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FRANCESCA FERRARA

HOW MULTIMODALITY WORKS IN MATHEMATICAL ACTIVITY: YOUNG CHILDREN GRAPHING MOTION

ABSTRACT. This paper aims to contribute to discussions on the multimodal nature of cognition through an elaboration of the ways multimodal aspects of thinking are exploited by learners doing mathematics. Moving beyond the fact *that* multimodality occurs, this paper focuses on *how* it occurs, with particular attention drawn to the complex network of perceptual, bodily and imaginary experiences of students. Through an analysis of 2 selected episodes of the work of 1 primary school child learning to graph motion, the paper shows how the notion of blending spaces is central to understanding the mechanism of multimodality.

KEY WORDS: graphing motion, imagination, multimodality, perceptuo–motor activity

Introduction

In recent years, mathematics education research has paid increased attention to the importance of bodily activity in mathematics teaching and learning. Psychological research on gestures has provided empirical support for the presence of the body in thinking (e.g. McNeill, 2000; Goldin-Meadow, 2003). In cognitive science, embodied cognition research has embraced theoretical perspectives that argue that cognitive processes are grounded in the body's interaction with the world, denying the traditional dualism between mind and body (Lakoff & Núñez, 2000; Seitz, 2000; Wilson, 2002). Informed by these trends, the "turn to bodily activity" in the theoretical stances adopted by researchers in mathematics education has brought with it more and more interest in the nature of bodily activity and the broad range of semiotic resources used in the classroom, and in the function that these may have in the doing of mathematics. The acknowledgement of the import of the body in mathematical cognition has led, on the one hand, to conjectures about the critical part played by perceptuo–motor activities in mathematical thinking (Nemirovsky, 2003) and, on the other hand, to attention to the relevance of the sociocultural factors involved in learning (Schiralli & Sinclair, 2003). Starting from the assumption that both these aspects relate strongly to knowing in mathematics, my primary concern in this paper is to contribute to the discussion on the multimodal nature of cognition.

A special issue of *Educational Studies in Mathematics*, entitled *Gestures and Multimodality in the Construction of Mathematical Meaning*, takes into account the complex range of cognitive, physical and perceptual resources that people utilize when working with mathematical ideas, moved by the awareness that, in our acts of knowing, different sensorial modalities—tactile, perceptual and kinaesthetic—become integral parts of our cognitive processes (Radford, Edwards & Arzarello, 2009). Arzarello, Paola, Robutti & Sabena (2009) analysed gestures in mathematics teaching and learning as semiotic resources used by students and teachers in a multimodal way, introducing the notion of the semiotic game by the teacher. Radford (2009) discussed why gestures matter from a cultural perspective, by sketching the sensuous cognition view that "thinking does not occur solely *in* the head but also *in* and *through* a sophisticated semiotic coordination of speech, body, gestures, symbols and tools" (p. 111, *emphasis in the original*). Roth (2009) proposes the phenomenological stance that mathematical concepts are not abstractions transcending bodily activity, but emerge "in and through experience, never consisting in anything else but activated prior experiences (embodied bodily traces thereof)" (p. 188).

While contemporary research may not be incompatible with the cognitive potential of the "multiple modalities", the problem of their role in learning cannot be reduced to admitting that they have something to do with cognition. This paper seeks to further elaborate on the ways multimodal aspects are exploited by learners thinking about mathematics by considering as relevant, not the fact that multimodality manifests itself, but *how* this happens. To do so, I will centre on the ways in which perceptual, bodily and imaginary experiences entwine in two episodes of the work of one child learning to graph motion at primary school.

Multimodal cognition and perceptuo–motor activities

According to the theory of embodiment, we do not simply inhabit our bodies, we literally use them to think (Seitz, 2000). Lakoff & Núñez (2000) stress that sensory–motor experiences ultimately realize thinking and understanding, through the activation of metaphorical mechanisms: e.g. in mathematics, we conceive sets as containers and numbers as locations in space. While Lakoff and Núñez rely on the idea of inferencepreserving mappings instantiated in the brain, I am interested in a more situated bodily engagement, as a natural ingredient of the generative processes of learning in mathematics, that is, bodily based ways of thinking that might originate from and shape activity in the classroom. Nemirovsky (2003) argues that mathematical meanings grow to a large extent out of *perceptuo–motor activities* having the potential to refer to things and events as well as to be self-referential. Bodily actions, gestures, manipulation of materials, tool use, acts of drawing, sensory–motor coordination, eye movements and facial expressions are perceptuo–motor activities. For Radford (2009), they correspond to the sensuous aspects of mathematical cognition, that is, those semiotic resources that are genuine constituents of abstract thinking. In this perspective, the actions that one engages in during mathematical work (like writing down an equation) are also perceptuo–motor acts relevant to the context at hand (Nemirovsky, 2003).

More recent neuroscientific results give us new insights into the role that perceptual, sensory and motor experiences may have in learning. Gallese & Lakoff (2005) point out that the sensory–motor system characterises the semantic content of concepts in terms of the way that we function with our bodies in the world. In particular, special neurons (like mirror neurons) exist that are inherently *multimodal*: the firing of a single neuron may correlate with both seeing and performing an action. Neurons matching both action observation and execution entail multi-modal integration in many different locations in the brain. This excludes the existence of separate brain areas for action and perception that are associated via a sort of central engine: instead, "sensory modalities like vision, touch, hearing, and so on are actually integrated with each other *and* with motor control and planning", (Gallese & Lakoff, 2005, p. 459, *emphasis in the original*). In addition, many of the same neurons also fire when imagining an action. We can imagine picking a flower without actually doing it, but the same part of the brain is used as when we really pick the flower. This happens even if we see or imagine seeing someone else picking the flower. For Gallese and Lakoff, imagination is a form of simulation, which shares a neural substrate both with performing and observing. I believe that there is possibly something more visceral than a mere simulation in the brain. For instance, I can perceive a rose in entirely different ways depending on the possible smells that I entertain for it. Similarly, I proceed to pick the rose in entirely different ways if I hold the possibility of being pricked by its thorns or not. Imagining plays a decisive role, being fully part of any perceptual and motor activity. Turning to mathematics, suppose that you have to solve a second-degree equation written on the blackboard. You can perceive its resolution in different ways according to the possibility of factoring the corresponding trinomial, of applying the algorithmic formula for roots and of plotting it on a graph. *Mathematical imagination* is "entertaining possibilities for action; entertaining (in the sense of 'holding' or 'keeping') a state of readiness for the enactment of possible actions" (Nemirovsky & Ferrara, 2009, p. 159). Perceptuo–motor–imaginary activities/experiences were used to underline this multimodal unit. The cognitive dimension just described is one side of the coin. The other side is that of social interaction and communication in the classroom, where students do not live alone but share ideas with peers. On this side, multimodality refers to the multiple means we use for making meanings, which Kress (2004) calls modes of representation. The dominant media in this century are those of the screen, with images being pervasive, and the new digital technologies offering new potential in the ways people communicate and interact, learn and know, hugely changing the visual mediators of mathematics (Rotman, 2008). In addition, the possibilities for a multimodal expression in mathematics are increased by the nature of the subject itself, for which ostensives make tangible the abstract mathematical concepts. I want to stress that mathematics learning involves and occurs through interaction with these ostensive forms, which are not only made of perceptuo–motor activities, but of experiences where imagining naturally emerges and allows learners to grasp and share meanings. To understand how this happens, what is relevant is not the consideration of single modalities in isolation, but the broader study of their interplay. Extant research on multimodality (both within and outside of mathematics education) recognises its valuable presence in conceptualization, but it fails to fully grasp the way it functions. Thus, the question underpinning this study is: How does multimodality work in mathematical activity? In particular: What is the role of imagination in multimodality? I will investigate these issues through a microanalysis of two selected episodes of the work of one child, who constructs meanings for new graphs with the aid of motion detectors. In the next section, I describe the context from which the episodes came.

The study background

This paper is the result of a 4-year study involving a class of primary school children undertaking some activities to introduce the concept of function through *graphing motion*. Graphing refers to drawing graphs, reading graphs, selecting and customising graphs for particular purposes and interpreting and using graphs as tools (Ainley, 2000). Graphing motion refers to a real-time position versus time curves corresponding to movements performed by the children in front of motion detectors. The research goal was to analyse the processes that intervene in learners' interaction with the graphs. From the didactic point of view, the purpose was to design an approach to function based on the visualization of graphs as the results of movements.

A graphical approach by means of motion phenomena can evoke the epistemological roots of function, stressing its dynamic nature (Edwards, 1979). Graphs are central since they force the visualization in real time of the relationships between variables that give rise to the shape of the curve. On the other hand, the mutual connection between the mathematics used to describe change (calculus) and kinematics in physics is often difficult for learners to understand, despite evidence of its historical development. Basing function on motion may be didactically effective, because the origins of the concept are awakened.

Other studies of learners' understanding of motion have focused on the representations that primary and middle school learners create to describe motion (e.g. diSessa, Hammer, Sherin & Kolpakowski, 1991; Sinclair & Armstrong, 2011) and on primary learners' interpretations of the graphs they obtain moving in front of sensors (Nemirovsky, Tierney & Wright, 1998). The study referred to in this paper involved young children in graphing motion, starting from the second year of primary school.

Students and Curriculum

The study was conducted with a primary school class for a period of 4 years, from 2006 to 2009. The setting was a little school in the north-west part of Italy. At the beginning of the study, the children were attending the second grade (aged 7 years). At that time, the class consisted of 15 children (seven females and eight males). In the next year (grade 3), a child moved to another school and a new one came. In grade 5, another student joined the group making a class of 16 children. One child with a handicap certification participated in the study in all 4 years. The class was taking regular mathematics lessons 2 days per week, for a total of 8 h a week. Each year the research activities were carried out in the period February–May, for about ten sessions on a weekly basis.

The study adhered to the national indications for primary school curricula, in force since 2007, and which included the primary school competency to use suitable data representations to get information in various situations. One of the learning objectives is to represent relations and data by diagrams, schemes and tables and to use representations to gain knowledge, to formulate conjectures and to make decisions.

Activities and Technologies

The children participated in activities in which two technological devices were used for graphing motion. The first was a one-dimensional motion detector, called *Calculator-Based Ranger* (CBR) linked to a graphic calculator, and the second (*Motion Visualizer DV*) was a computer software program that gathers motion in two spatial dimensions (horizontal and vertical). The former provides a single graph (a temporal graph), and the latter displays two graphs along motion directions (spatial or temporal by choice). We worked only with position–time graphs. We set the tools to produce real-time graphs and to project them. The children interacted with the tools both to interpret graphs related to given movements and, vice versa, to check movements associated to given graphs. Both phases were important in graphing motion. The tools enabled learners to, on the one hand, describe a certain movement by a Cartesian curve and, on the other hand, to make predictions about the shape of a curve on the basis of a certain movement experience. Figure 1 shows two examples of the graphs with which the children worked: (a) position–time graph of a back and forth movement in front of the CBR and (on the left) the position–time graphs (horizontal position *x* at the top, vertical position *z* at the bottom: *x*(*t*), *z*(*t*)) that are displayed by the Motion Visualizer as a result of a circular movement on the *xz* plane (on the right).

The DV feature of the Motion Visualizer supplies a digital video of the movement together with its trajectory (the small window at the bottom left of Fig. 1 right; the trajectory also appears in a model of the room: top left of Fig. 1 right). Offline, it is possible to replay the video, watching the trajectory being shaped and the graphs being revealed step by step.

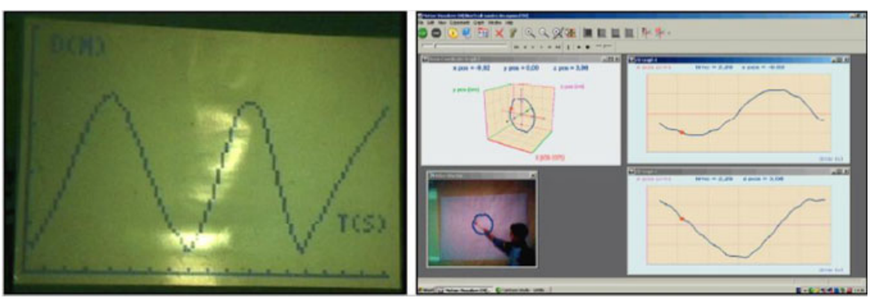

Figure 1. Graph obtained using the CBR and the window of the Motion Visualizer

From grade 1, the children have also encountered in their mathematics lessons fantasy characters like the Wizard of Numbers (a character from a famous book about numbers, *The Number Devil*, by the German author H.M. Enzensberger). The teacher read parts of the book with the class and often recalled the Wizard to introduce new topics. Working with the CBR, we often spoke of animals in order to distinguish their speeds and those of the children. In the case of the Motion Visualizer, we spoke of Movilandia and Cartesiolandia, respectively the Land of Motion and the Land of Descartes—created using a big piece of paper on one wall of the classroom. The computer screen, showing the viewing window of the Motion Visualizer, was projected on the left side (Cartesiolandia, where the graphs were displayed): on the right side, movements were performed using coloured objects, like an orange glove or an orange ball on a stick (Movilandia). A view-screen allowed having the graph displayed on the wall for the CBR. Two blackboards were always available for use. The class was seated in front of the projection and could watch the graph(s) unfolding on the Cartesian plane.

The presence of technology in the activities was fundamental. In the literature it has been shown that experiences, in which students interact with tools to create phenomena, help them to understand the mathematics connected to those phenomena (e.g. Ferrara, 2006; Nemirovsky, Tierney & Wright, 1998). The tools were attractive for the children and stimulated them to discover the relationships between motion and graphs. The real-time feedback enabled them to link their moves to the graphical shape. The sense of the graph did not grow separated from, but embodied in, the shape of the curve as it originated from a precise movement (say, with a certain trajectory, pace or speed).

Methodology

The instructional part of the study fits the methodology of a mathematics laboratory, in that it provided a structured set of activities aimed at the construction of meanings for graphs of motion. The idea of the laboratory encompasses all the situations in which the traditional lesson is modified by the introduction of specific artefacts and modelling activities. In this approach, understanding is strictly tied to tool use and to the interactions among learners who work together. The case of children graphing motion gives an example.

The activities were of different kinds: individual and pair activities, small group work and class discussions. The children were always asked to complete some worksheets and to make their reasoning explicit, through a final page entitled *Space of reasoning* on each worksheet.

The classroom teacher and I engaged together from the design stage to the implementation of activities. We conducted the study as both participants and observers in the classroom (Cobb, 2000) and facilitators for the collective discussions.

Concerning data analysis procedures, the multimodal investigation of perceptuo–motor–imaginary activities calls for a microgenetic analysis (or microanalysis) of the situation, that is, a detailed examination of the genesis of ideas by a subject over short periods of time, while ideas are occurring. Following Nemirovsky, Rasmussen, Sweeney & Wawro (2012), this method is inside the range of microethnography, which encompasses approaches tracing the moment-by-moment bodily and situated activity of subjects engaged in certain tasks and interactions, drawing attention to the many modalities at play (see, e.g. Streeck & Mehus, 2005). Microethnography is increasingly recognised as a suitable research methodology thanks to the notable interest in the study of gestures in mathematics learning (Arzarello et al., 2009; Nemirovsky & Ferrara, 2009; Radford, 2009; Roth, 2009).

Data Collection

The research design included data from audio and video recordings of the classroom. A qualitative analysis was completed of audio/video data and written materials like observation notes and students' work samples. All class sessions were videotaped, and the written material produced by students was collected. We used two cameras, one following the interactions of participants, the other capturing the projection of the calculator/computer screens and children's reactions to them. For the video analysis, transcripts were generated for each session, and we employed microanalyses to code them, attending to the lexicon, gesture and kind of interaction. The second type of data (written material) was relevant to detecting the complexity of children's multimodal thinking processes when constrained to working with paper and pencil. In such contexts perceptuo–motor–imaginary activities are expressed through words, shapes and diagrams.

For the purpose of this paper, in the next section, I closely analyse two examples from the work of one child, through which I exemplify the way the multimodality of thinking works, exploring the complex merging of perceptual, motor and imaginary activities that generate ways of understanding and communicating.

The first episode came from a class discussion of the shape of the position–time graph associated with a movement in front of the CBR. The second episode focuses on a written protocol concerning the Motion Visualizer and the connection of a given graph to the corresponding movement. Both examples concern the ways one child, Benny, acted, thought and communicated.

Benny was not the perfect student. He was a very intuitive child and liked maths, but he was quiet, timid and rather messy in his arguments, often struggling with written arguments. I choose to use his examples in this paper essentially for two reasons. On the one hand, Benny is a beautiful example of the fact that activities like those we are considering may change one's involvement and raise awareness of one's own understanding. The insight and sophistication of some of his written arguments were surprising. In addition, looking at the videos, Benny appears to have developed a positive attitude towards ways of reasoning and arguing that embodied practices acquired through experience with the motion detectors. His ways of acting and interacting exemplify to a great extent the pervading way in which the children in this study understand and communicate about the graphs they work with.

For each episode, I focus on Benny's experiences, looking at his gestures and actions guided by tool use, creation of written signs and diagrams.

Episode 1: the CBR and the First Experience

Overview. This episode concerned the starting activity of the whole experiment. It occurred the first year of the project when Benny and his classmates were in grade 2. We worked with the CBR. The motion detector was presented to the children as a little box that works by means of invisible waves bouncing off a body moving in front of it. The very first movement arose from the request that a volunteer perform a free move along a red line on the floor. The CBR works in real time detecting for 15 s, for about each tenth of a second, the position of someone that moves within its action cone from 0.5 to 6 m (this is why the line was there). Benny performed the move, and the rest of the group, seated on the floor, watched on the wall, the creation of the position–time graph given by the linked calculator. He walked back and forth covering the line five times, before stopping at the end farthest from the detector position.

Data. Benny's walk resulted in a graph in the shape of "mountains"—words spontaneously adopted by the children. A collective moment of knowledge construction started, with the aim of understanding the graph, with a question from the researcher (Res: in the excerpt below): “How can we explain this drawing?” Benny answered recalling his experience (in Fig. 2, the line on the floor is marked, and the arrows indicate the gestures):

Benny: *While I arrived to the end* [left hand pointing to the red line on the floor, with gaze and torso towards it; Fig. 2 left]*, the thing arrived on it* [right hand tracing an ascending line in front of his torso and stopping at the top, addressed to Res: Fig. 2 centre].
Res: *Can you also show me this there* [on the wall]*?*
Benny: *When* [gazing at and pointing to the line behind him, while walking to the wall]*, I pretend that this* [left hand pointing to the right end of *t*-axis] *is the start. I go* [left hand running *t*-axis to the left end, body shifting from right to left]*, I arrived here* [left hand pointing to the left end] *and this piece came* [right hand tracing the first ascent: Fig. 2 right].

“Pretend that This Is the Start”. Benny’s gestures serve to highlight initial relations between the shape of the graph and the movement he performed. These relations would be difficult to express only in words. At the beginning, Benny gesticulates in the space he can use to communicate, in the closest and most natural way, the extended bodily space he can reach in the surroundings of his body with his arms, that is, his peripersonal space (Rizzolatti, Fadiga, Fogassi & Gallese, 1997). In the effort of imagining why the shape of the graph is the way it is, Benny first uses gestures and words in a way that is not so informative. The initial pointing and tracing gestures already distinguish motion and graph in a rather clear way. The distinction is also marked by the use of both hands, with different roles. On the one side, pointing with the left hand, he conveys positions on the trajectory, as demonstrated by the head and the torso turned towards the line (by the way, the left hand is the closest one to the space where motion was performed). On the other side, his tracing with the right hand in the bodily space mimes the origin of the first piece of the graph (now gazing towards the graph and the researcher). The end of the line on the floor and the first peak of the graph perfectly correspond to each other in the gestures (the peak just coming in the final deictic

Figure 2. Benny’s gestures

phase of the miming gesture: it is made present, though). There is even an exact consonance in the temporal order of the repeated verb “to arrive” (“I arrived to the end”, “the thing arrived upon it”), associated in one case to the personal walk (“I”) and in the other case to that some-“thing”. The verb of motion “to arrive” is indistinctly used for the movement and for the graph, which effectively *moved* when originated point by point in real time. Even if the subjects of the verb differ from each other, the second subject is controversial within the context. In effect, “the thing” here—due to its generality—may either be interpreted as the motion detector or as the curve, although neither had really “arrived” anywhere. The action is clearly associated with the ascent of the first part of the curve (the gesture leaves no room for doubt of such a link). But the tool is what allows having that graph, and seeing its origin in real time, through a moving point. This is what gives the curve the dynamic character recalled by the action of “arriving upon”. It is a moment of exploration, as witnessed by reference to the physical space where motion occurred (the line, and one of its “end”-s)—even with the whole body turned towards this space. Benny tried to articulate the tension between motion and graph, kept distinct in gesturing so far by means of the two hands. However, in words, he still mixes the trajectory (“the end” of the line), the tool and the graph (“the thing”, “upon”). In this mixture, the CBR combines with the qualities of the graph and the qualities of the movement. So, when Benny says: “the thing arrived upon”, he is not yet able to discern the role of the CBR and the graph associated to motion: he merges the two.

As Benny repeats his reasoning in front of the wall, something changes. His bodily space actually blends with the space of the graph (the paper where the graph is projected). At this point, everything happens in that space: only memory of the movement remains (“when”). This marks how the physical space of motion transforms now into a cognitive space open to investigation. Benny spontaneously introduces a verb of temporary intentionality (“pretend”) and takes the point of view of the tool. He treats the horizontal axis of the graph (the axis of time) *as if* it were the motion trajectory, the line. The temporality supports him in recollecting the initial part of his move along the line, making it cognitively present in the physical space of the graph. This happens through the left hand running along the *t*-axis from right to left exactly as the real movement occurred from the point of view of the children (“this is the start”, “here”) and according to Benny’s experience (“I went”, “I arrived”). Imagining the occurrence of this first fraction of the movement (a past kinaesthetic experience) in the space of the graph allows Benny to explain the shape of the corresponding part of the graph (“this piece came”), traced with the right hand. The hands are again used with reference to the different characters inhabiting the scene: the left hand for motion, the right hand for the curve, without any confusion between the two. It is as if the running gesture (a present vivid experience) suddenly provided the space of the graph with an *imaginative apparel* that lasts just the time of a flashback to the motion. The apparel makes present the movement: perception goes back to the experience, leaving the curve on the background. But immediately after, the background comes to the fore making explicit the link between motion and graph. The graphical space becomes for a moment an imaginative space in which to reason. The previous confusion is overcome also in words, with distinct subjects now used to refer to the subject of motion ("I") and to the graph ("this piece"), together with different verbs ("to go" and "to arrive", in relation to the motion, and "to come", regarding the curve). Gestures and words are well coordinated. The function of the pointing gestures is matched in the deictic words associated with the ends of the line ("this", "here"). Similarly, the actions of "going" and "arriving" at the left end, which re-enact the movement, entail the creation ("came") of the first ascending piece of the graph.

Interpretation. In this way, Benny makes apparent the link between a moment of motion and the corresponding part of the mountains. The fact that he was the one who moved is not at all a secondary aspect: it is what helps him. In such a brief excerpt, we see the start of an evolution in understanding the relations between the physical motion experience and its graph. The evolution happens in three phases: recollection of moments of motion, imagination of qualities of motion in relation to qualities of the graph and the interpretation of its shape. The walk is recalled by means of initial pointing to the line and of further pointing and quick gazes back to the place where motion occurred while Benny approaches the graph. Recollection begins to merge with the representation when Benny imagines re-living his walk's first portion through the gesture on the horizontal axis of the graph, envisioned as if it were the real trajectory. The direction, in which the finger was moved, was not casual. Instead, Benny associated this imaginary motion with the same direction of the corresponding real walk as it was lived by him and mirrored by his classmates. The imaginative traits were emphasized by the independence of this action from the opposite direction of time in the graph. Benny made present in the graphical space not only the first walking section, but also one of its qualities: its actual orientation. The blend of physical and imaginary experiences pushed him to interpret the initial piece of the curve, an ascending one, as related to moving away from the CBR.

Episode 2: the Motion Visualizer and the First Worksheet

Overview. Episode 2 concerns the first individual worksheet on the Motion Visualizer. It occurred about 1 year later than Episode 1 (grade 3). The children had explored graphs related to various movements experienced with the CBR: back/forth moves, moves from/to the motion detector (both with constant and changing speed) and absence of motion. Around February of the grade 3 year, we introduced children to the Motion Visualizer. A few preliminary situations allowed moving coloured objects along simple trajectories, or keeping an object in one place, and seeing the origin of the corresponding graphs. Episode 2 followed the real-time experience in which children discussed the graphs of $x(t)$ and $z(t)$ associated with an orange glove fixed in a generic position (x, z) on the plane xz. Through storytelling, x and z were the fantasy characters Mister x and Mister z: two secret agents acting in Movilandia without being visible (*hidden* in the glove). Mister x and Mister z communicate with children through the *special language* of Cartesiolandia (in Italian, *Cartesiolandese*), which describes the movements performed with the glove. By a series of tasks, the Wizard of Numbers challenged children to decipher Cartesiolandese, so as to discover the movement associated with a pair of specific graphs. Mister x contributes to motion in the horizontal direction, and Mister z in the vertical direction.

The written task was structured in three pages. On the first page, a precise (upper left) position of the glove in Movilandia and the corresponding graphs were given: the children were asked to observe the situation in the two worlds (Fig. 3 left). On the second page, the children were required to look at the new (lower right) position of the glove in Movilandia and to represent what they would have expected to see in Cartesiolandia, explaining their reasoning (Fig. 3 right). The third page was for the *Space of reasoning*.

In both cases, the software would return two straight lines as graphs of $x(t)$ and $z(t)$. These lines differ in their position around about a reference line. According to where the glove is kept in Movilandia, out of four

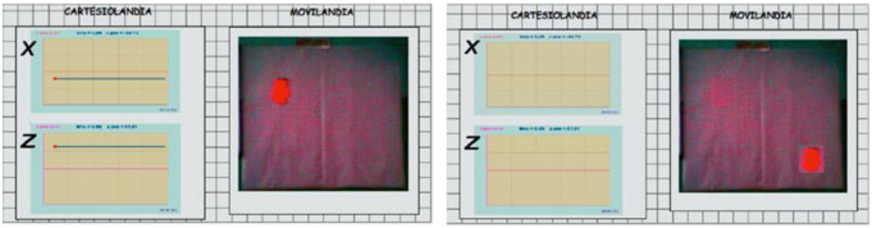

Figure 3. The situation on pages 1 (*on the left*) and 2 (*on the right*) of the worksheet

possibilities (upper left, upper right, lower left, lower right), in Cartesiolandia the lines of $x(t)$ and $z(t)$ can appear above and/or below the centre line. I did not consider here the uncommon place of the glove just in the centre of Movilandia (the point where diagonals meet), which would correspond in Cartesiolandia to two straight lines over the centre. The software takes as a reference for the graph, the position of the glove with respect to the centre of Movilandia. Being on the right/on the left of the centre in Movilandia gives information on the x-position, being above/below the centre on the z-position. In Cartesiolandia, x-position and z-position are represented on the vertical axes and time on the horizontal axis. For z the correspondence between Movilandia and Cartesiolandia is more immediate than for x. This is one difficulty encountered by the children.

The children had in general solved the task correctly, even if not all of them clearly expressed their reasoning. The correct solution is given by a straight line above the centre for $x(t)$ and a straight line below the centre for $z(t)$ (lower-right position of the glove in Movilandia).

Data. Benny proposed this argument (his words in italics: the recalled sketches in Fig. 4):

> *On page 2, I completed like that because in the "space" of M.x* [Mister x]*, placed this way* [sketch 1, Fig. 4]*, pretend that you turn it this way* [sketch 2, Fig. 4]*, pretend that you place the glove where it was before, that is placed this way* [sketch 3, Fig. 4]*, then I turn it again this way* [sketch 4, Fig. 4]*. Hence the line has to be placed this way* [sketch 5, Fig. 4] *and, in the other little scheme, only I don't turn it, but I move it* [the glove] *away and it* [the graph] *comes this way* [sketch 6, Fig. 4].

"Pretend that You Turn It This Way". Benny's argument is an example of a multimodal effort of explaining the reasoning he followed to solve the task. Benny used a short explanation, with many sketches entwined with words. The task forced him to produce a report. Nonetheless, he needed some way of expressing things he was not able to convey in

Figure 4. Space of reasoning: Benny's argument and his sketches

words (“this way” repeated many times). As a result, he drew a lot. Each “this way” is followed by a drawing that stands for the Cartesian plane (the small rectangle cut by a central segment, the centre). Some of the sketches have the same deictic function as words (sketches 1, 5 and 6). Some others have a really distinct and more complex cognitive function, being like gestures *crystallized* on paper (sketches 2, 3 and 4). It is as if Benny imagined making such gestures while writing. The space of reasoning blends in the moment with the *imaginative* space and past experiences with the tool turned into sketches. Benny’s peripersonal space embodies the piece of paper. The actions related to gestures and sketches are recalled in words and made present, even using the present tense for the verbs as if they were occurring now (“you turn it”, “you place the glove”, “I turn it again”, “I don’t turn it”).

Again, a temporary intentionality supported Benny in this imagination process (“pretend” restated), critical to distinguish x and z. He saw that the two variables have similar representations, but they do not behave equally. Understanding that position is always displayed on the vertical axis; Benny found a tactic to explain why the graph of $x(t)$ is given by the straight line that he had drawn above the centre (there is no trouble for $z(t)$, to which only few words are devoted at the end). He made present in written form an action of rotation that was not physically present before (in terms of the actual use of technology), even if it was already present in his mind. In effect, in a short passage of a previous discussion, Benny was striving to make sense of the graph obtained for $x(t)$ with a certain movement of the glove. He suddenly rotated his head clockwise 90°, and back, to search for suitable links with the graph, but he did not elaborate upon his actions. Here, Benny re-envisioned this action, perceiving it as crucial to his argument, and used it to establish an indissoluble connection between the two worlds. He thinks of the x-position in Cartesiolandia *as if* it were the same position in Movilandia just rotated clockwise 90° (“turn it this way”). He repeats the rotation twice, referring first to the passage from Cartesiolandia to Movilandia (“pretend that you turn it this way”), and then to the opposite passage (“pretend that you place the glove where it was before”, “then I turn it again this way”). The subjects Benny chose are also relevant to his reasoning. Introducing this temporality, Benny was first speaking to an imaginary interlocutor: the second person “you” is used as subject. As soon as the argument acquires a causal form, entailing the conclusion of the statement (“then”), he changes the subject to the first person (“I”). At this moment Benny justified his personal choice for the position of $x(t)$, with the turning back that marks the definitive position of the straight line above the centre. In so doing, he assumes a new perspective: the one of himself, no longer the one of *the other*, attentive to him.

Interpretation. This cognitive behaviour shows how Benny was able to master the tension between his understanding of the graph and the need for communicating it to others, in the social context of knowledge construction. It is a tension between the interpersonal dimension and the intrapersonal one, through which Benny involved his interlocutors in adopting the same view he took, to share his conclusion. A last remarkable feature of Benny's argument is the fact that, in its shortness, it is essential and clear. Each sketch he drew just contains the key elements, nothing more. Sketches 1 and 2 mark the centre, which is the pivotal factor to introducing the rotation image. Sketches 3 and 4 introduce the position of the glove, key to linking the situations in the two worlds. Sketches 5 and 6 close the argument ("hence"), revealing the positions of the lines. Thoughts about the line for $z(t)$ are restricted to a few words, given the immediacy of his understanding.

Discussion

The discussion of Episodes 1 and 2 highlights the way one child, Benny, made sense of position–time graphs related to movements. The two episodes are different for many reasons: the tools used, the experiential background, the complexity of the mathematics involved, the type of activities and the time in which they were carried out. However, in both cases, Benny was able to connect the movement with the graph(s) modelling it.

In the first episode, Benny's direct involvement as the one who moved is certainly crucial to making sense of the situation. But, also relevant to the advancement of the discussion is the way Benny explained this understanding. When Benny came to the blackboard, a blend of perceptual, motor and imaginary activities spontaneously emerged. His perception of both the movement and the graph pushed him to merge the two in the space in front: the space of the graph, projected on the wall. His bodily peripersonal space then entered the space of the graph as soon as he re-enacted motion through a specific action: running the t-axis so as to recall the experience of his walk along the red line. The action is physical, but in imagination it made present that there was something, which was now absent because it belonged to the past: the experience of movement. So, the t-axis becomes for *now* the red line, and Benny's finger moves exactly as he moved (in the initial part of his walk). Benny's thinking process was multimodal: communication was multimodal. Nothing was casual, not even the direction of the gesture. Each child saw the gesture occurring in the same direction as that of Benny's walk:

from right to left. In this way, Benny was inviting his interlocutor—to "pretend"—to engage in the same act of imagination. For the observer–interlocutor who watches the gesture, imagining that fraction of movement is like performing it. The space of the graph temporarily became the imaginative cognitive space where Benny and his interlocutor (the teacher, the researcher and every classmate) encountered each other, perceiving a shared sense of the graph.

In the second episode, traces of the multimodality through which Benny develops his sense making of the specific situation are in a written form. The argument was rich in perceptuo–motor–imaginary activities. A precise action acts as a thread of the argument: the 90° rotation, well expressed by sketch 2 in Fig. 4. The sketch is basic: it considers just the Cartesian axes and the centre, the only interesting aspects to explain the kind of *transformation*, illustrated with the aid of an arrow. By that action, Benny made present in the argument the link between the spatial situation and the space–time curve. He highlighted the connection between the worlds of Cartesiolandia and Movilandia concerning the behaviour of the horizontal position. Difficulties for this component of position depend on the conventional manner in which it is represented over time, i.e. on the vertical axis of the Cartesian plane (whereas the correspondence was more natural for the vertical position). Benny suggested a strategy to overcome the difficulty. He imagined transforming the given position of the glove in Movilandia by a rigid movement, in order to see where the horizontal component comes to be in the representation in Cartesiolandia. This strategy is always effective, regardless of where the glove is placed in Movilandia. It may be used for moving from Cartesiolandia to Movilandia, and for the opposite passage (both were present in Benny's words): to recognise the movement from which given graphs arise, as well as to anticipate graphs that describe a given movement. Benny's perception of the graphs and of the position of the glove pushed him to provide the space of reasoning with an imaginative nature, which crystallized on paper gestures expressed in words from the verbs "to turn" and "to place". Again, it is as if Benny was inviting the reader ("pretend") to engage in a process of imagination in order to experience the same rotations he describes. The rotations are hypothetical actions one could imagine executing with their hands, with their head and even with their eyes. This time, the (written) space of reasoning becomes the imaginative space where Benny and the reader have occasion to meet each other and share the strategy.

Three issues need underlining here before continuing with the rest of the discussion. First and foremost, after the association of the initial piece of the curve with the movement away from the CBR, in Episode 1, Benny did not continue his finger walk in the opposite direction, which might suggest that he could not interpret the remaining part of the graph. However, for him, this was sufficient to generalize the relationship between his movement and the graph, as witnessed by the ensuring dialogue. In fact, Benny immediately shifted attention to the "almost" equal height of the mountains. As soon as the researcher asked: "Why are they equal or almost equal in height?", he took into account both possibilities: "If it's equal [moving along the red line as in the first section of his walk], it's because I arrived here [pointing insistently with his foot to the end of the line farthest from the CBR] both times and then, then I returned there [finger pointing to the other end]"; "If it's not equal [looking at the line], it's because I didn't recognise it and I arrived here [pointing twice with his foot to a position rather close to the farthest end]". Benny's thinking was multimodal, involving his whole body in creative and genuine acts, in which sensory–motor and imaginary experiences are intricately entwined.

A second point is related to the fact that Benny was an exceptional example of the way multimodality generally works in mathematical activity. What Benny did occurred in real time as he interacted with the tool and the graph, and especially with the researcher and his classmates. This is a delicate but crucial aspect and can be depicted through the description of another episode of the ongoing dialogue. As soon as Benny explained why the mountains were of almost equal height, a question that the researcher posed to the entire class changed focus to how many times Benny moved back and forth (according to the graph). Arianna and Gaia both gave an answer that marked their bodily and imaginary engagement, through ways of expressing ideas that encompassed gestures both on the graphical space and toward the red line (Arianna: "Here [running with the hand on the last piece of the graph] there's a half, so he cannot always be going on [turning over and looking at the red line]", and Gaia: "He made only the first two [two fingers open indicating the two mountains on the graph], both back and forth [miming Benny's movement in the air], and the second time he stopped here [moving to the final end of the red line. Benny: 'Yeah']"). Arianna's and Gaia's unscripted ways of moving and thinking, back and forth between the physical space of the move and the graphical space, show that multimodality was present not only in Benny's thinking.

A third aspect refers to Episode 2. In that episode, one could not see any kind of interaction apart from the one with the tool. Even though Benny was working on an individual written task, he was inserted in the social context of the classroom, in which ideas (individual ideas also) are shared with others. During his written activity, it was as if he was interacting with an imaginary interlocutor, that is, the reader—being the reader, the teacher or his classmates. He knew that what he wrote was important because it would have been considered by the teacher and used in the classroom. So, exactly in the same way, the tool was not physically present, and even a *real* interlocutor was not present. But they were there for Benny. For the child, it was as if he was interacting with somebody else.

Conclusions

In both the episodes, physical spaces—such as that of the graph and that of the paper—are provided with an imaginative apparel, which also opens up cognitive possibilities for imaginary experiences for others. The imaginary nature and the temporality are striking components of this new cognitive dimension—along with the imperative use of the verb "to pretend" that marks an invitation for the others to enter this dimension. Actually, the imaginative space Benny created through his gestures and words was a *temporary possible* world where one may *pretend* something can happen in order to explain and understand something else. It is a multimodal world by its very origin, where perception, action and imagination fade gradually into each other in order to construct mathematical meanings. Through the introduction of this world, Benny guided the imaginative processes of the interlocutors or of the readers attentive to him, by establishing a ground of intersubjectivity from which others could develop understanding of what he was explaining—like someone who orchestrates a discussion by directing attention to the key elements of the argument and inviting everybody to be part of it.

The aim of this paper was to contribute to discussions on multimodal cognition through exploring the ways multimodality works in mathematical activity and, especially, the role of imagination in multimodality. The main result that addresses this issue is that multimodality manifests itself as a constitutive expression of thinking, which encompasses complex networks of perceptual, sensory–motor and imaginary experiences. Mathematics learning occurs, and meanings are shared in the classroom through a deep merging and overlapping of bodily and imaginary activities. In particular, imagination takes a decisive generative role in understanding and communication processes, making them genuinely creative, and as such, it is an essential ingredient in multimodality.

I better envision multimodality in mathematical activity offering the metaphor of *clouds of blending spaces*: physical and cognitive spaces, where—past and present—perceptual, motor and imaginary experiences by subjects emerge and merge to dissolve any eventual distance between body and mind (like clouds, which float encountering each other and transforming continuously, in shape, density, colour and transparency). Conceptualizing multimodality in this way helps strengthen views of mathematical cognition and conceptions in mathematics accepted in our field. It extends the idea of conceptions in mathematics "as networks of experiences that indeterminately emerge from lived (rather than intellectual) reorganizations of embodied bodily experiences" (Roth, 2009, p. 188). It also goes beyond the sensuous mathematical cognition discussed by Radford (2009) in terms of the sophisticated coordination of speech, body, gestures, symbols and tools, in and through which thinking occurs. In order not to lose that sophistication, I argue here that the multimodality of thinking, and of mathematical cognition, is not simply detected and expressed by a coordination of semiotic resources, but that it happens through the contemporary and entangled emergence of bodily, perceptual and imaginary activities, which shape mathematical thinking processes on the one hand, and, on the other hand, are shaped by the resources at play. The entanglement that constitutes this perceptuo–motor–imaginary unit is what gives learners a *sense of immersion* (Burbules, 2006) in the experience of doing mathematics, in which one forgets to be there as a passive learner and becomes a unique learner who actively knows, understands and interacts with the others in the social classroom.

Pedagogical Concerns

The blending of spaces is at the heart of the intricate interplay through which multimodality works in mathematical activity, intervening in shaping understandings and communications. Rethinking multimodality in the mathematics classroom in terms of blending spaces has a twofold consequence. On the one hand, it demands that mathematics education researchers recognise and grasp the complexity and intensity of mathematics learning in the classroom, reconsidering it as genuinely inventive (Sinclair, de Freitas & Ferrara, 2013). On the other hand, it makes space for a possible pedagogy of creativity. In fact, as imagination plays such an essential part in multimodality, approaches and tasks that provoke and fuel potential experiences together with bodily engagement may be pedagogically effective in mathematics. Educators should design instructional situations and settings in which the virtual is given space in order to favour inventiveness of learning.

For the activities in this study, their success depended to some extent on tool use that allowed the fostering of the multimodality of thinking and the recovering of the epistemological roots of the concepts making them cognitive roots for learners. In particular, the real-time influence of an action on the curve (displayed on the wall and visible to all the children) encouraged a perceptual–sensuous–imaginary readiness to look for a sense of the graph. Elsewhere it has been claimed that: "any perceptuo-motor activity is inscribed in a realm of possibilities encompassing all those for which the subject achieves a certain state of readiness" (Nemirovsky & Ferrara, 2009, p. 162). Suitable contexts might trigger a state of readiness to construct mathematical meanings, widening the realm of possibilities that are offered to learners. I do not intend here to "offer" in terms of a passive and individual exposure to these possibilities. Instead, learners actively intervene in the creation of this social virtuality of possibilities through which imagining makes mathematics a live inventive adventure.

But, as de Freitas & Sinclair (2012) argue, we need a philosophical shift to be able to conceptualize the learning of mathematics differently. We need a new ontology of mathematics, according to which "the mathematical subject comes into being (is always *becoming*) as an assemblage of material/social encounters" (de Freitas & Sinclair, 2012, p. 151, *emphasis in the original*). I believe that this shift might definitely change our point of view as educators, who are interested not only in the study of learning in the mathematics classroom but also in the emotional, immersive and animated experiences that mathematics students could live, converting, once and for all, their confidence in the discipline and their beliefs about it and its ontological status.

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Dipartimento di Matematica "Giuseppe Peano"
Università di Torino
Via Carlo Alberto 10, 10123 Torino, Italy
E-mail: francesca.ferrara@unito.it