Local variational problems and conservation laws

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Abstract

We investigate globality properties of conserved currents associated with local variational problems admitting global Euler–Lagrange morphisms. We show that the obstruction to the existence of a global conserved current is the difference of two conceptually independent cohomology classes: one coming from using the symmetries of the Euler–Lagrange morphism and the other from the system of local Noether currents.

Key words: fibered manifold, jet space, Lagrangian formalism, variational sequence, cohomology, symmetry, conservation law.
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1 Introduction

Local variational problems arise naturally from the various solutions of the so called global inverse problem in the calculus of variations. Talking about inverse problems in the calculus of variations refers to the question whether a given set of equations is variational or not; with the advent of the theory of manifolds and global analysis this question splits naturally in the local and the global case. From the modern point of view, the local case had its first solution a long time ago: equations are (locally) variational if and only if they satisfy Helmholtz conditions. The global case required an entire reformulation of the calculus of variations. From the seventies on, various authors gave differential formulations of the calculus of variations: since the variation of a Lagrangian has a lot in common with the n-th exterior differential in the de Rham complex they succeded, starting from the de Rham complex, to construct various types of differential complexes such that infact the degree n module consists of Lagrangians and taking their differential gives Euler–Lagrange equations. In fact, the geometrical formulations of the Calculus of Variations on fibered manifolds include a large class of theories for which the Euler–Lagrange operator is a morphism of an exact sequence [3, 4, 13, 16, 18, 20, 21, 22, 23]. The module in degree $n+1$, consequently contains 'equations', *i.e.* dynamical form, and the Helmholtz conditions are simply being closed with respect to the differential of the complex. At this point the global inverse problems becomes simple homological algebra: a given equation is an Euler–Lagrange equation if its dynamical form is the differential of a Lagrangian and this is equivalent to the 'equation' being closed in the complex and its cohomolgy class being trivial.

Once established a differential formulation of the Calculus of Variations, one is however immediately lead to a closer examination of 'equations' which are only locally variational, i.e. which are closed in the complex and define a non trivial cohomology class; a situation much needed once topologically non trivial spaces make their appearence in Theoretical Physics (see *e.g.* [5, 17, 19]): instead of a global Lagrangian these 'equations' admit a system of local Lagrangians, one for each open set in a suitable covering, which satisfy certain relations among them. Summing up: to be globally defined — but required only to be locally variational together with at least one system of local Lagrangians for them — is the minimal requirement for any set of equations to be considered of interest in the Calculus of Variations.

Of course, the first question which poses itself regards the existence and globality of conservation laws. Clearly, this means that one is looking for Noether type theorems. We tackle this by the following procedure. We give a explicit definition of local variational problem focused on the global, but only locally variational equations and not on the system of local Lagrangians. Consequently, the symmetries we choose for our Noether type theorems are those of the equations. Then we derive the local and global version of these theorems. We examine closely the motivation and consequences of our choices. In particular, we analyze the 'inner' structure of the obstruction to the existence of global conserved currents that arises.

While conservation laws for equations without global Lagrangians have been studied by several authors, among them [6, 8, 7, 17], a similar discussion, however, seems not to be in the literature and we hope it will clarify the highly involved situation somewhat.

2 Local variational problems

Let us consider a fibered manifold $\pi : \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n+m$. For $r \geq 0$ we have with the r–jet space $J_r\boldsymbol{Y}$ of jet prolongations of sections of the fibered manifold π. We have also the natural fiberings $\pi^r_s : J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$, $r \geq s$, and $\pi^r : J_r\boldsymbol{Y} \to \boldsymbol{X}$; among these the fiberings π^r_{r-1} are *affine bundles* which induce the natural fibered splitting [14]

$$J_r\boldsymbol{Y} \times_{J_{r-1}\boldsymbol{Y}} T^*J_{r-1}\boldsymbol{Y} = J_r\boldsymbol{Y} \times_{J_{r-1}\boldsymbol{Y}} (T^*\boldsymbol{X} \oplus V^*J_{r-1}\boldsymbol{Y} \,.$$

The above splitting induces also a decomposition of the exterior differential on $\boldsymbol{Y}$ in the *horizontal* and *vertical differential*, $(\pi^{r+1}_r)^* \circ d = d_H + d_V$. A *projectable vector field* on $\boldsymbol{Y}$ is defined to be a pair (Ξ, ξ), where the vector field $\Xi : \boldsymbol{Y} \to T\boldsymbol{Y}$ is a fibered morphism over the vector field $\xi : \boldsymbol{X} \to T\boldsymbol{X}$. By $(j_r\Xi, \xi)$ we denote the jet prolongation of (Ξ, ξ), and by $j_r\Xi_H$ and $j_r\Xi_V$ the horizontal and the vertical part of $j_r\Xi$, respectively.

For $q \leq s$, we consider the standard sheaves Λ^p_s of p–forms on $J_s\boldsymbol{Y}$, the sheaves $\mathcal{H}^p_{(s,q)}$ and $\mathcal{H}^p_s$ of *horizontal forms*, *i.e.* of local *fibered morphisms* over π^s_q and π^s of the type $\alpha : J_s\boldsymbol{Y} \to \overset{p}{\wedge} T^* J_q\boldsymbol{Y}$ and $\beta : J_s\boldsymbol{Y} \to \overset{p}{\wedge} T^*\boldsymbol{X}$, respectively. We also have the subsheaf $\mathcal{C}^p_{(s,q)} \subset \mathcal{H}^p_{(s,q)}$ of *contact forms*, *i.e.* of sections $\alpha \in \mathcal{H}^p_{(s,q)}$ with values into $\overset{p}{\wedge}(\mathcal{C}^*_q[\boldsymbol{Y}])$.

According to [13], the above fibered splitting yields the *sheaf splitting* $\mathcal{H}^p_{(s+1,s)} = \bigoplus_{t=0}^p \mathcal{C}^{p-t}_{(s+1,s)} \wedge \mathcal{H}^t_{s+1}$, which restricts to the inclusion $\Lambda^p_s \subset \bigoplus_{t=0}^p \mathcal{C}^{p-t}{}_s \wedge \mathcal{H}^{t,h}{}_{s+1}$, where $\mathcal{H}^{p,h}{}_{s+1} := h(\Lambda^p_s)$ for $0 < p \leq n$ and the map h is defined to be the restriction to Λ^p_s of the projection of the above splitting onto the non–trivial summand with the highest value of t. Starting from this splitting one can define the sheaves of contact forms, *i.e.* forms which 'do not contribute to the action integral along sections' of $\pi : \boldsymbol{Y} \to \boldsymbol{X}$.

By an abuse of notation, we denote by $d \ker h$ the sheaf generated by the presheaf $d \ker h$. Set then $\Theta*_r \equiv \ker h + d \ker h$.

Definition 1 The quotient sequence

$$0 \longrightarrow I\!\!R_{\boldsymbol{Y}} \longrightarrow \dots \overset{\mathcal{E}_{n-1}}{\longrightarrow} \Lambda^n_r/\Theta^n_r \overset{\mathcal{E}_n}{\longrightarrow} \Lambda^{n+1}_r/\Theta^{n+1}_r \overset{\mathcal{E}_{n+1}}{\longrightarrow} \Lambda^{n+2}_r/\Theta^{n+2}_r \overset{\mathcal{E}_{n+2}}{\longrightarrow} \dots \overset{d}{\longrightarrow} 0$$

is called the r–th order *variational sequence* associated with the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$. It turns out that it is an exact resolution of the constant sheaf $I\!\!R_{\boldsymbol{Y}}$ over $\boldsymbol{Y}$ [14].

The cohomology groups of the corresponding complex of global sections

$$0 \longrightarrow I\!\!R_{\boldsymbol{Y}} \longrightarrow \dots \overset{\mathcal{E}_{n-1}}{\longrightarrow} (\Lambda^n_r/\Theta^n_r)_{\boldsymbol{Y}} \overset{\mathcal{E}_n}{\longrightarrow} (\Lambda^{n+1}_r/\Theta^{n+1}_r)_{\boldsymbol{Y}} \overset{\mathcal{E}_{n+1}}{\longrightarrow} (\Lambda^{n+2}_r/\Theta^{n+2}_r)_{\boldsymbol{Y}} \overset{\mathcal{E}_{n+2}}{\longrightarrow} \dots \overset{d}{\longrightarrow} 0$$

will be denoted by $H^*_{\mathrm{VS}}(\boldsymbol{Y})$. Since the variational sequence is a soft resolution of the constant sheaf $I\!\!R_{\boldsymbol{Y}}$ over $\boldsymbol{Y}$, the cohomology of the complex of global sections is naturally isomorphic to both the Czech cohomology of $\boldsymbol{Y}$ with coefficients in the constant sheaf $I\!\!R$ and the de Rham cohomology $H^k_{\mathrm{dR}}\boldsymbol{Y}$ [13].

The quotient sheaves in the variational sequence can be represented as sheaves $\mathcal{V}^k_r$ of k-forms on jet spaces of higher order (see *e.g.* [10]). Lagrangians are $\lambda \in (\mathcal{V}^n_r)_{\boldsymbol{Y}}$, $\mathcal{E}_n(\lambda)$ is called a Euler–Lagrange form (being $\mathcal{E}_n$ the Euler–Lagrange morphism). Dynamical forms are $\eta \in (\mathcal{V}^{n+1}_r)_{\boldsymbol{Y}}$, $\mathcal{E}_{n+1}(\eta) := \tilde{H}_{d\eta}$ is a Helmohltz form (being
cE_{n+1} the Helmholtz morphism).

The formulations of the calculus of variations in terms of homological algebra of differential complexes were introduced in order to solve the so called global inverse problem. We will sketch it (and its solution) within the framework of variational sequences. This will lead naturally to what we call local variational problems.

Let $\boldsymbol{K}_r \equiv \mathrm{Ker}\ \mathcal{E}_n$. We call $\mathcal{E}_n(\mathcal{V}^n_r)$ the sheave of Euler–Lagrange morphisms. This is justified by the fact that for a global section $\eta \in (\mathcal{V}^{n+1}_r)_{\boldsymbol{Y}}$ we have $\eta \in (\mathcal{E}_n(\mathcal{V}^n_r))_{\boldsymbol{Y}}$ if and only if $\mathcal{E}_{n+1}(\eta) = 0$, which are the Helmholtz conditons of local variationality. The global inverse problem is now to find necessary and sufficient conditions for such a locally variational η to be globally variational.

Then the short exact sequence of sheaves

$$0 \longrightarrow \boldsymbol{K}_r \longrightarrow \mathcal{V}^n_r \stackrel{\mathcal{E}_n}{\longrightarrow} \mathcal{E}_n(\mathcal{V}^n_r) \longrightarrow 0$$

gives rise to the long exact sequence in Czech cohomology

$$0 \longrightarrow (\boldsymbol{K}_r)_{\boldsymbol{Y}} \longrightarrow (\mathcal{V}^n_r)_{\boldsymbol{Y}} \longrightarrow (\mathcal{E}_n(\mathcal{V}^n_r))_{\boldsymbol{Y}} \stackrel{\delta}{\longrightarrow} H^1(\boldsymbol{Y}, \boldsymbol{K}_r) \longrightarrow 0\,.$$

Hence, every $\eta \in (\mathcal{E}_n(\mathcal{V}^n_r))_{\boldsymbol{Y}}$ defines a cohmology class

$$\delta[\eta] \in H^1(\boldsymbol{Y}, \boldsymbol{K}_r) \simeq H^{n+1}_{VS}(\boldsymbol{Y}) \simeq H^{n+1}_{dR}(\boldsymbol{Y}) \simeq H^{n+1}(\boldsymbol{Y}, \boldsymbol{R})$$

The solution to global inverse problem is now simple and elegant: η is globally variational if and only if $\delta[\eta] = 0$, because only then exists a global section $\lambda \in (\mathcal{V}^n_r)_{\boldsymbol{Y}}$ with $\eta = \mathcal{E}_n(\lambda)$.

If instead $\delta[\eta] \neq 0$ then $\eta = \mathcal{E}_n(\lambda)$ can be solved only locally, i.e. for any countable good covering $\{\boldsymbol{U}_i\}_{i\in\mathbb{Z}}$ in $\boldsymbol{Y}$ there exist local Lagrangians λ_i over each subset $\boldsymbol{U}_i \subset \boldsymbol{Y}$ with $\eta_i = \mathcal{E}_n(\lambda_i)$. The local Lagrangians satisfy $\mathcal{E}_n((\lambda_i - \lambda_j)|_{U_i\cap U_j}) = 0$ and conversely any system of local sections of with this property gives rise to a Euler–Lagrange morphism $\eta \in (\mathcal{E}_n(\mathcal{V}^n_r))_{\boldsymbol{Y}}$ with cohomology class $\delta[\eta] \in H^1(\boldsymbol{Y}, \boldsymbol{K}_r)$.

Definition 2 A system of local sections λ_i of $(\mathcal{V}^n_r)U_i$ for an arbitrary covering $\{\boldsymbol{U}_i\}_{i\in\mathbb{Z}}$ in $\boldsymbol{Y}$ such that $\mathcal{E}_n((\lambda_i - \lambda_j)|_{U_i\cap U_j}) = 0$, is what we call a *local variational problem*. Two local variational problems are *equivalent* if and only if they give rise to the same Euler–Lagrange morphism. The covering $\{\boldsymbol{U}_i\}_{i\in\mathbb{Z}}$ in $\boldsymbol{Y}$ together with the local Lagrangians λ_i is called a *presentation* of the local variational problem.

Remark 1 This definition is fraught with problems. First, the dependence on the choice of a covering of Y makes the notion of equivalence rather cumbersome to deal with. To compare two local variational problems one has first to find their restriction to a common refinement of their respective coverings. Moreover, two equivalent systems of local Lagrangians already defined with respect to the same covering can differ by an arbitrary 0-cocycle of variationally trivial Lagrangians, i.e. an arbitrary collections of local sections (over the $\boldsymbol{U}_i \subset \boldsymbol{Y}$) of $\boldsymbol{K}_r$. In consequence, on a give open set, the local Lagrangian from one system will have in general infinitesimal symmetries different from those of the local Lagrangian from the other.

This means that the notion of (infinitesimal) symmetry of a Lagrangian does not carry over to the case of local variational systems. In particular it can be used no longer even in the case of Euler–Lagrange morphisms which admit a global Lagrangian. However, a reasonable more restrictive definition of 'equality' is not readily available. In the next section we will show a way of how to deal with the questions of symmetries. □

Note that every cohomology class in $H^{n+1}_{dR}(\boldsymbol{Y}) \simeq H^{n+1}(\boldsymbol{Y},\boldsymbol{R})$ gives rise to local variational problems. Non trivial $H^{n+1}(\boldsymbol{Y},\boldsymbol{R})$ can appear e.g. when dealing with symmetry breaking, $\boldsymbol{Y}$ will then be fibred (over $\boldsymbol{X}$) by homogeneous spaces. Geometrically the same situation arises also in the following example.

Example 1 Consider the fibering

$$\pi : S^2 \times I\!R^2 \times Gl(4)/O(1,3) \mapsto S^2 \times I\!R^2$$

Sections of this bundle are $(1,3)$-metrics on $S^2 \times \boldsymbol{R}^2$.These are, of course, the possible gravitational fields of a black hole. Here we have $H^5(\boldsymbol{Y},\boldsymbol{R}) \simeq \boldsymbol{R}$, so one actually gets equations for the gravitational field of a black hole,which are locally but not globally variational. Of course, we do not claim any physical relevance for any of them. But note that one can always 'add' a global Lagrangian to a local variational problem by simply adding its restrictions on each open set. The cohomolgy class of the local variational problem remains unchanged by this. Thus, for every cohmology class one can find Euler–Lagrange morphisms representing it, which give rise to equations of the type 'Einstein equations + constraints of some kind'.

3 Symmetries and conservation laws

Once the formalism of variational sequences (or any other differential formulations of the calculus of variations) is established, one is lead quite naturally to consider also equations which are not globally variational. To justify this, it is crucial that the differential formulations have a very rich mathematical structure which allows to derive a lot of additional information, both locally and globally.

Most of all, one *should have* reasonable conservation laws with preferably global conserved quantities. For this one wants, of course, Noether type theorems linking symmetries of the local variational problem to conserved quantities. The first question is, however, what the most natural choice for *symmetries of the local variational problem* might be.

We recall that, inspired by [12], for any projectable vector field (Ξ,ξ) one can define on $(\mathcal{V}^p_r)_{\boldsymbol{W}}$, $\boldsymbol{W}$ open in $\boldsymbol{Y}$, the *variational Lie derivative* operator $\mathcal{L}_{j_r\Xi}$ [10]. Then, we have

- if $p = n$ and $\lambda \in (\mathcal{V}^n_r)_{\boldsymbol{W}}$, then

$$\mathcal{L}_{j_r\Xi}\lambda = \Xi_V \rfloor \mathcal{E}_n(\lambda) + d_H(j_r\Xi_V \rfloor p_{d_V\lambda} + \xi \rfloor \lambda)\,; \tag{1}$$

- if $p = n+1$ and $\eta \in (\mathcal{V}^{n+1}_r)_{\boldsymbol{W}}$ then

$$\mathcal{L}_{j_r\Xi}\eta = \mathcal{E}_n(\Xi_V \rfloor \eta) + \tilde{H}_{d\eta}(j_{2r+1}\Xi_V)\,. \tag{2}$$

Hence, from (1) we see that to get conservation laws we need

$$0 = \mathcal{E}_n(\Xi_V \rfloor \eta)$$

for $\eta \in (\mathcal{E}_n(\mathcal{V}^n_r))_{\boldsymbol{W}}$ and by (2) this means

$$\mathcal{L}_{j_r\Xi}\eta = 0\,,$$

since $\tilde{H}_{d\eta}(j_{2r+1}\Xi_V)$ vanishes.

Thus, choosing as symmetries of *local variational problems* the symmetries of the corresponding Euler–Lagrange morphism $\eta \in (\mathcal{E}_n(\mathcal{V}^n_r))_{\boldsymbol{Y}}$ is the most natural choice.

Proposition 1 *Let η_λ be the Euler–Lagrange morphism of a local variational problem. Let $\mathcal{L}_{j_r\Xi}\eta_\lambda = 0$. Then, along the solutions, we have the following* local *conservation law*

$$0 = d_H(j_r\Xi_V \rfloor p_{d_V\lambda_i} + \xi \rfloor \lambda_i - \beta(\lambda_i, \Xi))\,,$$

and, in view of our defintion, it depends only on the local variational problem.

Proof. Locally we have $\Xi_V \rfloor \eta_\lambda = \Xi_V \rfloor \mathcal{E}_n(\lambda_i)$. From (1) above we have then

$$0 = \Xi_V \rfloor \eta_\lambda + d_H(j_r\Xi_V \rfloor p_{d_V\lambda_i} + \xi \rfloor \lambda_i) - \mathcal{L}_{j_r\Xi}\lambda_i\,.$$

Since all other terms are closed, also $\mathcal{L}_{j_r\Xi}\lambda_i$ is. In consequence, there is a $\beta(\lambda_i, \Xi)$ such that

$$0 = \Xi_V \rfloor \eta_\lambda + d_H(j_r\Xi_V \rfloor p_{d_V\lambda_i} + \xi \rfloor \lambda_i - \beta(\lambda_i, \Xi))$$

holds. This is, along the solutions, the desired local conservation law. The symmetries depend only on the Euler–Lagrange morphism, thus, by definition they depend only on the local variational system. Or, more explicitly, for a variationally trivial Lagrangian θ we have $\mathcal{L}_{j_r\Xi}\theta = d_H(j_r\Xi_V \rfloor p_{d_V\theta} + \xi \rfloor \theta)$. Thus, adding, after restriction to a suitbale refinement, an arbitray 0-cocyle of variationally trivial Lagrangians we get

$$\begin{aligned} 0 = {} & \Xi_V \rfloor \eta_\lambda + (d_H(j_r\Xi_V \rfloor p_{d_V\lambda_i} + \xi \rfloor \lambda_i) + d_H(j_r\Xi_V \rfloor p_{d_V\theta_i} + \xi \rfloor \theta_i)) + \\ & -(\mathcal{L}_{j_r\Xi}\lambda_i + \mathcal{L}_{j_r\Xi}\theta_i) = \Xi_V \rfloor \eta_\lambda + d_H(j_r\Xi_V \rfloor p_{d_V\lambda_i} + \xi \rfloor \lambda_i) - \mathcal{L}_{j_r\Xi}\lambda_i\,. \end{aligned}$$

Of course, $\epsilon(\lambda_i, \Xi) = (j_r\Xi_V \rfloor p_{d_V\lambda_i} + \xi \rfloor \lambda_i)$ is the usual *canonical* or *Noether current*. To clarify one point: the local conserved current is $\epsilon(\lambda_i, \Xi) - \beta(\lambda_i, \Xi)$; the Noether current $\epsilon(\lambda_i, \Xi)$ is conserved if and only if Ξ is also a symmetry of λ_i.

We will turn our attention to the global situation now. Note that in our definition a local variational problem is a global object in the sense that it has a global Euler–Lagrange morphism defining a topological invariant. Consequently, there is also a precise relation between our local conservation laws. We will summarize this in the following propositon.

Proposition 2 *Let η_λ be the Euler–Lagrange morphism of a local variational problem, λ_i the system of local Lagrangians of an arbitrary given presentation, then we have*

1. *the local currents satisfy* $d_H(\epsilon(\lambda_i, \Xi) - \beta(\lambda_i, \Xi) - \epsilon(\lambda_j, \Xi) + \beta(\lambda_j, \Xi)) = 0$*;*
2. *the local currents are the restrictions of a global conserved current,* i.e. $\epsilon(\lambda_i, \Xi) - \beta(\lambda_i, \Xi) - \epsilon(\lambda_j, \Xi) + \beta(\lambda_j, \Xi) = 0$*, if and only if the cohomlogy class* $[\Xi_V \rfloor \eta_\lambda] \in H^n_{dR}(\boldsymbol{Y})$ *vanishes.*

PROOF. $\Xi_V \rfloor \eta_\lambda$ defines a cohomology class, since $0 = \mathcal{L}_{j_r\Xi}\eta = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda)$. The local conserved currents are simply the negative of its local potentials. The first affirmation is one way to state this. Or, explicitly, the formula for the Lie derivative of the λ_i shows that the $d_H(j_r\Xi_V \rfloor p_{d_V\lambda_i} + \xi \rfloor \lambda_i - \beta(\lambda_i, \Xi))$ are the restrictions to the open sets of the corresponding covering of $-\Xi_V \rfloor \eta_\lambda$ and, hence, coincide on intersections.

If the local currents are the restrictions of a global one, then this global current is also a global potential of $-\Xi_V \rfloor \eta_\lambda$. This deals with the second affirmation.

Remark 2 Recall our above example regarding the gravitational field of the black hole. For our *hypothetical* Einstein equations + constraints on

$$\pi : S^2 \times I\!R^2 \times Gl(4)/O(1,3) \mapsto S^2 \times I\!R^2$$

the class $[\Xi_V \rfloor \eta_\lambda] \in H^4_{dR}(\boldsymbol{Y})$ always vanishes, since $H^4_{dR}(\boldsymbol{Y}) \simeq 0$. Thus, our conservation laws are always global. □

However, using the symmetries of the Euler-lagrange morphism to find conservation laws leads to serious practical problems when one is interested in the global case.

This comes from the fact that also in the case of a bona fide Lagrangian, when the cohomolgy class $[\mathcal{E}_n(\lambda)]$ is trivial the cohomolgy class $[\Xi_V \rfloor \mathcal{E}_n(\lambda)]$ may be non trivial: *the contraction of a closed, but cohomologically trivial form with a vector field is not necessarily cohomologically trivial itself; the simplest example is that* $H^n_{dR}(I\!R^{n+1} - 0)$ *can be generated in this way, i.e. we view* $(I\!R^{n+1} - 0)$ *as the total space of the fibre bundle* $S^n \times I\!R \mapsto S^n$ *and contract a volume form with a suitable vertical vector field.*

If, on the other hand, Ξ is a symmetry of all local Lagrangians λ_i of a given presentation of the local variational problem the Noether currents are conserved and form a system of local potentials of the cohomology class

$$[\Xi_V \rfloor \eta_\lambda] \in H^n_{dR}(\boldsymbol{Y})\,.$$

There is a global Noether current if and only if

$$0 = [\Xi_V \rfloor \eta_\lambda] \in H^n_{dR}(\boldsymbol{Y})\,.$$

In general we have

$$d_H(\epsilon(\lambda_i, \Xi) - \epsilon(\lambda_j, \Xi)) = \mathcal{L}_{j_r\Xi}\lambda_i - \mathcal{L}_{j_r\Xi}\lambda_j \neq 0\,,$$

thus neither the $\mathcal{L}_{j_r\Xi}\lambda_i$ nor the $d_H(\epsilon(\lambda_i, \Xi))$ are in general the restrictions of global closed n-forms. But since the obstruction to have a global closed form and, hence, a cohomology class is the same in both cases, it vanishes in the difference. Thus, we can summarize our discussion in the following propositon.

Proposition 3 *The cohomology class* $[\Xi_V \rfloor \eta_\lambda]$*,* i.e. *the obstruction to the existence of a global conserved current, is the difference of two conceptually independent cohomology classes. One coming from using the symmetries of the Euler–Lagrange morphism and the other from the system of local Noether currents.*

Or, in other words, the rather forced use of the symmetries of the Euler–Lagrange morphism introduces, compared to the use of Lagrangian symmetries, an additional type of obstruction, which makes it on the whole more difficult to keep the situation under control.

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References

[1] I. M. Anderson: Aspects of the Inverse Problem to the Calculus of Variations, *Arch. Math. Brno* **24** (1988) (4) 181–202.

[2] I. M. Anderson: Introduction to the variational bicomplex, *Contemp. Math., Am. Math. Soc.*, **132**, 1992, 51–73; see also *The variational bicomplex*, preprint (1989)

[3] I. M. Anderson, T. Duchamp: On the existence of global variational principles, *Amer. Math. J.* **102** (1980) 781–868.

[4] M. Bauderon: Le problème inverse du calcul des variations, *Ann. de l'I.H.P.* **36** (1982) (2) 159–179.

[5] A. Borowiec: On topological Lagrangians and the meaning of the Euler–Lagrange operator. *Hadronic mechanics* etc., H.Ch. Myung ed., Nova Sci. Publ. (1992) 139–147.

[6] A. Borowiec, M. Ferraris, M. Francaviglia, M. Palese: Conservation laws for non-global Lagrangians, *Univ. Iagel. Acta Math.* **41** (2003) 319331.

[7] J. Brajerčík, D. Krupka: Cohomology and local variational principles, *Publ. R. Soc. Mat. Esp.* **11** (2007) 119–124.

[8] J. Brajerčík, D. Krupka: Variational principles for locally variational forms, *J. Math. Phys.* **46** (5) (2005) 052903, 15 pp.

[9] P. Dedecker and W. M. Tulczyjew: Spectral sequences and the inverse problem of the calculus of variations, in *Lecture Notes in Mathematics* **836**, Springer–Verlag (1980), 498–503.

[10] M. Francaviglia, M. Palese, R. Vitolo: Symmetries in finite order variational sequences, *Czech. Math. J.* **52(127)** (1) (2002) 197213.

[11] I. Kolář: Lie Derivatives and Higher Order Lagrangians, *Proc. Diff. Geom. and its Appl.*; O. Kowalski ed., Univerzita Karlova (Praha, 1981) 117–123.

[12] D. Krupka: Some Geometric Aspects of Variational Problems in Fibred Manifolds, *Folia Fac. Sci. Nat. UJEP Brunensis* **14**, J. E. Purkyně Univ. (Brno, 1973) 1–65.

[13] D. Krupka: Variational Sequences on Finite Order Jet Spaces, *Proc. Diff. Geom. and its Appl.* (Brno, 1989); J. Janyška, D. Krupka eds., World Scientific (Singapore, 1990) 236–254.

[14] D. Krupka: Variational sequences and variational bicomplexes, *Proc. VII Conf. Diff. Geom. Appl., Satellite Conf. of ICM in Berlin*; I. Kolář *et al.* eds.; Masaryk University in Brno, 1999, 525–531.

[15] D. Krupka, J. Musilova: Trivial Lagrangians in field theory, *Differential Geom. Appl.* **9** (3) (1998) 293–305; *erratum, ibid.* **10** (3) (1999) 303.

[16] J.F. Pommaret: Spencer Sequence and Variational Sequence; *Acta Appl. Math.* **41** (1995) 285–296.

[17] G. Sardanashvily: Noether conservation laws issue from the gauge invariance of an Euler–Lagrange operator, but not a Lagrangian; arXiv: math-ph/0302012v1.

[18] F. Takens: A global version of the inverse problem of the calculus of variations, *J. Diff. Geom.* **14** (1979) 543–562.

[19] C.G. Torre: Some remarks on gravitational analogs of magnetic charge, (gr-qc/9411014) *Class. Quant. Grav.* **12** (1995) L43–L50.

[20] W. M. Tulczyjew: The Lagrange Complex, *Bull. Soc. Math. France* **105** (1977) 419–431.

[21] W. M. Tulczyjew: The Euler–Lagrange Resolution, *Lecture Notes in Mathematics* **836**, Springer–Verlag (Berlin, 1980) 22–48.

[22] A. M. Vinogradov: On the algebro–geometric foundations of Lagrangian field theory, *Soviet Math. Dokl.* **18** (1977) 1200–1204.

[23] A. M. Vinogradov: The $\mathcal{C}$–Spectral Sequence, Lagrangian Formalism, and Conservation Laws. I. The Linear Theory; II. The Nonlinear Theory, *Journ. Math. An. and Appl.* **100** (1984) (1) 1–40 and 41–129.

[24] R. Vitolo: Finite order Lagrangian bicomplexes, *Math. Proc. Cambridge Phil. Soc.* **125** (1) (1998) 321–333.