WADGE DEGREES AND POINTCLASSES
INTRODUCTION TO PART III

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§1. Introduction. One of the main objects of study in Descriptive Set Theory is that of **boldface pointclass**, that is a collection of subsets of the Baire space (or more generally: of a family of Polish spaces) closed under continuous preimages. Since in this paper we will have little use for the concept of lightface pointclass used in the effective theory, we will drop the 'boldface' and simply speak of pointclasses. Also, in order to avoid trivialities, we will always assume that a pointclass is non-empty and different from $\wp(\mathbb{R})$.

Despite the fact that the the concept of pointclass is both very simple and ubiquitous in modern Descriptive Set Theory, it is actually quite recent, at least in its modern conception. The French analysts at the turn of the twentieth century—Baire, Borel, and Lebesgue—and later Luzin, Suslin, Hausdorff, Sierpiński, Kuratowski, always worked with specific pointclasses (such as the collection of all Borel sets, or the collection of all projective sets) defined by closure under set theoretic operations, and stratified into a transfinite hierarchy, *e.g.*, the Baire classes $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$, and $\boldsymbol{\Delta}^0_\alpha$ for the Borel sets, and $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$, and $\boldsymbol{\Delta}^1_n$ for the projective sets. The fact that all these collections were closed under continuous preimages was probably considered a simple consequence of their definition, rather than a feature worth crystallizing into a mathematical definition. Even the fact that the Borel hierarchy (and similarly for the projective one) exhibited the well-known diamond-shape pattern

$$\begin{array}{ccccccccccc}
\boldsymbol{\Sigma}^0_1 & & & & \boldsymbol{\Sigma}^0_2 & & \boldsymbol{\Sigma}^0_\alpha & & & & \boldsymbol{\Sigma}^0_{\alpha+1} \\
& \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
& & \boldsymbol{\Delta}^0_2 & & \cdots\boldsymbol{\Delta}^0_\alpha & & & & \boldsymbol{\Delta}^0_{\alpha+1} & & \cdots \\
& \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\
\boldsymbol{\Pi}^0_1 & & & & \boldsymbol{\Pi}^0_2 & & \boldsymbol{\Pi}^0_\alpha & & & & \boldsymbol{\Pi}^0_{\alpha+1} \\
\end{array}$$
$$\longleftarrow\!\!\!\longleftarrow\!\!\!\longleftarrow\; \omega_1 \;\longrightarrow\!\!\!\longrightarrow\!\!\!\longrightarrow$$

apparently was not considered to be an indication of an underlying structure. Hausdorff showed that any $\boldsymbol{\Delta}^0_2$ set can be represented as a transfinite difference of open (or for that matter, closed) sets, and Kuratowski, by the trick of refining the topology, extended this to all $\boldsymbol{\Delta}^0_{\alpha+1}$ sets. Thus $\boldsymbol{\Delta}^0_{\alpha+1} = \bigcup_{\beta<\omega_1} \mathrm{D}_\beta\, \boldsymbol{\Sigma}^0_\alpha$

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where $\mathrm{D}_\beta\, \boldsymbol{\Gamma}$ denotes the class of β-differences of sets in $\boldsymbol{\Gamma}$, *i.e.*, sets of the form

$$\{x \in \textstyle\bigcup_{\gamma<\beta} A_\gamma \ :\ \text{the least } \gamma \text{ that } x \notin A_\gamma \text{ has parity different from } \beta\}$$

for some sequence $\langle A_\gamma \ :\ \gamma < \beta\rangle$ of sets in $\boldsymbol{\Gamma}$. Again we obtain a picture similar to the one for the Borel hierarchy:

$$\begin{array}{ccccccccc}
\boldsymbol{\Sigma}^0_\alpha = \mathrm{D}_1\,\boldsymbol{\Sigma}^0_\alpha & & & & \mathrm{D}_2\,\boldsymbol{\Sigma}^0_\alpha & & & & \mathrm{D}_3\,\boldsymbol{\Sigma}^0_\alpha \\
& \subseteq & & \supseteq & & \subseteq & & \supseteq & \\
& & \Delta_{\mathrm{D}_2\,\boldsymbol{\Sigma}^0_\alpha} & & & & \Delta_{\mathrm{D}_2\,\boldsymbol{\Sigma}^0_\alpha} & & \cdots \\
& \supseteq & & \subseteq & & \supseteq & & \subseteq & \\
\boldsymbol{\Pi}^0_\alpha = \mathrm{D}_1\,\boldsymbol{\Pi}^0_\alpha & & & & (\mathrm{D}_2\,\boldsymbol{\Sigma}^0_\alpha)\check{} & & & & (\mathrm{D}_3\,\boldsymbol{\Sigma}^0_\alpha)\check{} \\
\longleftarrow & & & & \omega_1 & & & & \longrightarrow
\end{array}$$

Wadge in his Ph.D. thesis [Wad84] was the first to investigate in a systematic manner the notion of **continuous reducibility** on the Baire space ${}^\omega\omega$. The motivation for his study, and the reason as to why these matters had not been studied before is explained in [Wad84, pp. 2–3]:

> The notion of reducibility, including many-one reducibility, plays an extremely important role in recursive function theory. One would expect the same to be true in descriptive set theory; but that has not (at least till recently) been the case. Of course, there are in the literature many instances in which continuous preimage is used to derive a particular result. In Sikorski (1957), for example, this approach is used to construct for each countable ordinal μ a set in the μth but no lower level of the Borel hierarchy. Luzin and Sierpiński (1929) used preimage to show that the collection of (codes for) wellorderings of ω is not Borel; and there are a number of other examples. Yet nowhere (to our knowledge) is the relation $A = f^{-1}(B)$ for some continuous f ever explicitly defined and studied as a partial order, not even in exhaustive work such as Kuratowski (1958) or Sierpiński (1952). In the latter, Sierpiński discusses preimage in general, continuous image and homeomorphic image, but not (explicitly) continuous preimage, which is perhaps the most natural. One possible explanation is that the investigation of $\leq$ naturally involves infinite games, and it is only recently that game methods have been fully understood and appreciated.[1]

Wadge's main objective was a complete analysis of all the Borel pointclasses, *i.e.*, boldface pointclasses contained in $\boldsymbol{\Delta}^1_1$. Working in ZF+DC, he defined a hierarchy of Borel sets refining the usual Borel hierarchy, he proved that it is well-founded and computed its length, and, assuming the determinacy of all Borel games, he could show that every Borel pointclass fits in this classification. As explained by Wadge in [Wad11] in the present volume and

[1]The relation $\leq$ is nowadays called Wadge reducibility and it is denoted by $\leq_{\mathrm{W}}$, and the references mentioned are, in order, [Sik58], [LS29], [Kur58], and [Sie52].

in [Wad84, pp. 10–11], all these results were obtained before Martin's proof of Borel determinacy [Mar75]. The problem whether Borel determinacy is needed to prove that all Borel pointclasses fall into Wadge's analysis remained open for over a decade, until Louveau and Saint-Raymond answered in the negative, by conducting Wadge's analysis within second order arithmetic (see the paper [LSR88B] in this volume).

As we mentioned, all pointclasses considered by early descriptive set theorists were defined in terms of operation on sets, like taking complements, countable intersections, countable unions, Suslin's operation $\mathscr{A}$, *etc.* All of these operations can be thought as operations

$$\mathcal{O}\colon \wp(\mathbb{R})^\omega \to \wp(\mathbb{R})$$

assigning a new set to a countable sequence of sets, and with the property that there is a $T \subseteq \wp(\omega)$ such that for any $\langle A_n : n \in \omega\rangle$

$$\forall x \in \mathbb{R}\,(x \in \mathcal{O}\langle A_n : n \in \omega\rangle \iff \{n \in \omega : x \in A_n\} \in T)\,.$$

A function $\mathcal{O}$ as above is said to be an **ω-ary Boolean operation**, or simply a **Boolean operation**, and the set $T = T_{\mathcal{O}}$ which completely determines $\mathcal{O}$, is called the truth table of $\mathcal{O}$. We will say that such an operation is Borel, or $\boldsymbol{\Sigma}^1_1$, *etc.*, if its truth table is Borel, or $\boldsymbol{\Sigma}^1_1$, *etc.*, as a subset of ${}^\omega 2$. For example: the operations of taking complements, countable intersections, or countable unions, as well as their compositions are all Borel, while Suslin's operation $\mathscr{A}$ is $\boldsymbol{\Sigma}^1_1$.

Wadge showed in ZFC that each non-self-dual Borel pointclass in ${}^\omega\omega$ is of the form

$$\{\mathcal{O}\langle A_n : n \in \omega\rangle : \forall n\,(A_n \text{ is open})\}$$

with $\mathcal{O}$ a Borel Boolean operation, and Van Wesep in [Van77], assuming AD and building on earlier results of Miller, Radin, and Steel, extended this result to all non-self-dual pointclasses, using of course arbitrary Boolean operations. Thus we have come to a full circle—non-self-dual pointclasses considered by early descriptive set theorists were defined in terms of (explicit) operations, and assuming AD every non-self-dual pointclass is defined in terms of operations on open sets.

Boolean operations are operations on the *collection* of open sets that allow us to construct all sets belonging to complicated pointclass $\boldsymbol{\Gamma}$, and they figure prominently in the work of Louveau and Saint-Raymond [Lou83, LSR87, LSR88B]. But Wadge also introduced certain specific operations on *sets* which yield *complete sets* for various $\boldsymbol{\Gamma}$. Thus a non-self-dual pointclass $\boldsymbol{\Gamma}$ can be described either as obtained via some some appropriate Boolean operation $\mathcal{O}$, $\boldsymbol{\Gamma} = \{\mathcal{O}\langle A_n : n \in \omega\rangle : A_n \in \boldsymbol{\Sigma}^0_1\}$, or else as the set of continuous preimages of a $\boldsymbol{\Gamma}$-complete set A, $\boldsymbol{\Gamma} = \{X : X \leq_{\mathrm{W}} A\}$. These operations on sets are quite useful to compute the Wadge rank of the various pointclasses, and were extensively used in [Ste81B] and [Van77]. Recently this approach to the Wadge hierarchy has been extended in the work of Duparc and others in connection with automata theory, see [Dup01, Dup03, DFR01].

In the next section we give a few basic definitions an review some basic results on the Wadge hierarchy.

§2. Some basic facts about the Wadge hierarchy. The relation of **Wadge reducibility**, $A \leq_{\mathrm{W}} B$, is defined as $A = f^{-1}(B)$ for some continuous function f. It can be defined for any pair of ambient topological spaces: $\mathcal{X}$ containing A and $\mathcal{Y}$ containing B, so that $f\colon \mathcal{X} \to \mathcal{Y}$, but the general theory becomes somewhat uninteresting if the spaces are not zero-dimensional, as there may be, in general, very few continuous maps. Following Wadge, from now on we will focus on the Baire space ${}^{\omega}\omega$, which—as customary in set theory—will be denoted by $\mathbb{R}$.

A continuous $f\colon \mathbb{R} \to \mathbb{R}$ is determined by a monotone $\varphi\colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ such that $\lim_n \operatorname{lh} \varphi(x{\restriction}n) = +\infty$. If we require that $\operatorname{lh} \varphi(x{\restriction}n) = n$, then the resulting f is a Lipschitz function with constant ≤ 1, where we use the usual distance on ${}^{\omega}\omega$. In this case we will say that A **is Lipschitz reducible to** B. Wadge introduced the Lipschitz game $G_{\mathrm{L}}(A, B)$: it is a game on ω

player I	a_0		a_1		$\cdots$
player II		b_0		b_1	$\cdots$

where player II wins iff

$$\langle a_i \,:\, i < \omega\rangle \in A \iff \langle b_i \,:\, i < \omega\rangle \in B\,.$$

Thus player II has a winning strategy for the game $G_{\mathrm{L}}(A, B)$ if and only if $A \leq_{\mathrm{L}} B$. Conversely, if player I has a winning strategy, then there is a Lipschitz map witnessing $B \leq_{\mathrm{L}} \neg A$. Note that in this case player I's strategy yields a $\varphi\colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ such that $\operatorname{lh}(\varphi(s)) = \operatorname{lh}(s) + 1$ hence the induced $f\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz map with constant $1/2$, and in fact the converse implication (if $B \leq_{\mathrm{L}} \neg A$ then player I wins $G_{\mathrm{L}}(A, B)$) in general does not hold.

Assuming determinacy we obtain the following simple—yet fundamental—result known as:

Wadge's Lemma. *Assume* AD. *Then*

$$\forall A, B \subseteq \mathbb{R}\,(A \leq_{\mathrm{L}} B \vee B \leq_{\mathrm{L}} \neg A)\,.$$

The gist of the result is that any two sets of reals are almost comparable, and that $\leq_{\mathrm{L}}$ is almost a linear order. Wadge dubbed this as the **Semi Linear Ordering** principle for **Lipschitz** reductions. As every Lipschitz reduction is, in particular, a Wadge reduction, Wadge's Lemma yields trivially the Semi Linear Ordering principle for **continuous** reductions:

$$\forall A, B \subseteq \mathbb{R}\,(A \leq_{\mathrm{W}} B \vee B \leq_{\mathrm{W}} \neg A)\,.$$

Since it turns out that these two versions are equivalent (assuming $\mathsf{DC}(\mathbb{R})$ and that all sets have the property of Baire—see [And03, And06]) we shall denote either version with SLO.

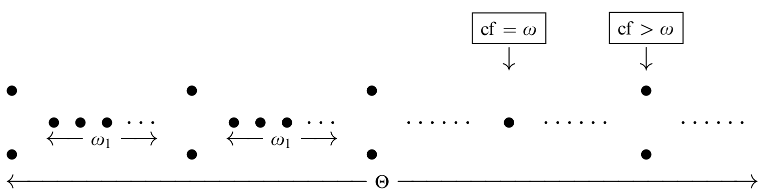
The **quasi-order**[2] $\leq_{\mathrm{W}}$ induces an equivalence relation on $\wp(\mathbb{R})$ whose equivalence classes are called **Wadge degrees**. The collection of all Wadge degrees together with the induced order is called the **Wadge hierarchy**. A set of reals A or its Wadge degree $[A]_{\mathrm{W}}$ is said to be self-dual if it $A \leq_{\mathrm{W}} \neg A$; otherwise it is said to be non-self-dual. Wadge's Lemma implies that a self-dual degree is comparable to any other degree, and that if two degrees are incomparable, then they must be dual to each other. In other words: all antichains have size at most 2. Martin—building on previous work of Monk—showed in 1973 that AD implies that the ordering $\leq$ is well-founded. In fact these results hold *verbatim* for the **Lipschitz hierarchy**, *i.e.*, the collection of degrees $[A]_{\mathrm{L}}$ obtained using Lipschitz reductions. By a result due independently to Steel [Ste77] and Van Wesep [Van77], AD implies that

$$A \leq_{\mathrm{W}} \neg A \iff A \leq_{\mathrm{L}} \neg A \tag{1}$$

and using this it is possible to completely determine the structure of the Lipschitz hierarchy: at the bottom of the hierarchy we have the non-self-dual pair $\{\varnothing\} = [\varnothing]_{\mathrm{L}}$ and $\{\mathbb{R}\} = [\mathbb{R}]_{\mathrm{L}}$, followed by an ω_1 chain of self-dual degrees formed by all clopen sets different from $\varnothing$ and $\mathbb{R}$. Above these there is the non-self-dual pair $\boldsymbol{\Sigma}^0_1 \setminus \boldsymbol{\Delta}^0_1$ and $\boldsymbol{\Pi}^0_1 \setminus \boldsymbol{\Delta}^0_1$ followed by an ω_1 chain of self-dual degrees. In general: at limit levels of uncountable cofinality we have a non-self-dual pair, while at all other levels we have a self-dual degree. The length of this hierarchy [Sol78B] is

$$\Theta \stackrel{\text{def}}{=} \sup\{\alpha : \exists f\ (f : \mathbb{R} \twoheadrightarrow \alpha)\}.$$

Thus the Lipschitz hierarchy looks like this:

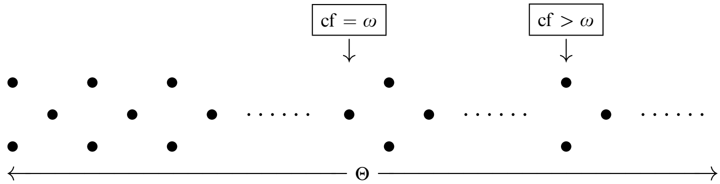


(2)

Each block of ω_1 consecutive self-dual Lipschitz degree is contained inside a single (necessarily self-dual) Wadge degree, and by the result of Steel and Van Wesep (1) nothing else is, so in the Wadge hierarchy self-dual degrees and non-self-dual pairs alternate, with the former appearing at levels of countable

[2] A quasi-order is a reflexive and transitive relation, and it is also known in the literature as a **pre-order**.

cofinality, and the latter appearing at the remaining limit levels:



(3)

The Wadge hierarchy is the ultimate analysis of $\wp(\mathbb{R})$ in terms of topological complexity, assigning to each set $A \subseteq \mathbb{R}$ an ordinal $\|A\|_{\mathrm{W}}$, the rank of $[A]_{\mathrm{W}}$ in the hierarchy. This is somewhat surprising, since AD forbids the existence of long transfinite sequences of reals. It is not hard to check that non-self-dual pointclasses are of the form

$$\{B \subseteq \mathbb{R} \,:\, B \leq_{\mathrm{W}} A\}$$

for some non-self-dual set A, while self-dual pointclasses are all of the form

$$\{B \subseteq \mathbb{R} \,:\, B <_{\mathrm{W}} A\}$$

for some arbitrary $A \neq \mathbb{R}, \varnothing$. (Here and below, $B <_{\mathrm{W}} A$ has the obvious meaning: $B \leq_{\mathrm{W}} A$ and $A \nleq_{\mathrm{W}} B$.) Thus Wadge's Lemma yields a semi-linear ordering principle for pointclasses: for any $\boldsymbol{\Gamma}$ and $\boldsymbol{\Lambda}$,

$$\boldsymbol{\Gamma} \subseteq \boldsymbol{\Lambda} \vee \breve{\boldsymbol{\Lambda}} \subseteq \boldsymbol{\Gamma}.$$

It is a classical fact that any pointclass $\boldsymbol{\Gamma}$ of the form $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$, $\boldsymbol{\Sigma}^1_n$, or $\boldsymbol{\Pi}^1_n$ has a **universal set**, *i.e.*, a set $U \subseteq \mathbb{R} \times \mathbb{R}$ that belongs to $\boldsymbol{\Gamma}$ (once it is coded as a subset of $\mathbb{R}$ via some canonical homeomorphism) and such that

$$\boldsymbol{\Gamma} = \{U_{(x)} \,:\, x \in \mathbb{R}\}$$

where $U_{(x)} = \{y \in \mathbb{R} \,:\, (x, y) \in U\}$ is the vertical section of U through x. This fact generalizes to *all* non-self-dual boldface pointclasses.

To see this, fix some canonical enumeration $\langle \boldsymbol{\ell}_x \,:\, x \in \mathbb{R}\rangle$ of all Lipschitz maps $\mathbb{R} \to \mathbb{R}$ with the further property that $(x, y) \mapsto \boldsymbol{\ell}_x(y)$ is continuous, and let

$$U = \{(x, y) \,:\, \boldsymbol{\ell}_x(y) \in A\},$$

where A is any set in $\boldsymbol{\Gamma} \setminus \breve{\boldsymbol{\Gamma}}$. Then U is in $\boldsymbol{\Gamma}$, and since

$$\begin{aligned} B \in \boldsymbol{\Gamma} &\iff B \leq_{\mathrm{W}} A \\ &\iff B \leq_{\mathrm{L}} A \qquad \text{(by (1))} \end{aligned}$$

we obtain that U is universal for $\boldsymbol{\Gamma}$.

Another property that generalizes under AD to arbitrary pointclasses is the following: a non-self-dual pointclass $\boldsymbol{\Gamma}$ is said to have the **separation property**, in symbols $\mathsf{Sep}(\boldsymbol{\Gamma})$ if for any pair of disjoint sets $A, B \in \boldsymbol{\Gamma}$ there is a set $C \in \boldsymbol{\Delta}_{\boldsymbol{\Gamma}} \stackrel{\text{def}}{=} \boldsymbol{\Gamma} \cap \breve{\boldsymbol{\Gamma}}$ that separates A from B, that is $A \subseteq C$ and $C \cap B = \varnothing$. By work of Sierpiński $\underset{\sim}{\Pi}^0_\alpha$ has the separation property, while $\underset{\sim}{\Sigma}^0_\alpha$ does not; assuming PD Moschovakis showed that $\underset{\sim}{\Sigma}^1_{2n+1}$ and $\underset{\sim}{\Pi}^1_{2n}$ have the separation, while neither $\underset{\sim}{\Sigma}^1_{2n}$ nor $\underset{\sim}{\Pi}^1_{2n+1}$ has it. (For $\underset{\sim}{\Sigma}^1_1$ this is the classical result of Suslin, and does not require PD.) Assuming AD, given a pair of non-self-dual pointclasses $\underset{\sim}{\Gamma}$ and $\breve{\underset{\sim}{\Gamma}}$, *at most one* of them has the separation property [Van78A], and *at least one* of them has the separation property [Ste81B], hence *exactly one* of them has the separation property.

The pointclasses $\underset{\sim}{\Sigma}^0_\alpha$ can be detected inside the Wadge hierarchy by means of the rank of their complete sets. Starting from the very bottom, $\mathbb{R}$ and $\varnothing$ have least possible rank, which for technical reasons is set to be equal to 1, then the clopen set have ranks 2, and thus sets in $\underset{\sim}{\Sigma}^0_1 \setminus \underset{\sim}{\Delta}^0_1$ have rank 3, From this point on the $\underset{\sim}{\Sigma}^0_\alpha$ are more and more spread apart. For example complete $\underset{\sim}{\Sigma}^0_2$ sets have Wadge rank ω_1, complete $\underset{\sim}{\Sigma}^0_3$ sets have Wadge rank $\omega_1^{\omega_1}$, and, in general, complete $\underset{\sim}{\Sigma}^0_{n+1}$ sets have Wadge rank ϑ_n where $\vartheta_1 = \omega_1$ and $\vartheta_{k+1} = \omega_1^{\vartheta_k}$. The rank ϑ_ω of a complete $\underset{\sim}{\Sigma}^0_\omega$ set is *not* the sup of the ϑ_ns, *i.e.*, the first fixed point of the map

$$\begin{aligned} E\colon \mathrm{Ord} &\to \mathrm{Ord} \\ \xi &\mapsto \omega_1^\xi, \end{aligned} \tag{4}$$

since this ordinal has countable cofinality, and hence it is the rank of a self-dual set. It turns out that ϑ_ω is the ω_1-st fixed point of the map E. The computation of the ranks of $\underset{\sim}{\Sigma}^0_\alpha$ with $\alpha \geq \omega$ is quite technical—see [Wad11] for a summary of the results and [Wad84, Chapter V] for complete proofs. For example, the length Ξ of the Wadge hierarchy of the Borel sets, or, equivalently, the rank of a complete $\underset{\sim}{\Sigma}^1_1$ or $\underset{\sim}{\Pi}^1_1$ set, is computed as follows: for any cub class $C \subseteq \mathrm{Ord}$ let

$$C' = \{\xi : \xi = F_C(\xi)\}$$

be the set of fixed points of F_C, where $F_C : \mathrm{Ord} \to C$ is the enumerating function, and consider the sequence of cub classes $C = C^{(0)} \supset C^{(1)} \supset C^{(2)} \supset \dots$ given by $C^{(\alpha+1)} = \left(C^{(\alpha)}\right)'$ and $C^{(\lambda)} = \bigcap_{\alpha<\lambda} C^{(\alpha)}$ when λ is limit. Then Ξ is the least element of $C^{(\omega_1)}$ where C is taken to be the class of fixed points of the map E defined in (4). Thus the length of the Wadge degrees of Borel sets is an ordinal of cofinality ω_1 strictly smaller than ω_2. This is not just an happenstance, since under AD the length of the hierarchy of $\underset{\sim}{\Delta}^1_{2n+1}$ degrees is $< \underset{\sim}{\delta}^1_{2n+2}$. On the other hand, by a theorem due independently to Martin and Steel, the length of the hierarchy of $\underset{\sim}{\Delta}^1_{2n}$ degrees is equal to $\underset{\sim}{\delta}^1_{2n+1}$.

§3. The papers in the volume.

Early investigations of the degrees of Borel sets by W. W. Wadge.

This paper is an overview of the results of the author's Ph.D. dissertation [Wad84] and gives a glimpse on how this area of Descriptive Set Theory was uncovered. Although it contains no proofs, this article gives a quick introduction to the techniques ((α, β)-homeomorphisms, $\underset{\sim}{\mathbf{\Sigma}}^0_{1+\mu}$-separated unions, etc.) used to give a complete analysis of the Wadge degrees of the Borel sets, and a computation of its length Ξ.

Wadge degrees and descriptive set theory by R. Van Wesep, and
A note on Wadge degrees by A. S. Kechris.

Van Wesep's paper provides a good introduction to the subject, with complete (albeit terse) proofs, surveying what was known at that time (1978). The reader will find the proof of several of the results stated in the preceding section, including Martin's proof of the well-foundedness of $\leq_{\mathrm{L}}$, the result by Steel and Van Wesep on self-dual degrees stated in (1), and the proof under AD that the hierarchy of $\underset{\sim}{\mathbf{\Delta}}^1_{2n}$ degrees has length $\underset{\sim}{\delta}^1_{2n+1}$. As this last fact is a result on projective sets, it is natural to ask for a proof assuming only PD. Such a proof is given in Kechris' paper, where $\mathsf{Det}(\underset{\sim}{\mathbf{\Delta}}^1_{2n})$ is shown to suffice.

The rest of Van Wesep's paper is devoted to the reduction and prewellordering properties. Recall that $\underset{\sim}{\mathbf{\Gamma}}$ is said to have the **reduction property**, in symbols $\mathsf{Red}(\underset{\sim}{\mathbf{\Gamma}})$, if given any two sets $A, B \in \underset{\sim}{\mathbf{\Gamma}}$ there are disjoint sets $A', B' \in \underset{\sim}{\mathbf{\Gamma}}$ such that $A' \subseteq A$, $B' \subseteq B$, and $A' \cup B' = A \cup B$; The prewellordering property $\mathsf{PWO}(\underset{\sim}{\mathbf{\Gamma}})$ means that every set in $\underset{\sim}{\mathbf{\Gamma}}$ admits a $\underset{\sim}{\mathbf{\Gamma}}$ norm [KM78B]. For any non-selfdual pointclass, $\mathsf{Red}(\underset{\sim}{\mathbf{\Gamma}}) \Longrightarrow \mathsf{Sep}(\check{\underset{\sim}{\mathbf{\Gamma}}})$ and if moreover $\underset{\sim}{\mathbf{\Gamma}}$ is closed under finite unions and intersections, then $\mathsf{PWO}(\underset{\sim}{\mathbf{\Gamma}}) \Longrightarrow \mathsf{Red}(\underset{\sim}{\mathbf{\Gamma}})$ [KM78B, Theorem 2.1]. Every $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ has the prewellordering and hence the reduction property, and Louveau and Saint-Raymond have shown that for Borel pointclasses, the reduction property and prewellordering properties are equivalent, and have given a complete description of which Borel pointclasses possess this property—see [LSR88A]. For the sake of brevity, we say that a non-self-dual pair of pointclasses $(\underset{\sim}{\mathbf{\Gamma}}, \check{\underset{\sim}{\mathbf{\Gamma}}})$ satisfies the prewellordering property if either $\mathsf{PWO}(\underset{\sim}{\mathbf{\Gamma}})$ or else $\mathsf{PWO}(\check{\underset{\sim}{\mathbf{\Gamma}}})$, and we follow a similar convention for the reduction property. In Van Wesep's paper it is shown that there are non-self-dual pairs $(\underset{\sim}{\mathbf{\Gamma}}, \check{\underset{\sim}{\mathbf{\Gamma}}})$ that fail to have the reduction property, and since (under AD, which will be tacitly assumed from now on) the separation property holds at every level of the Wadge hierarchy, this shows that the separation property is weaker that the reduction property. Determining which non-self-dual pairs $(\underset{\sim}{\mathbf{\Gamma}}, \check{\underset{\sim}{\mathbf{\Gamma}}})$ satisfy the reduction property is a non-trivial matter. In the paper under review it is shown that

If $\underset{\sim}{\mathbf{\Gamma}}$ is non-self-dual and closed under finite intersections then $\mathsf{Sep}(\check{\underset{\sim}{\mathbf{\Gamma}}}) \Longrightarrow \mathsf{Red}(\underset{\sim}{\mathbf{\Gamma}})$. (5)

(Notice that by the result mentioned below in (13), the hypothesis could be weakened to $\Delta_{\underset{\sim}{\mathbf{\Gamma}}}$.) A result of Steel is presented: If $\underset{\sim}{\mathbf{\Gamma}}$ is non-self-dual and closed under countable unions and intersections, then reduction holds for $(\boldsymbol{\Gamma}, \check{\boldsymbol{\Gamma}})$. This result was strengthened shortly afterwards by Steel himself in [Ste81B]:

> If $\boldsymbol{\Gamma}$ is non-self-dual and $\boldsymbol{\Delta}_{\boldsymbol{\Gamma}}$ is closed under finite unions and intersections, then the reduction property holds for $(\boldsymbol{\Gamma}, \check{\boldsymbol{\Gamma}})$. (6)

Finally the proof a theorem of Kechris and Solovay is given:

> Suppose $\boldsymbol{\Gamma} \subseteq \mathbf{L}(\mathbb{R})$ is non-self-dual and closed under countable unions and countable intersection. Suppose also $\exists^{\mathbb{R}}\boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}$ *and* $\forall^{\mathbb{R}}\boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}$. Then prewellordering holds for $(\boldsymbol{\Gamma}, \check{\boldsymbol{\Gamma}})$. (7)

The axiom of determinacy and the prewellordering property by A. S. Kechris, R. Solovay, and J. Steel.

This paper, as the title suggests, is devoted to the study of the prewellordering property under AD and, in a sense, it starts from where Van Wesep's paper ended. Firstly a criterion for PWO is established:

> Suppose $\boldsymbol{\Gamma}$ is non-self-dual, closed under countable unions and intersections, and *either* $\exists^{\mathbb{R}}\boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}$ *or else* $\forall^{\mathbb{R}}\boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}$. Then the prewellordering property holds for the non-self-dual pair $(\boldsymbol{\Gamma}, \check{\boldsymbol{\Gamma}})$ if and only if $\boldsymbol{\Delta}_{\boldsymbol{\Gamma}}$ is *not* closed under well-ordered unions. (8)

Recall that a pointclass $\boldsymbol{\Lambda}$ is closed under well-ordered unions if $\bigcup_{\alpha<\beta} A_\alpha \in \boldsymbol{\Lambda}$ for any sequence $\langle A_\alpha : \alpha < \beta \rangle$ of sets in $\boldsymbol{\Lambda}$. Note that if $A \in \boldsymbol{\Gamma} \setminus \check{\boldsymbol{\Gamma}}$ and $\varphi \colon A \twoheadrightarrow \kappa$ is a regular $\boldsymbol{\Gamma}$-norm, then each $A_\alpha = \{x \in A : \varphi(x) < \alpha\} \in \boldsymbol{\Delta}_{\boldsymbol{\Gamma}}$, but $A = \bigcup_{\alpha<\kappa} A_\alpha \notin \boldsymbol{\Delta}_{\boldsymbol{\Gamma}}$, so one of the two directions of the equivalence is immediate. The Theorem of Kechris and Solovay stated in (7) is thus extended to the case when $\boldsymbol{\Gamma}$ is closed under only one real quantifier:

> Suppose $\boldsymbol{\Gamma} \subseteq \mathbf{L}(\mathbb{R})$ is non-self-dual and closed under countable unions and countable intersection. Suppose also $\exists^{\mathbb{R}}\boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}$ *or* $\forall^{\mathbb{R}}\boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}$. Then prewellordering holds for $(\boldsymbol{\Gamma}, \check{\boldsymbol{\Gamma}})$. (9)

If $\boldsymbol{\Gamma}$ is $\boldsymbol{\Sigma}^1_n$ or $\boldsymbol{\Pi}^1_n$ then (9) says that exactly one among $\boldsymbol{\Gamma}$ and $\check{\boldsymbol{\Gamma}}$ has the prewellordering property—in fact by Moschovakis' First and Second Periodicity Theorems [KM78B] we can actually determine which of the two pointclasses has this property, namely $\mathsf{PWO}(\boldsymbol{\Gamma})$ iff $\boldsymbol{\Gamma} = \boldsymbol{\Pi}^1_{2n}$ or $\boldsymbol{\Gamma} = \boldsymbol{\Sigma}^1_{2n+1}$. The authors establish an analogous results for **projective-like pointclasses**, namely $\boldsymbol{\Gamma}$s which are contained in $\mathbf{L}(\mathbb{R})$, closed under countable unions and intersections, and closed under exactly one among $\exists^{\mathbb{R}}$ or $\forall^{\mathbb{R}}$. Any such pointclass can be taken to be the base of a hierarchy, obtained by taking complements and closure under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$, and if $\boldsymbol{\Gamma}$ itself is minimal, *i.e.*, it is not of the form $\exists^{\mathbb{R}}\boldsymbol{\Lambda}$ or $\forall^{\mathbb{R}}\boldsymbol{\Lambda}$ for some $\boldsymbol{\Lambda} \subset \boldsymbol{\Gamma}$, then the resulting hierarchy is maximal. Call such an object a **projective-like hierarchy**. The projective-like hierarchies are classified into four distinct types, and for each type the appropriate pattern for the prewellordering properties is established, first for the base level, and then for the higher levels by Moschovakis' periodicity. Since each projective-like pointclass is contained in a unique projective-like hierarchy, this yields a complete analysis of the prewellordering property for projective-like pointclasses.

Pointclasses and well-ordered unions by S. C. Jackson and D. A. Martin.

In this paper the general question of when a pointclass is closed under well-ordered unions is addressed. First a couple of easy facts are recalled: if $\mathsf{PWO}(\undertilde{\mathbf{\Gamma}})$ holds, then $\check{\undertilde{\mathbf{\Gamma}}}$ is not closed under well-ordered unions of length κ, where κ is the length of a $\undertilde{\mathbf{\Gamma}}$-norm; if moreover $\undertilde{\mathbf{\Gamma}}$ is closed under countable unions and intersections, and under $\exists^{\mathbb{R}}$, then $\undertilde{\mathbf{\Gamma}}$ is closed under well-ordered unions of length κ. Then Jackson and Martin prove under $\mathsf{AD}+\mathsf{DC}$ that

> Suppose $\undertilde{\mathbf{\Gamma}}$ is non-self-dual and closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$. Then either $\undertilde{\mathbf{\Gamma}}$ or $\check{\undertilde{\mathbf{\Gamma}}}$ is closed under well-ordered unions. (10)

Thus if $\undertilde{\mathbf{\Gamma}}$ is as above and moreover $\mathsf{PWO}(\undertilde{\mathbf{\Gamma}})$, then $\undertilde{\mathbf{\Gamma}}$ is closed under well-ordered unions. This last result complements Lemma 2.4.1 in the preceding paper by Kechris, Solovay, and Steel, which proves the same result[3] assuming that $\undertilde{\mathbf{\Gamma}}$ is closed under countable unions, countable intersections, under $\exists^{\mathbb{R}}$ but *not* under $\forall^{\mathbb{R}}$. Therefore

> If $\undertilde{\mathbf{\Gamma}}$ is non-self-dual and closed under countable intersections and $\exists^{\mathbb{R}}$, and $\mathsf{PWO}(\undertilde{\mathbf{\Gamma}})$ holds, then $\undertilde{\mathbf{\Gamma}}$ is closed under well-ordered unions. (11)

Clearly, for a pointclass $\undertilde{\mathbf{\Gamma}}$ to be closed under well-ordered unions is a meaningful property inasmuch there are well-ordered sequences of sets in $\undertilde{\mathbf{\Gamma}}$ to be considered. Moreover if $\langle A_\alpha : \alpha < \nu\rangle$ is a sequence of sets in such a $\undertilde{\mathbf{\Gamma}}$, then by replacing each A_α with $\bigcup_{\beta<\alpha} A_\beta$ and thinning out the sequence if needed, we may assume that the sets are strictly increasing. In this paper it is shown, assuming $\mathsf{AD}+\mathsf{DC}$, that

> If $\boldsymbol{S}(\kappa)$ has the scale property and $\operatorname{cf}(\kappa) > \omega$, then there is no strictly increasing sequence of sets in $\boldsymbol{S}(\kappa)$ of length κ^+, (12)

where $\boldsymbol{S}(\kappa)$ is the class of all κ-Suslin sets. The proof breaks down into two cases, depending whether κ is a successor or limit of uncountable cofinality.

The strength of Borel Wadge determinacy by A. Louveau and J. Saint-Raymond, and **Some results in the Wadge hierarchy of Borel sets** by A. Louveau.

Harrington proved in [Har78] that the semi-linear ordering principle restricted to the class of $\undertilde{\mathbf{\Pi}}^1_1$ sets, $\mathsf{SLO}(\undertilde{\mathbf{\Pi}}^1_1)$ for short, implies the existence of $x^{\#}$, for any

[3] Actually in that paper the assumption $\mathsf{PWO}(\undertilde{\mathbf{\Gamma}})$ is replaced by the weaker $\mathsf{Red}(\undertilde{\mathbf{\Gamma}})$.

real x, and therefore it implies $\mathsf{Det}(\underset{\sim}{\mathbf{\Pi}}^1_1)$. By work of Harrington and Martin $\mathsf{Det}(\underset{\sim}{\mathbf{\Pi}}^1_1)$ is equivalent to the determinacy of Boolean combinations of $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets, hence it follows that $\mathsf{SLO}(\underset{\sim}{\mathbf{\Pi}}^1_1)$, the determinacy of all $G_{\mathrm{W}}(A, B)$ and $G_{\mathrm{L}}(A, B)$, with $A, B \in \underset{\sim}{\mathbf{\Pi}}^1_1$, and $\mathsf{Det}(\underset{\sim}{\mathbf{\Pi}}^1_1)$, are all equivalent. In fact the determinacy of all Wadge games G_{W}, the determinacy of all Lipschitz games G_{L}, and SLO are all equivalent [And03, And06] and the same holds true when restricted to any pointclass with sufficient closure properties, such as the $\underset{\sim}{\mathbf{\Pi}}^1_n$'s; for these reason we shall refer to any one of these hypotheses as **Wadge determinacy**. By [Har78] and [Ste80], $\mathsf{Det}(\underset{\sim}{\mathbf{\Pi}}^1_1)$ is also equivalent to the following:

$$\forall A, B \in \underset{\sim}{\mathbf{\Pi}}^1_1 \setminus \underset{\sim}{\mathbf{\Delta}}^1_1 \, \exists f \, \left(f : \mathbb{R} \to \mathbb{R} \text{ is a Borel isomorphism and } f(A) = B \right).$$

All these results lent some credibility to the conjecture that a similar pattern should occur in the Borel context, namely that Wadge determinacy for Borel sets should imply Borel determinacy, which by work of Martin [Mar75] holds in ZFC and by work of Friedman [Fri71B] is not provable in second order arithmetic. But it is not so, as proved in the first paper by Louveau and Saint-Raymond: Wadge determinacy is provable in second order arithmetic. The proof relies heavily on Wadge's analysis of the Borel classes, together with a "ramification" technique which appeared in [LSR87] for the Borel classes: one associates to each non-self-dual Borel Wadge degree, as described by Wadge, a specific game, which is somewhat of an unfolding of a Wadge game. Its determinacy implies that any set which is of this degree is strategically complete, *i.e.*, player player II wins with it the Wadge game against any other set in the class.

The second paper [Lou83] is a bit different from the other papers, as it deals with the "lightface" aspects of the Wadge hierarchy. In a previous paper [Lou80], Louveau had proved that for hyperarithmetic sets, the Borel class can be witnessed hyperarithmetically. A similar feature is proved in the paper for each Borel class in the Wadge hierarchy of Borel sets. But a great deal of work is done on introducing operations in order to build all Borel Wadge classes, and define appropriate codings of both the classes and the sets in them so that the corresponding lightface statement makes sense. It was also the first—and for quite some time the only—place where a printed account of some of Wadge's work could be found.

Closure properties of pointclasses by J. Steel, and
More closure properties of pointclasses by H. Becker.

In several of the results mentioned in the paragraphs above, in order to prove that a pointclass $\underset{\sim}{\mathbf{\Gamma}}$ has some structural property, like reduction or prewellordering, we must require that $\underset{\sim}{\mathbf{\Gamma}}$ (or perhaps $\Delta_{\underset{\sim}{\mathbf{\Gamma}}}$) be closed under some simpler structural property, like closure under finite (or countable) unions or intersections. Notice that closure under finite union or intersections is never a problem with the Baire or projective classes, but in the realm or *arbitrary* pointclasses, closure under finite unions or intersections is a non-trivial matter. One might ask, for example: Under which assumptions on $\underset{\sim}{\mathbf{\Gamma}}$ does closure under finite unions imply closure for countable unions? Do closure properties of $\Delta_{\underset{\sim}{\mathbf{\Gamma}}}$ imply analogous properties for $\underset{\sim}{\mathbf{\Gamma}}$ or $\check{\underset{\sim}{\mathbf{\Gamma}}}$? The paper by Steel proves several theorems under AD that address these questions. Here is just a sample of such results:

$$\text{If } \Delta_{\underset{\sim}{\mathbf{\Gamma}}} \text{ is closed under finite (or countable) unions and } \mathsf{Sep}(\underset{\sim}{\mathbf{\Gamma}}) \text{ holds, then } \underset{\sim}{\mathbf{\Gamma}} \text{ is closed under finite (or countable) unions too.} \tag{13}$$

$$\text{If } \underset{\sim}{\mathbf{\Gamma}} \text{ is closed under finite unions and } \mathsf{Sep}(\check{\underset{\sim}{\mathbf{\Gamma}}}) \text{, then } \underset{\sim}{\mathbf{\Gamma}} \text{ is closed under countable unions.} \tag{14}$$

$$\text{Suppose } \underset{\sim}{\mathbf{\Gamma}} \text{ is closed under finite intersections and countable unions, but } \textit{not} \text{ under countable intersections. Then } \mathsf{PWO}(\underset{\sim}{\mathbf{\Gamma}}). \tag{15}$$

Thus (14) and (15) generalize a well-known fact about the Borel hierarchy, that is: $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ does not have the separation property but has the prewellordering property. Steel's paper contains also an interesting conjecture. Recall that Suslin's operation $\mathscr{A}$ is a Boolean operation with $\underset{\sim}{\mathbf{\Sigma}}^1_1$ truth table, and that the Boolean operations that generate the $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$s are just compositions of the operations of countable unions and countable intersections.

CONJECTURE 3.1. Assume AD and suppose $\underset{\sim}{\mathbf{\Gamma}}$ is non-self-dual and closed under both countable intersections and countable unions. Then either $\underset{\sim}{\mathbf{\Gamma}}$ or $\check{\underset{\sim}{\mathbf{\Gamma}}}$ is closed under $\mathscr{A}$.

Becker's paper deals with closure under measure and category quantifiers. If $A \subseteq {}^\omega 2 \times {}^\omega 2$ then let

$$\forall^* y\, A = \{x : A_{(x)} \text{ is comeager}\}$$

and

$$\forall^\mu y\, A = \{x : \mu({}^\omega 2 \setminus A_{(x)}) = 0\}$$

where μ is the Lebesgue measure on ${}^\omega 2$ and $A_{(x)} = \{y : (x, y) \in A\}$ is the vertical section of A through x. In other words, $\forall^* y\, A$ is the set of all x such that $(x, y) \in A$ for comeager many y, while $\forall^\mu y\, A$ is the set of all x such that $(x, y) \in A$ for μ-almost every y; their dual quantifiers are defined by

$$\exists^* y\, A = \{x : A_{(x)} \text{ is non-meager}\}$$

and

$$\exists^\mu y\, A = \{x : \mu(A_{(x)}) > 0\}.$$

The measure and category quantifiers are very useful in many parts of Descriptive Set Theory—see for example [BK96]. In the present paper it is shown that if $\underset{\sim}{\boldsymbol{\Gamma}}$ is nonselfdual and closed under countable unions and countable intersections, then it is closed under the category and measure quantifiers. In particular, $\underset{\sim}{\boldsymbol{\Delta}}^1_1$ is closed under measure and category quantifiers.

More measures from AD by J. Steel. One of the early consequences of determinacy is Martin's result that ω_1 has the strong partition property, $\omega_1 \to (\omega_1)^{\omega_1}$. This in turns implies Solovay's result that ω_1 is measurable. In the following years the study of the strong partition property for cardinals $< \Theta$ became one of the main research topics of the Cabal Seminar. The construction of the normal measure from the strong partition property is usually achieved via the Boundedness Lemma together with an appropriate coding of elements of ${}^{\kappa}\kappa$.

In the present paper it is shown that, assuming AD, for every regular $\kappa < \Theta$ there is a measure on ${}^{\kappa}\kappa$. The main technical twist is the use of the Recursion Theorem instead of the Boundedness Lemma.

§4. **Recent developments.** In this last section we will try to survey some of the development that occurred after the papers in this volume were originally written.

4.1. SLO **and weaker reducibilities.** As we already mentioned, Harrington proved in [Har78] that $\mathsf{SLO}(\underset{\sim}{\boldsymbol{\Pi}}^1_1)$ is equivalent to the determinacy of all $\underset{\sim}{\boldsymbol{\Pi}}^1_1$ games. This was extended by Hjorth [Hjo96] to the next level, *i.e.*, $\mathsf{SLO}(\underset{\sim}{\boldsymbol{\Pi}}^1_2)$ implies $\underset{\sim}{\boldsymbol{\Pi}}^1_2$-determinacy—generalizations of these results to all projective levels, and beyond, have been an elusive goal, as they seem to depend on further technical advancement of core model theory. Yet the results we have now seem to lend some evidence to the following conjecture, probably due to Solovay:

Conjecture 4.1. Assume $\mathbf{V} = \mathbf{L}(\mathbb{R})$. Then

$$\mathsf{SLO} \Longrightarrow \mathsf{AD}.$$

Note that there is no obvious natural way to reduce a general perfect information, zero-sum game on ω into a Wadge game, so the proof—if the conjecture is true—will probably be quite indirect. Although progress on this conjecture has been essentially nil after [Hjo96], the Semi-Linear Ordering principle and some generalizations of it have been investigated in recent years. In [And03] it is shown that SLO is strong enough to prove the basic structural results on the Wadge hierarchy as embodied in diagram (3), and in [AM03], the analogue of the Wadge hierarchy using Borel functions was introduced: for any $A, B \subseteq \mathbb{R}$ let

$$A \leq_{\underset{\sim}{\boldsymbol{\Delta}}^1_1} B \Longleftrightarrow \exists f \left(f : \mathbb{R} \to \mathbb{R} \text{ is Borel and } f^{-1}(B) = A \right).$$

The induced equivalence relation yields the notion of $\underset{\sim}{\boldsymbol{\Delta}}^1_1$ degree, and it turns out that the their structure is similar to the one of Wadge degrees, *i.e.*, it is well-founded, the self-dual degrees and non-self-dual pairs of degrees alternate, with self-dual degrees occupying the limit levels of countable cofinality, and since the length of this hierarchy is Θ, then its picture is just (3). Since all uncountable Polish spaces are Borel isomorphic, this hierarchy is independent of the underlying space, a feature sorely missing from the Wadge hierarchy. In this case there are no analogues of the games G_{W} or G_{L}, and the proofs use the principle $\mathsf{SLO}^{\undertilde{\Delta}^1_1}$, the analogue of SLO for Borel reductions,

$$\forall A, B \subseteq \mathbb{R}\big(A \leq_{\undertilde{\Delta}^1_1} B \vee \neg B \leq_{\undertilde{\Delta}^1_1} A\big). \qquad (\mathsf{SLO}^{\undertilde{\Delta}^1_1})$$

Note that $\mathsf{SLO}^{\undertilde{\Delta}^1_1}$ follows from SLO, hence from AD, and in [AM03] it is conjectured that $\mathsf{SLO}^{\undertilde{\Delta}^1_1} \implies \mathsf{SLO}$. In [And06] a similar analysis is carried out for the $\undertilde{\Delta}^0_2$ reducibility: again $\mathsf{SLO}^{\undertilde{\Delta}^0_2}$ is able to civilize this hierarchy and the familiar structure (3) is obtained, and moreover in this case it is shown that $\mathsf{SLO}^{\undertilde{\Delta}^0_2} \iff \mathsf{SLO}$. (A function is said to be $\undertilde{\Delta}^0_\alpha$ if the preimage of a $\undertilde{\Sigma}^0_\alpha$ is $\undertilde{\Sigma}^0_\alpha$.)

The results above seem to indicate that similar results should hold true of $\leq_{\mathcal{F}}$ reductions, *i.e.*,

$$A \leq_{\mathcal{F}} B \iff \exists f \in \mathcal{F}\left(A = f^{-1}(B)\right)$$

where $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$. Obviously the class $\mathcal{F}$ must satisfy some assumptions in order for us to obtain non trivial results, *e.g.*, $\mathcal{F}$ must be closed under composition, and must contain the identity, so that $\leq_{\mathcal{F}}$ is a quasi-order, $\mathcal{F} \neq {}^{\mathbb{R}}\mathbb{R}$, *etc.* Motto Ros in [MR07] has isolated a very general class of $\mathcal{F}$ as above, with $\mathcal{F}$ a collection of Borel functions, and has shown, assuming $\mathsf{AD}+\mathsf{DC}(\mathbb{R})$ that the structure of the $\mathcal{F}$-hierarchy can be either of Wadge-type or of Lipschitz type, *i.e.*, the ordering of the $\mathcal{F}$-degrees is as in (3) or as in (2). For example: when $\mathcal{F}$ is the collection of all $\undertilde{\Delta}^0_\xi$ functions, the resulting hierarchy is of Wadge type; when $\mathcal{F}$ is the collection of all $\undertilde{\Delta}^0_\alpha$ functions for some $\alpha < \lambda$, the resulting hierarchy is of Lipschitz type.

4.2. Connections with bqo theory. By Martin's result, Wadge reducibility $\leq_{\mathrm{W}}$ is one of a few examples of "natural" quasi-orderings which are well-quasi-orderings (wqo's), *i.e.*, which admit neither infinite antichains, nor infinite strictly decreasing sequences. Other famous examples are the countable linear orders with embeddability (Laver [Lav71]) and the finite graphs with the minor ordering (Robertson and Seymour [RS04]).

As the class of wqo's lacks nice closure properties, it is usual to consider the stronger notion of a better-quasi-ordering (bqo): A quasi-ordering $(Z, \leq_Z)$ is a bqo if, for any continuous (or equivalently Borel) map $h \colon [\omega]^\omega \to Z$ there is an $X \in [\omega]^\omega$ with $h(X) \leq h(X \setminus \{\min X\})$, where $[\omega]^\omega$ is the collection of all infinite subsets of ω identified with the set of all increasing elements of the Baire space, and Z is taken with the discrete topology. (For a nice introduction to bqo theory, see Simpson's contribution in [MW85, Chapter 9].)

It is not hard to check that under AD the quasi-orders $\leq_{\mathrm{L}}$ and $\leq_{\mathrm{W}}$ are indeed bqo's. But one can get by similar techniques other bqo results. For example, van Engelen, Miller, and Steel prove in [vEMS87] that if $(Z, \leq_Z)$ is a bqo and one orders $\mathcal{S}_Z$, the set of all functions $h\colon {}^{\omega}\omega \to Z$, by

$$h_1 \preceq h_2 \iff \exists \varphi\colon {}^{\omega}\omega \to {}^{\omega}\omega \text{ Lipschitz such that } \forall x \in {}^{\omega}\omega \left(h_1(x) \leq_Z h_2(\varphi(z))\right),$$

then $\mathcal{S}_Z$ is bqo too. ($\leq_{\mathrm{L}}$ corresponds to the case $Z = \{0, 1\}$, with 0 and 1 incomparable.)

This result in turn is used to prove other bqo results, in particular in [LSR90], where Louveau and Saint-Raymond extend Laver's result about countable linear orders to Borel (or projective) linear orders embeddable in $({}^{\omega}\mathbb{R}, \leq_{\mathrm{lex}})$, using AD.

4.3. Reducibility in higher dimension. An alternative way of looking at the Wadge hierarchy is to view subsets A of $\mathbb{R}$ as structures $(\mathbb{R}, A)$ in a language with a unary predicate, with the Wadge ordering being continuous homomorphisms between such structures. This of course opens the possibility of extending it to more complicated structures with domain $\mathbb{R}$ (or arbitrary Polish spaces) and, say, a n-ary relation on it. Concretely, in order to allow arbitrary Polish spaces as domains and still avoid purely topological difficulties, one prefers to consider Borel reductions rather than continuous in this context. So for $\mathcal{X}$, $\mathcal{Y}$ Polish spaces and $A \subseteq \mathcal{X}^n$, $B \subseteq \mathcal{Y}^n$, let

$$(\mathcal{X}, A) \leq_{\boldsymbol{B}} (\mathcal{Y}, B)$$

just in case

$$\exists f : \mathcal{X} \to \mathcal{Y} \text{ Borel, and } \forall \bar{a} \in \mathcal{X}^n \left(\bar{a} \in A \iff f(\bar{a}) \in B\right)$$

(where we follow the convention from model-theory and write $\bar{a}$ for the n-tuple $(a_1, \ldots, a_n)$ and $f(\bar{a})$ for $(f(a_1), \ldots, f(a_n))$; also, when the ambient spaces $\mathcal{X}$ and $\mathcal{Y}$ are understood, we simply write $A \leq_{\boldsymbol{B}} B$).

These considerations provide a natural descriptive complexity for relations. This notion was first introduced by Friedman and Stanley in [FS89], who used it to provide a classification for first order theories, by comparing their associated space of countable models with domain ω, endowed with isomorphism. It was extended soon after by Kechris and Louveau [Kec92, Lou92] to equivalence relations and even more complicated structures. It should be noted that many properties, like being an equivalence relation, a quasi-ordering, ... are downward preserved under $\leq_{\boldsymbol{B}}$, so that the subject breaks naturally into many sub-areas. And in each sub-area there is no satisfying alternative approach to descriptive complexity by using operations instead of reducibility, as in the one-dimensional case. This is because equivalence relations, for example, are not built from simpler equivalence relations, in general. And the only ways that have been proposed, like Louveau's notion of "potential Wadge class" (see [Lou94]), may be useful but are too coarse (*i.e.*, too close to the one-dimensional situation) to provide the right notion of descriptive complexity.

A lot of work has been done on linear orders, quasi-orders, even graphs, but the main part of the activity in Descriptive Set Theory over the last two decades has been to understand Polish spaces with Borel, or more generally analytic, equivalence relations. We won't try to give here an account of this theory, but refer the reader to the nice overview [HK01].

Let us just mention here that the situation for the higher dimensional theory is very different, and much more complicated that in dimension one: although some features of $\leq_{\boldsymbol{B}}$ are nice, it is a very complicated quasi-order, ill-founded and with large antichains. And games are of little use in the new situation, so that one cannot really work by analogy with $\leq_{\mathrm{W}}$.

4.3.1. *Definable cardinality.* In the context of AD and using arbitrary reductions rather than Borel ones, the classification results for equivalence relations become results on cardinality of quotients: if $\preccurlyeq$ denotes this coarser reducibility relation, any f witnessing $E \preccurlyeq F$ induces an injection $\hat{f} \colon \mathbb{R}/E \to \mathbb{R}/F$. Conversely, assuming $\mathsf{AD}_{\mathbb{R}}$, for any $g \colon \mathbb{R}/E \to \mathbb{R}/F$ we can uniformize the relation $\tilde{g} = \{(x, y) \in \mathbb{R}^2 : f([x]_E) = [y]_F\}$ by some $f \colon \mathbb{R} \to \mathbb{R}$: then f witnesses $E \preccurlyeq F$ and moreover $\hat{f} = g$. In other words, under AD the quasi-order $\preccurlyeq$ on equivalence relations yields an injection of the quotients, and under $\mathsf{AD}_{\mathbb{R}}$ any injection of the quotients lifts to a $\preccurlyeq$-reduction on $\mathbb{R}$. Many of the results on Borel or analytic equivalence relations using $\leq_{\boldsymbol{B}}$, can be recast under AD using $\preccurlyeq$ with essentially the same proof: for example the Silver [Sil80] and Harrington-Kechris-Louveau [HKL90] dichotomies become: If $E \in \boldsymbol{\Pi}^1_1$ is an equivalence relation on $\mathbb{R}$, then either $|\mathbb{R}/E| \leq \omega$, or else $|\mathbb{R}| \leq |\mathbb{R}/E|$; if $E \in \boldsymbol{\Delta}^1_1$ is an equivalence relation on $\mathbb{R}$, then either $|\mathbb{R}/E| \leq |\mathbb{R}|$ or else $|\wp(\omega)/\operatorname{Fin}| \leq |\mathbb{R}/E|$. But in fact these dichotomies of admit a more substantial generalization:

If E an arbitrary equivalence relation on $\mathbb{R}$, then either (16)
(a) $|\mathbb{R}/E| < \Theta$, or else
(b) $|\mathbb{R}| \leq |\mathbb{R}/E|$.

If $E \in$ an arbitrary equivalence relation on $\mathbb{R}$, then either (17)
(a) $|\mathbb{R}/E| \leq |\wp(\kappa)|$, for some $\kappa < \Theta$, or else
(b) $|\wp(\omega)/\operatorname{Fin}| \leq |\mathbb{R}/E|$.

Dichotomies (16) and (17) were first proved under $\mathsf{AD}_{\mathbb{R}}$ by Harrington and Sami [HS79], and Ditzen [Dit92], and, independently, by Foreman and Magidor (unpublished). The consistency strength was then reduced to AD + $\mathbf{V}{=}\mathbf{L}(\mathbb{R})$ by Woodin (unpublished), and Hjorth [Hjo95], respectively.

4.4. The Wadge hierarchy in set theory. Although most of the research on the notion of continuous pre-images is concerned with the general theory of pointclasses, the Wadge hierarchy has important applications in the study of models of AD^+, a generalization of AD defined by:

$$\left.\begin{array}{l} \bullet\ \text{every } A \subseteq \mathbb{R} \text{ is } \infty\text{-Borel, } i.e., A = \{y \in \mathbb{R} : \mathbf{L}_\alpha[S, y] \models \\ \quad \varphi[S, y]\} \text{ for some } S \subseteq \text{Ord and some formula } \varphi, \text{ and} \\ \bullet\ \text{for every } A \subseteq \mathbb{R}, \text{ every } \lambda < \Theta \text{ and every surjection } f : {}^\omega\lambda \twoheadrightarrow \\ \quad \mathbb{R}, \text{ the ordinal game on } \lambda \text{ with payoff } f^{-1}(A) \text{ is determined.} \end{array}\right. \qquad (\mathsf{AD}^+)$$

Clearly $\mathsf{AD}^+ \implies \mathsf{AD}$, and both $\mathsf{AD} + \mathbf{V}{=}\mathbf{L}(\mathbb{R})$ and $\mathsf{AD}_{\mathbb{R}}{+}\mathsf{DC}$ imply AD^+; in fact every known model of AD does satisfies AD^+, and the general consensus seems to be that AD^+ is the correct axiom for the study of models of determinacy. Assuming AD^+ let

$$\Theta(A) = \sup\{\|B\|_{\mathrm{W}} : B \text{ is ordinal definable from reals and } A\}$$

and let

$$\begin{aligned} \Theta_0 &= \Theta(\varnothing) \\ \Theta_{\alpha+1} &= \Theta(A) \text{ for some/any } A \text{ such that } \|A\|_{\mathrm{W}} = \Theta_\alpha \\ \Theta_\lambda &= \sup_{\alpha<\lambda} \Theta_\alpha. \end{aligned}$$

The sequence of the Θ_α's was introduced in [Sol78B] and it is called the Solovay sequence; note that it may not be defined for all α's. For example, if $\mathbf{V} = \mathbf{L}(\mathbb{R})$ then every set is ordinal definable from a real, hence $\Theta = \Theta_0$ and the Solovay sequence is not defined for larger indexes; assuming $\mathsf{AD}_{\mathbb{R}}$ will ensure that the sequence is defined up to some limit ordinal λ. In general, if Θ_α is defined then $\mathbf{L}(\wp_{\Theta_\alpha}(\mathbb{R}))$ is a model for $\mathsf{AD}^+ + \Theta{=}\Theta_\alpha$, where $\wp_\nu(\mathbb{R}) = \{X \subseteq \mathbb{R} : \|X\|_{\mathrm{W}} < \nu\}$. The smallest model of $\mathsf{AD}_{\mathbb{R}}$ is $\mathbf{L}(\wp_{\Theta_\omega}(\mathbb{R}))$, and in this model Θ has cofinality ω. Even stronger theories are obtained when the model satisfies '$\Theta = \Theta_\Theta$', or '$\Theta > \Theta_0$ is regular'—see [Woo99]. Thus the Wadge hierarchy and, in particular, the Solovay sequence of the Θ_α's can be used to measure the strength of models of AD^+. Unfortunately, this method of comparing AD^+ models is not always successful since Woodin, in unpublished work, has shown that it is consistent that there are two models M and N of AD^+ having the same reals and with divergent Wadge hierarchies.

Finally we mention a fairly recent application of the Wadge hierarchy to the study, under AD, of cardinalities of pointclasses. As any pointclass $\underset{\sim}{\boldsymbol{\Gamma}}$ is the surjective image of $\mathbb{R}$, *i.e.*, it is in bijection with $\mathbb{R}/E$ for some E, and as any $\mathbb{R}/E$ can be embedded into some $\underset{\sim}{\boldsymbol{\Gamma}}$, it follows that the cardinalities $|\underset{\sim}{\boldsymbol{\Gamma}}|$ are cofinal in the set of cardinalities of quotients of $\mathbb{R}$. The general problem is to determine which $\underset{\sim}{\boldsymbol{\Gamma}}$ are **cardinality pointclasses**, *i.e.*, such that $|\underset{\sim}{\boldsymbol{\Gamma}}| > |\underset{\sim}{\boldsymbol{\Delta}}|$, for any $\underset{\sim}{\boldsymbol{\Delta}} \subset \underset{\sim}{\boldsymbol{\Gamma}}$. Examples of self-dual cardinality pointclasses are $\underset{\sim}{\boldsymbol{\Delta}}^0_1$, or the pointclasses of the form $\bigcup_{\alpha<\lambda} \underset{\sim}{\boldsymbol{\Gamma}}_\alpha$ with $\underset{\sim}{\boldsymbol{\Gamma}}_\alpha$ increasing cardinality pointclasses and λ limit—call such a pointclass a **tower**, and say it has countable cofinality if $\mathrm{cf}(\lambda) = \omega$. In [AHN07] a complete description of the cardinality pointclasses is given, and an interesting feature of the proof is that it uses the detailed analysis of the Wadge hierarchy. Assuming $\mathsf{AD}+\mathsf{DC}(\mathbb{R})$, a non-self-dual $\boldsymbol{\Gamma}$ is a cardinality pointclass iff $\boldsymbol{\Gamma}$ is closed under pre-images of $\boldsymbol{\Delta}^0_2$ functions; a self-dual pointclass $\boldsymbol{\Delta}$ strictly larger than $\boldsymbol{\Delta}^0_1$ is a cardinality pointclass iff either it is a tower, or else it is the (necessarily self-dual) pointclass immediately above a tower of countable cofinality. Therefore assuming $\mathsf{AD}+\mathsf{DC}(\mathbb{R})$ the results of Hjorth [Hjo98, Hjo02]

$$\alpha < \beta \Longrightarrow |\boldsymbol{\Sigma}^0_\alpha| < |\boldsymbol{\Sigma}^0_\beta| \qquad \text{and} \qquad |\boldsymbol{\Delta}^1_n| < |\boldsymbol{\Sigma}^1_n| < |\boldsymbol{\Delta}^1_{n+1}|$$

are obtained as corollaries.

REFERENCES

ALESSANDRO ANDRETTA

[And03] *Equivalence between Wadge and Lipschitz determinacy*, ***Annals of Pure and Applied Logic***, vol. 123 (2003), no. 1–3, pp. 163–192.

[And06] *More on Wadge determinacy*, ***Annals of Pure and Applied Logic***, vol. 144 (2006), no. 1–3, pp. 2–32.

ALESSANDRO ANDRETTA, GREGORY HJORTH, AND ITAY NEEMAN

[AHN07] *Effective cardinals of boldface pointclasses*, ***Journal of Mathematical Logic***, vol. 7 (2007), no. 1, pp. 35–92.

ALESSANDRO ANDRETTA AND DONALD A. MARTIN

[AM03] *Borel-Wadge degrees*, ***Fundamenta Mathematicae***, vol. 177 (2003), no. 2, pp. 175–192.

HOWARD S. BECKER AND ALEXANDER S. KECHRIS

[BK96] ***The descriptive set theory of Polish group actions***, London Mathematical Society Lecture Note Series, vol. 232, Cambridge University Press, Cambridge, 1996.

ACHIM DITZEN

[Dit92] *Definable equivalence relations on Polish spaces*, ***Ph.D. thesis***, California Institute of Technology, 1992.

JACQUES DUPARC

[Dup01] *Wadge hierarchy and Veblen hierarchy. I. Borel sets of finite rank*, ***The Journal of Symbolic Logic***, vol. 66 (2001), no. 1, pp. 56–86.

[Dup03] *A hierarchy of deterministic context-free ω-languages*, ***Theoretical Computer Science***, vol. 290 (2003), no. 3, pp. 1253–1300.

JACQUES DUPARC, OLIVIER FINKEL, AND JEAN-PIERRE RESSAYRE

[DFR01] *Computer science and the fine structure of Borel sets*, ***Theoretical Computer Science***, vol. 257 (2001), no. 1–2, pp. 85–105.

HARVEY FRIEDMAN

[Fri71B] *Higher set theory and mathematical practice*, ***Annals of Mathematical Logic***, vol. 2 (1971), no. 3, pp. 325–357.

Harvey Friedman and Lee Stanley

[FS89] *A Borel reducibility theory for classes of countable structures*, ***The Journal of Symbolic Logic***, vol. 54 (1989), no. 3, pp. 894–914.

Leo A. Harrington

[Har78] *Analytic determinacy and* $0^{\#}$, ***The Journal of Symbolic Logic***, vol. 43 (1978), pp. 685–693.

Leo A. Harrington, Alexander S. Kechris, and Alain Louveau

[HKL90] *A Glimm–Effros dichotomy for Borel equivalence relations*, ***Journal of the American Mathematical Society***, vol. 3 (1990), pp. 902–928.

Leo A. Harrington and Ramez-Labib Sami

[HS79] *Equivalence relations, projective and beyond*, ***Logic Colloquium '78. Proceedings of the Colloquium held in Mons, August 24–September 1, 1978*** (Maurice Boffa, Dirk van Dalen, and Kenneth McAloon, editors), Studies in Logic and the Foundations of Mathematics, vol. 97, North-Holland, Amsterdam, 1979, pp. 247–264.

Gregory Hjorth

[Hjo95] *A dichotomy for the definable universe*, ***The Journal of Symbolic Logic***, vol. 60 (1995), no. 4, pp. 1199–1207.

[Hjo96] Π^1_2 *Wadge degrees*, ***Annals of Pure and Applied Logic***, vol. 77 (1996), no. 1, pp. 53–74.

[Hjo98] *An absoluteness principle for Borel sets*, ***The Journal of Symbolic Logic***, vol. 63 (1998), no. 2, pp. 663–693.

[Hjo02] *Cardinalities in the projective hierarchy*, ***The Journal of Symbolic Logic***, vol. 67 (2002), no. 4, pp. 1351–1372.

Gregory Hjorth and Alexander S. Kechris

[HK01] *Recent developments in the theory of Borel reducibility*, ***Fundamenta Mathematicae***, vol. 170 (2001), no. 1–2, pp. 21–52.

Alexander S. Kechris

[Kec92] *The structure of Borel equivalence relations in Polish spaces*, ***Set theory of the continuum. Papers from the workshop held in Berkeley, California, October 16–20, 1989*** (H. Judah, W. Just, and H. Woodin, editors), Mathematical Sciences Research Institute Publications, vol. 26, Springer, New York, 1992, pp. 89–102.

Alexander S. Kechris, Benedikt Löwe, and John R. Steel

[Cabal I] *Games, scales, and Suslin cardinals: the Cabal seminar, volume I*, Lecture Notes in Logic, vol. 31, Cambridge University Press, 2008.

Alexander S. Kechris, Donald A. Martin, and Yiannis N. Moschovakis

[Cabal iii] *Cabal seminar* 79–81, Lecture Notes in Mathematics, no. 1019, Berlin, Springer, 1983.

Alexander S. Kechris, Donald A. Martin, and John R. Steel

[Cabal iv] *Cabal seminar* 81–85, Lecture Notes in Mathematics, no. 1333, Berlin, Springer, 1988.

Alexander S. Kechris and Yiannis N. Moschovakis

[Cabal i] *Cabal seminar* 76–77, Lecture Notes in Mathematics, no. 689, Berlin, Springer, 1978.

[KM78B] *Notes on the theory of scales*, in *Cabal Seminar* 76–77 [Cabal i], pp. 1–53, reprinted in [Cabal I], p. 28–74.

Casimir Kuratowski

[Kur58] ***Topologie. Vol. I***, 4ème ed., Monografie Matematyczne, vol. 20, Państwowe Wydawnictwo Naukowe, Warsaw, 1958.

RICHARD LAVER

[Lav71] *On Fraïssé's order type conjecture*, ***Annals of Mathematics***, vol. 93 (1971), pp. 89–111.

ALAIN LOUVEAU

[Lou80] *A separation theorem for Σ^1_1 sets*, ***Transactions of the American Mathematical Society***, vol. 260 (1980), no. 2, pp. 363–378.

[Lou83] *Some results in the Wadge hierarchy of Borel sets*, this volume, originally published in Kechris et al. [CABAL iii], pp. 28–55.

[Lou92] *Classifying Borel structures*, ***Set Theory of the Continuum. Papers from the workshop held in Berkeley, California, October 16–20, 1989*** (H. Judah, W. Just, and H. Woodin, editors), Mathematical Sciences Research Institute Publications, vol. 26, Springer, New York, 1992, pp. 103–112.

[Lou94] *On the reducibility order between Borel equivalence relations*, ***Logic, Methodology and Philosophy of Science, IX. Proceedings of the Ninth International Congress held in Uppsala, August 7–14, 1991*** (Dag Prawitz, Brian Skyrms, and Dag Westerståhl, editors), Studies in Logic and the Foundations of Mathematics, vol. 134, North-Holland, Amsterdam, 1994.

ALAIN LOUVEAU AND JEAN SAINT-RAYMOND

[LSR87] *Borel classes and closed games: Wadge-type and Hurewicz-type results*, ***Transactions of the American Mathematical Society***, vol. 304 (1987), no. 2, pp. 431–467.

[LSR88A] *Les propriétés de réduction et de norme pour les classes de Boréliens*, ***Fundamenta Mathematicae***, vol. 131 (1988), no. 3, pp. 223–243.

[LSR88B] *The strength of Borel Wadge determinacy*, this volume, originally published in Kechris et al. [CABAL iv], pp. 1–30.

[LSR90] *On the quasi-ordering of Borel linear orders under embeddability*, ***The Journal of Symbolic Logic***, vol. 55 (1990), no. 2, pp. 537–560.

NIKOLAI LUZIN AND WACLAW SIERPIŃSKI

[LS29] *Sur les classes des constituantes d'un complémentaire analytique*, ***Comptes rendus hebdomadaires des séances de l'Académie des Sciences***, vol. 189 (1929), pp. 794–796.

RICHARD MANSFIELD AND GALEN WEITKAMP

[MW85] ***Recursive aspects of descriptive set theory***, Oxford Logic Guides, vol. 11, The Clarendon Press Oxford University Press, New York, 1985, With a chapter by Stephen Simpson.

DONALD A. MARTIN

[Mar75] *Borel determinacy*, ***Annals of Mathematics***, vol. 102 (1975), no. 2, pp. 363–371.

LUCA MOTTO ROS

[MR07] *General reducibilities for sets of reals*, ***Ph.D. thesis***, Politecnico di Torino, 2007.

NEIL ROBERTSON AND P. D. SEYMOUR

[RS04] *Graph minors. XX. Wagner's conjecture*, ***Journal of Combinatorial Theory. Series B***, vol. 92 (2004), no. 2, pp. 325–357.

WACLAW SIERPIŃSKI

[Sie52] ***General topology***, Mathematical Expositions, No. 7, University of Toronto Press, Toronto, 1952, Translated by C. Cecilia Krieger.

ROMAN SIKORSKI

[Sik58] *Some examples of Borel sets*, ***Colloquium Mathematicum***, vol. 5 (1958), pp. 170–171.

JACK SILVER

[Sil80] *Counting the number of equivalence classes of Borel and coanalytic equivalence relations*, ***Annals of Mathematical Logic***, vol. 18 (1980), no. 1, pp. 1–28.

ROBERT M. SOLOVAY
[Sol78B] *The independence of* DC *from* AD, in Kechris and Moschovakis [CABAL i], pp. 171–184.

JOHN R. STEEL
[Ste77] *Determinateness and subsystems of analysis*, ***Ph.D. thesis***, Berkeley, 1977.
[Ste80] *Analytic sets and Borel isomorphisms*, ***Fundamenta Mathematicae***, vol. 108 (1980), no. 2, pp. 83–88.
[Ste81B] *Determinateness and the separation property*, ***The Journal of Symbolic Logic***, vol. 46 (1981), no. 1, pp. 41–44.

FONS VAN ENGELEN, ARNOLD W. MILLER, AND JOHN R. STEEL
[vEMS87] *Rigid Borel sets and better quasi-order theory*, ***Proceedings of the AMS-IMS-SIAM joint summer research conference on applications of mathematical logic to finite combinatorics held at Humboldt State University, Arcata, Calif., August 4–10, 1985*** (Stephen G. Simpson, editor), Contemporary Mathematics, vol. 65, American Mathematical Society, Providence, RI, 1987, pp. 199–222.

ROBERT VAN WESEP
[Van77] *Subsystems of second-order arithmetic, and descriptive set theory under the axiom of determinateness*, ***Ph.D. thesis***, University of California, Berkeley, 1977.
[Van78A] *Separation principles and the axiom of determinateness*, ***The Journal of Symbolic Logic***, vol. 43 (1978), no. 1, pp. 77–81.

WILLIAM W. WADGE
[Wad84] *Reducibility and determinateness on the Baire space*, ***Ph.D. thesis***, University of California, Berkeley, 1984.
[Wad11] *Early investigations of the degrees of Borel sets*, 2011, this volume.

W. HUGH WOODIN
[Woo99] ***The axiom of determinacy, forcing axioms, and the nonstationary ideal***, De Gruyter Series in Logic and its Applications, Walter de Gruyter, Berlin, 1999.

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