Rend. Istit. Mat. Univ. Trieste
Volume 44 (2012), 1–13

A global bifurcation result for a second order singular equation[1]

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Dedicated, with gratefulness and friendship, to Professor Fabio Zanolin on the occasion of his 60th birthday

Abstract. *We deal with a boundary value problem associated to a second order singular equation in the open interval* $(0, 1]$. *We first study the eigenvalue problem in the linear case and discuss the nodal properties of the eigenfunctions. We then give a global bifurcation result for nonlinear problems.*

Keywords: self-adjoint singular operator, spectrum, nodal properties, global bifurcation
MS Classification 2010: 34C23, 34B09,35P05

1. Introduction

We are concerned with a second order ODE of the form

$$-u'' + q(x)u = \lambda u + g(x, u)u, \ \lambda \in \mathbb{R}, \ x \in (0, 1], \tag{1}$$

where $q \in C((0, 1])$ satisfies

$$\lim_{x\to 0^+} \frac{q(x)}{l/x^\alpha} = 1 \tag{2}$$

for some $l > 0$ and $\alpha \in (0, 5/4)$, and $g \in C([0, 1] \times \mathbb{R})$ is such that

$$\lim_{u\to 0} g(x, u) = 0, \quad \text{uniformly in } x \in (0, 1]. \tag{3}$$

The constant $5/4$ arises in a rather straightforward manner in the study of the differential operator in the left-hand side of (1) (cf. [13, p. 287-288]); details are given in Remark 2.3 below.

[1]Under the auspices of GNAMPA-I.N.d.A.M., Italy. The work has been performed in the frame of the M.I.U.R. Projects 'Topological and Variational Methods in the Study of Nonlinear Phenomena' and 'Nonlinear Control: Geometrical Methods and Applications'.

We will look for solutions u of (1) such that $u \in H_0^2(0,1)$.

When the x-variable belongs to a compact interval, problems of the form (1) have been very widely studied. A more limited number of contributions is available in the literature when the x-variable belongs to a (semi)-open interval, as it is the case in the present paper, or to an unbounded interval [7, 8].

We treat (1) in the framework of bifurcation theory. For this reason, we first discuss in Section 2 the eigenvalue problem

$$-u'' + q(x)u = \lambda u, \quad x \in (0,1], \; \lambda \in \mathbb{R}. \tag{4}$$

For such singular problems, the well-known embedding of (4) (by an elementary application of the integration by parts rule, together with the boundary condition $u(0) = 0 = u(1)$) in the setting of eigenvalue problems for compact self-adjoint operators cannot be performed. Thus, the questions of the existence of eigenvalues and of the nodal properties of the associated eigenfunctions have various delicate features. For a comprehensive account on the spectral properties of the Schrödinger operator we refer to the books [12] and [10]; for more specific results on singular problems in $(0,1)$ we refer, among many others, to [5, 15].

However, the linear spectral theory for singular problems is well-established and can be found, among others, in the classical book by Coddington and Levinson [4] and in the (relatively) more recent text by Weidmann [13]. The former monograph focuses on a generalization of the so-called "expansion theorem" valid for functions in $L^2([0,1])$ and, by doing this, a sort of "generalized shooting method" is performed. On the other hand, in [13] the singular problem is tackled from an abstract point of view; more precisely, it is considered the general question of the existence of a self-adjoint realization of the formal differential expression $\tau u = -u'' + q(x)u$ and the important Weyl alternative theorem [13, Theorem 5.6] is used. It is interesting to observe that the approach in [4] (based on more elementary ODE techniques) and the abstract one in [13] lead in different ways to the important concepts of "limit point case" and "limit circle case". The knowledge of one (or the other) case is ensured by suitable assumptions on q and lead to information on the boundary conditions to be added to (4) in order to have a self-adjoint realization of τ.

In the setting of the present paper, the operator τ is regular at $x = 1$; this implies that it is in the limit circle case. Moreover, under assumption (2), from [13, Theorem 6.4] it follows that τ is in the limit circle case also in $x = 0$. Thus, the differential operator $A : u \mapsto \tau u$ with

$$\begin{aligned} D(A) = \{u \in L^2(0,1) : \; u, u' \in AC(0,1), \; \tau u \in L^2(0,1), \\ \lim_{x \to 0^+} (xu'(x) - u(x)) = 0 = u(1)\} \end{aligned}$$

is a self-adjoint realization of τ ([13, p. 287-288]). We prove in Proposition 2.2 that in fact $D(A) = H_0^2(0,1)$; to do this, we need some knowledge of the behaviour of the solutions of (4) near zero. These estimates are developed in Proposition 2.1 by means of the classical Levinson theorem [6, Theorem 1.8.1]. Finally, at the end of Section 2 we focus on the nodal properties of a solution to (4); more precisely, in Proposition 2.4 we prove that (4) is non-oscillatory and conclude in Proposition 2.5 that the spectrum of A is purely discrete and that, for every $n \in \mathbb{N}$, the eigenfunction associated to the eigenvalue λ_n has $(n-1)$ simple zeros in $(0,1)$.

Section 3 contains a global bifurcation result (Theorem 3.2) which follows in a rather straightforward manner as an application of the celebrated Rabinowitz theorem in [11].

In order to exclude alternative (2) in Theorem 3.2, we use a technique that we already applied for Hamiltonian systems in $\mathbb{R}^{2N}$ in [2] and for planar Dirac-type systems in [3]. More precisely, we introduce a continuous integer-valued functional defined on the set of solutions to (1). Due to the singularity at $x = 0$, some care is necessary in order to prove its continuity; this is the content of Proposition 3.4. We can then state and prove our main result (Theorem 3.5).

In what follows, for a given function p we write $p(x) \sim \dfrac{m}{x^a}, x \to 0^+$, when

$$\lim_{x\to 0^+} \frac{p(x)}{m/x^a} = 1 \tag{5}$$

for some $m, a \in \mathbb{R}^+$.

Finally, we write

$$H_0^2(0,1) = \{u \in H^2(0,1) : \ u(0) = 0 = u(1)\},$$

equipped with the norm defined by

$$||u||^2 = ||u||^2_{L^2(0,1)} + ||u''||^2_{L^2(0,1)}, \quad \forall\ u \in H_0^2(0,1).$$

2. The linear equation

In this section we study a linear second order equation of the form

$$-u'' + q(x)u = \lambda u, \quad x \in (0,1],\ \lambda \in \mathbb{R}. \tag{6}$$

We will assume that $q \in C((0,1])$ and that

$$q(x) \sim \frac{l}{x^\alpha}, \quad x \to 0^+, \tag{7}$$

for some $l > 0$ and $\alpha \in (0, 5/4)$. Without loss of generality we may suppose that

$$q(x) > 0, \quad \forall\, x \in (0, 1]. \tag{8}$$

For every $u : (0, 1] \to \mathbb{R}$ we denote by τu the formal expression

$$\tau u = -u'' + q(x)u;$$

First of all, we study the asymptotic behaviour of solutions of (6) when $x \to 0^+$; to this aim, let us introduce the change of variables $t = -\log x$ and let

$$w(t) = u(e^{-t}), \quad \forall\, t > 0.$$

From the relations

$$\begin{aligned} w'(t) &= -e^{-t}u'(e^{-t}) \\ w''(t) &= e^{-t}u'(e^{-t}) + e^{-2t}u''(e^{-t}), \end{aligned} \tag{9}$$

we deduce that u is a solution of (6) on $(0, 1)$ if and only if w is a solution of

$$-w'' - w' + e^{-2t}q(e^{-t})w = \lambda e^{-2t} w \tag{10}$$

on $(0, +\infty)$. Equation (10) can be written in the form

$$Y' = (C + R(t))Y, \tag{11}$$

where $Y = (w, z)^T$ and

$$C = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad R(t) = \begin{pmatrix} 0 & 0 \\ e^{-2t}q(e^{-t}) - \lambda e^{-2t} & 0 \end{pmatrix}, \quad \forall\, t > 0. \tag{12}$$

Now, let us observe that C has eigenvalues $\lambda_1 = 0$, $\lambda_2 = -1$ and corresponding eigenvectors $u_1 = (1, 0)$, $u_2 = (1, -1)$ and that $R \in L^1(0, +\infty)$; therefore, an application of [6, Theorem 1.8.1] implies that (11) has two linearly independent solutions Y_1, Y_2 such that

$$\begin{aligned} Y_1(t) &= u_1 + o(1), \quad t \to +\infty, \\ Y_2(t) &= (u_2 + o(1))e^{-t}, \quad t \to +\infty. \end{aligned} \tag{13}$$

As a consequence, we obtain the following result:

Proposition 2.1. *For every $\lambda \in \mathbb{R}$ the equation* (6) *has two linearly independent solutions $u_{1,\lambda}$, $u_{2,\lambda}$ such that*

$$\begin{aligned} &u_{1,\lambda}(x) = 1 + o(1),\ u'_{1,\lambda}(x) = o\left(\frac{1}{x}\right) \quad x \to 0^+, \\ &u_{2,\lambda}(x) = x + o(x),\ u'_{2,\lambda}(x) = 1 + o(1), \quad x \to 0^+, \end{aligned} \tag{14}$$

and $u_{2,\lambda} \in H^2(0,1)$.

For every $f \in L^2(0,1)$ the solutions of $\tau u = f$ are given by

$$u(x) = c_1 u_{1,0}(x) + c_2 u_{2,0}(x) + u_f(x), \quad \forall\, x \in (0,1),\ c_1, c_2 \in \mathbb{R}, \tag{15}$$

where

$$\begin{aligned} &u_f(x) = \textstyle\int_0^x G(x,t) f(t)\, dt, \quad \forall\, x \in (0,1), \\ &G(x,t) = u_{1,0}(t) u_{2,0}(x) - u_{2,0}(t) u_{1,0}(x), \quad \forall\, x \in (0,1),\ t \in (0,1) \end{aligned} \tag{16}$$

fulfill $G \in L^\infty((0,1)^2)$, $u_f(0) = 0 = u'_f(0)$ and $u_f \in H^2(0,1)$.

Proof. The estimates in (14) follow from (9) and (13), while (16) is the usual variation of constants formula. Moreover, from (14) we obtain that $u_{2,\lambda}, u'_{2,\lambda} \in L^2(0,1)$. On the other hand we have

$$q(x) u_{2,\lambda}(x) \sim x^{1-\alpha}, \quad x \to 0^+, \tag{17}$$

which implies that $q u_{2,\lambda} \in L^2(0,1)$, since $\alpha < 5/4$ (cf. Remark 2.3 for comments on this restriction); using the fact that $\tau u_{2,\lambda} = \lambda u_{2,\lambda}$, we deduce that

$$u''_{2,\lambda} = \lambda u_{2,\lambda} - q u_{2,\lambda} \in L^2(0,1).$$

From now on, we will indicate $u_i = u_{i,0}$, $i = 1, 2$. The fact that the function G defined in (16) belongs to the space $L^\infty((0,1)^2)$ is a consequence of the asymptotic estimates (14). Moreover, from (16) we also deduce that $u_f(0) = 0$ and that

$$u'_f(x) = \int_0^x (u_1(t) u'_2(x) - u_2(t) u'_1(x)) f(t)\, dt, \quad \forall\, x \in (0,1), \tag{18}$$

which implies $u'_f(0) = 0$.

Finally, the condition $u_f(0) = 0 = u'_f(0)$ guarantees that $u_f, u'_f \in L^2(0,1)$; as far as the second derivative of u_f is concerned, let us observe that we have

$$\tau u_f = f$$

and so

$$u''_f = f - q u_f. \tag{19}$$

Using the fact that $u_f(0) = 0 = u'_f(0)$ and (7), it follows that $q u_f \in L^2(0,1)$; hence $u_f \in H^2(0,1)$. □

In what follows, we study the spectral properties of suitable self-adjoint realizations of τ; to this aim, let us first observe that the differential operator τ is regular at $x = 1$. As a consequence, it is in the limit circle case at $x = 1$; moreover, from (7), according to [13, Theorem 6.4], τ is in the limit circle case also in $x = 0$.

The differential operator A defined by

$$\begin{aligned} D(A) = \{u \in L^2(0,1) : u, u' \in AC(0,1),\ \tau u \in L^2(0,1), \\ \lim_{x\to 0^+} (xu'(x) - u(x)) = 0 = u(1)\} \\ Au = \tau u, \quad \forall\, u \in D(A), \end{aligned}$$

is then a self-adjoint realization of τ ([13, p. 287-288]). We can show the validity of the following Proposition:

Proposition 2.2. *The relation*

$$D(A) = H_0^2(0,1)$$

holds true. Moreover, A has a bounded inverse $A^{-1} : L^2(0,1) \to H_0^2(0,1)$.

Proof. 1. Let us start proving that $H_0^2(0,1) \subset D(A)$. It is well known that $H_0^2(0,1) \subset C^1(0,1)$; hence, for every $u \in H_0^2(0,1)$ we have $u, u' \in AC(0,1)$. Moreover, using the fact that $u(0) = 0$ we deduce that

$$u(x) = u'(0)x + o(x), \quad x \to 0^+$$

and

$$q(x)u(x) = u'(0)x^{1-\alpha} + o(x^{1-\alpha}), \quad x \to 0^+;$$

the condition $\alpha < 5/4$ guarantees again that $qu \in L^2(0,1)$ and therefore $\tau u = -u'' + qu \in L^2(0,1)$. Finally, the regularity of u and u' imply that

$$\lim_{x\to 0^+} (xu'(x) - u(x)) = 0$$

and so also the boundary condition in the definition of $D(A)$ is satisfied.

Now, let us prove that $D(A) \subset H_0^2(0,1)$; for every $u \in D(A)$ let $f = \tau u \in L^2(0,1)$. From (15) we deduce that u can be written as

$$u = c_1u_1 + c_2u_2 + u_f, \tag{20}$$

for some $c_1, c_2 \in \mathbb{R}$; it is easy to see that the function u_1 does not satisfy the boundary condition given in $x = 0$ in the definition of $D(A)$, while u_2 and u_f do. Hence $u \in D(A)$ if and only if $c_1 = 0$; the last statement of Proposition 2.1 implies then that $u \in H^2(0,1)$. As in the first part of the proof, the regularity of u allows to conclude that the boundary condition in $x = 0$ given in $D(A)$ reduces to $u(0) = 0$.

2. Let us study the invertibility of A; the existence of a bounded inverse of A is equivalent to the fact that $0 \in \rho_A$, being ρ_A the resolvent of A. Since A is self-adjoint on $H_0^2(0,1)$, this follows from the surjectivity of A (cf. [17, Theorem 5.24]); hence, it is sufficient to prove that A is surjective.

To this aim, let us first observe that condition (8) guarantees that 0 cannot be an eigenvalue of A. Now, let us fix $f \in L^2(0,1)$ and let us prove that there exists $u \in H_0^2(0,1)$ such that $Au = f$, i.e. $\tau u = f$; by applying Proposition 2.1 we deduce again that (20) holds true and the same argument of the first part of the proof implies that $c_1 = 0$.

Hence we obtain $u = c_2 u_2 + u_f$; from Proposition 2.1 we deduce that this function belongs to $H^2(0,1)$ and satisfies the boundary condition $u(0) = 0$. In order to prove that the missing condition $u(1) = 0$ is fulfilled for every $f \in L^2(0,1)$, let us observe that $u_2(1) \neq 0$, otherwise u_2 would be an eigenfunction of A associated to the zero eigenvalue. Therefore, $u(1) = 0$ is satisfied if

$$c_2 = -\frac{u_f(1)}{u_2(1)},$$

for every $f \in L^2(0,1)$. □

Remark 2.3. *As for the restriction $\alpha < 5/4$, we observe that for the proofs of Proposition 2.1 and Proposition 2.2 it is sufficient to require the milder condition $\alpha < 3/2$. The fact that $\alpha < 5/4$ is used (cf. [13, p. 287-288]) in order to obtain that $D(A)$ is the one described above. Finally, we observe that in the particular case when $\alpha < 1$ the problem is regular (cf., among others, [9]).*

The spectral properties of A are related to the oscillatory behaviour of solutions of (6). We first recall the following definition:

Definition 2.4. *The differential equation* (6) *is oscillatory if every solution u has infinitely many zeros in* $(0,1)$. *It is non-oscillatory when it is not oscillatory.*

We observe that the regularity assumptions on q imply that solutions of (6) have a finite number of zeros in any interval of the form $[a,1)$, for every $0 < a < 1$. Moreover, from (7) we infer that for every $\lambda \in \mathbb{R}$ there exists $c(\lambda) \in (0,1]$ such that

$$\lambda - q(x) < 0, \quad \forall\, x \in (0, c(\lambda)).$$

An application of the Sturm comparison theorem proves that every solution of (6) has at most one zero in $(0, c(\lambda))$; as a consequence, we obtain the following result:

Proposition 2.5. *For every $\lambda \in \mathbb{R}$ the differential equation* (6) *is non-oscillatory.*

Once Proposition 2.5 is obtained, we can provide in a straightforward way some useful information on the spectral properties of A; more precisely, denoting by σ_{ess} the essential spectrum of a given operator, we have:

Proposition 2.6. *([13, Theorem 14.3, Theorem 14.6 and Theorem 14.9], [12, Theorem XIII.1]) The differential operator A is bounded-below and satisfies*

$$\sigma_{ess}(A) = \varnothing.$$

Moreover, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of simple eigenvalues of A such that

$$\lim_{n\to+\infty} \lambda_n = +\infty$$

and for every $n \in \mathbb{N}$ the eigenfunction u_n of A associated to the eigenvalue λ_n has $(n-1)$ simple zeros in $(0,1)$.

Remark 2.7. *According to [13], operators of the form τ (defined on functions whose domain is $(0,+\infty)$) arise when the time independent Schrödinger equation with spherically symmetric potential*

$$-\Delta u(x) + V(|x|)u(x) = \lambda u(x), \quad u \in L^2(\mathbb{R}^m) \tag{21}$$

is reduced to an infinite system of eigenvalue problems associated to the ordinary differential operators in $L^2(0,+\infty)$

$$\tau_i = -\frac{d^2}{dr^2} + \frac{1}{r^2}\Big[i(i+m-2) + \frac{1}{4}(m-1)(m-3)\Big] + V(r)$$

$(i \in \mathbb{N})$. In Appendix 17.F of [13] it is treated the case of a potential V satisfying assumptions (which enable to consider Coulomb potentials) that lead to (7). *More precisely, it is shown that for $m = 3, i = 0$ the operator is in the limit circle case at zero and self-adjoint extensions of τ_0 are described.*

3. The main result

In this section we are interested in proving a global bifurcation result for a nonlinear eigenvalue problem of the form

$$-u'' + q(x)u = \lambda u + g(x,u)u, \ \lambda \in \mathbb{R}, \ x \in (0,1], \tag{22}$$

where $q \in C((0,1])$ satisfies (7) and $g \in C([0,1]\times\mathbb{R})$ is such that

$$\lim_{u\to 0} g(x,u) = 0, \quad \text{uniformly in } x \in [0,1]. \tag{23}$$

We will look for solutions u of (22) such that $u \in H_0^2(0,1)$. To this aim, let Σ denote the set of nontrivial solutions of (22) in $H_0^2(0,1) \times \mathbb{R}$ and let $\Sigma' = \Sigma \cup \{(0,\lambda) \in H_0^2(0,1) \times \mathbb{R} : \ \lambda \text{ is an eigenvalue of } A\}$, where A is as in Section 2.

Let M denote the Nemitskii operator associated to g, given by

$$M(u)(x) = g(x, u(x))u(x), \quad \forall\, x \in [0,1],$$

for every $u \in H_0^2(0,1)$. We can show the validity of the following:

Proposition 3.1. *Assume* $g \in C([0,1] \times \mathbb{R})$ *and* (23). *Then* $M : H_0^2(0,1) \longrightarrow L^2(0,1)$ *is a continuous map and satisfies*

$$M(u) = o(||u||), \quad u \to 0. \tag{24}$$

Proof. 1. We first show that $Mu \in L^2(0,1)$ when $u \in H_0^2(0,1)$. When this condition holds, $u \in L^\infty(0,1)$ and the continuity of g implies that there exists $C_u > 0$ such that

$$|g(x, u(x))u(x)| \le C_u, \quad \forall\, x \in [0,1].$$

As a consequence we obtain $Mu \in L^\infty(0,1) \subset L^2(0,1)$.

2. Let us prove that M is continuous. Let us fix $u_0 \in X$ and let $u_n \in X$ such that $u_n \to u_0$ when $n \to +\infty$; the continuous embedding

$$H_0^2(0,1) \subset L^\infty(0,1)$$

and the uniform continuity of g on compact subsets of $[0,1] \times \mathbb{R}$ ensure that

$$g(x, u_n(x)) \to g(x, u_0(x)) \quad \text{in} \quad L^\infty(0,1). \tag{25}$$

This is sufficient to conclude that $Mu_n \to Mu_0$ in $L^\infty(0,1)$ and hence $Mu_n \to Mu_0$ in $L^2(0,1)$.

3. Finally, let us prove (24): using again the fact that $H_0^2(0,1) \subset L^\infty(0,1)$, we have

$$||Mu||_{L^2(0,1)} \le ||g(x,u(x))||_{L^\infty(0,1)} ||u||_{L^2(0,1)} \le ||g(x,u(x))||_{L^\infty(0,1)} ||u||,$$

for all $u \in H_0^2(0,1)$; hence, we deduce that

$$\frac{||Mu||_{L^2(0,1)}}{||u||} \le ||g(x,u(x))||_{L^\infty(0,1)}, \quad \forall\, u \in H_0^2(0,1),\ u \ne 0.$$

Therefore the result follows from (23) and (25). $\square$

Now, let us observe that the search of solutions $u \in H_0^2(0,1)$ of (22) is equivalent to the search of solutions of the abstract equation

$$Au = \lambda u + M(u), \quad (u,\lambda) \in H_0^2(0,1) \times \mathbb{R}; \tag{26}$$

on the other hand, (26) can be written in the form

$$w = \lambda Rw + M(Rw), \ (w,\lambda) \in L^2(0,1) \times \mathbb{R}, \tag{27}$$

where $R : L^2(0,1) \to H_0^2(0,1)$ is the inverse of A (cf. Proposition 2.2).

Now, from [13, Theorem 7.10] we deduce that R is compact; this fact and the continuity of M guarantee that the operator $MR : L^2(0,1) \to H_0^2(0,1)$ is compact. Moreover, the condition

$$M(Rw) = o(||w||_{L^2(0,1)}), \quad w \to 0, \tag{28}$$

is a consequence of (24). From an application of the global bifurcation result of Rabinowitz (cfr. [11]) to (27) we then obtain the following result:

THEOREM 3.2. *Assume* (7) *and* (23). *Then, for every eigenvalue* λ_n *of* A *there exists a continuum* C_n *of nontrivial solutions of* (22) *in* $H_0^2(0,1)\times\mathbb{R}$ *bifurcating from* $(0,\lambda_n)$ *and such that one of the following conditions holds true:*

(1) C_n *is unbounded in* $H_0^2(0,1) \times \mathbb{R}$*;*

(2) C_n *contains* $(0,\lambda_{n'}) \in \Sigma'$, *with* $n' \neq n$.

Now, let us observe that a more precise description of the bifurcating branch, eventually leading to exclude condition (2), can be obtained when there exists a continuous functional $j : \Sigma' \to \mathbb{N}$ (cf. [2, Pr. 2.1]). In order to define such a functional, we will use the fact that nontrivial solutions of (22) have a finite number of zeros in $(0,1)$; this will be a consequence of our next result.

For every $\lambda \in \mathbb{R}$ and for every nontrivial solution $u \in H_0^2(0,1)$ of (22) let us define $q_{u,\lambda} : (0,1] \to \mathbb{R}$ by $q_{u,\lambda}(x) = q(x) - \lambda - g(x,u(x))$, for every $x \in (0,1]$. The following Lemma holds true:

LEMMA 3.3. *For every* $\lambda \in \mathbb{R}$ *and for every nontrivial solution* $u \in H_0^2(0,1)$ *of* (22) *there exists a neighborhood* $U \subset H_0^2(0,1)\times\mathbb{R}$ *of* (u,λ) *and* $x_{u,\lambda} \in (0,1)$ *such that*

$$q_{v,\mu}(x) > 0, \quad \forall\ (v,\mu) \in U, \ x \in (0, x_{u,\lambda}]. \tag{29}$$

Proof. Let $(u,\lambda) \in H_0^2(0,1)\times\mathbb{R}$, $u \not\equiv 0$, be fixed and let U be the neighborhood of radius 1 of (u,λ) in $H_0^2(0,1)\times\mathbb{R}$; from the continuous embedding $L^\infty(0,1) \subset H_0^2(0,1)$ we deduce that if $(w,\mu) \in \Sigma \cap U_1$ then

$$||w||_{L^\infty(0,1)} \leq 1 + ||u||_{L^\infty(0,1)}, \quad |\mu| \leq 1 + |\lambda|$$

and

$$q(x) - \mu - g(x, w(x)) \geq q(x) - |\lambda| - 1 - \max_{\substack{x \in [0,1], \\ |s| \leq 1 + ||u||_{L^\infty(0,1)}}} |g(x, s)|, \quad \forall\, x \in (0, 1).$$

From (7) we then deduce that there exists $x_{(u,\lambda)} \in (0, 1)$, depending only on (u, λ), such that

$$q(x) - \mu - g(x, w(x)) > 0, \quad \forall\, x \in (0, x_{(u,\lambda)}].$$

□

Now, let us observe that for every $\lambda \in \mathbb{R}$ and for every nontrivial solution $u \in H_0^2(0, 1)$ of (22) the function u is a nontrivial solution of the linear equation

$$-w'' + (q(x) - g(x, u(x)) - \lambda)w = 0. \tag{30}$$

From Lemma 3.3, with an argument similar to the one which led to Proposition 2.5, we deduce that all the nontrivial solutions of (30) (in particular u) have a finite number of zeros in $(0, 1)$. We denote by $n(u)$ this number.

We are then allowed to define the functional j by setting

$$j(u, \lambda) = \begin{cases} n(u) & \text{if } u \not\equiv 0 \\ \\ n - 1 & \text{if } u \equiv 0 \text{ and } \lambda = \lambda_n, \end{cases} \tag{31}$$

for every $(u, \lambda) \in \Sigma'$. Let us observe that the definition $j(0, \lambda_n) = n - 1$ is suggested by Proposition 2.6.

PROPOSITION 3.4. *The function $j : \Sigma' \to \mathbb{N}$ is continuous.*

Proof. 1. As for the continuity of j in every point of the form $(0, \lambda_n)$, $n \in \mathbb{N}$, we refer to [16, Lemma 2.5].

2. Let us now fix $(u_0, \lambda_0) \in \Sigma$ and let $(u, \lambda) \in U$, with U as in Lemma 3.3; this Lemma guarantees that both u and u_0 have no zeros in $(0, x_{u_0,\lambda_0})$.

On the other hand, in the interval $[x_{u_0,\lambda_0}, 1]$ a standard continuous dependence argument (cf. also [11]) ensures that u and u_0 have the same numbers of zeros if (u, λ) is in a sufficiently small neighborhood of (u_0, λ_0). As a consequence, we obtain that there exists a neighborhood U_0 of (u_0, λ_0) such that

$$j(u, \lambda) = j(u_0, \lambda_0), \quad \forall\, (u, \lambda) \in U_0.$$

□

As a consequence, from Theorem 3.2 and Proposition 3.4 we deduce the final result:

THEOREM 3.5. *Assume* (7) *and* (23). *Then, for every eigenvalue* λ_n *of* A *there exists a continuum* C_n *of nontrivial solutions of* (22) *in* $H_0^2(0,1)\times\mathbb{R}$ *bifurcating from* $(0,\lambda_n)$ *and such that condition (1) of Theorem 3.2 holds true and*

$$j(u,\lambda) = n-1, \quad \forall\ (u,\lambda)\in C_n. \tag{32}$$

REMARK 3.6. *Theorem 3.2 can be proved as an application of Stuart's result [16, Theorem 1.2] as well. However, since in the situation considered in this paper the singularity at zero does not affect the compactness of the operator* R *defined after* (27)*, we chose to apply Rabinowitz theorem [11]. We finally mention the interesting paper [1], where global branches of solutions, with prescribed nodal properties, are obtained for a second order degenerate problem in* $(0,1)$.

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Received May 28, 2012
Revised September 3, 2012