replaced by $\omega \sqcup \mathcal{N}$. Let $\hat{f}\colon \mathcal{N} \to \mathcal{C}$ be the well-known homeomorphism between $\mathcal{N}$ and $Y = \{y \in \mathcal{C} \mid \forall n \exists m \geq n\, (y(m) = 1)\}$ given by $\hat{f}(x) = 0^{x(0)}10^{x(1)}10^{x(2)}10^{x(3)}\ldots$ Since $\mathcal{C} \setminus Y$ is countable, we can fix an enumeration $\langle y_n \mid n \in \omega\rangle$ without repetitions of such a set. Extend $\hat{f}$ to a bijection $f\colon \omega \sqcup \mathcal{N} \to \mathcal{C}$ by setting $f(n) = y_n$ for every $n \in \omega$. Since each point of the spaces $\omega \sqcup \mathcal{N}$ and $\mathcal{C}$ is closed, $\mathcal{N}$ is (cl)open in $\omega \sqcup \mathcal{N}$, and Y is a (proper) $\mathbf{\Pi}_2^0(\mathcal{C})$ set, we have that $\langle \mathcal{N}, \{n\} \mid n \in \omega\rangle$ is a clopen partition of $\omega \sqcup \mathcal{N}$ witnessing $f \in \mathsf{D}_2^{\mathsf{W}}(\omega \sqcup \mathcal{N}, \mathcal{C})$, and $\langle Y, \{y_n\} \mid n \in \omega\rangle$ is a $\mathbf{\Pi}_2^0$-partition of $\mathcal{C}$ witnessing $f^{-1} \in \mathsf{D}_3^{\mathsf{W}}(\mathcal{C}, \omega \sqcup \mathcal{N})$.

(4) Let first $n = 1$. By part (1), it is enough to prove the claim with $\mathcal{N}$ replaced by $\omega \sqcup \mathcal{N}$. Let $\langle q_k \mid k \in \omega\rangle$ be an enumeration without repetition of the set of rational numbers $\mathbb{Q}$. It is well-known that $\mathcal{N}$ and $\mathbb{R} \setminus \mathbb{Q}$ are homeomorphic, so let $\hat{f}$ be a witness of this fact. Extend $\hat{f}$ to a bijection $f\colon \omega \sqcup \mathcal{N} \to \mathbb{R}$ by setting $f(k) = q_k$ for every $k \in \omega$. Since $\mathbb{R} \setminus \mathbb{Q}$ is a (proper) $\mathbf{\Pi}_2^0(\mathbb{R})$ set and each singleton of $\mathbb{R}$ (and hence of $\mathbb{Q}$) is closed, we have that $\langle \mathcal{N}, \{n\} \mid n \in \omega\rangle$ is a clopen partition of $\omega \sqcup \mathcal{N}$ witnessing $f \in \mathsf{D}_2^{\mathsf{W}}(\omega \sqcup \mathcal{N}, \mathbb{R})$, and $\langle \mathbb{R} \setminus \mathbb{Q}, \{q_n\} \mid n \in \omega\rangle$ is a $\mathbf{\Pi}_2^0$-partition of $\mathcal{C}$ witnessing $f^{-1} \in \mathsf{D}_3^{\mathsf{W}}(\mathbb{R}, \omega \sqcup \mathcal{N})$.

Now assume $n > 1$. First observe that $Z = \bigcup_{0<i<n}([n]^i \times \mathbb{Q}^i \times \mathcal{N})$ (where each $[n]^i \times \mathbb{Q}^i$ is endowed with the discrete topology) is homeomorphic to $\mathcal{N}$, hence by part (1) and the

fact that $\mathcal{N} \sqcup \mathcal{N}$ is homeomorphic to $\mathcal{N}$ it is enough to prove the claim with $\mathcal{N}$ replaced by $\omega \sqcup \mathcal{N} \sqcup Z$. For each $0 < i < n$, $a = \{a_0, \dots, a_{i-1}\} \in [n]^i$ and $s \in \mathbb{Q}^i$, let $f_{a,s}$ be an homeomorphism between $\mathcal{N}$ and

$$\mathbb{R}_{a,s} = \{x \in \mathbb{R}^n \mid \forall j < i\,(x(a_j) = s(j)) \wedge \forall k \notin a\,(x(k) \in \mathbb{R} \setminus \mathbb{Q})\}$$

(such an homeomorphism exists because $\mathbb{R}_{a,s}$ is homeomorphic to $(\mathbb{R} \setminus \mathbb{Q})^{n-i}$). Let also $\hat{f}$ be an homeomorphism between $\mathcal{N}$ and $(\mathbb{R} \setminus \mathbb{Q})^n$, and $\langle t_k \mid k \in \omega\rangle$ be an enumeration without repetitions of $\mathbb{Q}^n$. Then define

$$f \colon \omega \sqcup \mathcal{N} \sqcup Z \to \mathbb{R}^n$$

by setting $f(k) = t_k$, $f(x) = \hat{f}(x)$, and $f(a, s, x) = f_{a,s}(x)$ for every $k \in \omega$, $x \in \mathcal{N}$ and $(a, s) \in \bigcup_{0<i<n}([n]^i \times \mathbb{Q}^i)$. It is easy to check that f is in fact a bijection. Moreover, since all the $\mathbb{R}_{a,s}$ and $(\mathbb{R} \setminus \mathbb{Q})^n$ are $\mathbf{\Pi}^0_2(\mathbb{R}^n)$ sets and all points are closed in $\mathbb{R}^n$, we have that

$$\langle \mathcal{N}, \{k\}, \{(a, s, x) \mid x \in \mathcal{N}\} \mid k \in \omega, (a, s) \in \bigcup_{0<i<n} ([n]^i \times \mathbb{Q}^i)\rangle$$

is a clopen partition of $\omega \sqcup \mathcal{N} \sqcup Z$ witnessing $f \in \mathsf{D}^{\mathsf{W}}_2(\omega \sqcup \mathcal{N} \sqcup Z, \mathbb{R}^n)$, and

$$\langle \mathbb{Q}^n, (\mathbb{R} \setminus \mathbb{Q})^n, \mathbb{R}_{a,s} \mid (a, s) \in \bigcup_{0<i<n} ([n]^i \times \mathbb{Q}^i)\rangle$$

is a $\mathbf{\Pi}^0_2$-partition of $\mathbb{R}^n$ witnessing $f^{-1} \in \mathsf{D}^{\mathsf{W}}_3(\mathbb{R}^n, \omega \sqcup \mathcal{N} \sqcup Z)$.

(5) By part (1), it is again enough to prove the claim with $\mathcal{N}$ replaced by $\omega \sqcup \mathcal{N}$. Let $\langle \sigma_n \mid n \in \omega\rangle$ be an enumeration without repetition of ω^*. Define $f \colon \omega \sqcup \mathcal{N} \to \omega^{\leq\omega}$ by setting $f(n) = \sigma_n$ and $f(x) = x$ for all $n \in \omega$ and $x \in \mathcal{N}$. Since all the $\sigma \in \omega^*$ are compact elements of $\omega^{\leq\omega}$, their singletons are $\mathbf{\Delta}^0_2(\omega^{\leq\omega})$ subsets by Proposition 2.13(3), hence $\mathcal{N}$ is a $\mathbf{\Pi}^0_2(\omega^{\leq\omega})$ set. Therefore, $\langle \mathcal{N}, \{n\} \mid n \in \omega\rangle$ is a clopen partition of $\omega \sqcup \mathcal{N}$ witnessing $f \in \mathsf{D}^{\mathsf{W}}_2(\omega \sqcup \mathcal{N}, \omega^{\leq\omega})$, while $\langle \mathcal{N}, \{\sigma\} \mid \sigma \in \omega^*\rangle$ is a $\mathbf{\Pi}^0_2$-partition of $\omega^{\leq\omega}$ witnessing $f^{-1} \in \mathsf{D}^{\mathsf{W}}_3(\omega^{\leq\omega}, \omega \sqcup \mathcal{N})$. □

A natural way to compute the complexity of an isomorphism between two topological spaces is given by the following variant of the usual Schröder-Bernstein argument (see also (Jayne and Rogers 1979b)).

Lemma 4.8. Let $1 \leq \alpha < \omega_1$, $X, Y \in \mathscr{X}$, and $\mathcal{F}$ be a collection of functions between topological spaces closed under restrictions (i.e. $f \restriction X' \in \mathcal{F}$ for every $f \colon X \to Y \in \mathcal{F}$ and $X' \subseteq X$). If X is $\mathcal{F}$-isomorphic to a subset of Y via some $f \in \mathbf{\Pi}^0_\alpha[X, Y]$ and Y is $\mathcal{F}$-isomorphic to a subset of X via some $g \in \mathbf{\Pi}^0_\alpha[Y, X]$, then there are $X' \in \mathbf{\Delta}^0_{\alpha+1}(X)$ and $Y' \in \mathbf{\Delta}^0_{\alpha+1}(Y)$ such that $X' \simeq_{\mathcal{F}} Y'$ and $X \setminus X' \simeq_{\mathcal{F}} Y \setminus Y'$.

In particular, if X (respectively, Y) is homeomorphic to a $\mathbf{\Pi}^0_\alpha$ subset of Y (respectively, X), then $X \simeq^{\mathsf{W}}_{\alpha+1} Y$.

Proof. Inductively define $X_n \subseteq X$ and $Y_n \subseteq Y$, $n \in \omega$, by setting $X_0 = X$, $Y_0 = Y$, $X_{n+1} = g(Y_n)$, $Y_{n+1} = f(X_n)$. Let also $X_\infty = \bigcap_{n\in\omega} X_n$ and $Y_\infty = \bigcap_{n\in\omega} Y_n$. By our assumption on f and g, all of $X_n, Y_n, X_\infty, Y_\infty$ are in $\mathbf{\Pi}^0_\alpha$. Let $X' = X_\infty \cup \bigcup_{n\in\omega}(X_{2n} \setminus X_{2n+1})$ and $Y' = Y_\infty \cup \bigcup_{n\in\omega}(Y_{2n+1} \setminus Y_{2n+2})$. By their definition, X' and Y' are both in $\mathbf{\Sigma}^0_{\alpha+1}$. Since $X \setminus X' = \bigcup_{n\in\omega}(X_{2n+1} \setminus X_{2n})$ and $Y \setminus Y' = \bigcup_{n\in\omega}(Y_{2n} \setminus Y_{2n+1})$ are

both in $\mathbf{\Sigma}^0_{\alpha+1}$ as well, we have that $X', Y' \in \mathbf{\Delta}^0_{\alpha+1}$. Finally, $f \restriction X'$ and $g^{-1} \restriction (X \setminus X')$ witness that $X' \simeq_{\mathcal{F}} Y'$ and $X \setminus X' \simeq_{\mathcal{F}} Y \setminus Y'$ because $\mathcal{F}$ is closed under restrictions and $(f \restriction X')^{-1} = f^{-1} \restriction Y'$ and $(g^{-1} \restriction (X \setminus X'))^{-1} = g \restriction (Y \setminus Y')$. □

We can immediately derive some corollaries from Lemma 4.8. We need to recall the following definition from general topology: two spaces are of the same Fréchet dimension type if each one is homeomorphic to a subset of the other.

Corollary 4.9. If X, Y are two quasi-Polish spaces which are of the same Fréchet dimension type then $X \simeq^{\mathsf{W}}_3 Y$.

Proof. Apply the second part of Lemma 4.8 with $\alpha = 2$, using the fact that the class of quasi-Polish spaces is closed under homeomorphism and Proposition 2.9. □

The second part of the next corollary has been essentially already noticed in (Jayne and Rogers 1979b, Theorem 6.5).

Corollary 4.10. If X, Y are two quasi-Polish spaces such that X is homeomorphic to a closed subset of Y and Y is homeomorphic to a closed subset of X then $X \simeq^{\mathsf{W}}_2 Y$.

In particular, if X, Y are compact Hausdorff quasi-Polish spaces of the same Fréchet dimension type then $X \simeq^{\mathsf{W}}_2 Y$.

Proof. The first part follows from Lemma 4.8 with $\alpha = 1$. The second part follows from the first one and the classical facts that the class of compact spaces is closed under continuous images, and that a compact subset of an Hausdorff space is closed. □

Our next goal is to extend Proposition 4.7 (3)–(5) to a wider class of quasi-Polish spaces (see Theorem 4.21). Such generalization will involve the definition of the (inductive) topological dimension of a space X, denoted in this paper by $\dim(X)$ — see e.g. (Hurewicz and Wallman 1948, p. 24).

Definition 4.11. The empty set $\emptyset$ is the only space in $\mathscr{X}$ with *dimension* -1, in symbols $\dim(\emptyset) = -1$.

Let α be an ordinal and $\emptyset \neq X \in \mathscr{X}$. We say that X has *dimension* $\leq \alpha$, $\dim(X) \leq \alpha$ in symbols, if every $x \in X$ has arbitrarily small neighborhoods whose boundaries have dimension $< \alpha$, i.e. for every $x \in X$ and every open set U containing x there is an open $x \in V \subseteq U$ such that $\dim(\partial V) \leq \beta$ (where $\partial V = \mathrm{cl}(V) \setminus V$ and $\mathrm{cl}(V)$ is the closure of V in X) for some $\beta < \alpha$.

We say that a space X has *dimension* α, $\dim(X) = \alpha$ in symbols, if $\dim(X) \leq \alpha$ and $\dim(X) \not\leq \beta$ for all $\beta < \alpha$.

Finally, we say that a space X has *dimension* ∞, $\dim(X) = \infty$ in symbols, if $\dim(X) \not\leq \alpha$ for every $\alpha \in \mathsf{On}$.

It is obvious that the dimension of a space is a topological invariant (i.e. $\dim(X) = \dim(Y)$ whenever X and Y are homeomorphic). Moreover, one can easily check that $\dim(X) \leq \alpha$ (for α an ordinal) if and only if there is a base of the topology of X consisting of open sets whose boundaries have dimension $< \alpha$. Therefore, if X is countably based and $\dim(X) \neq \infty$, then $\dim(X) = \alpha$ for some *countable* ordinal α.

The following lemma shows that the notion of dimension is monotone.

Lemma 4.12. (Hurewicz and Wallman 1948, Theorem III 1) Let $X \in \mathscr{X}$ and α be an ordinal such that $\dim(X) \leq \alpha$. Then for every $Y \subseteq X$, $\dim(Y) \leq \alpha$ (where Y is endowed with the relative topology inherited from X).

Proof. This is proved by induction on α, using the fact that if $Y \subseteq X$ and U is open in X then the boundary in Y of $U \cap Y$ is contained in the boundary of U in X. □

It is a classical fact that for every $\alpha < \omega_1$ there is a compact Polish space of dimension α, and that the Hilbert cube $[0,1]^\omega$ is a compact Polish space of dimension ∞. Here we provide various examples of computations of the dimension of some concrete quasi-Polish spaces which are relevant for the results of this paper.

Example 4.13. *Finite dimension.*

(1) $\dim(\mathcal{N}) = \dim(\mathcal{C}) = 0$;
(2) $\dim(\mathbb{R}^n) = n$ for every $0 \neq n \leq \omega$;
(3) for $n < \omega$, let L_n be the (finite) quasi-Polish space obtained by endowing the dcpo $(n, \leq)$ with the Scott (equivalently, the Alexandrov) topology: then $\dim(L_n) = n - 1$.

Proof. (1) The canonical basis for $\mathcal{N}$ and $\mathcal{C}$ (namely, the collection of all sets of the form $\sigma \cdot \mathcal{N}$ for $\sigma \in \omega^*$ and, respectively, $\sigma \cdot \mathcal{C}$ for $\sigma \in 2^*$) consist of clopen sets, hence their elements have empty boundary.

(2) This is a classical (nontrivial) fact, see e.g. (Hurewicz and Wallman 1948, Theorem IV 1).

(3) This is proved by induction on $n \geq 0$. If $n = 0$, then $L_n = \emptyset$ and hence $\dim(L_0) = -1$ by definition. Now assume $\dim(L_i) = i$ for every $i \leq n$ and consider the space L_{n+1}. Every open set of L_n is of the form $U_i = \{j \in L_{n+1} \mid j \geq i\}$ for some $i \leq n$, and $\partial U_i = L_i$: hence by the inductive hypothesis $\dim(\partial U_i) < n + 1$ for every $i \leq n$, which implies $\dim(L_{n+1}) \leq n + 1$. Moreover, the set $\{n\}$ is open in L_{n+1}, and is obviously the minimal open set containing n. Since $\partial\{n\} = L_n$, $\dim(L_{n+1}) > \dim(L_n) = n$. Therefore $\dim(L_{n+1}) = n + 1$, as desired. □

Example 4.14. *Transfinite dimension.*

(1) the disjoint union $X = \bigsqcup_{0 \neq n \in \omega} [0,1]^n$ of the n-dimensional cubes $[0,1]^n$ is a Polish space of dimension ω;
(2) let $\omega^{\leq\omega}$ be the ω-algebraic domain $(\omega^{\leq\omega}, \sqsubseteq)$ endowed with the Scott topology: then $\dim(\omega^{\leq\omega}) = \omega$;
(3) for $\alpha < \omega_1$, let $L_{\alpha+1}$ be the quasi-Polish space obtained by endowing the dcpo[‡‡] $(\alpha + 1, \leq)$ with the Scott topology. Then $\dim(L_{\alpha+1}) = \alpha$.

Proof. (1) By part (2), each $[0,1]^n$ has dimension n. Since $[0,1]^n$ is topologically embedded in X, by Lemma 4.12 we have $\dim(X) \geq n$ for every $n \in \omega$, and hence $\dim(X) \geq \omega$. Let $\mathcal{B}_n = \{B_{n,m} \mid m \in \omega\}$ be a countable basis of $[0,1]^n$ such that $\dim(\partial B_{n,m}) < n$ for

[‡‡] Here we cannot consider the limit case, as if α is limit then the poset $(\alpha, \leq)$ is not directed-complete, and hence falls out of the scope of the spaces considered in this paper.

every $m \in \omega$. Then $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ is a basis for X with the property that for every $U \in \mathcal{B}$, $\dim(\partial U) < \omega$: hence $\dim(X) \leq \omega$, and therefore $\dim(X) = \omega$.

(2) Since every L_n can be topologically embedded in $\omega^{\leq\omega}$, $\dim(\omega^{\leq\omega}) \geq \omega$ by Lemma 4.12 and Example 4.13(2). Consider the basis $\mathcal{B}$ of $\omega^{\leq\omega}$ consisting of the open sets generated by its compact elements, i.e. of the sets $\sigma\cdot\omega^{\leq\omega}$ for $\sigma \in \omega^*$. Then $\partial\sigma\cdot\omega^{\leq\omega} = \{\tau \sqsubseteq \sigma \mid \tau \neq \sigma\}$. Therefore $\partial\sigma \cdot \omega^{\leq\omega}$ is homeomorphic to L_n, where n is the length of σ: this means that, by Example 4.13(2) again, $\dim(\partial U) < \omega$ for every $U \in \mathcal{B}$, and hence $\dim(\omega^{\leq\omega}) \leq \omega$. Therefore $\dim(\omega^{\leq\omega}) = \omega$.

(3) By an inductive argument similar to the proof of Example 4.13(3). □

Example 4.15. *Dimension* ∞.

(1) the Hilbert cube $[0,1]^\omega$, the space $\mathbb{R}^\omega$ (both endowed with the product topology), and the Scott domain $P\omega$ have all dimension ∞;
(2) Let C_∞ be the quasi-Polish space obtained by endowing the poset $(\omega, \geq)$ with the Scott (equivalently, the Alexandrov) topology. Then C_∞ is a (scattered) countable space with $\dim(\mathsf{C}_\infty) = \infty$. Hence the space $\mathsf{UC}_\infty = \mathsf{C}_\infty \times \mathcal{N}$, endowed with the product topology, is an (uncountable) quasi-Polish space of dimension ∞.

Proof. (1) It is a classical fact that $\dim([0,1]^\omega) = \dim(\mathbb{R}^n) = \infty$ — see e.g. Corollary on p. 51 of (Hurewicz and Wallman 1948). Since the Hilbert cube can be topologically embedded into $P\omega$ by Proposition 2.12, it follows from Lemma 4.12 that also the Scott domain has dimension ∞.

(2) To show that a topological space X has dimension ∞ it is enough to find a point $x \in X$ and an open neighborhood U of x such that X can be topologically embedded into ∂V for every open $x \in V \subseteq U$. Consider the point $0 \in \mathsf{C}_\infty$. Since 0 is a compact element, the basic open set $U = \uparrow 0 = \{0\}$ generated by 0 is a minimal (with respect to inclusion) open neighborhood for this point, hence it is enough to show that C_∞ can be topologically embedded into ∂U. Since C_∞ has a topmost element (i.e. 0 itself), $\partial U = \mathsf{C}_\infty \setminus U$; but then the map sending n into $n+1$ (for every $n \in \omega$) is clearly an homeomorphism between C_∞ and $\mathsf{C}_\infty \setminus U$, and hence $\dim(\mathsf{C}_\infty) = \infty$, as required. The second part of the claim follows from Lemma 4.12 and the fact that C_∞ can be topologically embedded into UC_∞ in the obvious way. □

Remark 4.16. The definition of dimension is usually formulated for separable metric spaces (Hurewicz and Wallman 1948) or for regular topological spaces. This is because the received opinion is that outside this scope this notion becomes somewhat pathological. Examples 4.13, 4.14 and 4.15 show e.g. that there are finite (quasi-Polish) spaces with nonzero dimension,[§§] and countable (quasi-Polish) spaces with arbitrarily high ordinal dimension, or even of dimension ∞: this seems to contradict our intuition of "geometric dimension". Nevertheless, Lemma 4.12 shows that some natural properties of the dimension function $\dim(\cdot)$ are preserved when considering arbitrary spaces, and Theorem 4.21 will show that it remains a quite useful notion also in this broader context.

[§§] Notice that there are also examples of *Hausdorff* countable spaces with nonzero dimension.

We now recall some classical results that will be used later.

Lemma 4.17. (see e.g. (Hurewicz and Wallman 1948, pp. 50-51)) Let X be a Polish space. Then the following are equivalent:

(1) $\dim(X) \neq \infty$;
(2) $X = \bigcup_{n<\omega} X_n$ with all the X_n of finite dimension (i.e. $\dim(X_n) < \omega$ for every $n < \omega$);
(3) $X = \bigcup_{n<\omega} X_n$ with all the X_n of dimension 0.

Notice that by Example 4.15(2), Lemma 4.17 cannot be extended to the context of arbitrary quasi-Polish spaces: the space C_∞ has dimension ∞, but can be decomposed into countably many zero-dimensional spaces (namely, its singletons). Similarly, UC_∞ can be decomposed into countably many copies of $\mathcal{N}$. On the other hand, we have the following corollary.

Corollary 4.18. The Scott domain $P\omega$ cannot be written as $\bigcup_{n<\omega} X_n$ with all the X_n of finite dimension. The same is true if $P\omega$ is replaced with any quasi-Polish space which is universal for (compact) Polish spaces.

Proof. Since $P\omega$ is universal for the class of all (quasi-)Polish spaces by Proposition 2.12, any decomposition into countably many finite dimensional spaces of $P\omega$ would induce a similar decomposition of e.g. $[0,1]^\omega$, contradicting Lemma 4.17. □

Lemma 4.19. (Jayne and Rogers 1979b, Theorem 6.1) Let X be an uncountable zero-dimensional Polish space:

(1) if X is σ-compact then $X \simeq_2^{\mathsf{W}} \mathcal{C}$;
(2) if X is not σ-compact then $X \simeq_2^{\mathsf{W}} \mathcal{N}$.

Proof. First assume that X is σ-compact. Since $\mathcal{C}$ is (σ-)compact as well, it is enough to show that $X \simeq_2^{\mathsf{W}} (\omega \times \mathcal{C}) \cup (\omega \times \{3^\omega\})$. Let K_n be compact sets such that $X = \bigcup_{n\in\omega} K_n$. Notice that since X is zero-dimensional we can assume that the K_n's are pairwise disjoint. (If not, replace the K_n's with any closed refinement of the partition of X given by the sets $D_n = K_n \setminus \bigcup_{i<n} K_i \in \mathbf{\Delta}^0_2$.) By the Cantor-Bendixson theorem (Kechris 1995, Theorem 6.4) and the obvious fact that every uncountable zero-dimensional Polish space can be partitioned into countably many closed sets such that infinitely many of them are uncountable and infinitely many of them are singletons, we can further assume that all the K_{2n} are nonempty and perfect and all the K_{2n+1} are singletons. By Brouwer's theorem (Kechris 1995, Theorem 7.4), for every $n \in \omega$ there is an homeomorphism f_{2n} between K_{2n} and $\{n\} \times \mathcal{C}$. Let f_{2n+1} be the constant function sending the unique point in K_{2n+1} to $(n, 3^\omega)$. Then the closed partitions $\langle K_n \mid n \in \omega\rangle$ and $\langle \{n\} \times \mathcal{C}, \{(n, 3^\omega)\} \mid n \in \omega\rangle$ of, respectively, X and $(\omega \times \mathcal{C}) \cup (\omega \times \{3^\omega\})$, together with the homeomorphisms $\langle f_n \mid n \in \omega\rangle$, witness that $X \simeq_{\mathsf{pw}(\mathbf{\Delta}^0_2)} (\omega \times \mathcal{C}) \cup (\omega \times \{3^\omega\})$, and hence $X \simeq_2^{\mathsf{W}} (\omega \times \mathcal{C}) \cup (\omega \times \{3^\omega\})$ by Lemma 4.2.

Now assume that X is not σ-compact. By Hurewicz' theorem (see e.g. (Kechris 1995, Theorem 7.10), X contains a closed set homeomorphic to $\mathcal{N}$. Conversely, by (Kechris 1995, Theorem 7.8) we have that X, being zero-dimensional, is homeomorphic to a closed subset of $\mathcal{N}$. Hence $X \simeq_2^{\mathsf{W}} \mathcal{N}$ by Lemma 4.8. □

Remark 4.20. Formally, (Jayne and Rogers 1979b, Theorem 6.1) is stated using D_2-isomorphisms instead of $\mathsf{D}_2^{\mathsf{W}}$-isomorphisms. However, our formulation can be recovered from that result *a posteriori* by using (Jayne and Rogers 1982, Theorem 5).

We are now ready to prove the main theorem of this section.

Theorem 4.21. Let X be an uncountable quasi-Polish space.

(1) if $\dim(X) \neq \infty$ then there is a bijection $f\colon \mathcal{N} \to X$ such that $f \in \mathsf{D}_2^{\mathsf{W}}(\mathcal{N}, X)$ and $f^{-1} \in \mathsf{D}_3^{\mathsf{W}}(X, \mathcal{N})$. In particular, $\mathcal{N} \simeq_3^{\mathsf{W}} X$;
(2) if $\dim(X) = \infty$ and X is Polish then $\mathcal{N} \not\simeq_\alpha^{\mathsf{W}} X$ for every $\alpha < \omega_1$ and $\mathcal{N} \not\simeq_n X$ for every $n < \omega$;
(3) $P\omega \not\simeq_\alpha^{\mathsf{W}} \mathcal{N}$ for every $\alpha < \omega_1$ and $P\omega \not\simeq_n \mathcal{N}$ for every $n < \omega$. The same result holds when replacing $P\omega$ with any other quasi-Polish space which is universal for (compact) Polish spaces;
(4) $\mathsf{UC}_\infty \simeq_2^{\mathsf{W}} \mathcal{N}$. Therefore $\mathsf{UC}_\infty \not\simeq_\alpha^{\mathsf{W}} X$ ($\alpha < \omega_1$) and $\mathsf{UC}_\infty \not\simeq_n X$ ($n \in \omega$) for X a Polish space of dimension ∞ (e.g. $X = [0,1]^\omega$ or $X = \mathbb{R}^\omega$) or $X = P\omega$.

Proof. (1) Since X is uncountable (hence nonempty) and countably based, we can assume $0 \leq \dim(X) < \omega_1$. We argue by induction on $\dim(X) = \alpha$. If $\alpha = 0$, then X is Hausdorff and regular. By the Urysohn's metrization theorem (Kechris 1995, Theorem 1.1), we have that X is metrizable and hence Polish by Proposition 2.14. Therefore the claim for X follows from Lemma 4.19 and Proposition 4.7(3). Now assume that $\alpha > 0$ and that the claim is true for every quasi-Polish space of dimension $< \alpha$. Let $\mathcal{B} = \{B_n \mid n \in \omega\}$ be a countable base for the topology of X such that $\dim(\partial B_n) < \alpha$ for every $n \in \omega$. Let $X' = X \setminus \bigcup_{n\in\omega} \partial B_n$ and inductively define $B'_n = \partial B_n \setminus \bigcup_{i<n} \partial B_i$. All of X', B'_n are $\mathbf{\Pi}^0_2(X)$ sets, so they are quasi-Polish by Proposition 2.9. Moreover, they clearly form a (countable) partition of X. The space X' is zero-dimensional, so by Lemma 4.19 and Proposition 4.7(3) there is a bijection $f_0\colon \{0\} \times \mathcal{N} \to X'$ such that $f_0 \in \mathsf{D}_2^{\mathsf{W}}(\{0\} \times \mathcal{N}, X')$ and $f_0^{-1} \in \mathsf{D}_3^{\mathsf{W}}(X', \{0\} \times \mathcal{N})$. By our hypothesis on $\mathcal{B}$ and Lemma 4.12, $\dim(B'_n) < \alpha$ for every $n \in \omega$, hence by inductive hypothesis for each $n \in \omega$ there is a bijection $f_{n+1}\colon \{n+1\} \times \mathcal{N} \to B'_n$ such that $f_{n+1} \in \mathsf{D}_2^{\mathsf{W}}(\{n+1\} \times \mathcal{N}, B'_n)$ and $f_{n+1}^{-1} \in \mathsf{D}_3^{\mathsf{W}}(B'_n, \{n+1\} \times \mathcal{N})$. Let h be an homeomorphism between $\mathcal{N}$ and $\omega \times \mathcal{N}$: then, using the fact that each $\{n\} \times \mathcal{N}$ is clopen in $\omega \times \mathcal{N}$ and all of X', B'_n are in $\mathbf{\Pi}^0_2(X) \subseteq \mathbf{\Delta}^0_3(X)$, it is straightforward to check that $f = \left(\bigcup_{n\in\omega} f_n\right) \circ h\colon \mathcal{N} \to X$ is a bijection such that $f \in \mathsf{D}_2^{\mathsf{W}}(\mathcal{N}, X)$ and $f^{-1} \in \mathsf{D}_3^{\mathsf{W}}(X, \mathcal{N})$.

(2) For the first part, assume toward a contradiction that $\mathcal{N} \simeq_\alpha^{\mathsf{W}} X$ for some $\alpha < \omega_1$. Then $\mathcal{N} \simeq_{\mathsf{pw}(\mathbf{\Delta}^0_\alpha)} X$ by Lemma 4.2, i.e. there would be countable partitions $\langle Y_n \mid n \in \omega\rangle, \langle X_n \mid n \in \omega\rangle$ in $\mathbf{\Delta}^0_\alpha$ sets of, respectively, $\mathcal{N}$ and X such that Y_n is homeomorphic to X_n for every $n \in \omega$. In particular, by Lemma 4.12 each of the Y_n would be zero-dimensional, and since homeomorphisms preserve dimension, we would also have $\dim(X_n) = 0$ for every $n \in \omega$, contradicting Lemma 4.17.

For the second part, we argue again by contradiction. Let f be a witness of $\mathcal{N} \simeq_n X$ (for some $n \in \omega$). By Proposition 3.11, both f and f^{-1} are decomposable, and hence

$\mathcal{N} \simeq_{\mathsf{pw}} X$ by Lemma 4.2. From this fact, arguing as in the first part, we again reach a contradiction with Lemma 4.17.

(3) For the first part, argue as in part (2), using Corollary 4.18 instead of Lemma 4.17.

For the second part, assume towards a contradiction that $P\omega \simeq_n \mathcal{N}$ (for some $n < \omega$). Since $P\omega$ is universal for quasi-Polish spaces by Proposition 2.12, there is a $\mathbf{\Pi}^0_2(P\omega)$ set X which is homeomorphic to $[0,1]^\omega$, and hence the image $Y \subseteq \mathcal{N}$ of X under any witness of $P\omega \simeq_n \mathcal{N}$ would be a Borel set such that $Y \simeq_n [0,1]^\omega$. Arguing as in part (2), we reach a contradiction with Lemma 4.17.

(4) Since $\mathcal{N}$ is homeomorphic to $\omega \times \mathcal{N}$, it is enough to prove that $\mathsf{UC}_\infty \simeq^{\mathsf{W}}_2 \omega \times \mathcal{N}$. Since each element of C_∞ is compact, $\{n\} \in \mathbf{\Delta}^0_2(\mathsf{C}_\infty)$ for every $n \in \omega$. Hence the sets $X_n = \{n\} \times \mathcal{N}$, which are all homeomorphic to $\mathcal{N}$, form a countable partition of UC_∞ in $\mathbf{\Delta}^0_2$ pieces. It follows that $\mathsf{UC}_\infty \simeq_{\mathsf{pw}(\mathbf{\Delta}^0_2)} \omega \times \mathcal{N}$, whence $\mathsf{UC}_\infty \simeq^{\mathsf{W}}_2 \omega \times \mathcal{N}$ by Lemma 4.2. The second part of the claim follows from the first one and parts (2) and (3). □

Remark 4.22.

(1) The special case of Theorem 4.21(1) in which X is assumed to be Polish essentially appeared in (Jayne and Rogers 1979b, Theorem 8.1). The unique differences are that in their statement Jayne and Rogers used the assumption that X is a countable union of spaces of finite dimension (which for X Polish is equivalent to $\dim(X) \neq \infty$ by Lemma 4.17), and that the conclusion is weakened in (Jayne and Rogers 1979b, Theorem 8.1) to $\mathcal{N} \simeq_3 X$ (but their proof already gives the more precise statement considered here). Concerning the proofs, at a first glance the Jayne-Rogers original argument could seem different from the one used here, as it does not involve any induction on the dimension. However, their argument heavily relies on (Hurewicz and Wallman 1948, Theorem III 3), whose proof already implicitly shows the result under consideration and is essentially the same as the one we used above. Therefore the instance of Theorem 4.21(1) concerning Polish spaces can be dated back at least to the work of Hurewicz and Wallman of 1948.

(2) Theorem 4.21(1) is optimal: in fact by (the obvious generalization to transfinite dimension of) (Jayne and Rogers 1982, Theorem 13), if $X \simeq_2 Y$ then $\dim(X) = \dim(Y)$. The converse is not true, as if X is a compact Polish space of dimension $\alpha < \omega_1$, then $\dim(X \sqcup \mathcal{N}) = \dim(X) = \alpha$ (by the generalization to transfinite dimensions of the Sum Theorem (Hurewicz and Wallman 1948, Theorem III 2)) but $X \sqcup \mathcal{N} \not\simeq_2 X$ by (Jayne and Rogers 1982, Theorem 5) and Remark 4.6. Nevertheless, in some specific cases one can get a better bound, e.g.:

   (a) Let X be an uncountable Polish space embedded in $\mathbb{R}^n$. Then X is D_2-isomorphic (equivalently, $\mathsf{D}^{\mathsf{W}}_2$-isomorphic) to $\mathbb{R}^n$ if and only if it is σ-compact and of dimension n (Jayne and Rogers 1979b, Theorem 6.3);

   (b) A Polish space that is locally Euclidean of dimension n is D_2-isomorphic (equivalently, $\mathsf{D}^{\mathsf{W}}_2$-isomorphic) to $\mathbb{R}^n$ (Jayne and Rogers 1979b, Theorem 6.4);

   (c) Let X, Y be two σ-compact metric spaces of dimension $\alpha < \omega_1$ that are universal for the compact metric spaces of dimension α. Then $X \simeq_2 Y$ by Theorem 7.1 in (Jayne and Rogers 1979b).

Theorem 4.21 has also several corollaries which generalize many results of various nature. The first one is related to Lemma 4.17. Recall that a zero-dimensional space X is called h-homogeneous if every clopen $U \subseteq X$ is homeomorphic to the entire space X. Examples of (uncountable) h-homogeneous spaces are $\mathcal{N}$ and $\mathcal{C}$.

Corollary 4.23. Let X be an uncountable Polish space. Then the following are equivalent:

(1) $\dim(X) \neq \infty$;
(2) $X = \bigcup_{n<\omega} X_n$ with all the X_n of finite dimension (equivalently, of dimension $\neq \infty$, or of dimension 0);
(3) $X = \bigcup_{n<\omega} X_n$ with each X_n a zero-dimensional h-homogeneous Polish space;
(4) $X \simeq_n \mathcal{N}$ for some $n < \omega$;
(5) $X \simeq_3^{\mathsf{W}} \mathcal{N}$.

Proof. (1) $\Rightarrow$ (5) by Theorem 4.21(1), (5) $\Rightarrow$ (4) is obvious, and (4) $\Rightarrow$ (1) by Theorem 4.21(2). Moreover (3) $\Rightarrow$ (2) is obvious, and (2) $\Rightarrow$ (1) by Lemma 4.17, so it is enough to show (5) $\Rightarrow$ (3). By Lemma 4.2, $\mathcal{N} \simeq_{\mathsf{pw}(\mathbf{\Delta}^0_3)} X$. It is a classical fact that every countable partition of $\mathcal{N}$ into $\mathbf{\Delta}^0_3$ pieces can be refined to a countable partition in $\mathbf{\Pi}^0_2$ pieces, hence there is a countable partition $\langle X'_n \mid n \in \omega\rangle$ of X such that each X_n is homeomorphic to a $\mathbf{\Pi}^0_2$-subset of $\mathcal{N}$. This means that each X'_n is a Polish zero-dimensional space. By e.g. (Ostrovsky 2011, Theorem 1), all of these X'_n can be written as countable unions $\bigcup_{m\in\omega} X_{n,m}$ of h-homogeneous spaces such that $X_{n,m} \in \mathbf{\Pi}^0_2(X_n)$ for all $m \in \omega$. Thus each $X_{n,m}$ is Polish by (Kechris 1995, Theorem 3.11), and hence any enumeration without repetitions $\langle X_n \mid n \in \omega\rangle$ of $\{X_{n,m} \mid n, m, \omega\}$ satisfies all the requirements of (3). □

Obviously, conditions (1)–(4) are true also when X is a countable Polish space and $\mathcal{N}$ is replaced by ω: however, the counterexamples C_∞ and UC_∞ of Example 4.15(2) show that this is no more true for countable *quasi-Polish* spaces, and that Corollary 4.23 cannot be extended in general to arbitrary quasi-Polish spaces. Nevertheless we have the following:

Corollary 4.24. Let X be an arbitrary quasi-Polish space of dimension $\neq \infty$. Then X can be written as a countable union of zero-dimensional h-homogeneous Polish spaces.

Proof. If X is countable the result is trivial (simply take the singletons of the elements of X as countable partition). If X is uncountable, it is enough to observe that the proofs of (1) $\Rightarrow$ (5) and (5) $\Rightarrow$ (3) of Corollary 4.23 are in fact valid for arbitrary uncountable quasi-Polish spaces. □

The next corollary extends Semmes' generalization (Theorem 3.4 in this paper) of Jayne-Rogers (Jayne and Rogers 1982, Theorem 5).

Corollary 4.25. Let X be any quasi-Polish space which is either countable or of dimension $\neq \infty$. Then $\mathsf{D}_3(X) = \mathsf{D}_3^{\mathsf{W}}(X)$.

Proof. If X is countable the result follows from Proposition 2.13 because then every function $f\colon X \to X$ is in $\mathsf{D}_3^{\mathsf{W}}(X)$. If X is uncountable, the result follows from Theo-

rem 4.21(1) and the fact that both the classes $\bigcup_{X,Y\in\mathscr{X}} \mathsf{D}_3(X,Y)$ and $\bigcup_{X,Y\in\mathscr{X}} \mathsf{D}_3^{\mathsf{W}}(X,Y)$ are closed under composition. □

This corollary shows an interesting phenomenon: Jayne-Rogers' original result stating that $\mathsf{D}_2(X) = \mathsf{D}_2^{\mathsf{W}}(X)$ (Jayne and Rogers 1982, Theorem 5) can arguably be considered to be simpler, even in the case $X = \mathcal{N}$, than Theorem 3.4. However, contrarily to the case of Corollary 4.25, it cannot be generalized to ω-algebraic domains, as shown by the following counterexample communicated to the authors by M. de Brecht. Let $X = (\omega+1, \leq)$ be endowed with the Scott topology (i.e. $X = L_{\omega+1}$ from Example 4.14(3)), and consider the function $f\colon X \to X$ defined by $f(\omega) = \omega$, $f(2i) = 2i+1$ and $f(2i+1) = 2i$ (for every $i \in \omega$): then $f \in \mathsf{D}_2(X) \setminus \mathsf{D}_2^{\mathsf{W}}(X)$. (A similar counterexample can of course be given also for uncountable quasi-Polish spaces — it is enough to consider the space $X \times \mathcal{N}$ and the map $(x,y) \mapsto (f(x),y)$, where f is the function defined above.)

The assumption that X be of dimension $\neq \infty$ in the uncountable case of Corollary 4.25 is not a true limitation: for example, by using Theorem 4.21(4), one gets that the corollary remains true also for $X = \mathsf{UC}_\infty$. However, the general case remains unclear.

Question 4.26. Is it possible to further generalize Corollary 4.25 to all uncountable (quasi-)Polish spaces of dimension ∞?

Notice that if Corollary 4.25 holds true for a space which is universal for all (quasi-)Polish spaces, then the answer to the previous question is automatically positive.

Finally, the next corollary generalizes a recent result of Ostrovsky (Ostrovsky 2011, p. 663) concerning the possibility of representing Borel sets as countable unions of h-homogeneous $\mathbf{\Pi}^0_2$ sets.

Corollary 4.27. Let X be a quasi-Polish space of dimension $\neq \infty$. Then every Borel set $B \subseteq X$ can be partitioned into countably many h-homogeneous $\mathbf{\Pi}^0_2$ subspaces of B.

Proof. By Theorem 4.21(1), $B \simeq_3^{\mathsf{W}} B'$ for some Borel $B' \subseteq \mathcal{N}$, hence by Lemma 4.2 there are two countable partitions $\langle B_n \mid n \in \omega\rangle$ and $\langle B'_n \mid n \in \omega\rangle$ of, respectively, B and B' into $\mathbf{\Delta}^0_3$ pieces such that B_n is homeomorphic to B'_n for every $n \in \omega$. Inspecting the proofs of the mentioned results, it is not hard to see that all of the B_n, B'_n can in fact be assumed to be in $\mathbf{\Pi}^0_2$. By (Ostrovsky 2011, Theorem 1), each B'_n can be partitioned into a countable union of h-homogeneous $\mathbf{\Pi}^0_2(B'_n)$ (hence also $\mathbf{\Pi}^0_2(B')$) sets. Therefore, a similar decomposition can be obtained also for the B_n's, and the union of these partitions gives the desired partition of B. □

Theorem 4.21(2)-(4) shows that the bound for the complexity of an isomorphism between *arbitrary* (quasi-)Polish spaces obtained in Proposition 4.3 cannot be improved, but leaves open the problem of computing the minimal complexity of an isomorphism between two *Polish* spaces with dimension ∞. In this direction, we can make some basic observation.

Proposition 4.28. Let X, Y be (quasi-)Polish spaces of dimension ∞.

(1) If X, Y are both universal for the class of all (quasi-)Polish spaces then $X \simeq_3^{\mathsf{W}} Y$.

(2) The Hilbert cube $[0,1]^\omega$ and the space $\mathbb{R}^\omega$ are both universal for Polish spaces, but $[0,1]^\omega \not\simeq_2^{\mathsf{W}} \mathbb{R}^\omega$ (i.e. $[0,1]^\omega \not\simeq_2 \mathbb{R}^\omega$): hence the bound of part (1) cannot be improved.
(3) If X and Y are compact Hausdorff spaces and they are both universal for compact Polish spaces then $X \simeq_2^{\mathsf{W}} Y$.

Proof. Parts (1) and (3) follow from Corollaries 4.9 and 4.10, respectively. Part (2) follows from Remark 4.6, the fact that the Hilbert cube $[0,1]^\omega$ is compact (by Tychonoff's theorem (Kechris 1995, Proposition 4.1 (vi))), and the fact that the space $\mathbb{R}^\omega$ is not σ-compact. To see this, consider $\bigcup_{n\in\omega} K_n \subseteq \mathbb{R}^\omega$ with all K_n compact: we will show that $\mathbb{R}^\omega \setminus \bigcup_{n\in\omega} K_n \neq \emptyset$. For every $n \in \omega$, let $\pi_n \colon \mathbb{R}^\omega \to \mathbb{R}$ be the projection on the n-th coordinate. Since π_n is continuous, $\pi_n(K_n) \subseteq \mathbb{R}$ is compact, hence there is $x_n \in \mathbb{R} \setminus \pi_n(K_n)$. Notice that every $y = \langle y_k \mid k \in \omega\rangle \in \mathbb{R}^\omega$ with $y_n = x_n$ does not belong to K_n: hence $x = \langle x_n \mid n \in \omega\rangle \in \mathbb{R}^\omega \setminus \bigcup_{n\in\omega} K_n$, as required. □

Question 4.29. Are there Polish spaces X, Y of dimension ∞ such that $X \not\simeq_3^{\mathsf{W}} Y$? Are there Polish spaces X, Y of dimension ∞ such that $X \not\simeq_n Y$ for every $n \in \omega$?

5. Degree structures in quasi-Polish spaces

In this section we will analyze the $\mathcal{F}$-hierarchies on X for various reducibilities $\mathcal{F}$ and various quasi-Polish spaces X. We start with some general ways to transfer results about the $\mathcal{F}$-hierarchy from one quasi-Polish space to another (for a fixed collection $\mathcal{F}$).

Definition 5.1. Let $\mathcal{F}$ be a collection of functions between topological spaces. For $X, Y \in \mathscr{X}$, denote by $\mathcal{F}(X,Y)$ the collection of functions from $\mathcal{F}$ with domain X and range included in Y (obviously, $Y \subseteq Y'$ implies $\mathcal{F}(X,Y) \subseteq \mathcal{F}(X,Y')$). The collection $\mathcal{F}$ is said *family of reducibilities* if the following conditions hold:

(1) $\mathcal{F}$ contains all the identity functions, i.e. $\mathrm{id}_X \in \mathcal{F}(X,X)$ for every $X \in \mathscr{X}$ (and hence $\mathrm{id}_X \in \mathcal{F}(X,Y)$ for every $X \subseteq Y \in \mathscr{X}$);
(2) $\mathcal{F}$ is closed under composition, i.e. if $X, Y, Z \in \mathscr{X}$, $f \in \mathcal{F}(X,Y)$ and $g \in \mathcal{F}(Y,Z)$ then $g \circ f \in \mathcal{F}(X,Z)$;
(3) F is closed under restrictions, i.e. for every $X, Y \in \mathscr{X}$, $f \in \mathcal{F}(X,Y)$ and $X' \subseteq X$, we have $f \restriction X' \in \mathcal{F}(X',Y)$.

For $X \in \mathscr{X}$, we will abbreviate $\mathcal{F}(X,X)$ with $\mathcal{F}(X)$. Notice that if $\mathcal{F}$ is a family of reducibilities then $\mathcal{F}(X)$ is a reducibility on X for every $X \in \mathscr{X}$ by conditions (1) and (2). Notice that all the classes of functions considered in Section 3, namely

- $\bigcup_{X,Y\in\mathscr{X}} \mathsf{D}_\alpha(X,Y)$ for $\alpha < \omega_1$,
- $\bigcup_{X,Y\in\mathscr{X}} \mathsf{D}_\alpha^{\mathsf{W}}(X,Y)$ for $\alpha < \omega_1$, and
- $\bigcup_{X,Y\in\mathscr{X}} \mathcal{B}_\gamma(X,Y) = \bigcup_{X,Y\in\mathscr{X}} \bigcup_{\beta<\gamma} \mathbf{\Sigma}^0_{\beta,1}(X,Y)$ for $\gamma < \omega_1$ additively closed,

are in fact families of reducibilities. With a little abuse of notation, in what follows we will denote the above families of reducibilities by, respectively, D_α, $\mathsf{D}_\alpha^{\mathsf{W}}$ and $\mathcal{B}_\gamma$ whenever this is not a source of confusion.

Proposition 5.2. Let $\mathcal{F}$ be a family of reducibilities and $X, Y \in \mathscr{X}$. If $X \simeq_{\mathcal{F}} Y$ then $(\mathscr{P}(X), \leq_{\mathcal{F}}^{X})$ is isomorphic to $(\mathscr{P}(Y), \leq_{\mathcal{F}}^{Y})$ and, moreover, $(S^X, \leq_{\mathcal{F}}^{S,X})$ is isomorphic to $(S^Y, \leq_{\mathcal{F}}^{S,Y})$ for every set S.

Proof. It is enough to consider the more general case of X- and Y-namings, so let us fix a set S. Let $f\colon X \to Y$ be a bijection such that both $f \in \mathcal{F}(X,Y)$ and $f^{-1} \in \mathcal{F}(Y,X)$. Then $\nu \mapsto \nu \circ f^{-1}$ is an isomorphism between $(S^X, \leq_{\mathcal{F}}^{S,X})$ and $(S^X, \leq_{\mathcal{F}}^{S,Y})$. In fact, if $\mu \leq_{\mathcal{F}}^{S,X} \nu$ via $g \in \mathcal{F}(X)$ then $\mu \circ f^{-1} \leq_{\mathcal{F}}^{S,Y} \nu \circ f^{-1}$ via $f \circ g \circ f^{-1}$, which is in $\mathcal{F}(Y)$ since $\mathcal{F}$ is closed under composition. Similarly, if $\mu \circ f^{-1} \leq_{\mathcal{F}}^{S,Y} \nu \circ f^{-1}$ via some $h \in \mathcal{F}(Y)$, then $\mu \leq_{\mathcal{F}}^{S,X} \nu$ via $f^{-1} \circ h \circ f \in \mathcal{F}(X)$. □

Notice that the converse to Proposition 5.2 does not hold in general. For example, it is not hard to check that (under AD) the D_2-hierarchies on, respectively, $\mathcal{N}$ and $\mathcal{C}$ are isomorphic (in fact, they are isomorphic to the Wadge hierarchy on $\mathcal{N}$), while $\mathcal{N} \not\simeq_{\mathsf{D}_2} \mathcal{C}$ by Proposition 4.7(2).

Definition 5.3. Let $\mathcal{F}$ be a family of reducibilities and $X \in \mathscr{X}$. We say that $Y \subseteq X$ is an *$\mathcal{F}$-retract* of X if there is $r \in \mathcal{F}(X,Y)$ such that $r \restriction Y = \mathrm{id}_Y$ (such an r will be called *$\mathcal{F}$-retraction* of X onto Y).

The next fact extends (Selivanov 2005, Proposition 2.3).

Proposition 5.4. Let $\mathcal{F}$ be a family of reducibilities and $X \in \mathscr{X}$. If Y is an $\mathcal{F}$-retract of X, then there is an injection from $(\mathscr{P}(Y), \leq_{\mathcal{F}}^{Y})$ into $(\mathscr{P}(X), \leq_{\mathcal{F}}^{X})$. Similarly, for every set S there is an injection from $(S^Y, \leq_{\mathcal{F}}^{S,Y})$ into $(S^X, \leq_{\mathcal{F}}^{S,X})$.

Proof. It is again enough to consider the more general case of X- and Y-namings. Let $r\colon X \to Y$ be an $\mathcal{F}$-retraction of X onto Y. Then the map $\nu \mapsto \nu \circ r$ is an injection of $(S^Y, \leq_{\mathcal{F}}^{S,Y})$ into $(S^X, \leq_{\mathcal{F}}^{S,X})$. In fact, if $\mu \leq_{\mathcal{F}}^{S,Y} \nu$ via some $g \in \mathcal{F}(Y)$, then it is easy to check that $\mu \circ r \leq_{\mathcal{F}}^{S,X} \nu$ via $g \circ r$, which is in $\mathcal{F}(X)$ because $\mathcal{F}$ is closed under composition. Conversely, if $\mu \circ r \leq_{\mathcal{F}}^{S,X} \nu \circ r$ via some $h \in \mathcal{F}(X)$ then $\mu \leq_{\mathcal{F}}^{S,Y} \nu$ via $r \circ (h \restriction Y)$, which is in $\mathcal{F}(Y,Y)$ because $\mathcal{F}$ is closed under composition and restrictions. □

Corollary 5.5. Let $\emptyset \neq Y \subseteq X$ be quasi-Polish spaces, S be a set, and $\mathcal{F}$ be a family of reducibilities. If $\mathsf{D}_3^{\mathsf{W}}(X,Y) \subseteq \mathcal{F}$ then there is an injection from $(\mathscr{P}(Y), \leq_{\mathcal{F}}^{Y})$ (respectively, $(S^Y, \leq_{\mathcal{F}}^{S,Y})$) into $(\mathscr{P}(X), \leq_{\mathcal{F}}^{X})$ (respectively, $(S^Y, \leq_{\mathcal{F}}^{S,Y})$).

The same conclusion holds also if $Y \in \mathbf{\Delta}_2^0(X)$ and $\mathsf{D}_2^{\mathsf{W}}(X,Y) \subseteq \mathcal{F}$.

Proof. The map $f\colon X \to Y$ such that $f \restriction Y = \mathrm{id}_Y$ and $f(x) = y_0$ for $x \in X \setminus Y$ (where y_0 is any fixed element of Y) is a $\mathsf{D}_3^{\mathsf{W}}$-retraction of X onto Y because $Y \in \mathbf{\Pi}_2^0(X)$ by Proposition 2.9. If moreover $Y \in \mathbf{\Delta}_2^0(X)$, then f is also a $\mathsf{D}_2^{\mathsf{W}}(X,Y)$-retraction of X onto Y. Hence the result follows from Proposition 5.4. □

Corollary 5.6. Let $\mathcal{F}$ be a family of reducibilities such that $\bigcup_{X,Y \in \mathscr{X}} \mathsf{D}_3^{\mathsf{W}}(X,Y) \subseteq \mathcal{F}$. The $\mathcal{F}$-hierarchy on $P\omega$ is (very) good if and only if for all quasi-Polish spaces X, the $\mathcal{F}$-hierarchy on X is (very) good. Similarly, the $\mathcal{F}$-hierarchy on $P\omega$ is (very) bad if and only if there exists a quasi-Polish space X such that the $\mathcal{F}$-hierarchy on X is (very) bad.

The same result holds when replacing $P\omega$ with the Hilbert cube or with the space $\mathbb{R}^\omega$, and letting X vary only on Polish spaces.

Proof. By Corollary 5.5 and the fact that $P\omega$ (respectively, $[0,1]^\omega$ or $\mathbb{R}^\omega$) is universal for the class of quasi-Polish (respectively, Polish) spaces. □

Lemma 5.7. (Folklore) Let $\leq, \preceq$ be preorders on an arbitrary set A such that $\preceq$ extends $\leq$ (i.e. $\leq \subseteq \preceq$).

(1) every antichain with respect to $\preceq$ is an antichain with respect to $\leq$;
(2) if $\leq$ is a wqo then so is $\preceq$.

Proof. Part (1) is obvious, so let us consider just part (2). By (1), $\preceq$ cannot contain infinite antichains. Assume towards a contradiction that $a_0 \succeq a_1 \succeq \dots$ is an infinite (countable) $\preceq$-decreasing sequence of elements from A. Clearly $a_i \not\leq a_j$ for every $i < j$. Define a coloring $c\colon [\omega]^2 \to \{0,1\}$ by setting (for $i < j \in \omega$) $c(\{i,j\}) = 0 \iff a_j \leq a_i$. By Ramsey's theorem, there is an infinite $H \subseteq \omega$ such that $c \restriction [H]^2$ is constantly equal to either 0 or 1. In the first case the sequence $\vec{a} = \langle a_i \mid i \in H\rangle$ is an infinite $\leq$-descending chain, while in the second case $\vec{a}$ is an infinite $\leq$ antichain: therefore, in both cases we reach a contradiction with the fact that $\leq$ is a wqo. □

Proposition 5.8. Suppose $\mathcal{F} \subseteq \mathcal{G}$ are families of reducibilities and X is quasi-Polish.

(1) If $(\mathscr{P}(X), \leq^X_{\mathcal{F}})$ is very good, then so is $(\mathscr{P}(X), \leq^X_{\mathcal{G}})$. Similarly, for every set S, if $(S^X, \leq^X_{\mathcal{F}})$ is very good, then so is $(S^X, \leq^X_{\mathcal{G}})$. The same results hold when replacing very good with good.
(2) If the $\mathcal{G}$-hierarchy on X is (very) bad, then the $\mathcal{F}$-hierarchy on X is bad. Similarly, for every set $S \in \mathcal{P}(X)$, if $(S^X, \leq^X_{\mathcal{G}})$ is (very) bad, then $(S^X, \leq^X_{\mathcal{F}})$ is bad.

Proof. Use Lemma 5.7. □

Notice that Proposition 5.8(2) cannot be strenghtened to the statement: "If the $\mathcal{G}$-hierarchy on X is very bad, then the $\mathcal{F}$-hierarchy on X is very bad". This is because it is possible that every $\leq^X_{\mathcal{G}}$-descending chain of subsets of X in fact consists of $\leq^X_{\mathcal{F}}$-incomparable elements.

Remark 5.9. Notice that if the so-called Semi-linear Ordering Principle for $\mathcal{F}$ on the space X

$$\forall A, B \subseteq X\ (A \leq^X_{\mathcal{F}} B \vee \overline{B} \leq^X_{\mathcal{F}} A) \qquad (\mathsf{SLO}^{\mathcal{F},X})$$

holds (which is the case, under AD, for every $\mathcal{F} \supseteq \mathsf{W}$ and X zero-dimensional Polish space), then the first part of Proposition 5.8(1) can be strengthened to the following:

- There is an injection from $(\mathscr{P}(X), \leq^X_{\mathcal{G}})$ into $(\mathscr{P}(X), \leq^X_{\mathcal{F}})$.

This is because in this case $(\mathcal{D}^X_{\mathcal{G}}, \leq)$ is a coarsification of $(\mathcal{D}^X_{\mathcal{F}}, \leq)$. To see this, it is enough to show that if $A, B \subseteq X$ are such that $A <^X_{\mathcal{G}} B$ then $A <^X_{\mathcal{F}} B$. Clearly, $B \not\leq^X_{\mathcal{F}} A$, so it remains to show that $A \leq^X_{\mathcal{F}} B$. Notice that $\mathsf{SLO}^{\mathcal{F},X} \Rightarrow \mathsf{SLO}^{\mathcal{G},X}$, hence $A <^X_{\mathcal{G}} \overline{B}$ as well. This implies that $\overline{B} \not\leq^X_{\mathcal{F}} A$, and hence $A \leq^X_{\mathcal{F}} B$ by $\mathsf{SLO}^{\mathcal{F},X}$.

Corollary 5.10. Let $\mathcal{F} \supseteq \mathsf{W}$ be a reducibility on $\mathcal{N}$ and $k \in \omega$. Then $((\mathbf{\Delta}^1_1(\mathcal{N}))_k, \leq_{\mathcal{F}})$ is good (but not very good if $k \geq 3$). Similarly, under AD we have that $((\mathscr{P}(\mathcal{N}))_k, \leq_{\mathcal{F}})$ is good (but not very good if $k \geq 3$).

Proof. The claim is well-known for $\mathcal{F} = \mathsf{W}$ (see the discussion in Subsection 2.6). For $\mathsf{W} \subsetneq \mathcal{F}$, apply Proposition 5.8(1). To see that $((\mathbf{\Delta}^1_1(\mathcal{N}))_k, \leq_{\mathcal{F}})$ is not very good when $k \geq 3$, notice that the k-partitions $\nu_i \colon \mathcal{N} \to k$ (for $i < k$) defined by $\nu_i(x) = i$ for every $x \in \mathcal{N}$ are in fact clopen partitions and are pairwise $\leq_{\mathcal{F}}$-incomparable. □

Remark 5.11. Notice that all the previous results hold "locally" i.e. when considering $(\mathbf{\Gamma}, \mathcal{F})$-hierarchies in place of $\mathcal{F}$-hierarchies, as long as $\mathbf{\Gamma}$ is a family of pointclasses closed under $\mathcal{F}$-preimages, i.e. such that for every $X, Y \in \mathscr{X}$ and $f \in \mathcal{F}(X, Y)$, if $A \in \mathbf{\Gamma}(Y)$ then $f^{-1}(A) \in \mathbf{\Gamma}(X)$.

5.1. *Degree structures in uncountable quasi-Polish spaces*

In this subsection we consider various reducibilities $\mathcal{F}$ and study the $\mathcal{F}$-hierarchies on arbitrary uncountable quasi-Polish spaces X. We begin with an important corollary to the results obtained in Section 4.

Theorem 5.12. Let X be an uncountable quasi-Polish space and $\mathcal{F}$ be a family of reducibilities.

(1) If $\dim(X) = 0$, then the $\mathcal{F}$-hierarchy on X can be embedded into the $\mathcal{F}$-hierarchy on $\mathcal{N}$ for every $\mathcal{F} \supseteq \mathsf{D}_1 = \mathsf{W}$. Hence the $(\mathbf{B}, \mathcal{F})$-hierarchy on X is very good, and assuming AD the entire $\mathcal{F}$-hierarchy on X is very good.

(2) Assume $\dim(X) = 0$ and that $\mathcal{F} \supseteq \mathsf{D}^{\mathsf{W}}_2 = \mathsf{D}_2$. If X is σ-compact then the $\mathcal{F}$-hierarchy on X is isomorphic to the $\mathcal{F}$-hierarchy on $\mathcal{C}$, while if X is not σ-compact then the $\mathcal{F}$-hierarchy on X is isomorphic to the $\mathcal{F}$-hierarchy on $\mathcal{N}$. Hence, if e.g. $\mathcal{F} = \mathsf{D}_\alpha$ for some $\alpha \geq 2$, then the $(\mathbf{B}, \mathcal{F})$-hierarchy on X is isomorphic to the $(\mathbf{B}, \mathsf{D}_1)$-hierarchy on $\mathcal{N}$, and assuming AD the entire $\mathcal{F}$-hierarchy on X is isomorphic to the Wadge hierarchy on $\mathcal{N}$.

(3) If $\dim(X) \neq \infty$ then the $\mathcal{F}$-hierarchy on X is isomorphic to the $\mathcal{F}$-hierarchy on $\mathcal{N}$ whenever $\mathcal{F} \supseteq \mathsf{D}^{\mathsf{W}}_3$. Hence the $(\mathbf{B}, \mathcal{F})$-hierarchy on X is very good, and assuming AD the $\mathcal{F}$-hierarchy on X is very good as well. Moreover, if e.g. $\mathcal{F} = \mathsf{D}_\alpha$ for some $\alpha \geq 3$, then the $\mathcal{F}$-hierarchy on X is isomorphic to the Wadge hierarchy on $\mathcal{N}$.

(4) If X is universal for Polish (respectively, quasi-Polish) spaces and $\mathcal{F} \supseteq \mathsf{D}^{\mathsf{W}}_3$, then the $\mathcal{F}$-hierarchy on X is isomorphic to the $\mathcal{F}$-hierarchy on $[0,1]^\omega$ (respectively, on $P\omega$). Moreover, the $\mathcal{F}$-hierarchy on X is (very) good if and only if the $\mathcal{F}$-hierarchy on every Polish (respectively, quasi-Polish) space is (very) good, and it is (very) bad if and only if the $\mathcal{F}$-hierarchy on some Polish (respectively, quasi-Polish) space is (very) bad.

(5) If $\mathcal{F} \supseteq \mathcal{B}_\omega$ (hence, in particular, if $\mathcal{F} = \mathsf{D}_\alpha$ for some $\alpha \geq \omega$) then the $\mathcal{F}$-hierarchy on X is isomorphic to the $\mathcal{F}$-hierarchy on $\mathcal{N}$. Hence the $(\mathbf{B}, \mathcal{F})$-hierarchy on X is very good, and assuming AD the $\mathcal{F}$-hierarchy on X is very good as well. In fact, if

e.g. $\mathcal{F} = \mathsf{D}_\alpha$ for some $\alpha \geq \omega$, then the $\mathcal{F}$-hierarchy on X is isomorphic to the Wadge hierarchy on $\mathcal{N}$.

Analogous results hold for k-partitions of X (for every $k \in \omega$) when replacing "very good" by "good" in all the statements above.

Proof. Let us first consider the first item of the list. Without loss of generality we can assume that X is a closed subset of $\mathcal{N}$ by (Kechris 1995, Theorem 7.8), and hence that X is a (W-)retract of $\mathcal{N}$ by (Kechris 1995, Theorem 7.3). Hence the claim follows from Proposition 5.4 and the results from (Wadge 1984).

The other claims of the list follow from Proposition 5.2 and, respectively, Lemma 4.19 and the fact that every zero-dimensional quasi-Polish space is Polish (see the proof of Theorem 4.21), Theorem 4.21(1), Proposition 4.28 and Corollary 5.6, and Proposition 4.3, together with the results from (Motto Ros 2009; Motto Ros 2010a).

The results about k-partitions can be obtained in a similar way using Corollary 5.10. □

Theorem 5.12 leaves open the problem of determining the $\mathcal{F}$-hierarchy on X for many reducibilities $\mathcal{F}$, most notably for $\mathcal{F} = \mathsf{D}_1 = \mathsf{W}$ and $\mathcal{F} = \mathsf{D}_2$ on quasi-Polish spaces X of dimension $\neq \infty$, and for $\mathcal{F} = \mathsf{D}_n$, $1 \leq n \in \omega$, for quasi-Polish spaces of dimension ∞: in the rest of this subsection we will give some partial answers to this problem.

Let us first consider the D_1- and D_2-hierarchies on uncountable Polish spaces. The case of the D_1-hierarchies is now quite well-understood. For example, we have the following results.

Theorem 5.13. (Hertling 1996) The D_1-hierarchy on $\mathbb{R}^n$ and $[0,1]^n$ (for $1 \leq n \leq \omega$) is very bad.

Theorem 5.14. (Ikegami 2010; Ikegami *et al.* 2012) Let $X = \mathbb{R}^n$ for $1 \leq n \leq \omega$.

(1) Any countable partial order can be embedded into the D_1-hierarchy on X.
(2) (ZFC) Any partial order of size ω_1 can be embedded into the D_1-hierarchy on X.

Notice that by Proposition 5.4 these results hold also for Polish spaces admitting a continuous retraction to $\mathbb{R}^n$, like e.g. $[0,1]^n$.

Theorem 5.15. (Schlicht 2012) Suppose X is a metric space with $\dim(X) \neq 0$: then the D_1-hierarchy on X is bad. In fact it contains uncountable antichains.

Notice that Theorem 5.15 cannot be improved by replacing bad with very bad, as by (Cook 1967, Theorem 11) there is an uncountable connected compact Polish space X with the property that all continuous maps $f\colon X \to X$ are either constant or the identity map; this implies that all nonempty subsets $A \neq X$ are Wadge incomparable, and hence that the Wadge hierarchy on X is bad but not very bad. Moreover, we cannot require X to be just quasi-Polish: for example, the Wadge hierarchy on the perfect ω-algebraic domain $L_{\omega+1}$ from Example 4.14(3), which has dimension ω, is good (but not very good), hence it does not contain infinite antichains.

Let us now consider the D_2-hierarchy on locally connected Polish spaces. Notice that for any $f \in \mathsf{D}_2(X, Y)$ between Polish spaces X, Y there is a nonempty open set $U \subseteq X$ such

that the restriction of f to the closure $\mathrm{cl}(U)$ of U is continuous. In fact, the Jayne-Rogers theorem $\mathsf{D}_2(X,Y) = \mathsf{D}_2^{\mathsf{W}}(X,Y)$ implies that for such an f there is a closed covering $\langle X_k \mid k \in \omega\rangle$ of X such that $f \restriction X_k$ is continuous for each $k \in \omega$. By Baire's category

theorem there is $k \in \omega$ such that X_k is not meager, hence $U \subseteq X_k$ for some nonempty open set U. Since X_k is closed, it follows that $\mathrm{cl}(U) \subseteq X_k$, and hence $f \restriction \mathrm{cl}(U) = (f \restriction X_k) \restriction \mathrm{cl}(U)$ is continuous.

Proposition 5.16. Suppose X is an uncountable locally connected Polish space. Then the $(\mathbf{B}, \mathsf{D}_2)$-hierarchy on X is not very good.

Proof. We will find Borel sets $A, B \subseteq X$ such that $\{A, \overline{A}, B, \overline{B}\}$ is an antichain with respect to $\mathsf{D}_2(X)$-reducibility. Since X is uncountable, there is a compact set $Y \subseteq X$ with $Y \simeq_1 \mathcal{C}$ by (Kechris 1995, Theorem 13.6). Let A be a proper $\mathbf{\Sigma}^0_2(X)$ set such that both A and $\overline{A}$ are dense in X, and let $B \subseteq Y$ be a proper $\mathbf{\Sigma}^0_3(Y)$ (so that, in particular, B is also a proper $\mathbf{\Sigma}^0_3(X)$ set). By their topological complexity (and the fact that all the pointclasses $\mathbf{\Sigma}^0_2(X), \mathbf{\Pi}^0_2(X), \mathbf{\Sigma}^0_3(X), \mathbf{\Pi}^0_3(X)$ are closed under D_2-preimages), we have that the pairs $(A, \overline{A})$ and $(B, \overline{B})$ are D_2-incomparable, and that $B, \overline{B} \not\leq_{\mathsf{D}_2} A, \overline{A}$. Hence it remains to show that $A, \overline{A} \not\leq_{\mathsf{D}_2} B, \overline{B}$. In fact, since $D_0 \leq_{\mathsf{D}_2} D_1 \iff \overline{D_0} \leq_{\mathsf{D}_2} \overline{D_1}$ for every $D_0, D_1 \subseteq X$, it suffices to show that $C \not\leq_{\mathsf{D}_2} B$ for every $C \in \{A, \overline{A}\}$. Suppose toward a contradiction that there is a reduction $f \in \mathsf{D}_2(X)$ of C to B. Then by the observation preceding this proposition, there is a nonempty open connected set $U \subseteq X$ such that $f \restriction U$ is continuous. Since $C \cap U$ is dense in U, its image $f(C \cap U)$ is also dense in $f(U)$. This implies $f(U) \subseteq Y$, since $f(C \cap U) \subseteq B \subseteq Y$ and Y is closed. Since $f(U)$ is connected and Y is totally disconnected, $f \restriction U$ is constant. But this contradicts our assumption that $C = f^{-1}(B)$ because both C and $\overline{C}$ are dense and hence have nonempty intersection with U. □

Using Corollary 5.5, we have also the following result.

Corollary 5.17. Suppose that X is quasi-Polish and that there is $Y \in \mathbf{\Delta}^0_2(X)$ which is an uncountable locally connected Polish space. Then the $(\mathsf{B}, \mathsf{D}_2)$-hierarchy on X is not very good.

We will now turn our attention to Euclidean spaces and show that their $(\mathbf{B}, \mathsf{D}_2)$-hierarchy is in fact bad. In what follows, we will crucially use two simple properties of the real line $\mathbb{R}$, namely the fact that, since $\mathbb{R}$ is connected, every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ maps intervals to (possibly degenerate) intervals, and the fact that $\mathbb{R}$ is σ-compact. Recall that in an arbitrary Polish space X the image of a closed subset of X under a continuous reduction can be analytic non-Borel: however, if X is σ-compact the situation becomes considerably simpler, as it is shown by the next lemma.

Lemma 5.18. Let $2 \leq \alpha < \omega_1$. Suppose X and Y are Polish spaces, $f\colon X \to Y$ is surjective, and $A \subseteq Y$.

(1) (Saint Raymond 1976, Theorem 5) If X is compact, f is continuous, and $f^{-1}(A) \in \mathbf{\Sigma}^0_\alpha(X)$, then $A \in \mathbf{\Sigma}^0_\alpha(Y)$.
(2) If X is σ-compact, $f \in \mathsf{D}_2(X,Y)$, and $f^{-1}(A) \in \mathbf{\Sigma}^0_\alpha(X)$, then $A \in \mathbf{\Sigma}^0_\alpha(Y)$.

Proof. The first part is proved in (Saint Raymond 1976). For the second part, if X is σ-compact then by the Jayne-Rogers theorem $\mathsf{D}_2(X,Y) = \mathsf{D}_2^{\mathsf{W}}(X,Y)$ there is a countable covering $\langle X_k \mid k \in \omega\rangle$ of X consisting of compact sets such that $f \restriction X_k$ is continuous for each $k \in \omega$. Since $(f \restriction X_k)^{-1}(A \cap f(X_k)) = f^{-1}(A) \cap X_k \in \mathbf{\Sigma}^0_\alpha(X)$, the first part applied to $f \restriction X_k \colon X_k \to f(X_k)$ implies that $A \cap f(X_k) \in \mathbf{\Sigma}^0_\alpha(f(X_k)) \subseteq \mathbf{\Sigma}^0_\alpha(Y)$ (because $f(X_k)$ is compact, and hence closed in Y). Hence $A = \bigcup_{k\in\omega}(A \cap f(X_k)) \in \mathbf{\Sigma}^0_\alpha(Y)$. □

To construct antichains in the $(\mathbf{B}, \mathsf{D}_2)$-hierarchies of a Polish spaces X, we will consider sets which are everywhere proper $\mathbf{\Sigma}^0_\alpha(X)$ for some $1 < \alpha < \omega_1$.

Lemma 5.19. Suppose X is a perfect Polish space and $1 < \alpha < \omega_1$. Then there is $A \in \mathbf{\Sigma}^0_\alpha(X)$ such that for all nonempty open sets $U \subseteq X$, $A \cap U \in \mathbf{\Sigma}^0_\alpha(X) \setminus \mathbf{\Pi}^0_\alpha(X)$.

Proof. Let d be a compatible metric for X. We first construct a sequence $\langle C_n \mid n \in \omega\rangle$ of disjoint nowhere dense closed subsets of X with $\lim_{n\to\infty} \operatorname{diam}(C_n) = 0$ (where the operator diam refers to the chosen metric d) and such that $\bigcup_{n\in\omega} C_n$ is dense. Let $\langle B_n \mid n \in \omega\rangle$ be an enumeration of a basis for the topology of X. Observe that for every nonempty open set $U \subseteq X$ there is a homeomorphic copy $C \subseteq U$ of $\mathcal{C}$ which is nowhere dense (in X), and that C is necessarily closed since $\mathcal{C}$ is compact and X is Hausdorff. By induction on $n \in \omega$, choose (using DC) a closed nowhere dense $C_n \subseteq B'_n = B_n \setminus \bigcup_{i<n} C_i$ such that $\operatorname{diam}(C_n) \leq 2^{-n}$: such a C_n exists by the observation above since, by the inductive hypothesis applied to the C_i's, B'_n is a nonempty open set, and if necessary B'_n can obviously be further shrunk to a nonempty open set of diameter $\leq 2^{-n}$. It is straightforward to check that the sequence of the C_n's constructed in this way has the desired properties.

Now choose sets $A_n \in \mathbf{\Sigma}^0_\alpha(C_n) \setminus \mathbf{\Pi}^0_\alpha(C_n)$ (using DC again). We claim that $A = \bigcup_{n\in\omega} A_n$ is as required. First notice that each $A_n \in \mathbf{\Sigma}^0_\alpha(X)$ since $\alpha \geq 2$ and C_n is closed, hence $A \in \mathbf{\Sigma}^0_\alpha(X)$ as well. Now assume towards a contradiction that there is an open $U \subseteq X$ such that $A \cap U \in \mathbf{\Pi}^0_\alpha(X)$. Let $x \in X$ and $0 < \varepsilon \in \mathbb{R}^+$ be such that $B_d(x,\varepsilon) \subseteq U$. Let $N \in \omega$ be such that $\operatorname{diam}(C_n) < \frac{\varepsilon}{2}$ for every $n \geq N$. Since each C_n was assumed nowhere dense, $V = B_d(x, \frac{\varepsilon}{2}) \setminus \bigcup_{n<N} C_n$ is a nonempty open set. By density of $\bigcup_{n\in\omega} C_n$ and the definition of V, there is $n \geq N$ such that $C_n \cap V \neq \emptyset$. Hence, by the choice of N and V we have $C_n \subseteq U$. By the assumption $A \cap U \in \mathbf{\Pi}^0_\alpha(X)$, $A_n = A \cap C_n = (A \cap U) \cap C_n \in \mathbf{\Pi}^0_\alpha(C_n)$, contradicting the choice of the A_n's. □

Proposition 5.20. There are uncountable antichains in the D_2-hierarchy on $[0,1]$.

Proof. Let $W_\alpha \subseteq \mathcal{C}$ denote the set of codes of well-orders on ω of order type at most α. Let $W = \bigcup_{\alpha<\omega_1} W_\alpha$. Since each W_α is Borel and W is $\mathbf{\Pi}^1_1$-complete, the Borel ranks of the sets W_α are unbounded in ω_1. Hence we obtain an unbounded set $C \subseteq \omega_1$ and a collection $(A_\alpha)_{\alpha\in C}$ of subsets of $\mathcal{C}$ such that A_α is $\mathbf{\Sigma}^0_\alpha(\mathcal{C})$-complete. It follows from the previous Lemma (using DC) that there is a collection $(A_\alpha)_{\alpha\in C}$ of Borel subsets of $[0,1]$ such that $A_\alpha \cap (a,b)$ is a proper $\mathbf{\Sigma}^0_\alpha([0,1])$ set for all nondegenerate open intervals $(a,b) \subseteq [0,1]$.

Suppose $\alpha, \beta \in C$ and $1 < \alpha < \beta$. Then $A_\beta \not\leq_{\mathsf{D}_2} A_\alpha$ because the pointclass $\mathbf{\Sigma}^0_\alpha(X)$ is closed under D_2-preimages. Conversely, assume towards a contradiction that there is a

reduction $f \in \mathsf{D}_2([0,1])$ of A_α to A_β. Let $[a,b]$ be a nondegenerate closed interval such that $f \restriction [a,b]$ is continuous (which exists by the observation preceding Proposition 5.16). Then there is an interval $[c,d]$ with $f([a,b]) = [c,d]$, and since $f \restriction [a,b]$ cannot be constant (as otherwise either $A_\alpha \cap [a,b] = [a,b]$ or $A_\alpha \cap [a,b] = \emptyset$, contradicting the choice of the A_α's), $[c,d]$ is nondegenerate. Since $(f \restriction [a,b])^{-1}[A_\beta \cap [c,d]] = A_\alpha \cap [a,b] \in \mathbf{\Sigma}^0_\alpha([0,1])$ and $A_\beta \cap [c,d] \notin \mathbf{\Sigma}^0_\alpha([0,1]) \subseteq \mathbf{\Pi}^0_\beta([0,1])$, this contradicts Lemma 5.18. □

Corollary 5.21.

(1) Suppose X is an Hausdorff quasi-Polish space with $[0,1] \subseteq X$. Then there are uncountable antichains in the D_2-hierarchy on X.
(2) The D_2-hierarchy on $\mathbb{R}^n$ and $[0,1]^n$ (for $1 \leq n \leq \omega$) is bad.

Proof. This follows from the fact that a compact subset of an Hausdorff quasi-Polish space is closed and from Corollary 5.5. □

Corollary 5.21 leaves open the following problem:

Question 5.22. Is the D_2-hierarchy on $\mathbb{R}$ *very* bad?

We do not know the answer to the above question, but we are at least able to show that the $(\mathbf{B}, \mathsf{D}_2)$-hierarchy on $\mathbb{R}^2$ (and hence, since $\mathbb{R}^2$ is σ-compact, on any Hausdorff quasi-Polish space containing $\mathbb{R}^2$, like the spaces $[0,1]^n$ and $\mathbb{R}^n$ for $2 \leq n \leq \omega$) is very bad. Unfortunately, our argument cannot be adapted to $\mathbb{R}$.

Theorem 5.23. The quasi-order $(\mathscr{P}(\omega), \subseteq^*)$ of inclusion modulo finite sets on $\mathscr{P}(\omega)$ embeds into $(\mathbf{\Sigma}^0_2(\mathbb{R}^2), \leq_{\mathsf{D}_2})$.

Before proving Theorem 5.23, we present some constructions and prove some technical lemmas which will be needed later. Call a map $f\colon \mathbb{R} \to \mathbb{R}$ *weakly increasing* if $a \leq b$ implies $f(a) \leq f(b)$ for all $a, b \in \mathbb{R}$. Let $\{A_x \mid x \subseteq \omega\}$ be a collection of $\mathbf{\Sigma}^0_2$ subsets of $[0,1]$ with the following properties:

(A1) $\inf(A_x) = 0$, $\sup(A_x) = 1$, and both $A_x \cap (0,1)$ and $\overline{A_x} \cap (0,1)$ are nonempty;
(A2) $x \subseteq^* y$ if and only if $A_x \leq^{\mathbb{R}}_{\mathsf{D}_1} A_y$;
(A3) If $x \subseteq^* y$, then there is a weakly increasing uniformly continuous surjective reduction $f\colon \mathbb{R} \to \mathbb{R}$ of A_x to A_y with $f(0) = 0$ and $f(1) = 1$.

Such a collection exists by (Ikegami *et al.* 2012).

Definition 5.24. A map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ is called *special* if it is uniformly continuous, surjective, and such that for every $a, a', b, b' \in \mathbb{R}$

(S1) if $a = a'$ then $\pi_0(h(a,b)) = \pi_0(h(a',b'))$,
(S2) $a \leq a' \iff \pi_0(h(a,b)) \leq \pi_0(h(a',b'))$, and
(S3) if $b \leq b'$ then $\pi_1(h(a,b)) \leq \pi_1(h(a,b'))$.

For $a \in \mathbb{R}$, let $\mathbb{R}_a$ denote $\{a\} \times \mathbb{R}$. Notice that for every map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfying (S1) and (S2) there is an order-preserving (hence injective) map $\pi(h)\colon \mathbb{R} \to \mathbb{R}$ such that $h(\mathbb{R}_a) \subseteq \mathbb{R}_{\pi(h)(a)}$.

Let $\langle \beta_n \mid n \in \omega \rangle$ be a decreasing sequence of reals with $\lim_{n\to\infty} \beta_n = 0$ and $\beta_0 < \frac{1}{12}$. Let $\{q_n \mid n \in \omega\}$ be an enumeration without repetitions of $\mathbb{Q}$, and let $\{p_n \mid n \in \omega\}$ be a collection in $\mathbb{Q}$ such that $\{(q_n, p_n) \in \mathbb{Q}^2 \mid n \in \omega\}$ is dense in $\mathbb{R}^2$. For $x \subseteq \omega$ and $n \in \omega$, let $B_{x,n} \subseteq \mathbb{R}$ be an affine image of A_x (with the same orientation as A_x) such that $\inf(B_{x,n}) = p_n$ and $\sup(B_{x,n}) = p_n + \beta_n$. Let I be a countable dense subset of $\mathbb{R} \setminus \mathbb{Q}$ (so that $\mathbb{R} \setminus (\mathbb{Q} \cup I)$ is dense in $\mathbb{R}$ as well), and let $\{i_k \mid k \in \omega\}$ be an enumeration without repetitions of I. Finally, define

$$C_x = (I \times \mathbb{R}) \cup \bigcup_{n\in\omega} (\{q_n\} \times B_{x,n}) \subseteq \mathbb{R}^2. \tag{$*$}$$

Thus $C_x \cap \mathbb{R}_a$ is empty if $a \in \mathbb{R} \setminus (I \cup \mathbb{Q})$, equals $\mathbb{R}_a$ if $a \in I$, and is an affine copy of A_x if $a \in \mathbb{Q}$.

In what follows we will tacitly assume that every *partial* function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is such that $\mathrm{dom}(f) = D_f \times \mathbb{R}$ and $\mathrm{range}(f) = R_f \times \mathbb{R}$ for some (possibly empty) $D_f, R_f \subseteq \mathbb{R}$, and moreover that it satisfies (S1)–(S3) when restricted to $a, a' \in D_f$. Note that necessarily D_f and R_f have the same cardinality. Given such an f, we set $f_a\colon \mathbb{R} \to \mathbb{R}\colon b \mapsto \pi_1(f(a,b))$ for every $a \in D_f$.

Given a partial function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ with finite D_f, we define a canonical extension $\bar{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ of f as follows. If $D_f = \emptyset$, let $\bar{f} = \mathrm{id}_{\mathbb{R}^2}$. Suppose now that $D_f = \{a_0, \dots, a_n\}$ with $a_0 < a_1 < \dots < a_n$ (for some $n < \omega$), and pick $(a,b) \in \mathbb{R}^2$. Then we set set $\bar{f}(a,b) = (a, f_{a_0}(b))$ if $a < a_0$, $\bar{f}(a,b) = (a, f_{a_n}(b))$ if $a > a_n$, and

$$\bar{f}(a,b) = \frac{a_{m+1} - a}{a_{m+1} - a_m} \cdot f(a_m, b) + \frac{a - a_m}{a_{m+1} - a_m} \cdot f(a_{m+1}, b) \tag{$\dagger$}$$

if $a_m < a < a_{m+1}$ for some $m < n$, where $+$ and $\cdot$ denote the usual operations of vector addition and multiplication by a scalar on $\mathbb{R}^2$; in other words, $\bar{f}(a,b)$ is the linear combination of $f(a_m, b)$ and $f(a_{m+1}, b)$.

Let d denote the usual Euclidean distance on $\mathbb{R}^2$, and for $f, g\colon \mathbb{R}^2 \to \mathbb{R}^2$ write

$$\|f - g\| = \sup\{d(f(a,b), g(a,b)) \mid a, b \in \mathbb{R}\}.$$

Definition 5.25. Given a (partial) function $f\colon D_f \times \mathbb{R} \to R_f \times \mathbb{R}$, we say that $(\delta, \varepsilon) \in (\mathbb{R}^+)^2$ is a *modulus* (of uniform continuity) for f if

$$\forall a \in D_f\, \forall b, b' \in \mathbb{R}\, (|b - b'| < \delta \Rightarrow d(f(a,b), f(a,b')) < \varepsilon).$$

Notice that in fact $d(f(a,b), f(a,b')) = |f_a(b) - f_a(b')|$ by our assumptions on f, and that if (δ, ε) is a modulus for a partial function $f\colon D_f \times \mathbb{R} \to R_f \times \mathbb{R}$ with finite D_f, then (δ, ε) is also a modulus for $\bar{f}$.

Definition 5.26. We let $\mathcal{Y}$ be the the class of all partial functions $f\colon D_f \times \mathbb{R} \to R_f \times \mathbb{R}$ such that

(Y1) D_f is finite;
(Y2) f satisfies (S1)–(S3) restricted to $a, a' \in D_f$;
(Y3) f is uniformly continuous (by (Y1) this is equivalent to require that f_a is uniformly continuous for every $a \in D_f$);
(Y4) there is $K \in \omega$ such that $|f_a(b) - f_{a'}(b)| < K$ for every $a, a' \in D_f$ and $b \in \mathbb{R}$;

(Y5) f is a partial reduction of C_x to C_y, i.e. for every $a \in D_f$ and every $b \in \mathbb{R}$, $(a,b) \in C_x \iff f(a,b) \in C_y$.

Lemma 5.27. For every $f \in \mathcal{Y}$, $\bar{f}$ is special.

Proof. All the computations in the proof rely on the very specific definition (†) of $\bar{f}$. It is not hard to check that $\bar{f}$ is surjective since $\mathrm{range}(f) = R_f \times \mathbb{R}$, and that $\bar{f}$ satisfies (S1)–(S3) by (Y2), so that $\pi(\bar{f})$ is well-defined. It remains to check that $\bar{f}$ is uniformly continuous. Fix $K \in \omega$ as in (Y4), and notice that from this property we get $|\bar{f}_a(b) - \bar{f}_{a'}(b)| < \frac{K}{\rho} \cdot |a - a'|$ for every $a, a', b \in \mathbb{R}$, where $\rho = \min\{|a - a'| \mid a, a' \in D_f\}$. Given $\varepsilon > 0$, let δ' be such that $(\delta', \frac{\varepsilon}{2})$ is a modulus for f (such a δ' exists by (Y3)), and δ'' be such that $\frac{K}{\rho} \cdot |a - a'| < \frac{\varepsilon}{4}$ and $|\pi(\bar{f})(a) - \pi(\bar{f})(a')| < \frac{\varepsilon}{4}$ whenever $|a - a'| < \delta''$ (such a δ'' exists by (Y1)). Finally, let $\delta = \min\{\delta', \delta''\}$. Fix $(a,b) \in \mathbb{R}^2$, and let $(a', b') \in \mathbb{R}^2$ be such that $d((a,b),(a',b')) < \delta$, so that in particular $|a - a'| < \delta''$ and $|b - b'| < \delta'$. Since $(\delta', \frac{\varepsilon}{2})$ is a modulus for $\bar{f}$ as well, we have that

$$\begin{aligned} d(\bar{f}(a,b), \bar{f}(a',b')) &\leq d(\bar{f}(a,b), \bar{f}(a',b)) + d(\bar{f}(a',b), \bar{f}(a',b')) \\ &\leq (|\pi(\bar{f})(a) - \pi(\bar{f})(a')| + |\bar{f}_a(b) - \bar{f}_{a'}(b)|) + |\bar{f}_{a'}(b) - \bar{f}_{a'}(b')| < \left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore $\bar{f}$ is uniformly continuous, and hence special. □

Lemma 5.28. Suppose that $f \in \mathcal{Y}$ and that (δ, ε) and (δ', ε') are moduli for f. Let $\{k_i \mid i < M\} \subseteq \omega$ and $\{l_j \mid j < N\} \subseteq \omega$ be such that $k_i \leq k_{i'}$ for $i \leq i' < M$, and assume that $\beta_{k_0} < \min\{\delta, \varepsilon, \varepsilon'\}$. Then there is an extension $g \in \mathcal{Y}$ of f such that

(1) $\{q_{k_i}, i_{l_j} \mid i < M, j < N\} \subseteq D_g \cap R_g$;
(2) $(\delta', 3\varepsilon')$ is a modulus for g;
(3) $\|\bar{f} - \bar{g}\| \leq 2\varepsilon$.

Proof. We will define g through some intermediate extensions $f \subseteq g' \subseteq g'' \subseteq g''' \subseteq g$. Let

$$H = \{q_{k_i}, \pi(\bar{f})(q_{k_i}), \pi(\bar{f})^{-1}(q_{k_i}), i_{l_j}, \pi(\bar{f})(i_{l_j}), \pi(\bar{f})^{-1}(i_{l_j}) \mid i < M, j < N\},$$

and let $\theta = \frac{1}{2} \min\{d(a,a') \colon a, a' \in H,\ a \neq a'\}$.

First we extend f to a partial function g' in such a way that $D_{g'} = D_f \cup \{q_{k_i} \mid i < M\}$. Fix $i < M$: if $\bar{f} \restriction \mathbb{R}_{q_{k_i}}$ is already a partial reduction of C_x to C_y, we let $g' \restriction \mathbb{R}_{q_{k_i}} = \bar{f} \restriction \mathbb{R}_{q_{k_i}}$. Otherwise, to simplify the notation let $\beta = \beta_{k_i}$, $a_0 = p_{k_i} = \inf(B_{x,k_i})$, and $a_1 = p_{k_i} + \beta_{k_i} = \sup(B_{x,k_i})$. Let $l \in \omega$ be such that

- $q_l \leq \pi(\bar{f})(q_{k_i})$,
- $|\pi(\bar{f})(q_{k_i}) - q_l| < \min\{\varepsilon, \theta\}$,
- $\beta_l < \varepsilon'$, and
- $\bar{f}_{q_{k_i}}(a_0) < p_l < p_l + \beta_l < \bar{f}_{q_{k_i}}(a_1)$.

Notice that such an l exists by the choice of the β_n's and the p_n's. Then we define $g' \restriction \mathbb{R}_{q_{k_i}}$ from $\mathbb{R}_{q_{k_i}}$ onto $\mathbb{R}_{q_l}$ as follows. First we "translate" the values of $\bar{f}_{q_i}$ preceding a_0 and following a_1 by suitable fixed constants so that $g'_{q_{k_i}}(a_0) = p_l$ and $g'_{q_{k_i}}(a_1) = p_l + \beta_l$. More precisely, let $u_0 = p_l - \bar{f}_{q_{k_i}}(a_0)$ and $u_1 = p_l + \beta_l - \bar{f}_{q_{k_i}}(a_1)$, and set $g'(q_{k_i}, b) =$

$(q_l, \bar{f}_{q_{k_i}}(b) + u_0)$ for $b \leq a_0$, and $g'(q_{k_i}, b) = (q_l, \bar{f}_{q_{k_i}}(b) + u_1)$ for $b \geq a_1$. Finally, we extend g' to the whole $\mathbb{R}_{q_{k_i}}$ by mapping $\{q_{k_i}\} \times [a_0, a_1]$ onto $\{q_l\} \times [p_l, p_l + \beta_l]$ by a uniformly continuous partial reduction of C_x to C_y which is weakly increasing in the second coordinate and maps a_0 to p_l and a_1 to $p_l + \beta_l$ (this is possible by (A3)).

By construction, g' is well-defined and satisfies (Y1)–(Y3) and (Y5) (for (Y2) notice that g' still satisfies (S2) by the condition $|\pi(\bar{f})(q_{k_i}) - q_l| < \theta$, while for (Y3) use the fact that $g' \restriction (-\infty, a_0]$, $g' \restriction [a_0, a_1]$ and $g' \restriction [a_1, +\infty)$ are all uniformly continuous). We now check that $(\delta', 3\varepsilon')$ is a modulus for g'. Obviously, it is enough to consider $a \in D_{g'} \setminus D_f$, i.e. $a = q_{k_i}$ for some $i < M$. Let $b < b'$ be such that $|b' - b| < \delta'$. First assume that $b, b' \leq a_0$. Then $d(g'(q_{k_i}, b), g'(q_{k_i}, b')) = |\bar{f}_{q_{k_i}}(b') + u_0 - (\bar{f}_{q_{k_i}}(b) + u_0)| = d(\bar{f}(q_{k_i}, b), \bar{f}(q_{k_i}, b')) < \varepsilon'$ since (δ', ε') is a modulus for f (and hence also for $\bar{f}$). The case $b, b' \geq a_1$ is treated in a similar way and gives that $d(g'(q_{k_i}, b), g'(q_{k_i}, b')) < \varepsilon'$ again. If $b \leq a_0 \leq b' \leq a_1$ then $|a_0 - b| < \delta'$: by the previous computation, it follows that $d(g'(q_{k_i}, b), g'(q_{k_i}, b')) \leq d(g'(q_{k_i}, b), g'(q_{k_i}, a_0)) + d(g'(q_{k_i}, a_0), g'(q_{k_i}, b')) = |g'_{q_{k_i}}(b) - g'_{q_{k_i}}(a_0)| + |g'_{q_{k_i}}(a_0) - g'_{q_{k_i}}(b')| < \varepsilon' + \varepsilon' = 2\varepsilon'$ since, by construction, $p_l = g'_{q_{k_i}}(a_0) \leq g'_{q_{k_i}}(b') \leq p_l + \beta_l$ and $\beta_l < \varepsilon'$. The case $a_0 \leq b \leq a_1 \leq b'$ is treated similarly, so let us consider the case $b \leq a_0 \leq a_1 \leq b'$. Since in this case $|a_0 - b| < \delta'$ and $|b' - a_1| < \delta'$, by the previous computations we get $d(g'(q_{k_i}, b), g'(q_{k_i}, b')) \leq d(g'(q_{k_i}, b), g'(q_{k_i}, a_0)) + d(g'(q_{k_i}, a_0), g'(q_{k_i}, a_1)) + d(g'(q_{k_i}, a_1), g'(q_{k_i}, b')) < 3\varepsilon'$ since $d(g'(q_{k_i}, a_0), g'(q_{k_i}, a_1)) = |g'_{q_{k_i}}(a_0) - g'_{q_{k_i}}(a_1)| = |p_l + \beta_l - p_l| = \beta_l < \varepsilon'$. Finally, the case $a_0 \leq b \leq b' \leq a_1$ is trivial since in this situation we have $d(g'(q_{k_i}, b), g'(q_{k_i}, b')) \leq d(g'(q_{k_i}, a_0), g'(q_{k_i}, a_1)) = \beta_l < \varepsilon'$. Hence $(\delta', 3\varepsilon')$ is a modulus for g', as required.

Finally, we check that $\sup\{d(\bar{f}(a, b), g'(a, b)) \mid a \in D_{g'}, b \in \mathbb{R}\} \leq 2\varepsilon$. It is obviously enough to consider the case $a = q_{k_i}$ and show that $d(\bar{f}(q_{k_i}, b), g'(q_{k_i}, b)) < 2\varepsilon$ for every $b \in \mathbb{R}$. Since $\beta_{k_i} \leq \beta_{k_0} < \delta$ it follows that $|\bar{f}_{q_{k_i}}(a_1) - \bar{f}_{q_{k_i}}(a_0)| = d(\bar{f}(q_{k_i}, a_0), \bar{f}(q_{k_i}, a_1)) < \varepsilon$ (because (δ, ε) is a modulus for $\bar{f}$), therefore $u_0, u_1 < \varepsilon$ by the choice of l. This implies that $d(\bar{f}(q_{k_i}, b), g'(q_{k_i}, b)) \leq |\pi(\bar{f})(q_{k_i}) - q_l| + |\bar{f}_{q_{k_i}}(b) - g'_{q_{k_i}}(b)| < 2\varepsilon$ for $b \leq a_0$ or $b \geq a_1$. If instead $a_0 \leq b \leq a_1$, notice that both $\bar{f}_{q_{k_i}}(b)$ and $g'_{q_{k_i}}(b)$ belong to the interval $[\bar{f}_{q_{k_i}}(a_0), \bar{f}_{q_{k_i}}(a_1)]$, whence $d(\bar{f}(q_{k_i}, b), g'(q_{k_i}, b)) \leq |\pi(\bar{f})(q_{k_i}) - q_l| + |\bar{f}_{q_{k_i}}(a_1) - \bar{f}_{q_{k_i}}(a_0)| < 2\varepsilon$.

We now extend g' to g'' in such a way that $D_{g''} = D_{g'} \cup \{i_{l_j} \mid j < N\}$. Fix $j < n$. If $\pi(\bar{f})(i_{l_j}) \in I$, then $\bar{f} \restriction \mathbb{R}_{i_{l_j}}$ is already a partial reduction of C_x to C_y and we define $g'' \restriction \mathbb{R}_{i_{l_j}} = \bar{f} \restriction \mathbb{R}_{i_{l_j}}$. Otherwise, we choose $l \in \omega$ such that

- $i_l < \pi(\bar{f})(i_{l_j})$ and
- $|\pi(\bar{f})(i_{l_j}) - i_l| < \min\{\varepsilon, \theta\}$,

and define $g''(i_{l_j}, b) = (i_l, \bar{f}_{i_{l_j}}(b))$ for every $b \in \mathbb{R}$. Then g'' satisfies (Y1)–(Y3) and (Y5) (for (Y2) use $|\pi(\bar{f})(i_{l_j}) - i_l| < \theta$), $(\delta', 3\varepsilon')$ is a modulus for g'' (since it is a modulus for g' and (δ', ε') is a modulus for $\bar{f}$) and $\sup\{d(\bar{f}(a, b), g''(a, b)) \mid a \in D_{g''}, b \in \mathbb{R}\} \leq 2\varepsilon$ by the analogous property for g' and $|\pi(\bar{f})(i_{l_j}) - i_l| < \varepsilon$.

We then extend g'' to g''' in such a way that $R_{g'''} = R_{g''} \cup \{q_{k_i} \mid i < M\}$. Fix $i < M$ and let $r = \pi(\bar{f})^{-1}(q_{k_i})$ (such an r exists because $\bar{f}$, and hence $\pi(\bar{f})$, is surjective). If $\bar{f} \restriction \mathbb{R}_r$ is already a partial reduction of C_x to C_y, we let $g''' \restriction \mathbb{R}_r = \bar{f} \restriction \mathbb{R}_r$. Otherwise, to

simplify the notation let $\beta = \beta_{k_i}$, $b_0 = p_{k_i} = \inf(B_{x,k_i})$, and $b_1 = p_{k_i} + \beta_{k_i} = \sup(B_{x,k_i})$. Let $l \in \omega$ be such that

- $q_l \leq r$,
- $|r - q_l| < \theta$,
- $d(\bar{f}(q_l, b), \bar{f}(r, b)) < \varepsilon$ for all $b \in \mathbb{R}$, and
- $\pi_1(\bar{f}^{-1}(q_{k_i}, b_0)) < p_l < p_l + \beta_l < \pi_1(\bar{f}^{-1}(q_{k_i}, b_1))$.

Notice that the third requirement is possible since $\bar{f}$ is uniformly continuous. Then we define $g''' \restriction \mathbb{R}_{q_l}$ from $\mathbb{R}_{q_l}$ onto $\mathbb{R}_{q_{k_i}}$ similarly to the case of $g' \restriction \mathbb{R}_{q_{k_i}}$. More precisely, let $u_0 = b_0 - \bar{f}_r(p_l)$ and $u_1 = b_1 - \bar{f}_r(p_l + \beta_l)$, and set $g(q_l, b) = (q_{k_i}, \bar{f}_r(b) + u_0)$ for $b \leq p_l$ and $g(q_l, b) = (q_{k_i}, \bar{f}_r(b) + u_1)$ for $b \geq p_l + \beta_l$. Finally, we extend g''' to the entire $\mathbb{R}_l$ by mapping $\{q_l\} \times [p_l, p_l + \beta_l]$ onto $\{q_{k_i}\} \times [b_0, b_1]$ via a uniformly continuous partial reduction of C_x onto C_y which is weakly increasing in the second coordinate and maps p_l to b_0 and $p_l + \beta_l$ to b_1 (this is possible by (A3)).

Arguing as for g', it is not hard to check that g''' satisfies (Y1)–(Y3) and (Y5), and that $(\delta', 3\varepsilon')$ is a modulus for g''' since it is a modulus for g'' and $|b_1 - b_0| = \beta_{k_i} \leq \beta_{k_0} < \varepsilon'$. Finally, we check that $\sup\{d(\bar{f}(a, b), g'''(a, b)) \mid a \in D_{g'''}, b \in \mathbb{R}\} \leq 2\varepsilon$. Since the analogous property for g'' holds, it is enough to check that given $i < M$ and $l \in \omega$ as above, $d(\bar{f}(q_l, b), g'''(q_l, b)) < 2\varepsilon$ for every $b \in \mathbb{R}$. First notice that $d(\bar{f}(q_l, b), g'''(q_l, b)) \leq d(\bar{f}(q_l, b), \bar{f}(r, b)) + d(\bar{f}(r, b), g'''(q_l, b))$: since $d(\bar{f}(q_l, b), \bar{f}(r, b)) < \varepsilon$ by the third property above, it is enough to check that $d(\bar{f}(r, b), g'''(q_l, b)) = |\bar{f}_r(b) - g'''_{q_l}(b)| < \varepsilon$. Since $|b_1 - b_0| = \beta_{k_i} \leq \beta_{k_0} < \varepsilon$, we get $u_0, u_1 < \varepsilon$, whence $|\bar{f}_r(b) - g'''_{q_l}(b)| < \varepsilon$ for $b \leq p_l$ and $b \geq p_l + \beta_l$. If instead $p_l \leq b \leq p_l + \beta_l$ we have that both $\bar{f}_r(b)$ and $g'''_{q_l}(b)$ belong to the interval $[b_0, b_1]$ which has length $\beta_{k_i} \leq \beta_{k_0} < \varepsilon$, whence $|\bar{f}_r(b) - g'''_{q_l}(b)| < \varepsilon$ again.

Finally, we extend g''' to g in such a way that $R_g = R_{g'''} \cup \{i_{l_j} \mid j < N\}$. Let $r = \pi(\bar{f})^{-1}(i_{l_j})$. If $r \in I$, then $\bar{f} \restriction \mathbb{R}_r$ is already a partial reduction of C_x to C_y and we simply define $g \restriction \mathbb{R}_r = \bar{f} \restriction \mathbb{R}_r$. Otherwise, we choose $l \in \omega$ with

- $i_l \leq r$,
- $d(i_l, r) < \theta$, and
- $d(\bar{f}(i_l, b), \bar{f}(r, b) < \varepsilon$ for all $b \in \mathbb{R}$.

As for the definition of g''', the last requirement is possible by the uniform continuity of $\bar{f}$. Then we define $g(i_l, b) = (i_{l_j}, \bar{f}_r(b))$ for all $b \in \mathbb{R}$. As for the previous steps, g satisfies (Y1)–(Y3) and (Y5), and $(\delta', 3\varepsilon')$ is a modulus for g since it is a modulus for g''' and (δ', ε') is a modulus for $\bar{f}$. Finally,

$$\sup\{d(\bar{f}(a, b), g(a, b)) \mid a \in D_g, b \in \mathbb{R}\} \leq 2\varepsilon \tag{‡}$$

by the last requirement in the definition of $l \in \omega$ and since the analogous property holds for g'''. A straightforward computation shows that (‡) implies $\|\bar{f} - \bar{g}\| \leq 2\varepsilon$ by (†), therefore g satisfies (1)–(3). Hence it remains to check that $g \in \mathcal{Y}$. Let $K \in \omega$ be a witness of the fact that f satisfies (Y4). As already observed, this implies that $|\bar{f}_a(b) - \bar{f}_{a'}(b)| < \frac{K}{\rho} \cdot |a - a'|$ for every $a, a', b \in \mathbb{R}$, where $\rho = \min\{|a - a'| \mid a, a' \in D_f\}$. Let

$K' = \frac{K}{\rho} \cdot \max\{|a - a'| \mid a, a' \in D_g\}$. Then for every $a, a' \in D_g$ and $b \in \mathbb{R}$ we have

$$\begin{aligned}|g_a(b) - g_{a'}(b)| &\leq |g_a(b) - \bar{f}_a(b)| + |\bar{f}_a(b) - \bar{f}_{a'}(b)| + |\bar{f}_{a'}(b) - g_{a'}(b)| \\ &\leq d(g(a,b), \bar{f}(a,b)) + K' + g(\bar{f}(a',b), g(a',b)) \leq 2\varepsilon + K' + 2\varepsilon = K' + 4\varepsilon\end{aligned}$$

by (3). This shows that $K' + 4\varepsilon$ witnesses that g satisfies (Y4), and hence $g \in \mathcal{Y}$ by Definition 5.26, as required. □

Proof of Theorem 5.23 We will show that the map $\mathscr{P}(\omega) \to \mathscr{P}(\mathbb{R}^2)\colon x \mapsto C_x$, where C_x is defined as in $(*)$, is the desired embedding.

Assume first that $x \subseteq^* y$: we claim that $C_x \leq_{\mathsf{D}_1} C_y$ (hence, in particular, $C_x \leq_{\mathsf{D}_2} C_y$). In fact, we will construct a continuous reduction of C_x to C_y as a *uniform* limit of a sequence of special maps $\langle \bar{f}_n \mid f_n \in \mathcal{Y}, n \in \omega\rangle$ with the property that for every $k \in \omega$ there is $N_k \in \omega$ such that:

(1) $\bar{f}_{N_k} \restriction \mathbb{R}_{q_k} = \bar{f}_n \restriction \mathbb{R}_{q_k}$ and $\bar{f}_{N_k} \restriction \mathbb{R}_{i_k} = \bar{f}_n \restriction \mathbb{R}_{i_k}$ for every $n \geq N_k$ (so that, in particular, $\pi(\bar{f}_{N_k})(q_k) = \pi(\bar{f}_n)(q_k)$ and $\pi(\bar{f}_{N_k})(i_k) = \pi(\bar{f}_n)(i_k)$);
(2) $\bar{f}_{N_k} \restriction \mathbb{R}_{q_k}$ is a partial reduction of C_x to C_y: in particular, $\pi(\bar{f}_{N_k})(q_k) \in \mathbb{Q}$ by property (A1);
(3) $\pi(\bar{f}_{N_k})(i_k) \in I$;
(4) for every $l \in \omega$ there is $m \in \omega$ such that $\pi(\bar{f}_{N_m})(q_m) = q_l$;
(5) for every $l \in \omega$ there is $m \in \omega$ such that $\pi(\bar{f}_{N_m})(i_m) = i_l$.

Assume that such a sequence exists, and let $\bar{f} = \lim_{n\to\infty} \bar{f}_n$, so that, in particular, $\bar{f}$ is continuous because the $\bar{f}_n$'s are continuous and are assumed to converge uniformly: then $\bar{f}$ witnesses $C_x \leq_{\mathsf{D}_1} C_y$. In fact, let $(a,b) \in \mathbb{R}^2$. If $a = q_k$ or $a = i_k$ for some $k \in \omega$, then $\bar{f}(a,b) = \bar{f}_{N_k}(a,b)$ by (1), and hence $(a,b) \in C_x \iff \bar{f}(a,b) \in C_y$ by, respectively, (2) and (3) (depending on whether $a \in \mathbb{Q}$ or $a \in I$). It remains to consider the case $a \in \mathbb{R} \setminus (\mathbb{Q} \cup I)$. Notice that since $\bar{f}$ is the limit of a sequence of special maps, $\bar{f}$ still satisfies (S1) and (S2), hence $\pi(\bar{f})\colon \mathbb{R} \to \mathbb{R}$ is a well-defined injective map. By (4), (5) and (1), $\mathbb{Q} \cup I \subseteq \pi(\bar{f})(\mathbb{Q} \cup I)$ (in fact, $\mathbb{Q} \cup I = \pi(\bar{f})(\mathbb{Q} \cup I)$ by (2) and (3)). By injectivity of $\pi(\bar{f})$, this implies $\pi(\bar{f})(a) \in \mathbb{R} \setminus (\mathbb{Q} \cup I)$, whence $(a,b) \in C_x \iff \bar{f}(a,b) \in C_y$ because, by construction, $(a,b) \notin C_x$ and $\bar{f}(a,b) \notin C_y$.

It remains to show that the above sequence of maps exists: the sequence $\langle \bar{f}_n \mid f_n \in \mathcal{Y}, n \in \omega\rangle$ will be defined recursively using a back-and-forth construction. Towards this aim, we will also define an auxiliary sequence $\langle \delta_n \mid n \in \omega\rangle$ of positive real numbers, and require that

(i) f_{n+1} extends f_n;
(ii) $q_m \in D_{f_n} \cap R_{f_n}$ for all m with $\beta_m \geq \min\{\delta_n, \frac{1}{3\cdot 2^{n+2}}\}$,
(iii) $i_m \in D_{f_n} \cap R_{f_n}$ for all $m < n$,
(iv) $(\delta_n, \frac{1}{2^{n+1}})$ is a modulus for f_n, and
(v) $\|\bar{f}_{n+1} - \bar{f}_n\| \leq \frac{1}{2^n}$.

Set $f_0 = \emptyset$ and $\delta_0 = 1$. Notice that f_0 fulfills all the requirements since $\beta_0 < \frac{1}{12}$. To define f_{n+1}, we first choose δ_{n+1} such that $(\delta_{n+1}, \frac{1}{3\cdot 2^{n+2}})$ is a modulus for $\bar{f}_n$: this is possible since $\bar{f}_n$ is uniformly continuous by Lemma 5.27. Next we extend f_n to f_{n+1} by applying Lemma 5.28 with $f = f_n$, $\delta = \delta_n$, $\varepsilon = \frac{1}{2^{n+1}}$, $\delta' = \delta_{n+1}$, and $\varepsilon' = \frac{1}{3\cdot 2^{n+2}}$ in such

a way that setting $f_{n+1} = g$ we get a function which fulfills (ii) and (iii). More precisely, we let $\{k_i \mid i < M\}$ be an increasing enumeration of those $m \in \omega$ such that

$$\min\left\{\delta_{n+1}, \frac{1}{3 \cdot 2^{n+3}}\right\} \leq \beta_m < \min\left\{\delta_n, \frac{1}{3 \cdot 2^{n+2}}\right\},$$

and set $N = 1$ and $l_0 = n$. Notice that Lemma 5.28 can be applied with these parameters because $\beta_{k_0} < \min\{\delta_n, \frac{1}{3 \cdot 2^{n+2}}\} = \min\{\delta, \varepsilon, \varepsilon'\}$, and that the resulting $f_{n+1} = g$ satisfies (ii) and (iii) because of the choice of the k_i's and l_0, together with the fact that f_n satisfies such conditions by inductive hypothesis. Then $(\delta_{n+1}, \frac{1}{2^{n+2}})$ is a modulus for f_{n+1} by Lemma 5.28(ii), and $\|\bar{f}_{n+1} - \bar{f}_n\| \leq \frac{1}{2^n}$ by Lemma 5.28(iii), hence f_{n+1} satisfies also (iv) and (v). This completes the recursive definition. It is immediate to check that conditions (1)–(5) are satisfied by construction. Moreover, using standard arguments one can easily show that (v) implies that the sequence $\langle \bar{f}_n \mid n \in \omega \rangle$ uniformly converges to some $\bar{f} \colon \mathbb{R}^2 \to \mathbb{R}^2$, hence we are done.

To complete the proof of Theorem 5.23, it remains to show that if $C_x \leq_{\mathsf{D}_2} C_y$ then $x \subseteq^* y$. Suppose that f witnesses $C_x \leq_{\mathsf{D}_2} C_y$. Then there is a nonempty open set $U \subseteq \mathbb{R}^2$ such that $f \restriction U$ is continuous by the observation preceding Proposition 5.16. We may assume that U is an open rectangle. Fix $a \in I$. If $b < b'$ are such that $(a, b), (a, b') \in U$, then $\{a\} \times [b, b'] \subseteq C_x \cap U$ is homeomorphic to $[0, 1] \subseteq \mathbb{R}$, hence $f(\{a\} \times [b, b'])$ is a (possibly degenerate) path totally contained in C_y (since $f \restriction U$ is continuous and f reduces C_x to C_y). By construction of C_y and the choice of I, this implies that $f(\{a\} \times [b, b'])$ is contained in a single vertical line, so that in particular $\pi_0(f(a, b)) = \pi_0(f(a, b'))$. We have thus shown that for every $a \in I$, $f(\mathbb{R}_a \cap U) \subseteq \mathbb{R}_d$ for some $d \in \mathbb{R}$. It follows that the same holds also for an arbitrary $a \in \mathbb{R}$, i.e. that $\pi_0(f(a, b)) = \pi_0(f(a, b'))$ for every $(a, b), (a, b') \in U$, because otherwise, by continuity of $f \restriction U$, we would have $\pi_0(f(a', b)) \neq \pi_0(f(a', b'))$ for some $a' \in I$ sufficiently close to a. Now notice that by the choice of the p_n's and the β_n's there is an index $n \in \omega$ with $\{q_n\} \times [p_n, p_n + \beta_n] \subseteq U$. Let $s \in \mathbb{R}$ be such that $f(\mathbb{R}_{q_n} \cap U) \subseteq \mathbb{R}_s$. Since f is a reduction of C_x to C_y and $C_x \cap \mathbb{R}_{q_n} = \{q_n\} \times B_{x,n} \subseteq U$ is neither empty nor the entire $\mathbb{R}_{q_n} \cap U$ by (A1), $s = q_m$ for some $m \in \omega$. Therefore, $C_y \cap \mathbb{R}_{q_m} = \{q_m\} \times B_{y,m}$. Let $\varepsilon > 0$ be such that $\{q_n\} \times [p_n - \varepsilon, p_n + \beta_n + \varepsilon] \subseteq U$. Then the map $g \colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = f(p_n - \varepsilon)$ if $x \leq p_n - \varepsilon$, $g(x) = f(p_n + \beta_n + \varepsilon)$ if $x \geq p_n + \beta_n + \varepsilon$, and $g(x) = f(x)$ if $p_n - \varepsilon \leq x \leq p_n + \beta_n + \varepsilon$ is a continuous reduction of $B_{x,n}$ to $B_{y,m}$. Since $B_{x,n}$ is an affine image of A_x and $B_{y,m}$ is an affine image of A_y, it follows that $A_x \leq^{\mathbb{R}}_{\mathsf{D}_1} A_y$, and hence $x \subseteq^* y$ by (A2). □

Corollary 5.29. Suppose that X is an Hausdorff quasi-Polish space such that either $\mathbb{R}^2 \subseteq X$ (in particular, we can take e.g. $X = \mathbb{R}^n$ or $X = [0, 1]^n$ for $2 \leq n < \omega$), or X is an n-dimensional σ-compact Polish space which is embeddable into $\mathbb{R}^n$ (for some $n \geq 2$), or X is locally Euclidean and of dimension $n \geq 2$.

(1) Any countable partial order embeds into the $(\mathbf{B}, \mathsf{D}_2)$-hierarchy on X.
(2) (ZFC) Any partial order of size ω_1 embeds into the $(\mathbf{B}, \mathsf{D}_2)$-hierarchy on X.
(3) The $(\mathbf{B}, \mathsf{D}_2)$-hierarchy on X is very bad.

Proof. First we consider the case $X = \mathbb{R}^2$. Assuming the axiom of choice, every partial order of size ω_1 embeds into $(\mathscr{P}(\omega), \subseteq^*)$ by Parovičenko's Theorem (Parovičenko 1963). The embedding of countable partial orders is constructed by diagonalizing over finitely many subsets of ω in each step, hence AC is not necessary. In particular, $(\mathscr{P}(\omega), \subseteq^*)$ contains infinite antichains and infinite decreasing sequences. Hence the result follows from Theorem 5.23.

Now consider the case of an Hausdorff quasi-Polish space X such that $\mathbb{R}^2 \subseteq X$. Since $\mathbb{R}^2$ is (quasi-)Polish, $\mathbb{R}^2 \in \mathbf{\Pi}^0_2(X)$ by Proposition 2.9. Since $\mathbb{R}^2$ is σ-compact and X is Hausdorff, $\mathbb{R}^2 \in \mathbf{\Sigma}^0_2(X)$, hence $\mathbb{R}^2 \in \mathbf{\Delta}^0_2(X)$. Therefore the result follows from Corollary 5.5.

Finally, let X be either an n-dimensional σ-compact Polish space which is embeddable into $\mathbb{R}^n$ (for some $n \geq 2$), or a locally Euclidean Polish space of dimension $n \geq 2$. Then $X \cong_2 \mathbb{R}^n$ by Remark 4.22(2)(a-b), and hence the desired claim follows from Proposition 5.2 and the fact that we already proved the analogous result for $\mathbb{R}^n$. □

Unfortunately, for what concerns the D_n-hierarchies ($n \geq 3$) on uncountable quasi-Polish spaces of dimension ∞, the situation is still completely unclear.

Question 5.30. Are the D_3-hierarchies on the Hilbert cube $[0,1]^\omega$ and on the Scott domain $P\omega$ (very) good? What about the D_n-hierarchies (for larger $n \in \omega$)? What about other (quasi-)Polish spaces of dimension ∞?

5.2. *Degree structures in countable spaces*

Here we consider the case of countable quasi-Polish spaces and show that even when considering the very restricted class of scattered countably based spaces, the Wadge hierarchy may not be very good.

It follows from Proposition 4.5(1) and Proposition 5.2 that all countable countably based T_0-spaces X have isomorphic $\mathcal{F}$-hierarchies whenever $\mathcal{F} \supseteq \mathsf{D}^{\mathsf{W}}_3$ is a family of reducibilities. Moreover, it follows from parts (2) and (3) of Proposition 4.5 and Proposition 5.2 that all countable T_1-spaces and all scattered countably based spaces X have isomorphic $\mathcal{F}$-hierarchies whenever $\mathcal{F} \supseteq \mathsf{D}^{\mathsf{W}}_2$. In fact, the proof of Proposition 4.5 shows that in all the above mentioned cases, the class $\mathcal{F}(X)$ coincides with the class of all functions from X to itself: hence the resulting $\mathcal{F}$-hierarchy is formed by two incomparable degrees consisting of $\emptyset$ and the whole space X, together with a single degree above them containing all other subsets of X (this in particular means that the $\mathcal{F}$-hierarchy on X is very good). Therefore only the Wadge reducibility and the D_2-reducibility are of interest when considering countable countably based T_0-spaces X, and when X is a countable T_1 space or a countably based scattered space, then only the Wadge reducibility needs to be considered.

The next result identifies two classes of countable spaces with a very good structure of Wadge degrees.

Proposition 5.31. Let X be a countable Polish space or a finite T_0-space. Then the D_1-structure on X is very good.

Proof. If X is countable Polish then it is zero-dimensional, hence the claim follows from Theorem 5.12 and the fact that $\mathscr{P}(X) \subseteq \mathbf{\Delta}^0_2(X)$.

Let now X be a finite T_0-space, and let $\leq$ be the *specialization order* on X defined by setting $x \leq y \iff x$ belongs to the closure of $\{y\}$ (for every $x, y \in X$). Then X coincides with the ω-algebraic domain $(X, \leq)$ (endowed with the Scott or, equivalently, the Alexandrov topology), and the continuous functions on X coincide with the monotone functions on $(X, \leq)$. Moreover, any subset of X is a finite Boolean combination of open sets, hence it is in $\bigcup_{n\in\omega} \mathbf{\Sigma}^{-1}_n = \bigcup_{n\in\omega} \mathbf{\Pi}^{-1}_n = \bigcup_{n\in\omega} \mathbf{\Delta}^{-1}_n$. By (the second part of) Theorem 2.7, this implies that every subset of X is Wadge complete in one of $\mathbf{\Sigma}^{-1}_n, \mathbf{\Pi}^{-1}_n, \mathbf{\Delta}^{-1}_n$, $n \in \omega$: therefore, $(\mathscr{P}(X), \leq_{\mathsf{W}})$ is semi-well-ordered. □

Remark 5.32. In both the cases considered in Proposition 5.31, the space X falls in one of the cases mentioned at the beginning of this subsection: if X is Polish then it is also T_1, while if X is finite then it is automatically scattered. Therefore the D_2-hierarchy on such an X is always (trivially) very good.

We now show that Proposition 5.31 cannot be extended to scattered quasi-Polish spaces by showing that there is a scattered ω-continuous domain whose Wadge hierarchy is good but not very good. Note that by Proposition 2.5, any scattered dcpo is in fact an algebraic domain and all of its elements are compact.

Proposition 5.33. There is a scattered ω-algebraic domain $(X, \leq)$ such that $(\mathscr{P}(X), \leq_{\mathsf{W}}$) has four pairwise incomparable elements (hence it is not very good).

Proof. For any $n \in \omega$, fix an $\leq$-chain $C_n = \{c^n_n < \ldots < c^n_0\}$ with $n+1$-many elements. Let $C = \bigsqcup_{n\in\omega} C_n$ be the disjoint union of these chains (so the elements of different chains are $\leq$-incomparable in C), and let X be obtained from C by adjoining a bottom element $\bot$ and a top element $\top$. By Proposition 2.5, X is a scattered ω-algebraic domain.

We inductively define the sets $D_k \subseteq X$, $k \in \omega$, by letting $D_0 = \{x \in X \mid \exists n(c^n_0 \leq x)\}$ and $D_{i+1} = D_i \cup \{x \in X \mid \exists n > i(c^n_{i+1} \leq x)\}$. Then $D_0 \subseteq D_1 \subseteq \ldots$ and any D_k is open in X, so the sets $A = D_0 \cup \bigcup_k (D_{2k+2} \setminus D_{2k+1})$ and $B = A \setminus \{\top\}$ are in the pointclass $\mathbf{\Sigma}^{-1}_\omega(X)$, i.e. in the ω-th level of the Hausdorff difference hierarchy over the open sets in X (see Subsection 2.3). Observe that for all $k < n \in \omega$ we have $\bot \notin D_k$, $c^n_0 \in D_0$, and $c^n_{k+1} \in D_{k+1} \setminus D_k$. Therefore, for $k \leq n \in \omega$ we have $\bot \notin A \cup B$, $c^n_k \in A \cap B$ if k is even, and $c^n_k \notin A \cup B$ if k is odd. This means that for each of A and B one can construct in the obvious way a 0-alternating tree onto $X \setminus \{\top\} = \{\bot, c^n_k \mid k \leq n \in \omega\}$, so $A, B \notin \mathbf{\Pi}^{-1}_\omega$ by (the first part of) Theorem 2.7.

We claim that the sets $A, \overline{A}, B, \overline{B}$ are pairwise Wadge incomparable. Since any of $\mathbf{\Sigma}^{-1}_\alpha, \mathbf{\Pi}^{-1}_\alpha$, being a family of pointclasses, is closed under continuous preimages, the fact that $A, B \in \mathbf{\Sigma}^{-1}_\omega \setminus \mathbf{\Pi}^{-1}_\omega$ implies that each of A, B is Wadge incomparable with both $\overline{A}$ and $\overline{B}$. So it remains only to check that A is Wadge incomparable with B: we will just show that $A \not\leq_{\mathsf{W}} B$, as the fact that $B \not\leq_{\mathsf{W}} A$ can be proved in the same way. Assume towards a contradiction that $A = f^{-1}(B)$ for a continuous (i.e. monotone) function f on X. Since $\top \in A$ and $\top \notin B$, $f(\top) \neq \top$, hence $f(\top) \in X \setminus \{\top\} = \{\bot, c^n_k \mid k \leq n \in \omega\}$. Since $f(x) \leq f(\top)$ for all $x \in X$, the range of f is contained in $C_m \cup \{\bot\}$ for some $m < \omega$. Choose an alternating chain $a_0 < \ldots < a_{m+2}$ for A of length $m+3$ (e.g. we can

take $a_i = c^{m+2}_{m+2-i}$ for $i \leq m+2$). Then, since f is supposed to be a reduction of A to B, the image under f of this chain must be alternating for B. Since, as already observed in Subsection 2.4, this implies $f(a_0) < \ldots < f(a_{m+2})$, we have $|\{f(a_i) \mid i \leq m+2\}| = m+3 > m+2 = |C_m \cup \{\bot\}| \geq |\{f(x) \mid x \in X\}|$, a contradiction. $\square$

Notice that in the example above the four Wadge incomparable elements have (almost) the minimal possible complexity, as by (the second part of) Theorem 2.7 the Wadge hierarchy on any ω-algebraic domain is semi-well-ordered when restricted to $\bigcup_{n\in\omega} \mathbf{\Sigma}^{-1}_n$. Moreover, since the structure of the poset $(X, \leq)$ in the previous proof is very simple, it is possible to completely describe the Wadge hierarchy on X, as it is shown in the next proposition.

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be arbitrary posets. By $P+Q$ we denote the poset $(P \sqcup Q, \leq)$ where $\leq \restriction P$ (respectively, on $\leq \restriction Q$) coincides with $\leq_P$ (respectively, with $\leq_Q$), and $p \leq q$ for all $p \in P$ and $q \in Q$. By $P \cdot Q$ we denote the poset $(P \times Q, \leq)$ where $(p_0, q_0) < (p_1, q_1)$ if and only if $q_0 <_Q q_1$ or $q_0 = q_1 \wedge p_0 <_P p_1$. For any integer $n \geq 2$, let $\bar{n}$ denote the poset consisting of exactly one antichain with n elements, and identify ω with the poset $(\omega, \leq)$.

Proposition 5.34. The quotient-poset of $(\mathscr{P}(X), \leq_{\mathsf{W}})$ is isomorphic to $(\bar{2}\cdot\omega)+\bar{4}$. Hence the W-hierarchy on X is good but not very good.

Proof. Let us first make a basic observation which is intimately related to the very particular structure of the poset $(X, \leq)$. Given $S \subseteq X$, call an alternating chain $a_0 < \ldots < a_n$ for S *maximal* if for every $b < a_0$ and $b' > a_n$ none of $b < a_0 < \ldots < a_n$ and $a_0 < \ldots < a_n < b'$ is alternating for S. Notice that if $a_0 < \ldots < a_n$ is a maximal alternating chain for S, then $a_0 \in S \iff \bot \in S$ and $a_n \in S \iff \top \in S$ (otherwise one of $\bot < a_0 < \ldots < a_n$ or $a_0 < \ldots < a_n < \top$ would be alternating, contradicting the maximality of the chain). This also implies that all maximal alternating chains are compatible, i.e. that if $a'_0 < \ldots < a'_{n'}$ is another maximal alternating chain for S then $a'_0 \in S \iff a_0 \in S$ and $a'_{n'} \in S \iff a_n \in S$.

Let now $\mathcal{S}$ be the class of all $\emptyset \neq S \subsetneq X$ such that there is a natural number n bounding the lengths of all alternating chains for S. We will use Theorem 2.7 to show that each $S \in \mathcal{S}$ belongs to some finite level of the difference hierarchy over the open sets of X. First notice that every $S \in \mathcal{S}$ (in fact, every $S \subseteq X$) is approximable because all elements of X are compact. Given $S \in \mathcal{S}$, let $m(S)+1$ be the maximal length of an alternating chain for S, so that, in particular, there is an alternating tree for S of rank $m(S)$ but no alternating tree for S of rank $m(S)+1$. Notice that by definition of $\mathcal{S}$, we necessarily have $m(S) \geq 1$. Let $a_0 < \ldots < a_{m(S)}$ be an alternating chain for S of length $m(S)+1$. By definition of $m(S)$, such a chain is necessarily maximal, hence by the compatibility of the maximal alternating chains for S there is no alternating chain $b_0 < \ldots < b_{m(S)}$ for S such that $a_0 \in S \iff b_0 \notin S$. This means that all alternating trees for S of rank $m(S)$ are of the same type, i.e. either they are all 1-alternating or they are all 0-alternating (depending on whether $\bot \in S$ or not). Then by Theorem 2.7, either $S \in \mathbf{\Pi}^{-1}_{m(S)}(X) \setminus \mathbf{\Sigma}^{-1}_{m(S)}(X)$ or $S \in \mathbf{\Sigma}^{-1}_{m(S)}(X) \setminus \mathbf{\Pi}^{-1}_{m(S)}(X)$, hence S is also Wadge complete in (exactly) one of $\mathbf{\Sigma}^{-1}_{m(S)}(X)$ or $\mathbf{\Pi}^{-1}_{m(S)}(X)$.

Conversely, it is not hard to check that all the possibilities are realized, i.e. that $\mathbf{\Sigma}^{-1}_n(X) \setminus \mathbf{\Pi}^{-1}_n(X) \neq \emptyset$ for every $1 \leq n \in \omega$. In fact, let A, B be defined as in the proof of Proposition 5.33, and put $C'_n = C_n \cup \{\bot, \top\}$ for every $n \in \omega$. Then one can straightforwardly check using Theorem 2.7 that $\overline{A} \cap C'_{2i} \in \mathbf{\Sigma}^{-1}_{2i+1}(X) \setminus \mathbf{\Pi}^{-1}_{2i+1}(X)$, $A \cap C'_{2i} \in \mathbf{\Pi}^{-1}_{2i+1}(X) \setminus \mathbf{\Sigma}^{-1}_{2i+1}(X)$, $B \cap C'_{2i} \in \mathbf{\Sigma}^{-1}_{2i+2}(X) \setminus \mathbf{\Pi}^{-1}_{2i+2}(X)$, and $\overline{B} \cap C'_{2i} \in \mathbf{\Pi}^{-1}_{2i+2}(X) \setminus \mathbf{\Sigma}^{-1}_{2i+2}(X)$ for every $i \in \omega$.

Therefore, letting $h(\emptyset) = (0, 0)$, $h(X) = (1, 0)$, and, for $S \in \mathcal{S}$, $h(S) = (0, m)$ (respectively, $h(S) = (1, m)$) if and only if S is Wadge complete in $\mathbf{\Sigma}^{-1}_m(X)$ (respectively, in $\mathbf{\Pi}^{-1}_m(X)$), we get that the function $h\colon \mathcal{S} \cup \{\emptyset, X\} \to \bar{2} \cdot \omega$ induces an isomorphism between the quotient-poset of $(\mathcal{S} \cup \{\emptyset, X\}, \leq_{\mathsf{W}})$ and $\bar{2} \cdot \omega$.

Now assume that $S \subseteq X$ has arbitrarily long finite alternating chains, i.e. that $S \neq \emptyset, X$ and $S \notin \mathcal{S}$. Obviously, $\emptyset, X \leq_{\mathsf{W}} S$. We claim that also every set from $\mathcal{S}$ is Wadge reducible to S. In fact, let $S' \in \mathcal{S}$ and $m = m(S')$. Let $a_0 < \ldots a_{m+1}$ be an alternating chain for S such that $a_0 \in S \iff \bot \in S'$ (such a chain exists by the choice of S). Consider the function $f\colon X \to \{a_0, \ldots, a_{m+1}\}$ defined in the following way: first, $f(\bot) = a_0$ and $f(\top) = a_{m+1}$. Then fix $n \in \omega$, and define f on c^n_{n-i} by induction on $i \leq n$ as follows. For $i = 0$, set $f(c^n_n) = a_0$ if $c^n_n \in S' \iff \bot \in S'$ and $f(c^n_n) = a_1$ otherwise. For the inductive step, let $f(c^n_{n-i}) = a_j$, and set $f(c^n_{n-(i+1)}) = a_j$ if $c^n_{n-i} \in S' \iff c^n_{n-(i+1)} \in S'$ and $f(c^n_{n-(i+1)}) = a_{j+1}$ otherwise. Then f is clearly monotone (hence continuous) and reduces S' to S.

Using essentially the same method as in the previous paragraph, for every $C \in \{A, B, \overline{A}, \overline{B}\}$ (where A, B are again defined as in the proof of Proposition 5.33) one can easily define two monotone functions $f', g'\colon X \setminus \{\bot, \top\} \to X \setminus \{\bot, \top\}$ such that $x \in C \iff f'(x) \in S$ and $x \in S \iff g'(x) \in C$ for all $x \in X \setminus \{\bot, \top\}$ (i.e. f' and g' are partial continuous reduction of, respectively, C to S and S to C). Extend f' and g', respectively, to the functions $f, g\colon X \to X$ by setting $f(\bot) = g(\bot) = \bot$ and $f(\top) = g(\top) = \top$. Then it is straightforward to check that f and g are continuous, and that they witness exactly one of the following four possibilities:

(1) $S \equiv_{\mathsf{W}} A$ (in case $\top \in S, \bot \notin S$);
(2) $S \equiv_{\mathsf{W}} B$ (in case $\top \notin S, \bot \notin S$);
(3) $S \equiv_{\mathsf{W}} \overline{A}$ (in case $\top \notin S, \bot \in S$);
(4) $S \equiv_{\mathsf{W}} \overline{B}$ (in case $\top \in S, \bot \in S$).

Therefore, we can extend h in the obvious way to the desired isomorphism between the quotient-poset of $(\mathscr{P}(X), \leq_{\mathsf{W}})$ and the poset $(\bar{2} \cdot \omega) + \bar{4}$. □

By the previous proof, the n-th level of the Wadge hierarchy on X (for $n \in \omega$) is occupied by the pair of Wadge degrees $(\mathbf{\Sigma}^{-1}_n(X) \setminus \mathbf{\Pi}^{-1}_n(X), \mathbf{\Pi}^{-1}_n(X) \setminus \mathbf{\Sigma}^{-1}_n(X))$, while the four Wadge degrees on the top of the hierarchy are exactly $[A]_{\mathsf{W}}$, $[B]_{\mathsf{W}}$, $[\overline{A}]_{\mathsf{W}}$, and $[\overline{B}]_{\mathsf{W}}$, where $A, B \subseteq X$ are defined as in the proof of Proposition 5.33.

Remark 5.35. Contrarily to the case of Polish spaces, there is no obvious relation between the dimension of a scattered ω-algebraic domain and the Wadge hierarchy on it. For example, the dimension of the space X considered in Propositions 5.33 and 5.34 is ω, and the Wadge hierarchy on X is (good but) not very good. On the other hand,

the dimension of the space C_∞ from Example 4.15(2) is ∞, while the Wadge hierarchy on C_∞ is very good, as one can easily check using an argument similar to the one of Proposition 5.34.

We end this subsection with some natural questions that are left open by Propositions 5.33 and 5.34.

Question 5.36.

(1) Is there a countable quasi-Polish space X with a (very) bad D_1-hierarchy?
(2) Is there a (necessarily non scattered) quasi-Polish space whose D_2-hierarchy is not very good? If yes, can it be even (very) bad?
(3) Is there a (necessarily uncountable) *Polish* space whose D_α-hierarchy is good but not very good (for some $1 \leq \alpha < \omega_1$)? Notice that by Theorem 5.15 the D_1-hierarchy on any Polish space X is either very good or bad, depending on whether $\dim(X) = 0$ or not.

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