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Thomas–Fermi Approximation for Coexisting Two Component Bose–Einstein Condensates and Nonexistence of Vortices for Small Rotation

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Abstract: We study minimizers of a Gross–Pitaevskii energy describing a two-component Bose–Einstein condensate confined in a radially symmetric harmonic trap and set into rotation. We consider the case of coexistence of the components in the Thomas–Fermi regime, where a small parameter ε conveys a singular perturbation. The minimizer of the energy without rotation is determined as the positive solution of a system of coupled PDEs, for which we show uniqueness. The limiting problem for $\varepsilon = 0$ has degenerate and irregular behavior at specific radii, where the gradient blows up. By means of a perturbation argument, we obtain precise estimates for the convergence of the minimizer to this limiting profile, as ε tends to 0. For low rotation, based on these estimates, we can show that the ground states remain real valued and do not have vortices, even in the region of small density.

1. Introduction

1.1. The problem. In this paper, we study the behavior of the minimizers of the following energy functional describing a two component Bose–Einstein condensate

$$E_\varepsilon^\Omega(u_1,u_2) = \sum_{j=1}^{2}\int_{\mathbb{R}^2}\left\{\frac{|\nabla u_j|^2}{2} + \frac{|x|^2}{2\varepsilon^2}|u_j|^2 + \frac{g_j}{4\varepsilon^2}|u_j|^4 - \Omega x^\perp\cdot(iu_j,\nabla u_j)\right\}dx + \frac{g}{2\varepsilon^2}\int_{\mathbb{R}^2}|u_1|^2|u_2|^2\,dx \tag{1.1}$$

in the space

$$\mathcal{H}=\left\{(u_1,u_2):\ u_j\in H^1(\mathbb{R}^2,\mathbb{C}),\ \int_{\mathbb{R}^2}|x|^2|u_j|^2\,dx<\infty,\ \|u_j\|_{L^2(\mathbb{R}^2)}=1,\ j=1,2\right\}. \tag{1.2}$$

The parameters g_1, g_2, g, ε and Ω are positive: Ω is the angular velocity corresponding to the rotation of the condensate, $x^\perp = (-x_2, x_1)$ and $\cdot$ is the scalar product for vectors, whereas $(\ ,\ )$ is the complex scalar product, so that we have

$$x^\perp \cdot (iu, \nabla u) = x^\perp \cdot \frac{iu\nabla\bar{u} - i\bar{u}\nabla u}{2} = -x_2 \frac{iu\partial_{x_1}\bar{u} - i\bar{u}\partial_{x_1}u}{2} + x_1 \frac{iu\partial_{x_2}\bar{u} - i\bar{u}\partial_{x_2}u}{2}.$$

Here, g_j is the self interaction of each component (intracomponent coupling), while g measures the effect of interaction between the two components (intercomponent coupling). We are interested in studying the existence and behavior of the minimizers in the limit when ε is small, describing strong interactions, also called the Thomas–Fermi limit. We assume the condition

$$g^2 < g_1 g_2, \tag{1.3}$$

which implies that the two components u_1 and u_2 of the minimizers can coexist, as opposed to the segregation case $g^2 > g_1 g_2$. Additionally, we can assume without loss of generality that

$$0 < g_1 \le g_2. \tag{1.4}$$

We start with the analysis of the minimizers of the energy functional E_ε^0 without rotation, namely with $\Omega = 0$. Up to multiplication by a complex number of modulus 1, the minimizers $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$ of E_ε^0 in $\mathcal{H}$ are positive solutions of the following system of coupled Gross–Pitaevskii equations:

$$-\varepsilon^2\Delta\eta_{1,\varepsilon} + |x|^2\eta_{1,\varepsilon} + g_1|\eta_{1,\varepsilon}|^2\eta_{1,\varepsilon} + g\eta_{1,\varepsilon}|\eta_{2,\varepsilon}|^2 = \lambda_{1,\varepsilon}\eta_{1,\varepsilon} \quad \text{in } \mathbb{R}^2, \tag{1.5a}$$

$$-\varepsilon^2\Delta\eta_{2,\varepsilon} + |x|^2\eta_{2,\varepsilon} + g_2|\eta_{2,\varepsilon}|^2\eta_{2,\varepsilon} + g|\eta_{1,\varepsilon}|^2\eta_{2,\varepsilon} = \lambda_{2,\varepsilon}\eta_{2,\varepsilon} \quad \text{in } \mathbb{R}^2, \tag{1.5b}$$

$$\eta_{i,\varepsilon}(x) \to 0 \quad \text{as } |x| \to \infty,\ i = 1, 2, \tag{1.5c}$$

where $\lambda_{1,\varepsilon}, \lambda_{2,\varepsilon}$ are the Lagrange multipliers due to the constraints. We will also refer to the positive minimizers as ground state solutions. Formally setting $\varepsilon = 0$ in (1.5) gives rise to the nonlinear algebraic system

$$\begin{cases} |x|^2\eta_1 + g_1\eta_1^3 + g\eta_1\eta_2^2 = \lambda_{1,0}\eta_1 & \text{in } \mathbb{R}^2, \\ |x|^2\eta_2 + g_2\eta_2^3 + g\eta_1^2\eta_2 = \lambda_{2,0}\eta_2 & \text{in } \mathbb{R}^2, \end{cases} \tag{1.6}$$

where $\eta_i \ge 0$ satisfy $\|\eta_i\|_{L^2(\mathbb{R}^2)} = 1$, $i = 1, 2$. In the region where neither η_i is identically zero, this yields the system

$$\begin{cases} g_1\eta_1^2 + g\eta_2^2 = \lambda_{1,0} - |x|^2, \\ g\eta_1^2 + g_2\eta_2^2 = \lambda_{2,0} - |x|^2. \end{cases} \tag{1.7}$$

This leads to the condition (1.3) and the fact that the supports of η_i are compact sets: more precisely, the supports of η_i are either 2 disks or a disk and an annulus. The limiting geometry is two disks when

$$0 < g < \frac{g_1 + \sqrt{g_1^2 + 8g_1g_2}}{4}. \tag{1.8}$$

This condition is more restrictive than (1.3), in the sense that (1.4) and (1.8) together imply (1.3). If, on the contrary, we assume that

$$g > \frac{g_1 + \sqrt{g_1^2 + 8g_1g_2}}{4}, \tag{1.9}$$

then the limiting configuration consists of a disk and an annulus; in this case, the assumption $g_1 \leq g_2$ implies $g > g_1$. It is helpful to introduce the following quantities:

$$\Gamma_1 = 1 - \frac{g}{g_1}, \qquad \Gamma_2 = 1 - \frac{g}{g_2}, \qquad \Gamma = 1 - \frac{g^2}{g_1g_2}. \tag{1.10}$$

Under the assumptions (1.3)–(1.4) and (1.8), we prove that

$$\eta_{i,\varepsilon}^2 \to a_i \text{ uniformly in } \mathbb{R}^2 \text{ as } \varepsilon \to 0,\ i = 1, 2, \tag{1.11}$$

where $\sqrt{a_1}$, $\sqrt{a_2}$ are the solutions to (1.6) with L^2 constraint 1, so that

$$a_1(x) = \begin{cases} a_{1,0}(x), & |x| \leq R_{1,0}, \\ 0, & |x| \geq R_{1,0}, \end{cases} \qquad a_2(x) = \begin{cases} a_{2,0}(x), & |x| \leq R_{1,0}, \\ a_{2,0}(x) + \frac{g}{g_2}a_{1,0}(x), & R_{1,0} \leq |x| \leq R_{2,0}, \\ 0, & |x| \geq R_{2,0}, \end{cases} \tag{1.12}$$

with $R_{1,0} \leq R_{2,0}$ determined explicitly in terms of g, g_1, g_2 [see (3.5)], and

$$a_{1,0}(x) = \frac{\Gamma_2}{g_1\Gamma}(R_{1,0}^2 - |x|^2), \quad a_{2,0}(x) = \frac{R_{2,0}^2 - R_{1,0}^2}{g_2} + \frac{\Gamma_1}{g_2\Gamma}(R_{1,0}^2 - |x|^2), \tag{1.13}$$

$$a_{2,0}(x) + \frac{g}{g_2}a_{1,0}(x) = \frac{1}{g_2}(R_{2,0}^2 - |x|^2). \tag{1.14}$$

We note that $R_{1,0} < R_{2,0}$ if $g_1 < g_2$ and $a_1 \equiv a_2$ if $g_1 = g_2$. Moreover, we show that $\lambda_{i,\varepsilon} \to \lambda_{i,0}$, $i = 1, 2$, where

$$\lambda_{1,0} = \frac{g}{g_2}R_{2,0}^2 + \Gamma_2 R_{1,0}^2, \quad \lambda_{2,0} = R_{2,0}^2. \tag{1.15}$$

Because of (1.3)–(1.4), we always have that Γ and Γ_2 are positive. On the other hand, Γ_1 can have either sign: if $g < g_1$, the singular limits a_i consist of two decreasing functions, and in the case $g > g_1$, a_2 is increasing near the origin and up to $R_{1,0}$ [though it remains strictly positive under assumption (1.8)] and then decreasing, while a_1 is decreasing. If $g = g_1$, we have that a_2 is constant on the ball of radius $R_{1,0}$. We remark that the first derivatives of $\sqrt{a_1}$ and $\sqrt{a_2}$ have an *infinite* jump discontinuity across the circles $|x| = R_{1,0}$ and $|x| = R_{2,0}$ respectively, while the first derivative of $\sqrt{a_2}$ has a *finite* jump discontinuity across $|x| = R_{1,0}$ (if $g_1 < g_2$). In particular, neither function belongs to the Sobolev space $H^1(\mathbb{R}^2)$. Actually, their maximal regularity is that of the Hölder space $C^{\frac{1}{2}}(\mathbb{R}^2)$.

In the case of (1.3)–(1.4) and (1.9), that is when a_1 is supported in a disk and a_2 in an annulus, we also define the corresponding functions a_i and prove (1.11).

Based on the estimates for the convergence in (1.11), we will show that for a large range of velocities Ω, the minimizers of E_ε^Ω in $\mathcal{H}$ coincide with the minimizers of E_ε^0, provided that $\varepsilon > 0$ is sufficiently small.

The aim of this paper is threefold:

(1) prove the uniqueness of the positive solution $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$ of (1.5) (given any $\lambda_{1,\varepsilon}$ and $\lambda_{2,\varepsilon}$), and of the minimizer of E_ε^0 in $\mathcal{H}$ (modulo a constant complex phase),
(2) get precise estimates on the convergence, as $\varepsilon \to 0$, of $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$, the positive minimizer of E_ε^0 in $\mathcal{H}$, to the singular limit $(\sqrt{a_1}, \sqrt{a_2})$ defined in (1.12),
(3) prove that for Ω below a critical velocity, the minimizers of E_ε^Ω in $\mathcal{H}$ have no vortices in $\mathbb{R}^2$, provided that $\varepsilon > 0$ is sufficiently small.

Point 1 relies on the division of two possible positive solutions componentwise, and proving that each quotient is equal to a constant of modulus 1.

Point 2 is the extension to the system of the results of [27] for a single equation. The idea is to apply a perturbation argument to construct a positive solution to (1.29), "near" $(\sqrt{a_1}, \sqrt{a_2})$. Then, the uniqueness result in Point 1 allows us to conclude that the constructed solution is indeed the ground state. Therefore, we are able to obtain precise asymptotic estimates for the behavior of the ground state as $\varepsilon \to 0$. We emphasize that, even though the system (1.5) is coupled, we are going to reduce it, at leading order, to two independent Gross–Pitaevskii equations. The proof of Point 2, when both condensates are disks, with different techniques, is the topic of a paper in preparation by Gallo [20].

Point 3 relies on fine estimates for the Jacobian from [25]. It consists in extending the proof of [3] for a single equation to the system, which works well once the difficult results of points 1 and 2 have been established.

1.2. Motivation and known results. Two component condensates can describe a single isotope in two different hyperfine spin states, two different isotopes of the same atom or isotopes of two different atoms. We refer to [4] for more details on the modeling and the experimental references.

According to the respective values of g_1, g_2, g and Ω, the minimizers exhibit very different properties in terms of shape of the bulk, defects and coexistence of the components or spatial separation, as $\varepsilon \to 0$. In a recent paper, Aftalion and Mason [4] have produced phase diagrams to classify the types of minimizers according to the parameters of the problem. Below, we summarize their findings.

- Coexisting condensates with vortex lattices: each condensate is a disk, and, for sufficiently large rotation, displays a vortex lattice. The specificity is that each vortex in component 1 creates a peak in component 2 and vice versa. It is this interaction between peaks and vortices that governs the shape of the vortex lattice. For some parameter regimes, the square lattice gets stabilized because it has less energy than the triangular lattice [5].
- Phase separation with radial symmetry: component 1 is a disk while component 2 is an annulus. New defects emerge, such as giant skyrmions and the presence of peaks inside the annulus, corresponding to vortices in the disk.
- Phase separation and complete breaking of symmetry with either droplets or vortex sheets.

It turns out that the sign of the parameter Γ defined in (1.10) plays an important role: if $\Gamma > 0$, the two components coexist while if $\Gamma < 0$, they separate or segregate (case of droplets, vortex sheets). In the case of no rotation, the segregation behavior in two component condensates has been studied by many authors: regularity of the wave function [30], regularity of the interface [16], asymptotic behavior near the interface [12,13,18], Γ-convergence to a Modica–Mortola type energy [6] in the case of a trapped condensate. On the other hand, the case of coexistence is the topic of emerging works in terms of vortices: [5] for a trapped condensate and [8] for a homogeneous condensate. Among other things, the results of this paper are the first step to get a description of vortices for a trapped two component condensate. Indeed, in order to understand the behaviour of vortices in a trapped condensate, one has first to understand the effect of the trap at leading order on the profile. Therefore, one requires very precise estimates on the ground state at $\Omega = 0$ for small ε. This is the analogue of what has been obtained for the single component case that we now recall. Many papers [1,3,21,24,27] have studied the one component analogue of the energy functional (1.1), namely the functional

$$J_\varepsilon^\Omega(u) = \int_{\mathbb{R}^2} \left\{ \frac{|\nabla u|^2}{2} + \frac{|x|^2}{2\varepsilon^2}|u|^2 + \frac{\gamma}{4\varepsilon^2}|u|^4 - \Omega x^\perp \cdot (iu, \nabla u) \right\} dx \tag{1.16}$$

under the constraint $\int_{\mathbb{R}^2} |u|^2\, dx = 1$, where γ is some positive constant.

In the following theorem we have collected various results from [3,21,27] (see also Appendix A herein) concerning the minimizers of the energy J_ε^0 without rotation.

Theorem 1.1. *For every $\varepsilon > 0$, there exists a unique positive minimizer η_ε of J_ε^0 with L^2 constraint 1, and any minimizer has the form $e^{i\alpha}\eta_\varepsilon$ for some $\alpha \in \mathbb{R}$. The minimizer η_ε is radial and there is a unique pair $(\eta_\varepsilon, \lambda_\varepsilon)$, which is a solution of*

$$-\varepsilon^2\Delta\eta + |x|^2\eta + \gamma\eta^3 = \lambda_\varepsilon\eta, \quad x \in \mathbb{R}^2,\ \eta(x) \to 0 \text{ as } |x| \to \infty, \tag{1.17}$$

with η positive. Let

$$a_0(x) = \begin{cases} \frac{\lambda_0 - |x|^2}{\gamma}, & |x| \le R_0, \\ 0, & |x| \ge R_0, \end{cases}$$

where $\lambda_0 > 0$ is uniquely determined from the condition $\int_{\mathbb{R}^2} a_0(x)\, dx = 1$ and $R_0 = \sqrt{\lambda_0}$.

There exist constants $c, C, \delta > 0$, with $\delta < \frac{R_0}{4}$, such that the following properties hold:

$$|\lambda_\varepsilon - \lambda_0| \le C|\log\varepsilon|\varepsilon^2,$$
$$\|\eta_\varepsilon - \sqrt{a_0}\|_{L^\infty(\mathbb{R}^2)} \le C\varepsilon^{\frac{1}{3}},$$
$$\left|\eta_\varepsilon(r) - \sqrt{a_0(r)}\right| \le C\varepsilon^{\frac{1}{3}}\sqrt{a_0(r)}, \quad 0 \le r = |x| \le R_0 - \varepsilon^{\frac{1}{3}},$$
$$\|\eta_\varepsilon - \sqrt{a_0}\|_{L^\infty(|x| \le R_0 - \delta)} \le C|\log\varepsilon|\varepsilon^2,$$

and

$$\eta_\varepsilon(r) \le C\varepsilon^{\frac{1}{3}} \exp\left\{-c\frac{r - R_0}{\varepsilon^{\frac{2}{3}}}\right\}, \quad r \ge R_0,$$

for sufficiently small $\varepsilon > 0$.

In fact, given $\lambda > 0$, the assertions of the above theorem hold for the unique positive solution of equation (1.17) with λ in place of λ_ε. Using these estimates and a Jacobian estimate from [25], the following theorem is proven in [3]:

Theorem 1.2. *Assume that u_ε, η_ε minimize respectively J_ε^Ω, J_ε^0 under the constraint of L^2 norm 1. There exist $\tilde{\varepsilon}_0, \tilde{\omega}_0, \tilde{\omega}_1 > 0$ such that if $0 < \varepsilon < \tilde{\varepsilon}_0$ and $\Omega \leq \tilde{\omega}_0 |\log\varepsilon| - \tilde{\omega}_1 \log|\log\varepsilon|$ then $u_\varepsilon = e^{i\alpha}\eta_\varepsilon$ in $\mathbb{R}^2$ for some constant α.*

This leads, in particular, to the uniqueness of the ground state for small Ω.

1.3. Main results. Our first result concerns uniqueness and radial symmetry for the problem without rotation.

Theorem 1.3. *Assume that* (1.3) *holds.*

(1) *Let us fix some $\lambda_{i,\varepsilon} > 0$, $i = 1, 2$. Then, the positive solution of* (1.5) *is unique, if it exists.*
(2) *The positive minimizer $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$ of E_ε^0 in $\mathcal{H}$ is unique and radially symmetric. Every other minimizer has the form $(e^{i\alpha_1}\eta_{1,\varepsilon}, e^{i\alpha_2}\eta_{2,\varepsilon})$, where α_1, α_2 are constants. If $g_1 = g_2$ then $\eta_{1,\varepsilon} \equiv \eta_{2,\varepsilon}$.*

In the case $g_1 = g_2$, the system reduces to a single equation. Therefore, for the next result, we can assume that

$$g_1 < g_2. \tag{1.18}$$

Our main result provides estimates on the convergence of the positive minimizer $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$ to its limiting profile $\left(\sqrt{a_1}, \sqrt{a_2}\right)$ as $\varepsilon \to 0$, first in the case of two disks.

Theorem 1.4. *Assume that* (1.3), (1.8) *and* (1.18) *hold. Recall that the a_i are defined by* (1.12)–(1.13) *and $\lambda_{i,0}$ by* (1.15). *Let $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$ be the positive minimizer of E_ε^0 in $\mathcal{H}$. Then, $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$ is a solution of* (1.5), *for some positive Lagrange multipliers $\lambda_{1,\varepsilon}$, $\lambda_{2,\varepsilon}$, and there exist constants $c, C, \delta > 0$, with $\delta < \min\left\{\frac{R_{1,0}}{4}, \frac{R_{2,0}-R_{1,0}}{4}\right\}$, such that the following estimates hold:*

$$|\lambda_{i,\varepsilon} - \lambda_{i,0}| \leq C|\log\varepsilon|\varepsilon^2, \tag{1.19}$$

$$\|\eta_{i,\varepsilon} - \sqrt{a_i}\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^{\frac{1}{3}}, \tag{1.20}$$

$$\left|\eta_{i,\varepsilon}(r) - \sqrt{a_i(r)}\right| \leq C\varepsilon^{\frac{1}{3}}\sqrt{a_i(r)}, \quad |x| \leq R_{i,0} - \varepsilon^{\frac{1}{3}}, \tag{1.21}$$

$$\sum_{i=1}^{2} \|\eta_{i,\varepsilon} - \sqrt{a_i}\|_{L^\infty(|x|\leq R_{1,0}-\delta)} + \|\eta_{2,\varepsilon} - \sqrt{a_2}\|_{L^\infty(R_{1,0}+\delta\leq|x|\leq R_{2,0}-\delta)} \leq C|\log\varepsilon|\varepsilon^2, \tag{1.22}$$

and

$$\eta_{i,\varepsilon}(r) \leq C\varepsilon^{\frac{1}{3}} \exp\left\{-c\frac{r - R_{i,0}}{\varepsilon^{\frac{2}{3}}}\right\}, \quad r \geq R_{i,0}, \quad i = 1, 2, \tag{1.23}$$

for sufficiently small $\varepsilon > 0$.

This theorem is the natural extension of what is known for a single condensate described in Theorem 1.1. The fine behavior of the minimizer near $R_{1,0}$ and $R_{2,0}$, as $\varepsilon \to 0$, is established in Theorem 4.1 in Sect. 4. It is based on a perturbation argument, which proves very powerful in this system case, where we have not managed to extend the sub and super solutions techniques of [2,3].

Based on the above, we can show the absence of vortices for the minimizers of E_ε^Ω with small rotation.

Theorem 1.5. *Assume that* (1.3), (1.4) *and* (1.8) *hold. Let* $(u_{1,\varepsilon}, u_{2,\varepsilon})$ *be a minimizer of* E_ε^Ω *in* $\mathcal{H}$. *There exist* $\varepsilon_0, \omega_0, \omega_1 > 0$ *such that if* $0 < \varepsilon < \varepsilon_0$ *and* $\Omega \le \omega_0 |\log\varepsilon| - \omega_1 \log|\log\varepsilon|$ *then* $u_{i,\varepsilon} = e^{i\alpha_i}\eta_{i,\varepsilon}$ *in* $\mathbb{R}^2$ *for some constants* α_i, $i = 1, 2$.

In the case of a disk and an annlus, that is, when (1.9) holds instead of (1.8), we can prove the equivalent of Theorem 1.4. Generalizing Theorem 1.5 is harder, because vortices may exist in the central hole of component 2 and this has to be tackled by techniques other than the ones in this paper.

Theorem 1.6. *Assume that* (1.3), (1.9) *and* (1.18) *hold. We define the functions* a_i *by*

$$a_1(r) = \begin{cases} a_{1,0}(r) + \frac{g}{g_1} a_{2,0}(r), & 0 \le r \le R_{2,0}^-, \\ a_{1,0}(r), & R_{2,0}^- \le r \le R_{1,0}, \\ & \text{where } a_{i,0}(r) = \frac{1}{g_i \Gamma}\left(\lambda_{i,0} - \frac{g}{g_{i+1}}\lambda_{i+1,0} - \Gamma_{i+1} r^2\right), \\ 0, & r \ge R_{1,0}, \end{cases} \tag{1.24}$$

$$a_2(r) = \begin{cases} 0, & 0 \le r \le R_{2,0}^-, \\ a_{2,0}(r), & R_{2,0}^- \le r \le R_{1,0}, \\ a_{2,0}(r) + \frac{g}{g_2} a_{1,0}(r), & R_{1,0} \le r \le R_{2,0}^+, \\ 0 & r \ge R_{2,0}^+. \end{cases} \tag{1.25}$$

and

$$\lambda_{2,0} = (R_{2,0}^+)^2, \quad \lambda_{1,0} - \frac{g}{g_2}\lambda_{2,0} = \Gamma_2 R_{1,0}^2 \quad \text{and} \quad \lambda_{2,0} - \frac{g}{g_1}\lambda_{1,0} = \Gamma_1 (R_{2,0}^-)^2 \tag{1.26}$$

where $\lambda_{1,0}$, $\lambda_{2,0}$ *will be given by* (3.8). *Let* $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$ *be the positive minimizer of* E_ε^0 *in* $\mathcal{H}$. *Then there exist constants* $c, C, \delta > 0$, *such that, for sufficiently small* $\varepsilon > 0$, (1.19), (1.20) *hold,* (1.21) *holds for* $i = 1$ *and is replaced, for* $i = 2$ *by*

$$\left|\eta_{2,\varepsilon}(r) - \sqrt{a_2(r)}\right| \le C\varepsilon^{\frac{1}{3}}\sqrt{a_2(r)}, \quad \text{for } R_{2,0}^- + \varepsilon^{\frac{1}{3}} < |x| \le R_{2,0}^+ - \varepsilon^{\frac{1}{3}}, \tag{1.27}$$

(1.23) *holds with* $R_{2,0}^+$ *instead of* $R_{2,0}$, *and on fixed compact sets away from* $|x| = R_{1,0}$ *and* $|x| = R_{2,0}^\pm$, $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$ *is close to* $(\sqrt{a_1}, \sqrt{a_2})$ *with an error of order* $\mathcal{O}(|\log\varepsilon|)\varepsilon^2$.

We point out that if $g_2 < g_1$, then an analogous theorem holds exchanging the subscript 1 and 2 in the formulae.

1.4. Methods of proof and outline of the paper. Theorem 1.3 is proved by assuming that there are two solutions, studying their ratio componentwise and writing the system satisfied by the ratio, as inspired by [15,28]. For the first part of the theorem, we need decay properties at infinity of the solutions of (1.5) that we prove in a similar way to a Liouville theorem in [11]. We point out that the system is non-cooperative, and the usual moving plane method does not seem to apply easily in this case to derive radial symmetry of positive solutions. Nevertheless, our result implies that since positive solutions of (1.5) are unique, they are thus radial.

For the second part of the theorem, we use the decay of finite energy solutions and extra estimates for radial functions. A key relation is the following splitting of energy: if (η_1, η_2) is a ground state among radial functions, then for any (u_1, u_2), $E_\varepsilon^0(u_1, u_2) = E_\varepsilon^0(\eta_1, \eta_2) + F_\varepsilon^0(v_1, v_2)$, where $v_i = u_i/\eta_i$ and

$$F_\varepsilon^0(v_1, v_2) = \sum_{i=1}^{2} \int_{\mathbb{R}^2} \left\{ \frac{\eta_i^2}{2} |\nabla v_i|^2 + \frac{g_i}{4\varepsilon^2} \eta_i^4 (|v_i|^2 - 1)^2 \right\} dx$$
$$+ \frac{g}{2\varepsilon^2} \int_{\mathbb{R}^2} \eta_1^2 \eta_2^2 (1 - |v_1|^2)(1 - |v_2|^2)\, dx. \tag{1.28}$$

The condition $\Gamma > 0$, that is, $g > \sqrt{g_1 g_2}$, implies that $F_\varepsilon^0(v_1, v_2) \geq 0$. If we assume that (u_1, u_2) is a ground state of E_ε^0, then using the sign of F_ε^0, we find that (u_1, u_2) is equal, up to a multiplication by a complex number of modulus 1, to (η_1, η_2). Thus, any ground state is radially symmetric and equal, up to a multiplication by a complex number of modulus 1, to (η_1, η_2).

Theorem 1.4 contains a fine asymptotic behaviour of the ground state $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$ as ε tends to zero. The difficulty is especially in the regions near $R_{1,0}$, $R_{2,0}$, where the approximate inverted parabola matches an exponentially small function in a region of size $\varepsilon^{2/3}$. The general procedure is to first construct a sufficiently good approximate solution to the problem (1.5), for small $\varepsilon > 0$, with coefficients $\lambda_{1,\varepsilon}$ and $\lambda_{2,\varepsilon}$ being equal to the unique Lagrange multipliers that are provided by Theorem 1.3. Then, using the invertibility properties of the linearized operator about this approximate solution, we perturb it to a genuine solution. The first uniqueness result of Theorem 1.3 implies that this constructed solution coincides with the positive minimizer of E_0^ε in $\mathcal{H}$. The method is a generalization to the system case of the tools developed in [27] for the single equation.

In order to construct this approximate solution, we rewrite the system (1.5) as

$$-\varepsilon^2 \Delta \eta_1 + g_1 \eta_1 \left(\eta_1^2 - a_{1,\varepsilon}(x) \right) + g \eta_1 \left(\eta_2^2 - a_{2,\varepsilon}(x) \right) = 0, \tag{1.29a}$$

$$-\varepsilon^2 \Delta \eta_2 + g_2 \eta_2 \left(\eta_2^2 - a_{2,\varepsilon}(x) \right) + g \eta_2 \left(\eta_1^2 - a_{1,\varepsilon}(x) \right) = 0, \tag{1.29b}$$

$$\eta_i(x) \to 0 \quad \text{as } |x| \to \infty, \quad i = 1, 2, \tag{1.29c}$$

where $a_{1,\varepsilon}$, $a_{2,\varepsilon}$ are the ε equivalent to (1.13), that is,

$$a_{1,\varepsilon}(x) = \frac{1}{g_1 \Gamma} \left(\lambda_{1,\varepsilon} - \frac{g}{g_2} \lambda_{2,\varepsilon} - \Gamma_2 |x|^2 \right), \quad a_{2,\varepsilon}(x) = \frac{1}{g_2 \Gamma} \left(\lambda_{2,\varepsilon} - \frac{g}{g_1} \lambda_{1,\varepsilon} - \Gamma_1 |x|^2 \right), \tag{1.30}$$

and

$$R_{1,\varepsilon}^2 = \frac{1}{\Gamma_2} \left(\lambda_{1,\varepsilon} - \frac{g}{g_2} \lambda_{2,\varepsilon} \right), \quad R_{2,\varepsilon}^2 = \lambda_{2,\varepsilon}. \tag{1.31}$$

At leading order, in the regions where neither η_i is close to zero, we expect that the $\varepsilon^2\Delta\eta_i$ terms are negligible so that at leading order,

$$g_1\left(\eta_1^2 - a_{1,\varepsilon}(x)\right) + g\left(\eta_2^2 - a_{2,\varepsilon}(x)\right) = 0, \tag{1.32a}$$

$$g_2\left(\eta_2^2 - a_{2,\varepsilon}(x)\right) + g\left(\eta_1^2 - a_{1,\varepsilon}(x)\right) = 0. \tag{1.32b}$$

Near $R_{1,0}$, the term $\varepsilon^2\Delta\eta_1$ cannot be neglected so that we use (1.32b) to express $\eta_2^2 - a_{2,\varepsilon}$ and insert it into (1.29a) to find a scalar equation for η_1. In the region of coexistence, that is, in the disk of radius $R_{1,0}$, η_2 is obtained from η_1 by (1.32b), while outside this disk, η_1 is small and can be neglected in (1.29b). This reduces the system (1.29) to two independent approximate scalar problems:

$$-\varepsilon^2\Delta\eta_1 + \left(g_1 - \frac{g^2}{g_2}\right)\eta_1\left(\eta_1^2 - a_{1,\varepsilon}(x)\right) = 0, \quad x \in \mathbb{R}^2; \ \eta_1(x) \to 0 \text{ as } |x| \to \infty, \tag{1.33a}$$

$$-\varepsilon^2\Delta\eta_2 + g_2\eta_2\left(\eta_2^2 - a_{2,\varepsilon}(x) - \frac{g}{g_2}a_{1,\varepsilon}(x)\right) = 0, \quad x \in \mathbb{R}^2; \ \eta_2(x) \to 0 \text{ as } |x| \to \infty, \tag{1.33b}$$

whose unique positive solutions are called $\hat{\eta}_{1,\varepsilon}$ and $\hat{\eta}_{2,\varepsilon}$. The properties of $\hat{\eta}_{1,\varepsilon}$ and $\hat{\eta}_{2,\varepsilon}$ can be deduced from an analogue of Theorem A.1 (see Proposition 4.2 below). We point out that they are linearly nondegenerate, which implies that the spectrum of the associated linearized operators to (1.33a) and (1.33b) consists only of positive eigenvalues. The main features of $a_{1,\varepsilon}$ and $a_{2,\varepsilon}$ that are used for studying $\hat{\eta}_{1,\varepsilon}$ and $\hat{\eta}_{2,\varepsilon}$ are that $a_{1,\varepsilon}$ and $a_{2,\varepsilon} + \frac{g}{g_2}a_{1,\varepsilon}$ change sign once, from positive to negative as $|x|$ crosses $R_{1,\varepsilon}$ and $R_{2,\varepsilon}$ respectively and that

$$a_{1,\varepsilon}'(R_{1,\varepsilon}) \to -c < 0 \ \text{ and } \ \left(a_{2,\varepsilon} + \frac{g}{g_2}a_{1,\varepsilon}\right)'(R_{2,\varepsilon}) \to -c' < 0 \ \text{ as } \varepsilon \to 0. \tag{1.34}$$

In particular, $\hat{\eta}_{1,\varepsilon}^2$ and $\hat{\eta}_{2,\varepsilon}^2$ converge to $\left(a_{1,0}\right)^+$ and $\left(a_{2,0} + \frac{g}{g_2}a_{1,0}\right)^+$, uniformly on $\mathbb{R}^2$, as $\varepsilon \to 0$.

Equation (1.33a) provides an effective approximation for (1.29a) up to a neighborhood of $R_{2,0}$, where the term $\varepsilon^2\Delta\eta_2$ is expected to be of equal or higher order than $\varepsilon^2\Delta\eta_1$ as $\varepsilon \to 0$. The approximate solutions to (1.29) that are constructed in this way match in C^k, in fixed intervals contained between $R_{1,0}$ and $R_{2,0}$. Therefore, we can pick any point in $(R_{1,0}, R_{2,0})$, for instance the middle point, and we can smoothly glue the solutions together, via a standard interpolation argument to create a global approximate solution to the problem (1.29). We remark that, in order to estimate the remainder that this approximation leaves in (1.29), we have to prove some new estimates for the behavior of the derivatives of the ground states of (1.33a) and (1.33b) near $R_{1,0}$ and $R_{2,0}$ respectively, as $\varepsilon \to 0$.

The next step is to study the associated linearized operator to the system (1.29) about this approximation [see (4.41) below]. To this end, as before, our approach is to reduce the corresponding coupled linearized system to the two independent scalar linearized problems that are associated to (1.33a) and (1.33b). As we have already remarked, the spectrum of the latter scalar operators consists only of strictly positive eigenvalues. This property allows us to apply a domain decomposition argument to handle the “two center” difficulty of the problem. We are able to show that the associated linearized operator to the system (1.29) about the approximate solution is invertible, for small $\varepsilon > 0$, and estimate its inverse in various suitable Sobolev L^2-norms. We point out that this is in contrast to the scalar case, where it is more convenient to estimate the inverse in uniform norms (see [21,27]). Armed with these estimates, we can apply a contraction mapping argument to prove the existence of a true solution to (1.29), near the approximate one with respect to a suitable ε-dependent Sobolev norm, for small $\varepsilon > 0$. Finally, we can show that the solution is positive and obtain the uniform estimates of Theorem 1.4 by building on the Sobolev estimates and making use of the equation. In particular, we also make use of some carefully chosen weighted uniform norms, see (4.109), which are partly motivated by [31].

We note that our approach can be extended to cover the more complicated case of the disk and annulus configuration, that is, when (1.9) is assumed.

In order to prove Theorem 1.5, we need to study some auxiliary functions $\xi_{i,\varepsilon}$, involving the primitive of $s\eta_{i,\varepsilon}^2(s)$, where $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$ is the positive minimizer of E_ε^0. For this purpose, we use the estimates that link (1.29) to (1.33), obtained in the proof of Theorem 1.4, to derive the estimates for $\xi_{i,\varepsilon}$ as perturbations of those for the scalar equation. Then, we use a division trick which splits the energy as the sum of the energy of the minimizer without rotation plus a reduced energy (see Lemma 7.5). The reduced energy bears similarities to a weighted coupled Ginzburg-Landau energy and, after integrating by parts, includes a Jacobian (see Lemma 7.6). Assumption (1.3) allows us to treat this coupled energy as two uncoupled ones of analogous form, and conclude by following the arguments of [3], which are based on the control of the auxiliary functions and a Jacobian estimate due to [25].

The organization of the paper is as follows: in Sect. 2, we prove Theorem 1.3. In Sect. 3, we obtain the first rough estimates for the asymptotic behavior of $(\eta_{1,\varepsilon}^2, \eta_{2,\varepsilon}^2)$ as $\varepsilon \to 0$. In Sect. 4, we apply the perturbation argument to prove Theorem 1.4. In Sect. 5, we prove Theorem 1.6. In Sect. 6, we study some auxiliary functions that will be useful for the estimates with rotation. In Sect. 7, we prove Theorem 1.5. We close the paper with two appendixes. In Appendix A, we summarize some known results about the scalar ground states of Theorem 1.1, in a more general setting, and derive estimates for their derivatives. In Appendix B, we have postponed the proof of a technical estimate from Sect. 4 that is related to our use of the weighted norms.

1.5. Notation. By c/C we denote small/large positive generic constant, whose values may decrease/increase from line to line. By $\mathcal{O}(\cdot)$ and $o(\cdot)$, we denote the standard Landau symbols. We write $r = |x|$ to denote the Euclidean distance of a point x from the origin. By B_R we denote the Euclidean ball of radius R and center 0.

2. Uniqueness Issues

In this section, we prove Theorem 1.3. Since the result holds for every $\varepsilon > 0$, we often omit the subscript ε.

2.1. Uniqueness of positive solutions of (1.5). Given positive $\lambda_{1,\varepsilon}$, $\lambda_{2,\varepsilon}$, we want to prove the uniqueness of the positive solutions of (1.5). We use some ideas from [15] which deals with a class of scalar equations in bounded domains. In order to extend it to the entire space, we have to establish some control on the decay of positive, possibly non-radial, solutions.

Lemma 2.1. *Let* (u_1, u_2) *be a positive solution of* (1.5a)–(1.5b)*, then*

$$u_i^2 \le \lambda_{i,\varepsilon}/g_i, \quad \|\nabla u_i\|_{L^\infty(\mathbb{R}^2)} \le C\frac{\sqrt{\lambda_{i,\varepsilon}}(\lambda_{i,\varepsilon}+\lambda_{j,\varepsilon}+1)}{\varepsilon}, \quad i=1,2, \quad j\neq i.$$

The proof is adapted from [3] and [24].

Proof. Let $w_i = (\sqrt{g_i}u_i - \sqrt{\lambda_{i,\varepsilon}})/\varepsilon$, then Kato's inequality yields

$$\Delta w_i^+ \ge \chi_{\{w_i\ge 0\}}\Delta w_i \ge \chi_{\{w_i\ge 0\}}\frac{\sqrt{g_i}}{\varepsilon^3}u_i(g_iu_i^2 - \lambda_{i,\varepsilon}),$$

where χ is the characteristic function of a set. Then we obtain

$$\Delta w_i^+ \ge \chi_{\{w_i\ge 0\}}\frac{\varepsilon w_i + \sqrt{\lambda_{i,\varepsilon}}}{\varepsilon^2}w_i(\varepsilon w_i + 2\sqrt{\lambda_{i,\varepsilon}}) \ge (w_i^+)^3.$$

A non-existence result by Brezis [14] implies that $w_i^+ \equiv 0$, so that the first bound is proved. In fact, since u_i is bounded, it follows by a standard barrier argument that (1.5c) is also satisfied.

Now fix $x \in \mathbb{R}^2$, $L > 0$ and for $y \in B_{2L}(x)$, let $z_i(y) = u_i(\varepsilon(y-x))$. Then

$$-\Delta z_i = -z_i(\varepsilon^2|y-x|^2 + g_iz_i^2 + gz_j^2 - \lambda_{i,\varepsilon}) =: h_{i,\varepsilon}(y), \quad (i\neq j).$$

We have proved above that there exists $C > 0$ independent of ε and of x such that $\|h_{i,\varepsilon}\|_{L^\infty(B_{2L}(x))} \le C\sqrt{\lambda_{i,\varepsilon}}(\lambda_{i,\varepsilon}+\lambda_{j,\varepsilon}+1)$. Standard regularity theory for elliptic equations implies $\|\nabla z_i\|_{L^\infty(B_L(x))} \le C\sqrt{\lambda_{i,\varepsilon}}(\lambda_{i,\varepsilon}+\lambda_{j,\varepsilon}+1)$, and in turn the second part of the statement. □

This implies in particular uniform bounds for the solutions of (1.5). In the following lemma, we prove that positive solutions of (1.5) decay super-exponentially fast as $|x| \to \infty$.

Lemma 2.2. *Let* (u_1, u_2) *be a positive solution of* (1.5)*. For every* $k > 0$*, let* $r_i = \sqrt{(1+k)\lambda_{i,\varepsilon}}$ *and*

$$W_i(s) = \max_{\partial B_r} u_i \cdot \exp\left(-\frac{1}{2\varepsilon}\sqrt{\frac{k}{1+k}}(s^2-r^2)\right) \quad \textit{for } s\ge r\ge r_i, \quad i=1,2.$$

Then we have $u_i(x) \le W_i(|x|)$ *for* $|x| \ge r \ge r_i$*,* $i = 1, 2$*. Moreover,*

$$|u_i(x)| + |\nabla u_i(x)| \le C_\varepsilon e^{-c_\varepsilon|x|^2}, \quad x\in\mathbb{R}^2, \quad i=1,2. \tag{2.1}$$

Proof. Given $k > 0$, let $r \ge r_i = \sqrt{(1+k)\lambda_{i,\varepsilon}}$. For $|x| \ge r$ we have $\lambda_{i,\varepsilon} \le \frac{|x|^2}{1+k}$, and hence $|x|^2 - \lambda_{i,\varepsilon} \ge \frac{k}{1+k}|x|^2$, so that u_i satisfies

$$-\Delta u_i + \frac{u_i}{\varepsilon^2}\frac{k}{1+k}|x|^2 \le -\Delta u_i + \frac{u_i}{\varepsilon^2}(|x|^2-\lambda_{i,\varepsilon}) = -\frac{u_i}{\varepsilon^2}(g_iu_i^2+gu_j^2) \le 0$$

for every $|x| \geq r$ (here $j \neq i$). On the other hand, it is easy to check that

$$-\Delta W_i + \frac{W_i}{\varepsilon^2}\frac{k}{1+k}|x|^2 > 0 \quad \text{for } |x| \geq r, \quad W_i(r) \geq u_i(x) \text{ for } |x| = r.$$

Suppose by contradiction that $W_i - u_i$ is negative somewhere in the exterior of $\bar{B}_r$. Since both functions go to zero at infinity, there exists $\bar{x}$, with $|\bar{x}| > r$, where $W_i - u_i$ reaches its minimum: $W_i(|\bar{x}|) - u_i(\bar{x}) < 0$ and $\Delta W_i(|\bar{x}|) - \Delta u_i(\bar{x}) \geq 0$. By subtracting the two differential inequalities satisfied by u_i and W_i, and then evaluating at $\bar{x}$, we obtain a contradiction.

Lemma 2.1 implies a uniform bound for $\max u_i$. Using (1.5), we obtain that

$$|\Delta u_i(y)| \leq C_\varepsilon e^{-c_\varepsilon |x|^2}, \quad y \in B_1(x), \quad x \in \mathbb{R}^2.$$

By standard interior elliptic estimates, we deduce (2.1). □

Proposition 2.3. *Assume* (1.3)*. Given* $\lambda_{i,\varepsilon} > 0$*, then problem* (1.5) *has at most one positive solution.*

Proof. Let (η_1, η_2) and (u_1, u_2) be two positive solutions of (1.5) with the same $\lambda_{1,\varepsilon}$ and $\lambda_{2,\varepsilon}$. Then, the function $\psi_i = u_i/\eta_i$ solves the following equation with $j \neq i$:

$$\begin{aligned} -\nabla \cdot (\eta_i^2 \nabla \psi_i) &= u_i \Delta \eta_i - \eta_i \Delta u_i \\ &= \frac{\eta_i^2 \psi_i}{\varepsilon^2}\left[g_i \eta_i^2 (1-\psi_i^2) + g\eta_j^2(1-\psi_j^2)\right]. \end{aligned} \tag{2.2}$$

We want to show that ψ_i is identically equal to 1. To this end, we multiply Eq. (2.2) by $(\psi_i^2 - 1)/\psi_i$ in a ball of radius R to obtain

$$\begin{aligned} &\int_{B_R} \left\{ \eta_i^2 |\nabla \psi_i|^2 \left(1 + \frac{1}{\psi_i^2}\right) + \frac{\eta_i^2}{\varepsilon^2}\left[g_i \eta_i^2 (\psi_i^2 - 1)^2 + g\eta_j^2 (\psi_i^2 - 1)(\psi_j^2 - 1)\right]\right\} dx \\ &= \int_{\partial B_R} \left\{ \left(u_i - \frac{\eta_i^2}{u_i}\right)\nabla u_i - \left(\frac{u_i^2}{\eta_i} - \eta_i\right)\nabla \eta_i \right\} \cdot \nu \, d\sigma, \end{aligned}$$

where ν denotes the outer unit normal vector to ∂B_R. We sum the previous identities for $i = 1, 2$ and then we use the following inequality

$$|2g\eta_1^2\eta_2^2(\psi_1^2 - 1)(\psi_2^2 - 1)| \leq (g_1 - \gamma)\eta_1^4(\psi_1^2 - 1)^2 + (g_2 - \gamma)\eta_2^4(\psi_2^2 - 1)^2, \tag{2.3}$$

where $0 < \gamma < \min\{g_1, g_2\}$ is such that

$$g \leq \sqrt{g_1 - \gamma}\sqrt{g_2 - \gamma}, \tag{2.4}$$

which exists by (1.3). We get

$$\begin{aligned} &\sum_{i=1}^{2} \int_{B_R} \left\{ \eta_i^2 |\nabla \psi_i|^2 \left(1 + \frac{1}{\psi_i^2}\right) + \frac{\gamma}{\varepsilon^2}\eta_i^4 (\psi_i^2 - 1)^2 \right\} dx \\ &\leq \sum_{i=1}^{2} \int_{\partial B_R} \left\{ \left(u_i - \frac{\eta_i^2}{u_i}\right)\nabla u_i - \left(\frac{u_i^2}{\eta_i} - \eta_i\right)\nabla \eta_i \right\} \cdot \nu \, d\sigma. \end{aligned} \tag{2.5}$$

To conclude that $\psi_i \equiv 1$, it is enough to show that there exist $R_k \to \infty$ such that the right-hand side of (2.5), with $R = R_k$, tends to zero as $k \to \infty$. This task will take up the rest of the proof.

Let χ be a smooth cutoff function in $\mathbb{R}^2$ which is identically equal to 1 in the unit ball and identically equal to 0 outside of the ball of radius 2. For all $R \geq 1$, we define $\chi_R = \chi(\cdot/R)$. In the remaining part of this proof, we denote by C_ε a positive generic constant which is independent of $R \geq 1$. We multiply (2.2) by $r^\alpha \chi_R^2 \psi_i$, where $\alpha > 2$ ($r = |x|$), and integrate the resulting identity by parts over $\mathbb{R}^2$ to find that

$$\left| \int_{B_{2R}} \chi_R^2 \left(r^\alpha \nabla \psi_i + \alpha r^{\alpha-1} \frac{x}{r} \psi_i \right) \eta_i^2 \nabla \psi_i \, dx + \int_{B_{2R}} r^\alpha \psi_i 2 \chi_R \nabla \chi_R \eta_i^2 \nabla \psi_i \, dx \right| \leq C_\varepsilon,$$

where we have also made use of (2.1) and of the definition of ψ_i. Our motivation for including χ_R^2 comes from [11, Thm. 1.8]. Thanks to the elementary inequalities

$$\left| \chi_R^2 r^{\alpha-1} \frac{x}{r} \psi_i \eta_i^2 \nabla \psi_i \right| \leq d \chi_R^2 r^\alpha \eta_i^2 |\nabla \psi_i|^2 + \frac{1}{2d} \chi_R^2 r^{\alpha-2} \psi_i^2 \eta_i^2 \quad \forall \, d > 0,$$

and

$$\left| 2 r^\alpha \psi_i \chi_R \nabla \chi_R \eta_i^2 \nabla \psi_i \right| \leq d \chi_R^2 r^\alpha \eta_i^2 |\nabla \psi_i|^2 + \frac{1}{d} r^\alpha \psi_i^2 |\nabla \chi_R|^2 \eta_i^2 \quad \forall \, d > 0,$$

choosing a sufficiently small $d > 0$ (independent of R), via (2.1), we infer that

$$\int_{\mathbb{R}^2} r^\alpha \chi_R^2 \eta_i^2 |\nabla \psi_i|^2 \, dx \leq C_\varepsilon.$$

By Lebesgue's monotone convergence theorem, letting $R \to \infty$ in the above relation, we obtain that

$$\int_{\mathbb{R}^2} r^\alpha \eta_i^2 |\nabla \psi_i|^2 \, dx \leq C_\varepsilon.$$

Replacing ψ_i by its value and using (2.1), we find that

$$\int_{\mathbb{R}^2} r^\alpha u_i^2 \frac{|\nabla \eta_i|^2}{\eta_i^2} \, dx \leq C_\varepsilon.$$

Reversing the roles of u_i and η_i, and summing, we reach

$$\int_{\mathbb{R}^2} r^\alpha \sum_{i=1}^{2} \left(u_i^2 \frac{|\nabla \eta_i|^2}{\eta_i^2} + \eta_i^2 \frac{|\nabla u_i|^2}{u_i^2} \right) dx \leq C_\varepsilon.$$

Therefore, by the co-area formula, there exists a sequence $R_k \to \infty$ such that

$$R_k^\alpha \int_{\partial B_{R_k}} \sum_{i=1}^{2} \left(u_i^2 \frac{|\nabla \eta_i|^2}{\eta_i^2} + \eta_i^2 \frac{|\nabla u_i|^2}{u_i^2} \right) d\sigma \to 0 \quad \text{as } k \to \infty.$$

To conclude, we note that the above relation, the Cauchy-Schwarz inequality, and (2.1), imply that the right-hand side of (2.5) at $R = R_k$ tends to zero as $k \to \infty$, as desired. We remark that, in the case of radial symmetry, one can argue directly, analogously to [3], by making use of Lemma 2.8 below. □

Remark 2.4. If $g_1 = g_2$ and $\lambda_{1,\varepsilon} = \lambda_{2,\varepsilon}$, then $\eta_1 = \eta_2$. Indeed, (η_1, η_2) and (η_2, η_1) are both positive solutions to (1.5) and we can apply Proposition 2.3.

Remark 2.5. The uniqueness result of Proposition 2.3 yields radial symmetry of u_1 and u_2.

We observe that the proof of Proposition 2.3 applies to provide also the following local uniqueness result, since in this case, the boundary terms vanish.

Proposition 2.6. *Assume* (1.3)*. Given* $\lambda_{i,\varepsilon} > 0$*,* $i = 1, 2$*, and a bounded domain* $B \subset \mathbb{R}^2$ *with Lipschitz continuous boundary, if* (η_1, η_2) *and* (u_1, u_2) *are positive solutions to the elliptic system in* (1.5) *on* $\bar{B}$ *such that* $u_i = \eta_i$ *on* ∂B *for* $i = 1, 2$*, then* $u_i \equiv \eta_i$ *in* B*.*

2.2. Uniqueness and radial symmetry of the ground state. We now turn to the uniqueness and radial symmetry of the positive minimizer of the energy without rotation

$$E^0_\varepsilon(u_1, u_2) = \sum_{i=1}^{2} \int_{\mathbb{R}^2} \left\{ \frac{|\nabla u_i|^2}{2} + \frac{|x|^2}{2\varepsilon^2}|u_i|^2 + \frac{g_i}{4\varepsilon^2}|u_i|^4 \right\} dx + \frac{g}{2\varepsilon^2} \int_{\mathbb{R}^2} |u_1|^2|u_2|^2\, dx$$

in the space $\mathcal{H}$ which is defined in (1.2).

If (u_1, u_2) minimizes E^0_ε in $\mathcal{H}$, then the diamagnetic inequality implies $E^0_\varepsilon(|u_1|, |u_2|) \le E^0_\varepsilon(u_1, u_2)$, so that u_i differs from $|u_i|$ by a constant complex phase. In fact, by the strong maximum principle, we can assume that u_i are positive functions. By elliptic regularity, positive minimizers of E^0_ε in $\mathcal{H}$ lead to smooth solutions of (1.5a)–(1.5b) for some positive Lagrange multipliers $\lambda_{1,\varepsilon}, \lambda_{2,\varepsilon}$. Nevertheless, we have to prove that (1.5c) holds too. A priori, we only know that it holds for radial functions by the Strauss lemma [32]. In the subsequent lemma, we provide a lower bound for the decay rate of positive solutions to (1.5) as $|x| \to \infty$. The following proof is adapted from [3] and [24].

Lemma 2.7. *Let* (u_1, u_2) *be a positive solution of* (1.5)*. Let*

$$w_i(s) = \min_{\partial B_r} u_i \cdot \exp\left(-\frac{\alpha_i}{2}(s^2 - r^2)\right) \qquad \text{for } s \ge r > \sqrt{\lambda_{i,\varepsilon}},$$

where, for $i, j = 1, 2$ *and* $j \neq i$*, we have defined*

$$\alpha_i = \frac{1}{\lambda_{i,\varepsilon}} + \sqrt{\frac{1}{\lambda_{i,\varepsilon}^2} + \frac{1}{\varepsilon^2}\left(1 + \frac{g\lambda_{j,\varepsilon}}{g_j\lambda_{i,\varepsilon}}\right)}.$$

Then $u_i(x) \ge w_i(|x|)$ *for* $|x| \ge r > \sqrt{\lambda_{i,\varepsilon}}$*.*

Proof. We know from Lemma 2.1 that $u_i^2 \le \lambda_{i,\varepsilon}/g_i$ for $i = 1, 2$. Thus, u_i satisfies

$$-\varepsilon^2\Delta u_i + u_i\left(|x|^2 + g\frac{\lambda_{j,\varepsilon}}{g_j}\right) \ge 0, \quad x \in \mathbb{R}^2, \quad (j \neq i).$$

On the other hand, our choice of α_i implies that

$$-\varepsilon^2\Delta w_i + w_i\left(|x|^2 + g\frac{\lambda_{j,\varepsilon}}{g_j}\right) < w_i\left(2\alpha_i\varepsilon^2 + \frac{g}{g_j}\lambda_{j,\varepsilon} + (1 - \varepsilon^2\alpha_i^2)\lambda_{i,\varepsilon}\right) = 0, \quad |x| > r,$$

where we have used that $1 - \varepsilon^2\alpha_i^2 < 0$. The maximum principle can now be applied, as in Lemma 2.2, to yield the desired lower bound. □

In the case of radial solutions, we can show the following lemma analogous to the one in [3].

Lemma 2.8. *Let* (u_1, u_2) *be a positive radial solution of* (1.5)*. There exists* $C_\varepsilon > 0$ *independent of* r *such that, for* $i = 1, 2$*, we have*

$$|u_i'(r)| \le C_\varepsilon r u_i(r) \quad \text{for } r > 2\sqrt{\lambda_{i,\varepsilon}}.$$

Proof. Since u_i is radial, and $r > 2\sqrt{\lambda_{i,\varepsilon}}$, an application of Lemma 2.2 (with $k = 3$) yields that

$$u_i(r) = W_i(r) \quad \text{and} \quad u_i(s) \le W_i(s) \ \text{ for } \ s > r.$$

It follows that $u_i'(r) \le W_i'(r) \le 0$. Similarly, an application of Lemma 2.7 yields that $u_i'(r) \ge w_i'(r) = -\alpha_i r u_i(r)$, completing the proof. □

In order to proceed, we need the following splitting of the energy:

Proposition 2.9. *Assume* (1.3)*. Let* (η_1, η_2) *be a minimizer among radial functions in* $\mathcal{H}$ *with* $\eta_i > 0$*. Let* $(u_1, u_2) \in \mathcal{H}$*. Then the splitting of energy* (1.28) *holds.*

Proof. We test the equation for η_i by $\eta_i(|v_i|^2 - 1)$ in a ball of radius R and then integrate by parts. As a result, we get the term

$$\int_{B_R} \left\{ |\nabla \eta_i|^2 (|v_i|^2 - 1) + 2\eta_i \nabla \eta_i \cdot (v_i, \nabla v_i) \right\} dx = \int_{B_R} -\Delta \eta_i \eta_i (|v_i|^2 - 1)\, dx + \int_{\partial B_R} \left(\frac{|u_i|^2}{\eta_i} - \eta_i \right) \nabla \eta_i \cdot \nu \, d\sigma,$$

Lemmas 2.1 and 2.8 apply to η_i to provide

$$\left| \int_{\partial B_R} \left(\frac{|u_i|^2}{\eta_i} - \eta_i \right) \nabla \eta_i \cdot \nu \, d\sigma \right| \le C_\varepsilon \int_{\partial B_R} R|u_i|^2 \, d\sigma + C_\varepsilon \int_{\partial B_R} R\eta_i^2 \, d\sigma,$$

Note that the conditions $(\eta_1, \eta_2), (u_1, u_2) \in \mathcal{H}$ imply the existence of a sequence $R_k \to \infty$ such that the integrals above vanish along R_k via the co-area formula. Therefore, we have

$$\int_{\mathbb{R}^2} (|\nabla \eta_i|^2 (|v_i|^2 - 1) + 2\eta_i \nabla \eta_i \cdot (\nabla v_i, v_i))\, dx = -\frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \eta_i^2 (|v_i|^2 - 1)(|x|^2 + g_i \eta_i^2 + g\eta_j^2)\, dx,$$

where the Lagrange multiplier term has disappeared because $\int_{\mathbb{R}^2} \eta_i^2 (|v_i|^2 - 1)\, dx = 0$. We replace the last equality into the definition of $E_\varepsilon^0(u_1, u_2)$ to find

$$E_\varepsilon^0(u_1, u_2) = E_\varepsilon^0(\eta_1 v_1, \eta_2 v_2) = \sum_{i=1}^{2} \int_{\mathbb{R}^2} \left\{ \frac{|\nabla \eta_i|^2}{2} |v_i|^2 + \eta_i \nabla \eta_i \cdot (\nabla v_i, v_i) + \eta_i^2 \frac{|\nabla v_i|^2}{2} + \frac{|x|^2}{2\varepsilon^2} \eta_i^2 |v_i|^2 + \frac{g_i}{4\varepsilon^2} \eta_i^4 |v_i|^4 \right\} dx$$

$$
\begin{aligned}
&+\frac{g}{2\varepsilon^2}\int_{\mathbb{R}^2}\eta_1^2\eta_2^2|v_1|^2|v_2|^2\,dx\\
&=\sum_{i=1}^{2}\int_{\mathbb{R}^2}\left\{\frac{|\nabla\eta_i|^2}{2}+\eta_i^2\frac{|\nabla v_i|^2}{2}+\frac{|x|^2}{2\varepsilon^2}\eta_i^2+\frac{g_i}{4\varepsilon^2}\eta_i^4+\frac{g_i}{4\varepsilon^2}\eta_i^4(|v_i|^2-1)^2\right\}dx\\
&+\frac{g}{2\varepsilon^2}\int_{\mathbb{R}^2}\eta_1^2\eta_2^2(|v_1|^2|v_2|^2-|v_1|^2-|v_2|^2+2)\,dx.
\end{aligned}
$$

By collecting the term $E_\varepsilon^0(\eta_1,\eta_2)$ in the previous expression, the result follows. □

2.3. Proof of Theorem 1.3.

Proof. Given $\lambda_{i,\varepsilon}>0, i=1,2$, the first assertion of the theorem is proven in Proposition 2.3.

Now, let (η_1,η_2) be a minimizer of E_ε^0 in $\mathcal{H}$ among radial functions, and let (u_1,u_2) be a minimizer of E_ε^0 in $\mathcal{H}$. Since (η_1,η_2) is an admissible test function, we have $E_\varepsilon^0(u_1,u_2)\le E_\varepsilon^0(\eta_1,\eta_2)$. Consequently, the quantity $F_\varepsilon^0(v_1,v_2)$, defined in (1.28), satisfies

$$F_\varepsilon^0(v_1,v_2)=E_\varepsilon^0(u_1,u_2)-E_\varepsilon^0(\eta_1,\eta_2)\le 0.$$

On the other hand, recalling (2.4), we find that

$$F_\varepsilon^0(v_1,v_2)\ge\sum_{i=1}^{2}\int_{\mathbb{R}^2}\left(\frac{\eta_i^2}{2}|\nabla v_i|^2+\frac{\gamma}{4\varepsilon^2}\eta_i^4(|v_i|^2-1)^2\right)dx\ge 0.$$

This implies that $F_\varepsilon^0(v_1,v_2)=0$ and that $|v_i|\equiv 1$ for $i=1,2$, which implies the second assertion of the theorem. If $g_1=g_2$ then $\eta_1\equiv\eta_2$, as (η_1,η_2) and (η_2,η_1) are both minimizers. □

3. Preliminary Estimates for the Energy Minimizer Without Rotation

In this section we prove that, under assumptions (1.3), (1.8) and (1.18), the positive minimizer (η_1,η_2) provided by Theorem 1.3 satisfies

$$\eta_i^2\to a_i\quad\text{in } L^2(\mathbb{R}^2)\quad\text{and}\quad\lambda_{i,\varepsilon}\to\lambda_{i,0}\ \text{ as }\ \varepsilon\to 0.$$

This result is achieved through the estimate of the energy of the minimizer.

3.1. Limiting profiles. We recall briefly how to calculate the limiting configuration (1.12). We first assume the case of two disks

$$D_i=\{x\in\mathbb{R}^2:\ |x|<R_{i,0}\}.$$

where $R_{1,0}<R_{2,0}$ to be determined later. If $x\in D_1$, formally let $\varepsilon=0$ in (1.5) and solve the resulting algebraic system in η_1^2,η_2^2. This provides, for $x\in D_1$,

$$a_{1,0}(x)=\frac{1}{g_1\Gamma}\left(\lambda_{1,0}-\frac{g}{g_2}\lambda_{2,0}-\Gamma_2|x|^2\right),\tag{3.1}$$

$$a_{2,0}(x) = \frac{1}{g_2\Gamma}\left(\lambda_{2,0} - \frac{g}{g_1}\lambda_{1,0} - \Gamma_1|x|^2\right), \tag{3.2}$$

and also the value of $R_{1,0}$, which is the radius at which $a_{1,0}$ vanishes:

$$R_{1,0}^2 = \frac{1}{\Gamma_2}\left(\lambda_{1,0} - \frac{g}{g_2}\lambda_{2,0}\right). \tag{3.3}$$

If $x \in D_2\backslash D_1$, then $\eta_1 = 0$ and formally with $\varepsilon = 0$ in (1.5), we solve the resulting equation for η_2^2, to obtain the following limiting behavior for η_2^2:

$$\frac{\lambda_{2,0} - |x|^2}{g_2}, \quad \text{with} \quad R_{2,0}^2 = \lambda_{2,0}. \tag{3.4}$$

Notice that $(\lambda_{2,0} - |x|^2)/g_2 = a_{2,0} + \frac{g}{g_2}a_{1,0}$, in agreement with our definition of a_2 in (1.12). Finally, by imposing the normalization conditions $\|a_1\|_{L^2(\mathbb{R}^2)} = \|a_2\|_{L^2(\mathbb{R}^2)} = 1$, we obtain

$$\lambda_{2,0}^2 = \frac{2(g_2+g)}{\pi}, \quad \lambda_{1,0} - \frac{g}{g_2}\lambda_{2,0} = \sqrt{\frac{2g_1\Gamma\Gamma_2}{\pi}},$$

and hence

$$R_{1,0}^4 = \frac{2g_1\Gamma}{\pi\Gamma_2}, \quad R_{2,0}^4 = \frac{2(g_2+g)}{\pi}. \tag{3.5}$$

Notice that, in our setting, the condition (1.18) is equivalent to $R_{1,0} < R_{2,0}$, as can be deduced from (3.5). Next observe that the monotonicity of $a_{2,0}$ depends on the sign of Γ_1. If $\Gamma_1 > 0$, then $a_{2,0}$ is decreasing and

$$a_{2,0}(x) \geq a_{2,0}(R_{1,0}) = (R_{2,0}^2 - R_{1,0}^2)/g_2 > 0, \quad x \in D_1. \tag{3.6}$$

If $\Gamma_1 = 0$ then $a_{2,0}$ is constant, whereas it is increasing when $\Gamma_1 < 0$. In this last case, we have, for $x \in \mathbb{R}^2$,

$$a_{2,0}(x) \geq a_{2,0}(0) = \frac{1}{g_2\Gamma}\left(\lambda_{2,0} - \frac{g}{g_1}\lambda_{1,0}\right), \tag{3.7}$$

which is a positive constant thanks to (1.8). Condition (1.8) is thus equivalent to having two disks.

In the case of a disk plus annulus, we assume that a_1 is supported in a disk D_1 of radius $R_{1,0}$ and a_2 on an annulus

$$D_2 = \{x \in \mathbb{R}^2 : \ R_{2,0}^- < |x| < R_{2,0}^+\}$$

with $R_{2,0}^- < R_{1,0} < R_{2,0}^+$. Other rearrangements of $R_{1,0}$, $R_{2,0}^-$, $R_{2,0}^+$ can be excluded, see [5]. In the coexistence region, that is $R_{2,0}^- < |x| < R_{1,0}$, $(\sqrt{a_{1,0}}, \sqrt{a_{2,0}})$ given by (3.1)–(3.2) is the solution of (1.7). The fact that a_1 vanishes at $R_{1,0}$ and a_2 at $R_{2,0}^-$ and $R_{2,0}^+$ yields (1.26). If

$$r \leq R_{2,0}^- \leq R_{2,0}^+, \quad a_2 = 0 \text{ and } a_1 = \frac{\lambda_{1,0} - r^2}{g_1}$$

$$R_{1,0} \leq r, \quad a_1 = 0 \text{ and } a_2 = \frac{\lambda_{2,0} - r^2}{g_2}$$

which are consistent with (1.24)–(1.25). The computations of the L^2 norms provide

$$\lambda_{1,0} = \sqrt{2\frac{g_1(1+\frac{g_2^2}{g^2}(1-\Gamma_2)^2)}{\pi}} \quad \text{and} \quad \lambda_{2,0}-\lambda_{1,0} = \sqrt{-\Gamma_1\Gamma_2}\sqrt{2\frac{g_1g_2^2(1-\Gamma_2)}{\pi g^2}}. \tag{3.8}$$

3.2. Energy estimates. In order to obtain some energy estimates, we first rewrite the energy functional in a different form.

Lemma 3.1. *Assume* (1.3), (1.8) *and* (1.18). *Let* $(u_1,u_2)\in\mathcal{H}$, *then* $E_\varepsilon^0(u_1,u_2) = \tilde{E}_\varepsilon^0(u_1,u_2)+K$, *where*

$$\begin{aligned}\tilde{E}_\varepsilon^0(u_1,u_2) = &\sum_{i=1}^2\int_{\mathbb{R}^2}\left\{\frac{|\nabla u_i|^2}{2}+\frac{g_i}{4\varepsilon^2}(|u_i|^2-a_i)^2\right\}dx\\ &+\frac{g}{2\varepsilon^2}\int_{\mathbb{R}^2}(|u_1|^2-a_1)(|u_2|^2-a_2)\,dx\\ &+\frac{g_1\Gamma}{2\varepsilon^2}\int_{\mathbb{R}^2\setminus D_1}|u_1|^2a_{1,0}^-\,dx\\ &+\frac{1}{2\varepsilon^2}\int_{\mathbb{R}^2\setminus D_2}(g|u_1|^2+g_2|u_2|^2)\left(a_{2,0}+\frac{g}{g_2}a_{1,0}\right)^-\,dx\end{aligned}$$

and K *is the following constant* (*depending on* ε)

$$K = \frac{\lambda_{1,0}+\lambda_{2,0}}{2\varepsilon^2}-\frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2}(g_1a_1^2+g_2a_2^2+2ga_1a_2)\,dx.$$

Proof. We note that

$$|x|^2+g_1a_1+ga_2 = \begin{cases}\lambda_{1,0}, & x\in D_1,\\ \Gamma_2|x|^2+\frac{g}{g_2}\lambda_{2,0}, & x\in D_2\setminus D_1,\\ |x|^2, & x\in\mathbb{R}^2\setminus D_2,\end{cases}$$

and

$$|x|^2+g_2a_2+ga_1 = \begin{cases}\lambda_{2,0}, & x\in D_2,\\ |x|^2, & x\in\mathbb{R}^2\setminus D_2.\end{cases}$$

Therefore, we have:

$$\begin{aligned}&g_1|u_1|^4+2|x|^2|u_1|^2+g_2|u_2|^4+2|x|^2|u_2|^2+2g|u_1|^2|u_2|^2\\ &\quad= g_1(|u_1|^2-a_1)^2+g_2(|u_2|^2-a_2)^2+2g(|u_1|^2-a_1)(|u_2|^2-a_2)\\ &\quad\quad+2|u_1|^2(|x|^2+g_1a_1+ga_2)+2|u_2|^2(|x|^2+g_2a_2+ga_1)\\ &\quad\quad-g_1a_1^2-g_2a_2^2-2ga_1a_2.\end{aligned}$$

Inserting the above in the definition of E_ε^0, and rearranging the terms, gives the statement. □

The following proposition provides some estimates for the minimizer which will be used in the sequel for estimating the associated Lagrange multipliers.

Proposition 3.2. *Assume* (1.3)*,* (1.8) *and* (1.18)*. Let* (η_1, η_2) *be the positive minimizer of* E_ε^0 *in* $\mathcal{H}$ *that is provided by Theorem* 1.3*. If* $\varepsilon > 0$ *is sufficiently small, for* $i = 1, 2$*, we have*

$$\int_{\mathbb{R}^2} |\nabla \eta_i|^2 \, dx \le C|\log \varepsilon|, \tag{3.9}$$

$$\int_{\mathbb{R}^2} (\eta_i^2 - a_i)^2 \, dx \le C\varepsilon^2 |\log \varepsilon|, \tag{3.10}$$

$$\int_{\mathbb{R}^2 \setminus D_1} \eta_1^2 a_{1,0}^- \, dx + \int_{\mathbb{R}^2 \setminus D_2} (g\eta_1^2 + g_2\eta_2^2) \left(a_{2,0} + \frac{g}{g_2} a_{1,0} \right)^- \, dx \le C\varepsilon^2 |\log \varepsilon|. \tag{3.11}$$

In particular, $\eta_i^2 \to a_i$ *in* $L^2(\mathbb{R}^2)$ *as* $\varepsilon \to 0$.

Proof. First, we claim that, for small $\varepsilon > 0$, we have

$$\tilde{E}_\varepsilon^0(\eta_1, \eta_2) \le C|\log \varepsilon|, \tag{3.12}$$

with $\tilde{E}_\varepsilon^0$ defined in Lemma 3.1. This is proved as in [2] and [3], therefore we only give a sketch here. Consider the competitor functions

$$\tilde{\eta}_i = \frac{h_\varepsilon(a_i)}{\|h_\varepsilon(a_i)\|_{L^2(\mathbb{R}^2)}}, \qquad \text{where} \qquad h_\varepsilon(s) = \begin{cases} s/\varepsilon & \text{if } 0 \le s \le \varepsilon^2, \\ \sqrt{s} & \text{if } s > \varepsilon^2. \end{cases}$$

It is proved in the aforementioned papers that

$$1 - C\varepsilon^2 \le \int_{\mathbb{R}^2} h_\varepsilon(a_i)^2 \, dx \le 1$$

$$\int_{\mathbb{R}^2} |\nabla h_\varepsilon(a_i)|^2 \, dx \le C|\log \varepsilon|$$

$$\int_{\mathbb{R}^2} (h_\varepsilon(a_i)^2 - a_i)^2 \, dx \le C\varepsilon^2.$$

Here we are implicitly using assumption (1.8) which ensures that a_i are positive [recall (3.6), (3.7)]. In addition notice that

$$2\int_{\mathbb{R}^2} (h_\varepsilon(a_1)^2 - a_1)(h_\varepsilon(a_2)^2 - a_2) \, dx \le \sum_{i=1}^{2} \int_{\mathbb{R}^2} (h_\varepsilon(a_i)^2 - a_i)^2 \, dx$$

and that

$$\int_{\mathbb{R}^2 \setminus D_1} \tilde{\eta}_1^2 a_{1,0}^- \, dx = \int_{\mathbb{R}^2 \setminus D_2} (g\tilde{\eta}_1^2 + g_2\tilde{\eta}_2^2) \left(a_{2,0} + \frac{g}{g_2} a_{1,0} \right)^- \, dx = 0.$$

Therefore, we have obtained $\tilde{E}_\varepsilon^0(\tilde{\eta}_1, \tilde{\eta}_2) \le C|\log \varepsilon|$. Finally, let (η_1, η_2) be the positive minimizer, the decomposition proved in the previous lemma provides

$$\tilde{E}_\varepsilon^0(\eta_1, \eta_2) + K \le \tilde{E}_\varepsilon^0(\tilde{\eta}_1, \tilde{\eta}_2) + K,$$

so that (3.12) is proved.

On the other hand, relation (2.4) implies

$$2g\left|\int_{\mathbb{R}^2}(\eta_1^2-a_1)(\eta_2^2-a_2)\,dx\right|\le\int_{\mathbb{R}^2}\left\{(g_1-\gamma)(\eta_1^2-a_1)^2+(g_2-\gamma)(\eta_2^2-a_2)^2\right\}\,dx.$$

The result follows by combining this inequality with (3.12). □

In the following proposition, we derive a preliminary estimate for the Lagrange multipliers. Even though this estimate is far from optimal, its form will play an important role when we improve it in Proposition 4.18.

Proposition 3.3. *Assume* (1.3)*,* (1.8) *and* (1.18)*. Let* (η_1,η_2) *be the positive minimizer of* E_ε^0 *in* $\mathcal{H}$*. Let* $\lambda_{i,\varepsilon}$ *be the associated Lagrange multipliers in* (1.5)*. There exists* $C>0$ *independent of* ε *such that, for* $i=1,2$*,*

$$|\lambda_{i,\varepsilon}-\lambda_{i,0}|\le C\varepsilon|\log\varepsilon|^{1/2} \tag{3.13}$$

where $\lambda_{i,0}$ *are defined in* (1.15)*. Given* (1.31)*, this implies*

$$|R_{i,\varepsilon}-R_{i,0}|\le C\varepsilon|\log\varepsilon|^{\frac{1}{2}},\quad i=1,2. \tag{3.14}$$

Proof. We test the equation for η_1 in (1.5) by η_1 itself, integrate by parts [since $\eta_1\in H^1(\mathbb{R}^2)$], and then subtract $\lambda_{1,0}$ from both sides to obtain

$$\lambda_{1,\varepsilon}-\lambda_{1,0}=\int_{\mathbb{R}^2}\left\{\varepsilon^2|\nabla\eta_1|^2+\eta_1^2(|x|^2+g_1\eta_1^2+g\eta_2^2-\lambda_{1,0})\right\}\,dx.$$

With calculations similar to the one used in the proof of Lemma 3.1, we rewrite the right hand side of the previous expression in the following form:

$$\int_{\mathbb{R}^2}\left\{\varepsilon^2|\nabla\eta_1|^2+g_1(\eta_1^2-a_1)^2+g(\eta_1^2-a_1)(\eta_2^2-a_2)+g_1a_1(\eta_1^2-a_1)\right.$$
$$\left.+ga_1(\eta_2^2-a_2)\right\}\,dx+g_1\Gamma\int_{\mathbb{R}^2\setminus D_1}a_{1,0}^-\eta_1^2\,dx+g\int_{\mathbb{R}^2\setminus D_2}\left(a_{2,0}+\frac{g}{g_2}a_{1,0}\right)^-\eta_1^2\,dx.$$

We notice that

$$\left|\int_{\mathbb{R}^2}a_1\left\{g_1(\eta_1^2-a_1)+g(\eta_2^2-a_2)\right\}\,dx\right|\le\|a_1\|_{L^2(\mathbb{R}^2)}\left(g_1\|\eta_1^2-a_1\|_{L^2(\mathbb{R}^2)}\right.$$
$$\left.+g\|\eta_2^2-a_2\|_{L^2(\mathbb{R}^2)}\right).$$

Hence by applying Proposition 3.2 we obtain the convergence of $\lambda_{1,\varepsilon}$. The convergence of $\lambda_{2,\varepsilon}$ can be proved similarly. □

Remark 3.4. The equivalent of Proposition 3.2 and 3.3 hold when (1.9) is assumed instead of (1.8). The only difference is that (3.11) has to be replaced by

$$\int_{\{|x|\ge R_{1,0}\}}\eta_{1,\varepsilon}^2a_{1,0}^-\,dx+\int_{\{|x|\le R_{2,0}^-\}}\eta_{2,\varepsilon}^2a_{2,0}^-\,dx$$
$$+\int_{\{|x|\ge R_{2,0}^+\}}(g\eta_{1,\varepsilon}^2+g_2\eta_{2,\varepsilon}^2)\left(a_{2,0}+\frac{g}{g_2}a_{1,0}\right)^-\,dx\le C\varepsilon^2|\log\varepsilon|.$$

4. Refined Estimates for the Energy Minimizer Without Rotation

In this section we capture the fine behavior of the minimizer $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$, as $\varepsilon \to 0$, by means of a perturbation argument. Since this type of approach is in principle not applicable to problems with integral constraints, we argue indirectly as follows. First, given $(\lambda_{1,\varepsilon}, \lambda_{2,\varepsilon})$ as in the previous section, for small $\varepsilon > 0$, we construct a positive radial solution of (1.5) "near" (a_1, a_2) by a perturbation argument. Then, the uniqueness result in Theorem 1.3 will imply that this solution coincides with the unique positive minimizer of E_ε^0 in $\mathcal{H}$.

4.1. The main result concerning the minimizer without rotation.

Theorem 4.1. *Assume that* (1.3)*,* (1.8) *and* (1.18) *hold. Let* (η_1, η_2) *be the positive minimizer of* E_ε^0 *in* $\mathcal{H}$*. Let* $\lambda_{i,\varepsilon}$ *be the associated Lagrange multipliers in* (1.5)*. There exist constants* $c, C, D > 0$ *and* $\delta \in \left(0, \frac{1}{4}\min\{R_{1,0}, R_{2,0} - R_{1,0}\}\right)$ *such that the following estimates hold:* Estimates for the Lagrange multipliers

$$|\lambda_{i,\varepsilon} - \lambda_{i,0}| \le C|\log\varepsilon|\varepsilon^2, \text{ which implies that } |R_{i,\varepsilon} - R_{i,0}| \le C|\log\varepsilon|\varepsilon^2, \quad i = 1, 2; \tag{4.1}$$

Outer estimates

$$\|\eta_{1,\varepsilon} - \sqrt{a_{1,\varepsilon}}\|_{L^\infty(|x|\le R_{1,\varepsilon}-\delta)} + \|\eta_{2,\varepsilon} - \sqrt{a_{2,\varepsilon}}\|_{L^\infty(|x|\le R_{1,\varepsilon}-\delta)} \le C\varepsilon^2, \tag{4.2}$$

and

$$\left\|\eta_{2,\varepsilon} - \sqrt{a_{2,\varepsilon} + \frac{g}{g_2}a_{1,\varepsilon}}\right\|_{L^\infty(R_{1,\varepsilon}+\delta\le|x|\le R_{2,\varepsilon}-\delta)} \le C\varepsilon^2, \tag{4.3}$$

uniformly as $\varepsilon \to 0$*;*
Algebraic decay estimates

$$\eta_{1,\varepsilon}(r) - \sqrt{a_{1,\varepsilon}(r)} = \mathcal{O}\left(\varepsilon^2|r - R_{1,\varepsilon}|^{-\frac{5}{2}}\right), \quad \eta_{2,\varepsilon}(r) - \sqrt{a_{2,\varepsilon}(r)} = \mathcal{O}\left(\varepsilon^2|r - R_{1,\varepsilon}|^{-2}\right), \tag{4.4}$$

if $r \in [R_{1,\varepsilon} - \delta, R_{1,\varepsilon} - D\varepsilon^{\frac{2}{3}}]$*, and*

$$\eta_{2,\varepsilon}(r) - \sqrt{a_{2,\varepsilon}(r) + \frac{g}{g_2}a_{1,\varepsilon}(r)} = \mathcal{O}\left(\varepsilon^2|r - R_{2,\varepsilon}|^{-\frac{5}{2}}\right) \quad \text{if} \quad r \in [R_{2,\varepsilon} - \delta, R_{2,\varepsilon} - D\varepsilon^{\frac{2}{3}}], \tag{4.5}$$

uniformly as $\varepsilon \to 0$*;*
Exponential decay estimates

$$\eta_{i,\varepsilon}(r) \le C\varepsilon^{\frac{1}{3}} \exp\left\{c\frac{R_{i,\varepsilon} - r}{\varepsilon^{\frac{2}{3}}}\right\}, \quad r \ge R_{i,\varepsilon}, \quad i = 1, 2, \tag{4.6}$$

and

$$\eta_{2,\varepsilon}(r) = \sqrt{a_{2,\varepsilon}(r) + \frac{g}{g_2}a_{1,\varepsilon}(r)} + \mathcal{O}(\varepsilon^{\frac{2}{3}}) \exp\left\{c\frac{R_{1,\varepsilon} - r}{\varepsilon^{\frac{2}{3}}}\right\} + \mathcal{O}(\varepsilon^2) \text{ if } r \in [R_{1,\varepsilon}, R_{1,\varepsilon} + \delta], \tag{4.7}$$

uniformly as $\varepsilon \to 0$*;*

Inner estimates

$$\eta_{1,\varepsilon}(r) = \varepsilon^{\frac{1}{3}}(g_1\Gamma)^{-\frac{1}{6}}\beta_{1,\varepsilon}V\left((g_1\Gamma)^{\frac{1}{3}}\beta_{1,\varepsilon}\frac{r-R_{1,\varepsilon}}{\varepsilon^{\frac{2}{3}}}\right) + \begin{cases} \mathcal{O}\left(\varepsilon+|r-R_{1,\varepsilon}|^{\frac{3}{2}}\right) & \textit{if } r\in[R_{1,\varepsilon}-\delta, R_{1,\varepsilon}], \\ \mathcal{O}(\varepsilon)\exp\left\{c\frac{R_{1,\varepsilon}-r}{\varepsilon^{\frac{2}{3}}}\right\} & \textit{if } r\in[R_{1,\varepsilon}, R_{1,\varepsilon}+\delta], \end{cases} \tag{4.8}$$

$$\eta_{2,\varepsilon}(r) = \sqrt{a_{2,\varepsilon}(r)+\frac{g}{g_2}a_{1,\varepsilon}(r)-\frac{g}{g_2}(g_1\Gamma)^{-\frac{1}{3}}\varepsilon^{\frac{2}{3}}\beta_{1,\varepsilon}^2V^2\left((g_1\Gamma)^{\frac{1}{3}}\beta_{1,\varepsilon}\frac{r-R_{1,\varepsilon}}{\varepsilon^{\frac{2}{3}}}\right)} + \begin{cases} \mathcal{O}\left(\varepsilon^{\frac{4}{3}}+|r-R_{1,\varepsilon}|^2+\varepsilon|r-R_{1,\varepsilon}|^{\frac{1}{2}}+\varepsilon^{\frac{1}{3}}|r-R_{1,\varepsilon}|^{\frac{3}{2}}\right) & \textit{if } r\in[R_{1,\varepsilon}-\delta, R_{1,\varepsilon}], \\ \mathcal{O}(\varepsilon^{\frac{4}{3}})\exp\left\{c\frac{R_{1,\varepsilon}-r}{\varepsilon^{\frac{2}{3}}}\right\}+\mathcal{O}(\varepsilon^2) & \textit{if } r\in[R_{1,\varepsilon}, R_{1,\varepsilon}+\delta], \end{cases} \tag{4.9}$$

and

$$\eta_{2,\varepsilon}(r) = \varepsilon^{\frac{1}{3}}g_2^{-\frac{1}{6}}\beta_{2,\varepsilon}V\left(g_2^{\frac{1}{3}}\beta_{2,\varepsilon}\frac{r-R_{2,\varepsilon}}{\varepsilon^{\frac{2}{3}}}\right) + \begin{cases} \mathcal{O}\left(\varepsilon+|r-R_{2,\varepsilon}|^{\frac{3}{2}}\right) & \textit{if } r\in[R_{2,\varepsilon}-\delta, R_{2,\varepsilon}], \\ \mathcal{O}(\varepsilon)\exp\left\{c\frac{R_{2,\varepsilon}-r}{\varepsilon^{\frac{2}{3}}}\right\} & \textit{if } r\in[R_{2,\varepsilon}, R_{2,\varepsilon}+\delta], \end{cases} \tag{4.10}$$

uniformly as $\varepsilon \to 0$*, where* V *is the Hastings–McLeod solution* [23] *to the Painlevé-II equation* [19]*, namely the unique solution of the boundary value problem*

$$v'' = v(v^2+s),\ s\in\mathbb{R};\ v(s)-\sqrt{-s}\to 0 \text{ as } s\to-\infty,\ v(s)\to 0 \text{ as } s\to\infty, \tag{4.11}$$

and

$$\beta_{1,\varepsilon} = \left(-a'_{1,\varepsilon}(R_{1,\varepsilon})\right)^{\frac{1}{3}},\quad \beta_{2,\varepsilon} = \left(-a'_{2,\varepsilon}(R_{2,\varepsilon})-\frac{g}{g_2}a'_{1,\varepsilon}(R_{2,\varepsilon})\right)^{\frac{1}{3}}.$$

The proof of this theorem will be completed in Sect. 4.8. Note that in the case $g_1 = g_2$, then an analogous of this theorem holds. It is simpler since $\eta_{1,\varepsilon} = \eta_{2,\varepsilon}$ so that $R_{1,\varepsilon} = R_{2,\varepsilon}$. The result is just a consequence of Theorem A.1 in the appendix.

4.2. An approximate solution. In this subsection we construct a sufficiently good approximate solution $(\check{\eta}_{1,\varepsilon}, \check{\eta}_{2,\varepsilon})$ to the problem (1.29) such that $\check{\eta}_{i,\varepsilon} > 0$ and $\check{\eta}_{i,\varepsilon} \to \sqrt{a_i}$, uniformly on $\mathbb{R}^2$, as $\varepsilon \to 0$, $i = 1, 2$. The building blocks of our construction will be the unique positive solutions $\hat{\eta}_{1,\varepsilon}$ and $\hat{\eta}_{2,\varepsilon}$ of the reduced problems (1.33a) and (1.33b) respectively.

4.2.1. The reduced problems. The asymptotic behavior of $\hat{\eta}_{i,\varepsilon}$, $i = 1, 2$, as $\varepsilon \to 0$, can be deduced, after a proper constant re-scaling, from the following proposition which is a special case of the more general result that we prove in Appendix A.

Proposition 4.2. *Suppose that λ_ε satisfy $\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$, for some $\lambda_0 > 0$, and let*

$$A_\varepsilon(x) = \lambda_\varepsilon - \mu|x|^2, \quad x \in \mathbb{R}^2,$$

with $\mu > 0$ independent of ε.

There exists a unique positive solution u_ε of the problem

$$\varepsilon^2 \Delta u = u\left(u^2 - A_\varepsilon(x)\right), \quad x \in \mathbb{R}^2; \quad u(x) \to 0 \text{ as } |x| \to \infty.$$

This solution is radially symmetric and, for small $\varepsilon > 0$, satisfies the following properties:

$$\|u_\varepsilon - \sqrt{A_\varepsilon^+}\|_{L^\infty(\mathbb{R}^2)} \le C\varepsilon^{\frac{1}{3}}, \quad \|u_\varepsilon - \sqrt{A_\varepsilon}\|_{C^2(|x| \le r_\varepsilon - \delta)} \le C\varepsilon^2, \tag{4.12}$$

where $r_\varepsilon = \left(\mu^{-1}\lambda_\varepsilon\right)^{\frac{1}{2}}$, for some $\delta \in \left(0, \frac{1}{4}r_0\right)$, $r_0 = (\mu^{-1}\lambda_0)^{1/2}$, and

$$u_\varepsilon(r) \le C\varepsilon^{\frac{1}{3}} \exp\left\{c\varepsilon^{-\frac{2}{3}}(r_\varepsilon - r)\right\}, \quad r \ge r_\varepsilon. \tag{4.13}$$

In fact, the potential of the associated linearized operator satisfies the lower bound

$$3u_\varepsilon^2(r) - A_\varepsilon(r) \ge \begin{cases} c|r - r_\varepsilon| + c\varepsilon^{\frac{2}{3}} & \text{if } |r - r_\varepsilon| \le \delta, \\ c & \text{otherwise}. \end{cases} \tag{4.14}$$

More precisely, we have

$$u_\varepsilon(r) = \varepsilon^{\frac{1}{3}}\beta_\varepsilon V\left(\beta_\varepsilon \frac{r - r_\varepsilon}{\varepsilon^{\frac{2}{3}}}\right) + \begin{cases} \mathcal{O}\left(\varepsilon + |r - r_\varepsilon|^{\frac{3}{2}}\right) & \text{if } r_\varepsilon - \delta \le r \le r_\varepsilon, \\ \mathcal{O}(\varepsilon)\exp\left\{-c\frac{|r - r_\varepsilon|}{\varepsilon^{\frac{2}{3}}}\right\} & \text{if } r_\varepsilon \le r \le r_\varepsilon + \delta, \end{cases} \tag{4.15}$$

where V is the Hastings–McLeod solution, as described in (4.11), and

$$\beta_\varepsilon = \left(-A_\varepsilon'(r_\varepsilon)\right)^{\frac{1}{3}}.$$

Furthermore, we have

$$u_\varepsilon'(r) = \varepsilon^{-\frac{1}{3}}\beta_\varepsilon^2 V'\left(\beta_\varepsilon \frac{r - r_\varepsilon}{\varepsilon^{\frac{2}{3}}}\right) + \mathcal{O}\left(\varepsilon^{\frac{1}{3}} + |r - r_\varepsilon|^{\frac{1}{2}}\right) \quad \text{if } |r - r_\varepsilon| \le \delta, \tag{4.16}$$

uniformly, as $\varepsilon \to 0$. Moreover, there exists $D > 0$ such that the following estimates hold for $r \in [r_\varepsilon - \delta, r_\varepsilon - D\varepsilon^{\frac{2}{3}}]$:

$$\begin{gathered} u_\varepsilon(r) - \sqrt{A_\varepsilon(r)} = \varepsilon^2\mathcal{O}(|r - r_\varepsilon|^{-\frac{5}{2}}), \quad u_\varepsilon' - \left(\sqrt{A_\varepsilon}\right)' = \varepsilon^2\mathcal{O}(|r - r_\varepsilon|^{-\frac{7}{2}}), \\ \Delta u_\varepsilon - \Delta\left(\sqrt{A_\varepsilon}\right) = \varepsilon^2\mathcal{O}(|r - r_\varepsilon|^{-\frac{9}{2}}), \end{gathered} \tag{4.17}$$

uniformly, as $\varepsilon \to 0$.

4.2.2. Gluing approximate solutions. Consider a one-dimensional smooth cutoff function ζ such that

$$\zeta(t) = 1 \quad \text{if } t \le R_\varepsilon - \delta; \quad \zeta(t) = 0 \quad \text{if } t \ge R_\varepsilon, \tag{4.18}$$

where, for convenience, we have denoted

$$R_\varepsilon = \frac{R_{1,\varepsilon} + R_{2,\varepsilon}}{2}, \tag{4.19}$$

and $\delta > 0$ is a small number that is independent of small $\varepsilon > 0$. Note that ζ can be chosen independent of $\varepsilon > 0$ as well. We recall that $\hat{\eta}_{1,\varepsilon}, \hat{\eta}_{2,\varepsilon}$ are the solutions of (1.33). In view of (4.13), let

$$\check{\eta}_{1,\varepsilon}(x) = \zeta(|x|)\hat{\eta}_{1,\varepsilon}(x), \quad x \in \mathbb{R}^2. \tag{4.20}$$

Then, let

$$\tilde{\eta}_{2,\varepsilon}(x) = \left(a_{2,\varepsilon}(x) + \frac{g}{g_2}a_{1,\varepsilon}(x) - \frac{g}{g_2}\check{\eta}_{1,\varepsilon}^2\right)^{\frac{1}{2}}, \quad |x| \le R_\varepsilon + \delta. \tag{4.21}$$

The motivation for this comes from neglecting the term $\varepsilon^2\Delta\eta_2$ in (1.29b), since it is expected to be of higher order, compared to the other terms, in the region $|x| \le R_{2,\varepsilon} - \delta$.

From (4.12), it follows that

$$\hat{\eta}_{2,\varepsilon}(x) - \left(a_{2,\varepsilon}(x) + \frac{g}{g_2}a_{1,\varepsilon}(x)\right)^{\frac{1}{2}} = \mathcal{O}(\varepsilon^2), \quad \text{in } C^2\,(R_\varepsilon \le |x| \le R_\varepsilon + \delta), \text{ as } \varepsilon \to 0.$$

In other words, recalling (4.18) and (4.21), we have that

$$\hat{\eta}_{2,\varepsilon}(x) - \tilde{\eta}_{2,\varepsilon}(x) = \mathcal{O}(\varepsilon^2), \quad \text{in } C^2\,(R_\varepsilon \le |x| \le R_\varepsilon + \delta), \quad \text{as } \varepsilon \to 0. \tag{4.22}$$

Thus, we can smoothly interpolate between $\tilde{\eta}_{2,\varepsilon}$ and $\hat{\eta}_{2,\varepsilon}$ to obtain a new approximation $\check{\eta}_{2,\varepsilon}$ such that

$$\check{\eta}_{2,\varepsilon}(x) = \begin{cases} \tilde{\eta}_{2,\varepsilon}(x), & |x| \le R_\varepsilon, \\ \tilde{\eta}_{2,\varepsilon}(x) + \mathcal{O}_{C^2}(\varepsilon^2), & R_\varepsilon \le |x| \le R_\varepsilon + \delta, \\ \hat{\eta}_{2,\varepsilon}(x), & |x| \ge R_\varepsilon + \delta. \end{cases} \tag{4.23}$$

To conclude, we define our approximate solution of the system (1.29), for small $\varepsilon > 0$, to be the pair $(\check{\eta}_{1,\varepsilon}, \check{\eta}_{2,\varepsilon})$, as described by (4.20) and (4.23). We point out that this approximation satisfies the desired limiting behavior

$$(\check{\eta}_{1,\varepsilon}, \check{\eta}_{2,\varepsilon}) \to (\sqrt{a_1}, \sqrt{a_2}), \text{ uniformly in } \mathbb{R}^2, \text{ as } \varepsilon \to 0, \tag{4.24}$$

where a_1 and a_2 are as in (1.12). Moreover, estimates that quantify this convergence can be derived easily from the corresponding ones that are available for the ground states $\hat{\eta}_{1,\varepsilon}$ and $\hat{\eta}_{2,\varepsilon}$ from Proposition 4.2.

4.3. Estimates for the error on the approximate solution. The remainder that is left when substituting the approximate solution $(\check{\eta}_{1,\varepsilon}, \check{\eta}_{2,\varepsilon})$ to the system (1.29) is

$$\mathcal{E}(\check{\eta}_{1,\varepsilon}, \check{\eta}_{2,\varepsilon}) \equiv \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \equiv \begin{pmatrix} -\varepsilon^2 \Delta\check{\eta}_{1,\varepsilon} + g_1 \check{\eta}_{1,\varepsilon}\left(\check{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon}\right) + g\check{\eta}_{1,\varepsilon}\left(\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon}\right) \\ -\varepsilon^2 \Delta\check{\eta}_{2,\varepsilon} + g_2 \check{\eta}_{2,\varepsilon}\left(\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon}\right) + g\check{\eta}_{2,\varepsilon}\left(\check{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon}\right) \end{pmatrix}. \tag{4.25}$$

The next proposition provides estimates for the L^2-norms of E_i, $i = 1, 2$, which follow from some delicate pointwise estimates that will be established in the process.

Proposition 4.3. *The following estimates hold for small $\varepsilon > 0$:*

$$\|E_1\|_{L^2(\mathbb{R}^2)} \leq Ce^{-c\varepsilon^{-\frac{2}{3}}}, \quad \|E_2\|_{L^2(|x|<R_\varepsilon)} \leq C\varepsilon^{\frac{5}{3}} \quad \textit{and} \quad \|E_2\|_{L^2(|x|>R_\varepsilon)} \leq C\varepsilon^2. \tag{4.26}$$

Proof. It follows from the construction of $\check{\eta}_{1,\varepsilon}$ and $\check{\eta}_{2,\varepsilon}$, via (4.13), that

$$E_1 = 0 \quad \text{if } |x| \leq R_\varepsilon - \delta \text{ or } |x| \geq R_\varepsilon; \quad |E_1| \leq Ce^{-c\varepsilon^{-\frac{2}{3}}} \quad \text{if } R_\varepsilon - \delta \leq |x| \leq R_\varepsilon. \tag{4.27}$$

On the other side, we have

$$\begin{aligned} &E_2 = -\varepsilon^2 \Delta\tilde{\eta}_{2,\varepsilon} \ \text{ if } |x| \leq R_\varepsilon - \delta; \\ &E_2 = -\varepsilon^2 \Delta\tilde{\eta}_{2,\varepsilon} + \mathcal{O}(e^{-c\varepsilon^{-\frac{2}{3}}}) \text{ uniformly if } R_\varepsilon - \delta \leq |x| \leq R_\varepsilon, \text{ as } \varepsilon \to 0; \\ &E_2 = 0 \text{ if } |x| \geq R_\varepsilon + \delta; \ \ |E_2| \leq C\varepsilon^2 \text{ if } R_\varepsilon \leq |x| \leq R_\varepsilon + \delta \text{ (recall (4.22))}. \end{aligned} \tag{4.28}$$

In view of the previous observations, we only have to show the second relation in (4.26). In fact, since $\Delta\tilde{\eta}_{2,\varepsilon}$ remains uniformly bounded if $|r - R_{1,\varepsilon}| \geq \delta$ as $\varepsilon \to 0$, it suffices to show that

$$\|\Delta\tilde{\eta}_{2,\varepsilon}\|_{L^2(|r-R_{1,\varepsilon}|<\delta)} \leq C\varepsilon^{-\frac{1}{3}} \quad \text{for small } \varepsilon > 0. \tag{4.29}$$

It follows readily from (4.21) and (4.23) that

$$|\Delta\tilde{\eta}_{2,\varepsilon}| \leq C\hat{\eta}_{1,\varepsilon}^2 |\nabla\hat{\eta}_{1,\varepsilon}|^2 + C\left|\hat{\eta}_{1,\varepsilon}\Delta\hat{\eta}_{1,\varepsilon} + |\nabla\hat{\eta}_{1,\varepsilon}|^2\right| + C \ \text{ if } |r - R_{1,\varepsilon}| < \delta. \tag{4.30}$$

Next, we estimate the terms in the right-hand side by making use of Proposition 4.2. To this end, we need to derive a relation for $\Delta\hat{\eta}_{1,\varepsilon}$ in terms of the Hastings–McLeod solution near $R_{1,\varepsilon}$. Making use of (1.33a), (4.15) and the natural bound $|V(s)| \leq C(|s|^{\frac{1}{2}}+1)$, $s \in \mathbb{R}$, setting $\tilde{\varepsilon} = (g_1\Gamma)^{-\frac{1}{2}}\varepsilon$, after a tedious calculation, we arrive at

$$\begin{aligned} \hat{\eta}_{1,\varepsilon}\Delta\hat{\eta}_{1,\varepsilon} = {} & \tilde{\varepsilon}^{-\frac{2}{3}}\beta_\varepsilon^4 V^2\left(\beta_\varepsilon \frac{r - R_{1,\varepsilon}}{\tilde{\varepsilon}^{\frac{2}{3}}}\right)\left[V^2\left(\beta_\varepsilon \frac{r - R_{1,\varepsilon}}{\tilde{\varepsilon}^{\frac{2}{3}}}\right) + \beta_\varepsilon \frac{r - R_{1,\varepsilon}}{\tilde{\varepsilon}^{\frac{2}{3}}}\right] \\ & + \mathcal{O}\left(\varepsilon^{-2}|r - R_{1,\varepsilon}|^3 + \varepsilon^{-\frac{4}{3}}|r - R_{1,\varepsilon}|^2 + \varepsilon^{-1}|r - R_{1,\varepsilon}|^{\frac{3}{2}}\right) \\ & + \mathcal{O}\left(\varepsilon^{-\frac{1}{3}}|r - R_{1,\varepsilon}|^{\frac{1}{2}} + \varepsilon^{-\frac{2}{3}}|r - R_{1,\varepsilon}| + 1 + \varepsilon^{-\frac{5}{3}}|r - R_{1,\varepsilon}|^{\frac{5}{2}}\right), \end{aligned} \tag{4.31}$$

uniformly if $|r - R_{1,\varepsilon}| \le \delta$, as $\varepsilon \to 0$, where $\beta_\varepsilon^3 = -a'_{1,\varepsilon}(R_{1,\varepsilon})$. Similarly, but with considerably less effort, it follows from (4.16), and the bound $\left|V'(s)\right| \le C\,(|s|+1)^{-\frac{1}{2}}$, that

$$|\nabla\hat{\eta}_{1,\varepsilon}(r)|^2 = \tilde{\varepsilon}^{-\frac{2}{3}}\beta_\varepsilon^4\left[V'\left(\beta_\varepsilon\frac{r-R_{1,\varepsilon}}{\tilde{\varepsilon}^{\frac{2}{3}}}\right)\right]^2 + \mathcal{O}(1),$$
$$\text{uniformly if } |r-R_{1,\varepsilon}| \le \delta, \text{ as } \varepsilon \to 0. \tag{4.32}$$

It follows readily from the estimates in (4.17) that

$$\hat{\eta}_{1,\varepsilon}^2|\nabla\hat{\eta}_{1,\varepsilon}|^2 \le C, \tag{4.33}$$

and

$$\hat{\eta}_{1,\varepsilon}\Delta\hat{\eta}_{1,\varepsilon} + |\nabla\hat{\eta}_{1,\varepsilon}|^2 = \mathcal{O}\left(1+\varepsilon^2|r-R_{1,\varepsilon}|^{-4}\right), \tag{4.34}$$

uniformly in $-\delta \le r - R_{1,\varepsilon} \le -C\varepsilon^{\frac{2}{3}}$, as $\varepsilon \to 0$. Keep in mind that our eventual goal is to show (4.29). In view of (4.30), the above relations imply the partial estimate

$$\|\Delta\tilde{\eta}_{2,\varepsilon}\|_{L^\infty(-\delta\le r-R_{1,\varepsilon}\le -\varepsilon^{\frac{7}{12}})} \le C\varepsilon^{-\frac{1}{3}} \quad \text{for small } \varepsilon > 0. \tag{4.35}$$

On the other side, by the exponential decay of $\hat{\eta}_{1,\varepsilon}$ for $r > R_{1,\varepsilon}$, we certainly have that

$$\|\Delta\tilde{\eta}_{2,\varepsilon}\|_{L^\infty(\varepsilon^{\frac{7}{12}}\le r-R_{1,\varepsilon}\le\delta)} \le C\varepsilon^{-\frac{1}{3}} \quad \text{for small } \varepsilon > 0. \tag{4.36}$$

In the remaining interval $(R_{1,\varepsilon} - \varepsilon^{\frac{7}{12}}, R_{1,\varepsilon} + \varepsilon^{\frac{7}{12}})$ we use the inner estimates (4.15), (4.31), and (4.32), which in particular imply that

$$\hat{\eta}_{1,\varepsilon}^2|\nabla\hat{\eta}_{1,\varepsilon}|^2 \le C, \tag{4.37}$$

and

$$\begin{aligned}\hat{\eta}_{1,\varepsilon}\Delta\hat{\eta}_{1,\varepsilon} + |\nabla\hat{\eta}_{1,\varepsilon}|^2 = \tilde{\varepsilon}^{-\frac{2}{3}}\beta_\varepsilon^4\Bigg\{&V\left(\beta_\varepsilon\frac{r-R_{1,\varepsilon}}{\tilde{\varepsilon}^{\frac{2}{3}}}\right)V''\left(\beta_\varepsilon\frac{r-R_{1,\varepsilon}}{\tilde{\varepsilon}^{\frac{2}{3}}}\right)\\ &+\left[V'\left(\beta_\varepsilon\frac{r-R_{1,\varepsilon}}{\tilde{\varepsilon}^{\frac{2}{3}}}\right)\right]^2\Bigg\} + \mathcal{O}\left(\varepsilon^{-\frac{1}{4}}\right),\end{aligned} \tag{4.38}$$

uniformly if $|r - R_{1,\varepsilon}| \le \varepsilon^{\frac{7}{12}}$, as $\varepsilon \to 0$ [for obtaining the last relation, we have also used (4.11)]. In order to proceed, we need the following easy estimate:

$$V(s)V''(s) + \left[V'(s)\right]^2 = \mathcal{O}(|s|^{-4}) \text{ as } |s| \to \infty, \tag{4.39}$$

which follows from the asymptotic behavior $V(s) = (-s)^{\frac{1}{2}} + \mathcal{O}\left(|s|^{-\frac{5}{2}}\right)$ as $s \to -\infty$, and from the super-exponential decay of V and its derivatives as $s \to \infty$. Now, by (4.37), (4.38), and (4.39), via (4.30), we deduce that

$$\|\Delta\tilde{\eta}_{2,\varepsilon}\|_{L^2(|r-R_{1,\varepsilon}|\le\varepsilon^{\frac{7}{12}})} \le C\varepsilon^{-\frac{1}{3}} \quad \text{for small } \varepsilon > 0. \tag{4.40}$$

Finally, the desired estimate (4.29) follows directly from (4.35), (4.36) and (4.40). □

4.4. Linear analysis. In this part of the paper we are going to study the linearization of (1.29) about the approximate solution $(\check{\eta}_{1,\varepsilon}, \check{\eta}_{2,\varepsilon})$, namely the linear operator

$$\mathcal{L}(\varphi, \psi) \equiv \begin{pmatrix} -\varepsilon^2 \Delta\varphi + \left[g_1(3\check{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon}) + g(\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon})\right]\varphi + 2g\check{\eta}_{1,\varepsilon}\check{\eta}_{2,\varepsilon}\psi \\ -\varepsilon^2 \Delta\psi + \left[g_2(3\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon}) + g(\check{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon})\right]\psi + 2g\check{\eta}_{1,\varepsilon}\check{\eta}_{2,\varepsilon}\varphi \end{pmatrix}, \tag{4.41}$$

for $(\varphi, \psi) \in D(\mathcal{L}) = \left\{(u, v) \in H^2(\mathbb{R}^2) \times H^2(\mathbb{R}^2) \ : \ \int_{\mathbb{R}^2} |x|^2(u^2 + v^2)dx < \infty\right\}$. By Friedrichs extension, the operator $\mathcal{L}$ is self-adjoint in $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ with domain $D(\mathcal{L})$.

4.4.1. Energy estimates for $\mathcal{L}$. We estimate from below the quotient

$$\frac{(\mathcal{L}(\varphi, \psi), (\varphi, \psi))}{\langle(\varphi, \psi), (\varphi, \psi)\rangle}, \quad (\varphi, \psi) \in D(\mathcal{L})\backslash(0, 0),$$

which turns out to be positive, where $(\cdot, \cdot)$ symbolizes the usual inner product in $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ while $< \cdot, \cdot >$ is a suitably weighted one. In turn, these lower bounds provide a-priori estimates for the problem $\mathcal{L}(\varphi, \psi) = (f_1, f_2)$.

Energy estimates in B_{R_ε}. In the sequel, we carry out this plan in detail in the domain B_{R_ε}. Analogous results can be deduced in $\mathbb{R}^2\backslash B_{R_\varepsilon}$ which we will describe later.

In view of (4.21) and (4.23), which imply that $\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon} = -\frac{g}{g_2}(\check{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon})$ in B_{R_ε}, we can conveniently rewrite (4.41) in B_{R_ε} as

$$\mathcal{L}(\varphi, \psi) = \begin{pmatrix} -\varepsilon^2 \Delta\varphi + \left[\left(g_1 - \frac{g^2}{g_2}\right)(3\check{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon})\right]\varphi + 2\frac{g^2}{g_2}\check{\eta}_{1,\varepsilon}^2\varphi + 2g\check{\eta}_{1,\varepsilon}\check{\eta}_{2,\varepsilon}\psi \\ -\varepsilon^2 \Delta\psi + 2g_2\check{\eta}_{2,\varepsilon}^2\psi + 2g\check{\eta}_{1,\varepsilon}\check{\eta}_{2,\varepsilon}\varphi \end{pmatrix}, \tag{4.42}$$

for $\varphi, \psi \in H^2(B_{R_\varepsilon})$ (the reason for not adding the similar terms in the first row is to keep the linearization of (1.33a) about $\check{\eta}_{1,\varepsilon}$ in the beginning).

Proposition 4.4. *The following a-priori estimates hold: Suppose that*

$$\varphi, \ \psi \in H_N^2(B_{R_\varepsilon}) \equiv \left\{v \in H^2(B_{R_\varepsilon}) \ : \ \nu \cdot \nabla v = 0 \ \text{ on } \partial B_{R_\varepsilon}\right\},$$

where ν denotes the outer unit normal vector to $\partial B_{R_\varepsilon}$, satisfy

$$\mathcal{L}(\varphi, \psi) = \lambda(\varepsilon^{\frac{2}{3}}\varphi, \psi) \quad \text{and} \quad \int_{B_{R_\varepsilon}} \left(\varepsilon^{\frac{2}{3}}\varphi^2 + \psi^2\right) dx = 1, \tag{4.43}$$

or

$$\mathcal{L}(\varphi, \psi) = \lambda(\check{\eta}_{1,\varepsilon}^2\varphi, \psi) \quad \text{and} \quad \int_{B_{R_\varepsilon}} \left(\check{\eta}_{1,\varepsilon}^2\varphi^2 + \psi^2\right) dx = 1, \tag{4.44}$$

then

$$\lambda \geq c.$$

Proof. We use the following estimates:

$$3\check{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon} \geq \begin{cases} c|r - R_{1,\varepsilon}| + c\varepsilon^{\frac{2}{3}}, & \text{if } |r - R_{1,\varepsilon}| \leq \delta, \\ c, & \text{otherwise}, \end{cases} \tag{4.45}$$

and

$$\check{\eta}_{1,\varepsilon}^2 \leq \begin{cases} C\varepsilon^{\frac{2}{3}} + C|r - R_{1,\varepsilon}|, & \text{if } |r - R_{1,\varepsilon}| \leq \delta, \\ C, & \text{otherwise}, \end{cases} \tag{4.46}$$

which are inherited from (4.12), (4.14) and (4.15). In particular, observe that

$$3\check{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon} \geq c\check{\eta}_{1,\varepsilon}^2, \quad x \in \mathbb{R}^2. \tag{4.47}$$

Note also that

$$2g_2\check{\eta}_{2,\varepsilon}^2 \geq c \ \text{ on } \bar{B}_{R_\varepsilon}. \tag{4.48}$$

Suppose that (4.43) holds. Testing by (φ, ψ), in the usual sense of $L^2(B_{R_\varepsilon}) \times L^2(B_{R_\varepsilon})$, and integrating by parts the resulting identity, we obtain that

$$\int_{B_{R_\varepsilon}} \left[\varepsilon^2|\nabla\varphi|^2 + \varepsilon^2|\nabla\psi|^2 + \left(g_1 - \frac{g^2}{g_2}\right)(3\check{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon})\varphi^2 \right. \\ \left. + 2\left(\frac{g}{\sqrt{g_2}}\check{\eta}_{1,\varepsilon}\varphi + \sqrt{g_2}\check{\eta}_{2,\varepsilon}\psi\right)^2\right] dx = \lambda. \tag{4.49}$$

In turn, using (4.45) and (4.47), we find that

$$\int_{B_{R_\varepsilon}} \left[\varepsilon^2|\nabla\varphi|^2 + \varepsilon^2|\nabla\psi|^2 + c\varepsilon^{\frac{2}{3}}\varphi^2 + c\check{\eta}_{1,\varepsilon}^2\varphi^2\right] dx \leq \lambda. \tag{4.50}$$

On the other side, the second equation of the system in (4.43) can be written as

$$-\varepsilon^2\Delta\psi + 2g_2\check{\eta}_{2,\varepsilon}^2\psi = \lambda\psi - 2g\check{\eta}_{2,\varepsilon}\check{\eta}_{1,\varepsilon}\varphi.$$

Then, testing the above relation by ψ, integrating by parts, using (4.48) and Young's inequality, we obtain

$$\int_{B_{R_\varepsilon}} \left(\varepsilon^2|\nabla\psi|^2 + c\psi^2\right) dx \leq \int_{B_{R_\varepsilon}} \left(\lambda\psi^2\, dx + C\check{\eta}_{1,\varepsilon}^2\varphi^2\right) dx \\ \overset{(4.43),(4.50)}{\leq} C\lambda. \tag{4.51}$$

Finally, by adding (4.50) and (4.51), recalling the integral constraint in (4.43), we deduce that $\lambda \geq c$, as desired. The case where (4.44) holds can be treated analogously. □

A direct consequence of Proposition 4.4, and of (4.49), is the following

Corollary 4.5. *If ε is sufficiently small, there exists $c > 0$ such that*

$$(\mathcal{L}(\varphi, \psi), (\varphi, \psi)) \geq c \int_{B_{R_\varepsilon}} \left(\varepsilon^2|\nabla\varphi|^2 + \varepsilon^2|\nabla\psi|^2 + \varepsilon^{\frac{2}{3}}\varphi^2 + \check{\eta}_{1,\varepsilon}^2\varphi^2 + \psi^2\right) dx \quad \forall\varphi, \psi \in D(\mathcal{L}), \tag{4.52}$$

where $(\cdot, \cdot)$ denotes the usual inner product in $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$.

Energy estimates in $\mathbb{R}^2 \backslash B_{R_\varepsilon}$. Since $\check{\eta}_{1,\varepsilon} = 0$ in $\mathbb{R}^2 \backslash B_{R_\varepsilon}$, the operator $\mathcal{L}$ in $\mathbb{R}^2 \backslash B_{R_\varepsilon}$ has the simple "decoupled" form

$$\mathcal{L}(\varphi, \psi) = \begin{pmatrix} -\varepsilon^2\Delta\varphi + \left[g(\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon}) - g_1 a_{1,\varepsilon}\right]\varphi \\ -\varepsilon^2\Delta\psi + g_2\left(3\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon} - \frac{g}{g_2}a_{1,\varepsilon}\right)\psi \end{pmatrix},$$

for $\varphi, \psi \in H^2(\mathbb{R}^2 \setminus B_{R_\varepsilon})$. Note that

$$g(\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon}) - g_1 a_{1,\varepsilon} = g\left(\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon} - \frac{g}{g_2}a_{1,\varepsilon}\right) + \left(\frac{g^2}{g_2} - g_1\right)a_{1,\varepsilon} \geq c,$$

in $\mathbb{R}^2 \setminus B_{R_\varepsilon}$, because $a_{1,\varepsilon} \leq -c$ and $\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon} - \frac{g}{g_2}a_{1,\varepsilon} > -C\varepsilon^{\frac{2}{3}}$ therein, which follows from (3.13), (4.15), (4.17) and (4.23).

In analogy to (4.52), for small $\varepsilon > 0$, using that $3\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon} - \frac{g}{g_2}a_{1,\varepsilon} > c(\varepsilon^{\frac{2}{3}} + \check{\eta}_{2,\varepsilon}^2)$, which follows analogously to (4.45) and (4.47), one can show rather straightforwardly that

$$(\mathcal{L}(\varphi, \psi), (\varphi, \psi)) \geq c \int_{\mathbb{R}^2 \backslash B_{R_\varepsilon}} \left(\varepsilon^2|\nabla\varphi|^2 + \varepsilon^2|\nabla\psi|^2 + \varphi^2 + \varepsilon^{\frac{2}{3}}\psi^2 + \check{\eta}_{2,\varepsilon}^2\psi^2\right) dx, \tag{4.53}$$

for every $(\varphi, \psi) \in D(\mathcal{L})$.

Energy estimates in $\mathbb{R}^2$. It follows at once from (4.52) and (4.53) that, for small $\varepsilon > 0$, we have

$$(\mathcal{L}(\varphi, \psi), (\varphi, \psi)) \geq c\varepsilon^2 \int_{\mathbb{R}^2} \left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) dx + c\int_{B_{R_\varepsilon}} \left(\varepsilon^{\frac{2}{3}}\varphi^2 + \check{\eta}_{1,\varepsilon}^2\varphi^2 + \psi^2\right) dx$$
$$+ c\int_{\mathbb{R}^2 \backslash B_{R_\varepsilon}} \left(\varphi^2 + \varepsilon^{\frac{2}{3}}\psi^2 + \check{\eta}_{2,\varepsilon}^2\psi^2\right) dx, \tag{4.54}$$

for every $(\varphi, \psi) \in D(\mathcal{L})$.

4.4.2. Invertibility properties of $\mathcal{L}$. We are now in position to obtain estimates for the solution of the inhomogeneous problem $\mathcal{L}(\varphi, \psi) = (f_1, f_2)$ in $\mathbb{R}^2$, with suitable right-hand side.

Proposition 4.6. *Let $f_i \in L^2(\mathbb{R}^2)$, $i = 1, 2$. The equation*

$$\mathcal{L}(\varphi, \psi) = (f_1, f_2) \text{ in } \mathbb{R}^2, \tag{4.55}$$

has a unique solution $(\varphi, \psi) \in H^2(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$*, provided that* $\varepsilon > 0$ *is sufficiently small, independently of* f_i*. Moreover, that solution satisfies*

$$|||(\varphi,\psi)|||^2 \le C\int_{B_{R_\varepsilon}} \left(\varepsilon^{-\frac{2}{3}} f_1^2 + f_2^2\right) dx + C\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}} \left(f_1^2 + \varepsilon^{-\frac{2}{3}} f_2^2\right) dx, \tag{4.56}$$

with $C > 0$ *independent of* ε *and* f_i*, where the norm* $|||(\cdot,\cdot)|||$ *in* $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ *is defined by*

$$|||(\varphi,\psi)|||^2 = \varepsilon^2\left(\|\nabla\varphi\|^2_{L^2(\mathbb{R}^2)} + \|\nabla\psi\|^2_{L^2(\mathbb{R}^2)}\right) + \varepsilon^{\frac{2}{3}}\|\varphi\|^2_{L^2(B_{R_\varepsilon})} + \|\psi\|^2_{L^2(B_{R_\varepsilon})} \\ + \|\varphi\|^2_{L^2(\mathbb{R}^2\setminus B_{R_\varepsilon})} + \varepsilon^{\frac{2}{3}}\|\psi\|^2_{L^2(\mathbb{R}^2\setminus B_{R_\varepsilon})}. \tag{4.57}$$

If $\varepsilon > 0$ *is sufficiently small, there exist* $c, C > 0$ *such that, for any* $f \in L^2(\mathbb{R}^2)$*, the solution of*

$$\mathcal{L}(\varphi,\psi) = (\check{\eta}_{1,\varepsilon} f, 0) \ \ in\ \mathbb{R}^2, \tag{4.58}$$

satisfies

$$|||(\varphi,\psi)|||^2 \le C\int_{B_{R_\varepsilon}} f^2\, dx + Ce^{-c\varepsilon^{-\frac{2}{3}}}\int_{\mathbb{R}^2} f^2\, dx. \tag{4.59}$$

If $\varepsilon > 0$ *is sufficiently small, there exists* $C > 0$ *such that, for any* $f \in L^2(\mathbb{R}^2)$*, the solution of*

$$\mathcal{L}(\varphi,\psi) = (0, \check{\eta}_{2,\varepsilon} f) \ \ in\ \mathbb{R}^2, \tag{4.60}$$

satisfies

$$|||(\varphi,\psi)||| \le C\|f\|_{L^2(\mathbb{R}^2)}. \tag{4.61}$$

Proof. As we have already discussed, the linear operator $\mathcal{L}$ is self-adjoint in $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ with domain $D(\mathcal{L})$. Relation (4.54) certainly implies that the kernel of $\mathcal{L}$ is empty for small $\varepsilon > 0$. Hence, the existence and uniqueness of a solution $(\varphi, \psi) \in H^2(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$ to (4.55) are clear. We now turn our attention to establishing estimate (4.56). Testing (4.55) by (φ, ψ), and using part of (4.54), we find that

$$\varepsilon^2\int_{\mathbb{R}^2}\left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) dx + \int_{B_{R_\varepsilon}}\left(\varepsilon^{\frac{2}{3}}\varphi^2 + \psi^2\right) dx + \int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}\left(\varphi^2 + \varepsilon^{\frac{2}{3}}\psi^2\right) dx \\ \le C\int_{B_{R_\varepsilon}}\left(|f_1\varphi| + |f_2\psi|\right) dx + C\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}\left(|f_1\varphi| + |f_2\psi|\right) dx.$$

Using Young's inequality, we can bound the first integral in the right-hand side by

$$\int_{B_{R_\varepsilon}}\left(\frac{\varepsilon^{\frac{2}{3}}}{2}\varphi^2 + C\varepsilon^{-\frac{2}{3}} f_1^2 + \frac{1}{2}\psi^2 + Cf_2^2\right) dx,$$

and analogously we can bound the second integral. By absorbing into the left-hand side the terms that involve φ or ψ, we get (4.56).

Suppose now that (4.58) holds. As before, but this time making more use of (4.54), we find that

$$\varepsilon^2\int_{\mathbb{R}^2}\left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) dx + \int_{B_{R_\varepsilon}}\left(\varepsilon^{\frac{2}{3}}\varphi^2 + \check{\eta}^2_{1,\varepsilon}\varphi^2 + \psi^2\right) dx \\ + \int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}\left(\varphi^2 + \varepsilon^{\frac{2}{3}}\psi^2\right) dx \le C\int_{B_{R_\varepsilon}} \check{\eta}_{1,\varepsilon}|f\varphi|\, dx + C\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}} \check{\eta}_{1,\varepsilon}|f\varphi|\, dx.$$

The desired estimate (4.59) follows readily as before, using Young's inequality to absorb a term of the form $\frac{1}{2}\int_{B_{R_\varepsilon}} \check{\eta}_{1,\varepsilon}^2\varphi^2\,dx$ into the left-hand side, and recalling the exponential decay (4.13) of $\check{\eta}_{1,\varepsilon}$ for $r > R_{1,\varepsilon}$.

Finally, suppose that (4.60) holds. As before, making use of (4.54) once more, we arrive at

$$\varepsilon^2\int_{\mathbb{R}^2}\left(|\nabla\varphi|^2+|\nabla\psi|^2\right)dx+\int_{B_{R_\varepsilon}}\left(\varepsilon^{\frac{2}{3}}\varphi^2+\psi^2\right)dx$$
$$+\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}\left(\varphi^2+\varepsilon^{\frac{2}{3}}\psi^2+\check{\eta}_{2,\varepsilon}^2\psi^2\right)dx$$
$$\le C\int_{B_{R_\varepsilon}}\check{\eta}_{2,\varepsilon}|f\psi|\,dx+C\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}\check{\eta}_{2,\varepsilon}|f\psi|\,dx.$$

The desired estimate (4.61) follows readily as before, using Young's inequality to absorb terms of the form $\frac{1}{2}\int_{\mathbb{R}^2\setminus B_{R_\varepsilon}}\check{\eta}_{2,\varepsilon}^2\psi^2\,dx$ and $\frac{1}{2}\int_{B_{R_\varepsilon}}\psi^2\,dx$ into the left-hand side. □

4.5. Existence and properties of a positive solution of the system (1.29). We seek a true solution of (1.29) in the form

$$(\eta_{1,\varepsilon},\eta_{2,\varepsilon})=(\check{\eta}_{1,\varepsilon},\check{\eta}_{2,\varepsilon})+(\varphi,\psi),\tag{4.62}$$

with $\varphi,\psi\in H^2_{rad}(\mathbb{R}^2)$.

In terms of (φ,ψ), system (1.29) becomes

$$-\mathcal{L}(\varphi,\psi)=\mathcal{N}(\varphi,\psi)+\mathcal{E}(\check{\eta}_{1,\varepsilon},\check{\eta}_{2,\varepsilon}),\tag{4.63}$$

where $\mathcal{L}$ is the linear operator in (4.41), the nonlinear operator $\mathcal{N}$ is

$$\mathcal{N}(\varphi,\psi)=\begin{pmatrix}N_1(\varphi,\psi)\\ N_2(\varphi,\psi)\end{pmatrix}=\begin{pmatrix}g_1\varphi^3+3g_1\check{\eta}_{1,\varepsilon}\varphi^2+g\check{\eta}_{1,\varepsilon}\psi^2+2g\check{\eta}_{2,\varepsilon}\psi\varphi+g\psi^2\varphi\\ g_2\psi^3+3g_2\check{\eta}_{2,\varepsilon}\psi^2+g\check{\eta}_{2,\varepsilon}\varphi^2+2g\check{\eta}_{1,\varepsilon}\varphi\psi+g\varphi^2\psi\end{pmatrix},\tag{4.64}$$

and the remainder $\mathcal{E}(\check{\eta}_{1,\varepsilon},\check{\eta}_{2,\varepsilon})$ is as in (4.25).

In view of Proposition 4.6, for small $\varepsilon > 0$, we can define a nonlinear operator $\mathcal{T}: H^2_{rad}(\mathbb{R}^2)\times H^2_{rad}(\mathbb{R}^2)\to H^2_{rad}(\mathbb{R}^2)\times H^2_{rad}(\mathbb{R}^2)$ via the relation

$$\mathcal{T}(\varphi,\psi)=(\bar{\varphi},\bar{\psi}),$$

where $(\bar{\varphi},\bar{\psi})\in H^2_{rad}(\mathbb{R}^2)\times H^2_{rad}(\mathbb{R}^2)$ is uniquely determined from the equation

$$-\mathcal{L}(\bar{\varphi},\bar{\psi})=\mathcal{N}(\varphi,\psi)+\mathcal{E}(\check{\eta}_{1,\varepsilon},\check{\eta}_{2,\varepsilon}).\tag{4.65}$$

Note that Sobolev's inequality implies that functions in $H^2(\mathbb{R}^2)$ are bounded, in particular $\mathcal{N}(\varphi,\psi)\in L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$ for every $(\varphi,\psi)\in H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$.

For $\varepsilon > 0$, $M > 1$, let

$$\mathcal{B}_{\varepsilon,M}=\left\{(\varphi,\psi)\in H^2_{rad}(\mathbb{R}^2)\times H^2_{rad}(\mathbb{R}^2)\ :\ |||(\varphi,\psi)|||\le M\varepsilon^{\frac{5}{3}}\right\}.$$

The following proposition contains the main properties of the operator $\mathcal{T}$.

Proposition 4.7. *If $M > 1$ is sufficiently large, the operator $\mathcal{T}$ maps $\mathcal{B}_{\varepsilon,M}$ into itself, and its restriction to $\mathcal{B}_{\varepsilon,M}$ is a contraction with respect to the $|||(\cdot,\cdot)|||$ norm, provided that $\varepsilon > 0$ is sufficiently small.*

Proof. Let $(\varphi,\psi) \in \mathcal{B}_{\varepsilon,M}$, and $(\bar\varphi,\bar\psi) = \mathcal{T}(\varphi,\psi)$. In view of (4.64) and (4.65), we have

$$(\bar\varphi,\bar\psi) = \sum_{i=1}^{4}(\bar\varphi_i,\bar\psi_i), \tag{4.66}$$

where $(\bar\varphi_i,\bar\psi_i) \in H^2_{rad}(\mathbb{R}^2)\times H^2_{rad}(\mathbb{R}^2)$, $i = 1,\dots,4$, satisfy

$$-\mathcal{L}(\bar\varphi_1,\bar\psi_1) = \begin{pmatrix} g_1\varphi^3 + 2g\check\eta_{2,\varepsilon}\psi\varphi + g\psi^2\varphi \\ g_2\psi^3 + 2g\check\eta_{1,\varepsilon}\varphi\psi + g\varphi^2\psi \end{pmatrix},$$

$$-\mathcal{L}(\bar\varphi_2,\bar\psi_2) = \begin{pmatrix} \check\eta_{1,\varepsilon}\left(3g_1\varphi^2 + g\psi^2\right) \\ 0 \end{pmatrix}, \quad -\mathcal{L}(\bar\varphi_3,\bar\psi_3) = \begin{pmatrix} 0 \\ \check\eta_{2,\varepsilon}\left(3g_2\psi^2 + g\varphi^2\right) \end{pmatrix},$$

and

$$-\mathcal{L}(\bar\varphi_4,\bar\psi_4) = \mathcal{E}(\check\eta_{1,\varepsilon},\check\eta_{2,\varepsilon}).$$

Using Proposition 4.6, and the Gagliardo–Nirenberg interpolation inequality in order to estimate the L^2-norms of the nonlinear terms, it follows readily that

$$\left|\left|\left|(\bar\varphi_1,\bar\psi_1)\right|\right|\right| \le CM^3\varepsilon^{\frac{7}{3}} + CM^2\varepsilon^{\frac{11}{6}} \text{ and } \left|\left|\left|(\bar\varphi_i,\bar\psi_i)\right|\right|\right| \le CM^2\varepsilon^2,\ i = 2,3, \tag{4.67}$$

where $C > 0$ is independent of both ε and M, provided that $\varepsilon > 0$ is sufficiently small. In order to illustrate the procedure, let us present in detail the proof of the second bound ($i = 2$): Estimate (4.59) implies that

$$\left|\left|\left|(\bar\varphi_2,\bar\psi_2)\right|\right|\right|^2 \le C\int_{B_{R_\varepsilon}}(\varphi^4+\psi^4)\,dx + Ce^{-c\varepsilon^{-\frac{2}{3}}}\int_{\mathbb{R}^2}(\varphi^4+\psi^4)\,dx,$$

with constants $c, C > 0$ independent of ε, M, provided that $\varepsilon > 0$ is sufficiently small. Since $(\varphi,\psi) \in \mathcal{B}_{\varepsilon,M}$, it follows that

$$\|\varphi\|_{L^2(B_{R_\varepsilon})} \le M\varepsilon^{\frac{4}{3}},\ \ \|\varphi\|_{H^1(\mathbb{R}^2)} \le M\varepsilon^{\frac{2}{3}};\ \ \|\psi\|_{L^2(B_{R_\varepsilon})} \le M\varepsilon^{\frac{5}{3}},\ \ \|\psi\|_{H^1(\mathbb{R}^2)} \le M\varepsilon^{\frac{2}{3}}. \tag{4.68}$$

Now, the desired bound follows via the Gagliardo-Nirenberg inequality

$$\|u\|_{L^p(\Omega)} \le C_p\|u\|_{H^1(\Omega)}^{1-\frac{2}{p}}\|u\|_{L^2(\Omega)}^{\frac{2}{p}},\quad p \ge 2, \tag{4.69}$$

for $\Omega \subseteq \mathbb{R}^2$ regular, which implies that

$$\|\varphi\|_{L^4(B_{R_\varepsilon})} \le C\|\varphi\|_{H^1(B_{R_\varepsilon})}^{\frac{1}{2}}\|\varphi\|_{L^2(B_{R_\varepsilon})}^{\frac{1}{2}} \le CM\varepsilon, \tag{4.70}$$

with constant $C > 0$ independent of ε, M, provided that $\varepsilon > 0$ is sufficiently small, and analogous estimates can be derived for $\|\psi\|_{L^4(B_{R_\varepsilon})}$, $\|\varphi\|_{L^4(\mathbb{R}^2)}$ and $\|\psi\|_{L^4(\mathbb{R}^2)}$. The remaining bounds in (4.67) can be proven analogously. On the other side, by (4.25), (4.26), and Proposition 4.6, we obtain that

$$\left|\left|\left|(\bar{\varphi}_4, \bar{\psi}_4)\right|\right|\right| \leq C\varepsilon^{\frac{5}{3}}, \tag{4.71}$$

for small $\varepsilon > 0$ (here C is clearly independent of M as well). Hence, by (4.66), (4.67) and (4.71), we deduce that

$$\left|\left|\left|(\bar{\varphi}, \bar{\psi})\right|\right|\right| \leq C\varepsilon^{\frac{5}{3}} \left(M^3\varepsilon^{\frac{2}{3}} + M^2\varepsilon^{\frac{1}{6}} + 1\right),$$

with $C > 0$ independent of ε, M, provided that $\varepsilon > 0$ is sufficiently small. Consequently, if we choose $M = 2C$, and fix it from now on, decreasing $\varepsilon > 0$ further if necessary so that $M^3\varepsilon^{\frac{2}{3}} + M^2\varepsilon^{\frac{1}{6}} \leq 1$, it follows that

$$\left|\left|\left|(\bar{\varphi}, \bar{\psi})\right|\right|\right| = |||\mathcal{T}(\varphi, \psi)||| \leq M\varepsilon^{\frac{5}{3}}.$$

We conclude that, if $\varepsilon > 0$ is sufficiently small, the operator $\mathcal{T}$ maps $\mathcal{B}_{\varepsilon,M}$ into itself, as asserted.

It remains to show that the restriction of $\mathcal{T}$ to $\mathcal{B}_{\varepsilon,M}$ is a contraction with respect to the $|||(\cdot,\cdot)|||$ norm, provided that $\varepsilon > 0$ is sufficiently small. To this end, let

$$(\varphi_i, \psi_i) \in \mathcal{B}_{\varepsilon,M}, \ i = 1, 2, \ \text{ and } \ (\bar{\varphi}_i, \bar{\psi}_i) = \mathcal{T}(\varphi_i, \psi_i), \ i = 1, 2. \tag{4.72}$$

Then, set

$$(\bar{w}, \bar{z}) = (\bar{\varphi}_1 - \bar{\varphi}_2, \bar{\psi}_1 - \bar{\psi}_2). \tag{4.73}$$

As before, it is convenient to write

$$(\bar{w}, \bar{z}) = \sum_{i=1}^{5} (\bar{w}_i, \bar{z}_i), \tag{4.74}$$

where $(\bar{w}_i, \bar{z}_i) \in H^2_{rad}(\mathbb{R}^2) \times H^2_{rad}(\mathbb{R}^2)$, $i = 1, \cdots, 5$, satisfy

$$-\mathcal{L}(\bar{w}_1, \bar{z}_1) = \begin{pmatrix} g_1(\varphi_1^2 + \varphi_1\varphi_2 + \varphi_2^2)(\varphi_1 - \varphi_2) \\ g_2(\psi_1^2 + \psi_1\psi_2 + \psi_2^2)(\psi_1 - \psi_2) \end{pmatrix},$$

$$-\mathcal{L}(\bar{w}_2, \bar{z}_2) = 2g\left[\psi_2(\varphi_1 - \varphi_2) + \varphi_1(\psi_1 - \psi_2)\right] \begin{pmatrix} \check{\eta}_{2,\varepsilon} \\ \check{\eta}_{1,\varepsilon} \end{pmatrix},$$

$$-\mathcal{L}(\bar{w}_3, \bar{z}_3) = g \begin{pmatrix} \psi_2^2(\varphi_1 - \varphi_2) + \varphi_1(\psi_1 + \psi_2)(\psi_1 - \psi_2) \\ \varphi_2^2(\psi_1 - \psi_2) + \psi_1(\varphi_1 + \varphi_2)(\varphi_1 - \varphi_2) \end{pmatrix},$$

$$-\mathcal{L}(\bar{w}_4, \bar{z}_4) = \begin{pmatrix} \check{\eta}_{1,\varepsilon}\left[3g_1(\varphi_1 + \varphi_2)(\varphi_1 - \varphi_2) + g(\psi_1 + \psi_2)(\psi_1 - \psi_2)\right] \\ 0 \end{pmatrix},$$

and

$$-\mathcal{L}(\bar{w}_5, \bar{z}_5) = \begin{pmatrix} 0 \\ \check{\eta}_{2,\varepsilon}\left[3g_2(\psi_1 + \psi_2)(\psi_1 - \psi_2) + g(\varphi_1 + \varphi_2)(\varphi_1 - \varphi_2)\right] \end{pmatrix}.$$

As before, using Proposition 4.6, the Gagliardo–Nirenberg inequality (4.69), and the inequalities

$$\|\varphi\|_{L^2(B_{R_\varepsilon})} \le \varepsilon^{-\frac{1}{3}} |||(\varphi,\psi)|||, \quad \|\varphi\|_{L^2(\mathbb{R}^2\setminus B_{R_\varepsilon})} \le |||(\varphi,\psi)|||, \|\varphi\|_{H^1(\mathbb{R}^2)} \le \varepsilon^{-1} |||(\varphi,\psi)|||,$$
$$\|\psi\|_{L^2(B_{R_\varepsilon})} \le |||(\varphi,\psi)|||, \quad \|\psi\|_{L^2(\mathbb{R}^2\setminus B_{R_\varepsilon})} \le \varepsilon^{-\frac{1}{3}} |||(\varphi,\psi)|||, \|\psi\|_{H^1(\mathbb{R}^2)} \le \varepsilon^{-1} |||(\varphi,\psi)|||, \tag{4.75}$$

for every $\varphi, \psi \in H^1(\mathbb{R}^2)$, we can show that

$$\begin{aligned} |||(\bar{w}_1,\bar{z}_1)||| &\le C\varepsilon^{\frac{2}{3}} |||(\varphi_1-\varphi_2,\psi_1-\psi_2)|||, \\ |||(\bar{w}_2,\bar{z}_2)||| &\le C\varepsilon^{\frac{1}{6}} |||(\varphi_1-\varphi_2,\psi_1-\psi_2)|||, \\ |||(\bar{w}_3,\bar{z}_3)||| &\le C\varepsilon^{\frac{5}{6}} |||(\varphi_1-\varphi_2,\psi_1-\psi_2)|||, \\ |||(\bar{w}_i,\bar{z}_i)||| &\le C\varepsilon^{\frac{1}{3}} |||(\varphi_1-\varphi_2,\psi_1-\psi_2)|||, \quad i=4,5, \end{aligned} \tag{4.76}$$

provided that $\varepsilon > 0$ is sufficiently small. In order to illustrate the procedure, let us present in detail the proof of the bound for $|||(\bar{w}_2,\bar{z}_2)|||$: From Proposition 4.6, we obtain that

$$\begin{aligned} |||(\bar{w}_2,\bar{z}_2)||| \le\ & C\varepsilon^{-\frac{1}{3}} \|\psi_2(\varphi_1-\varphi_2)\|_{L^2(B_{R_\varepsilon})} + C\varepsilon^{-\frac{1}{3}} \|\varphi_1(\psi_1-\psi_2)\|_{L^2(B_{R_\varepsilon})} \\ & + C\varepsilon^{-\frac{1}{3}} \|\psi_2(\varphi_1-\varphi_2)\|_{L^2(\mathbb{R}^2\setminus B_{R_\varepsilon})} + C\varepsilon^{-\frac{1}{3}} \|\varphi_1(\psi_1-\psi_2)\|_{L^2(\mathbb{R}^2\setminus B_{R_\varepsilon})}. \end{aligned}$$

The second term in the right-hand side of the above relation can be estimated as follows: by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \varepsilon^{-\frac{1}{3}} \|\varphi_1(\psi_1-\psi_2)\|_{L^2(B_{R_\varepsilon})} &\le \varepsilon^{-\frac{1}{3}} \|\varphi_1\|_{L^4(B_{R_\varepsilon})} \|\psi_1-\psi_2\|_{L^4(B_{R_\varepsilon})} \\ &\overset{(4.70)}{\le} C\varepsilon^{\frac{2}{3}} \|\psi_1-\psi_2\|_{L^4(B_{R_\varepsilon})} \\ &\overset{(4.69),(4.75)}{\le} C\varepsilon^{\frac{1}{6}} |||(\varphi_1-\varphi_2,\psi_1-\psi_2)|||. \end{aligned}$$

The remaining terms can be estimated in a similar fashion, giving the desired bound for $|||(\bar{w}_2,\bar{z}_2)|||$. The other bounds in (4.76) can be verified analogously. Consequently, combining relations (4.72), (4.73), (4.74), and (4.76), we infer that

$$|||\mathcal{T}(\varphi_1,\psi_1) - \mathcal{T}(\varphi_2,\psi_2)||| \le C\varepsilon^{\frac{1}{6}} |||(\varphi_1,\psi_1)-(\varphi_2,\psi_2)||| \ \ \forall\, (\varphi_i,\psi_i) \in \mathcal{B}_{\varepsilon,M},\ i=1,2.$$

We therefore conclude that, for sufficiently small $\varepsilon > 0$, the restriction of $\mathcal{T}$ to $\mathcal{B}_{\varepsilon,M}$ is a contraction with respect to the $|||(\cdot,\cdot)|||$ norm, as asserted. □

The above proposition implies the main result of this section:

Proposition 4.8. *There exists a constant $M > 0$, such that the system (1.29) has a unique solution $(\eta_{1,\varepsilon},\eta_{2,\varepsilon})$ such that*

$$\left|\left|\left|(\eta_{1,\varepsilon}-\check{\eta}_{1,\varepsilon},\eta_{2,\varepsilon}-\check{\eta}_{2,\varepsilon})\right|\right|\right| \le M\varepsilon^{\frac{5}{3}}, \tag{4.77}$$

if $\varepsilon > 0$ is sufficiently small, where the above norm is as in (4.57).

Proof. In view of Proposition 4.7, for small $\varepsilon > 0$, we can define iteratively a sequence $(\varphi_n, \psi_n) \in \mathcal{B}_{\varepsilon,M}$ such that

$$(\varphi_{n+1}, \psi_{n+1}) = \mathcal{T}(\varphi_n, \psi_n), \quad n \geq 0, \ (\varphi_0, \psi_0) = (0, 0). \tag{4.78}$$

Moreover, the same proposition implies that (φ_n, ψ_n) is a Cauchy sequence in $H^1_{rad}(\mathbb{R}^2) \times H^1_{rad}(\mathbb{R}^2)$. Hence, we infer that

$$(\varphi_n, \psi_n) \to (\varphi_\infty, \psi_\infty) \ \text{ in } H^1_{rad}(\mathbb{R}^2) \times H^1_{rad}(\mathbb{R}^2), \ \text{ as } n \to \infty,$$

for some $(\varphi_\infty, \psi_\infty) \in H^1_{rad}(\mathbb{R}^2) \times H^1_{rad}(\mathbb{R}^2)$ such that

$$|||(\varphi_\infty, \psi_\infty)||| \leq M\varepsilon^{\frac{5}{3}}.$$

In turn, letting $n \to \infty$ in the weak form of (4.78) [recall (4.65)] yields that $(\varphi_\infty, \psi_\infty)$ is a weak solution of (4.63). Then, by standard elliptic regularity theory, we deduce that $(\varphi_\infty, \psi_\infty) \in H^2_{rad}(\mathbb{R}^2) \times H^2_{rad}(\mathbb{R}^2)$ (i.e. $(\varphi_\infty, \psi_\infty) \in \mathcal{B}_{\varepsilon,M}$) and is smooth (i.e. a classical solution). The point being that $\mathcal{B}_{\varepsilon,M}$ is not closed in the $|||(\cdot,\cdot)|||$ norm. In fact, Eq. (4.63) has a unique solution in $\mathcal{B}_{\varepsilon,M}$, as the restriction of $\mathcal{T}$ to $\mathcal{B}_{\varepsilon,M}$ is a contraction with respect to the $|||(\cdot,\cdot)|||$ norm, provided that $\varepsilon > 0$ is sufficiently small. Consequently, recalling the equivalence of (1.29) to (4.63) via (4.62), we conclude that the assertions of the proposition hold. □

A direct consequence of (4.77) is the following

Corollary 4.9. *If $\varepsilon > 0$ is sufficiently small, the solutions $\eta_{1,\varepsilon}$ and $\eta_{2,\varepsilon}$ of the system* (1.29) *that are provided by Proposition* 4.8 *satisfy*

$$\|\eta_{i,\varepsilon} - \check{\eta}_{i,\varepsilon}\|_{L^\infty(|x|\geq\delta)} \leq C\varepsilon, \ \ \textit{and} \ \ \eta_{i,\varepsilon}(x) \to 0 \ \textit{as} \ |x| \to \infty, \ \ i = 1, 2. \tag{4.79}$$

Proof. Consider the fluctuations

$$\varphi = \eta_{1,\varepsilon} - \check{\eta}_{1,\varepsilon} \ \text{ and } \ \psi = \eta_{2,\varepsilon} - \check{\eta}_{2,\varepsilon}.$$

It follows from (4.57) and (4.77) that

$$\|\nabla\varphi\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon^{\frac{2}{3}}, \ \|\varphi\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon^{\frac{4}{3}} \ \text{ and } \ \|\nabla\psi\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon^{\frac{2}{3}}, \ \|\psi\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon^{\frac{4}{3}}, \tag{4.80}$$

[note that we did not make full use of (4.77)]. In order to transform the above into uniform estimates, we need the following inequality which can be traced back to [32]: There exists a constant $C > 0$ such that

$$|x|^{\frac{1}{2}}\, |v(x)| \leq C\|\nabla v\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)} \|v\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)} \ \text{ for a.e } x \in \mathbb{R}^2, \tag{4.81}$$

and all $v \in H^1_{rad}(\mathbb{R}^2) \equiv \{v \in H^1(\mathbb{R}^2) \ : \ v \text{ is radial}\}$. The desired asymptotic behavior in (4.79) follows at once. Making use of this inequality for $|x| \geq \delta$, we obtain that

$$\|\varphi\|_{L^\infty(|x|\geq\delta)} \leq C\varepsilon \ \text{ and } \ \|\psi\|_{L^\infty(|x|\geq\delta)} \leq C\varepsilon,$$

which are exactly the desired uniform estimates in (4.79). □

We now turn our attention to establishing uniform estimates on $\bar{B}_\delta$. The following lemma will come in handy in the proof of Corollary 4.11 below.

Lemma 4.10. *There exists a constant* $C > 0$ *such that the solutions that are provided by Proposition 4.8 satisfy*

$$\|\eta_{i,\varepsilon}\|_{L^\infty(B_\delta)} \le C, \quad i = 1, 2,$$

if $\varepsilon > 0$ *is sufficiently small.*

Proof. Suppose that the assertion is false. We use a blow-up argument to arrive at a contradiction (see also [22]). Without loss of generality, we may assume that there exist $\varepsilon_n \to 0$ and $x_n \in B_\delta$ such that

$$\eta_{1,\varepsilon_n}(x_n) = \|\eta_{1,\varepsilon_n}\|_{L^\infty(B_\delta)} = M_n \to \infty.$$

We may further assume that $x_n \to x_\infty \in \bar{B}_\delta$. Now, we re-scale η_{1,ε_n} by setting

$$v_n(y) = \mu_n \eta_{1,\varepsilon_n}(x_n + \varepsilon_n \mu_n y) \text{ with } \mu_n = M_n^{-1} \to 0.$$

The function v_n satisfies

$$-\Delta v_n + g_1 v_n^3 - g_1 \mu_n^2 a_{1,\varepsilon_n}(x_n + \varepsilon_n \mu_n y) v_n + g \mu_n^2 \left(\eta_{2,\varepsilon_n}^2(x_n + \varepsilon_n \mu_n y) \right.$$
$$\left. - a_{2,\varepsilon_n}(x_n + \varepsilon_n \mu_n y) \right) v_n = 0.$$

By using elliptic L^p estimates and standard imbeddings, exploiting the bound

$$\|\eta_{2,\varepsilon}^2 - a_{2,\varepsilon}\|_{L^p(B_\delta)} \le C_p \varepsilon^{\frac{2}{3}+\frac{1}{p}}, \ p \ge 2, \ \text{(readily derivable from Proposition 4.8)},$$

we deduce that a subsequence of v_n converges uniformly on compact sets to a bounded nontrivial solution v_∞ of the problem

$$\Delta v = g_1 v^3, \qquad v(0) = 1, \tag{4.82}$$

in the entire space $\mathbb{R}^2$ or in an open half-space H containing the origin, with zero boundary conditions on ∂H. Actually, in the latter scenario, one has to perform a rotation and stretching of coordinates in the resulting limiting equation to get (4.82), see [22]. In any case, by reflecting v_∞ oddly across ∂H if necessary, we have been led to a *nontrivial* solution of (4.82) in the whole space $\mathbb{R}^2$. On the other hand, this contradicts a well known Liouville type theorem of Brezis [14]. □

The following corollary provides additional information to Corollary 4.9, but will be considerably improved in Proposition 4.14.

Corollary 4.11. *If* $\varepsilon > 0$ *is sufficiently small, the solutions provided by Proposition 4.8 satisfy*

$$\|\eta_{i,\varepsilon} - \sqrt{a_{i,\varepsilon}}\|_{L^\infty(B_\delta)} \le C\varepsilon^{\frac{1}{3}}, \quad i = 1, 2. \tag{4.83}$$

Proof. Let

$$\phi = \eta_{1,\varepsilon} - \sqrt{a_{1,\varepsilon}}, \quad x \in \bar{B}_{2\delta}. \tag{4.84}$$

Observe that estimates (4.12) and (4.77) imply that

$$\|\phi\|_{L^2(B_{2\delta})} \le C\varepsilon^{\frac{4}{3}}. \tag{4.85}$$

From the first equation in (1.29), by rearranging terms, in $B_{2\delta}$ we obtain that

$$-\varepsilon^2\Delta\phi = -g_1\eta_{1,\varepsilon}\left(\eta_{1,\varepsilon}+\sqrt{a_{1,\varepsilon}}\right)\phi + \varepsilon^2\Delta\sqrt{a_{1,\varepsilon}} - g\eta_{1,\varepsilon}(\eta_{2,\varepsilon}^2 - a_{2,\varepsilon}) =: f.$$

By interior elliptic regularity theory, we deduce that

$$\|\phi\|_{H^2(B_\delta)} \le C\left(\varepsilon^{-2}\|f\|_{L^2(B_{2\delta})} + \|\phi\|_{L^2(B_{2\delta})}\right) \le C\varepsilon^{-\frac{2}{3}}, \tag{4.86}$$

where we also used Lemma 4.10 and (4.85). Now, by the two-dimensional Agmon inequality [7, Lem. 13.2], we infer that

$$\|\phi\|_{L^\infty(B_\delta)} \le C\|\phi\|_{H^2(B_\delta)}^{\frac{1}{2}}\|\phi\|_{L^2(B_\delta)}^{\frac{1}{2}} \overset{(4.85),(4.86)}{\le} C\varepsilon^{\frac{1}{3}}.$$

The desired bound for $\eta_{1,\varepsilon} - \sqrt{a_{1,\varepsilon}}$ follows at once from (4.84) and the above relation. The corresponding bound for $\eta_{2,\varepsilon} - \sqrt{a_{2,\varepsilon}}$ can be shown analogously. □

We are now in position to show that the solutions in Proposition 4.8 are in fact positive.

Proposition 4.12. *If $\varepsilon > 0$ is sufficiently small, the solutions in Proposition 4.8 satisfy*

$$\eta_{i,\varepsilon} > 0 \quad in\ \mathbb{R}^2, \quad i = 1, 2.$$

Proof. By virtue of (4.12), (4.15), (4.79), and Corollary 4.11, given $D \ge 1$, we deduce that

$$\eta_{1,\varepsilon} \ge c_D\varepsilon^{\frac{1}{3}} > 0 \ \text{ if } |x| \le R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}, \tag{4.87}$$

provided that $\varepsilon > 0$ is sufficiently small; where throughout this proof, unless specified otherwise, the generic constants $c, C > 0$ are also independent of $D \ge 1$. From (1.29a), we observe that $\eta_{1,\varepsilon}$ satisfies a linear equation of the form

$$-\varepsilon^2\Delta\eta_{1,\varepsilon} + Q(x)\eta_{1,\varepsilon} = 0, \tag{4.88}$$

where

$$Q(x) = g_1(\eta_{1,\varepsilon}^2 - a_{1,\varepsilon}) + g(\eta_{2,\varepsilon}^2 - a_{2,\varepsilon}).$$

If $R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}} \le |x| \le R_{2,\varepsilon}$, recalling (4.13) and (4.79), we find that

$$\begin{aligned}
Q(x) &= g_1(\eta_{1,\varepsilon}^2 - \check{\eta}_{1,\varepsilon}^2 + \check{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon}) + g(\eta_{2,\varepsilon}^2 - \check{\eta}_{2,\varepsilon}^2 + \check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon})\\
&\ge -g_1a_{1,\varepsilon} + g(\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon}) - C\varepsilon\\
&\ge \left(\frac{g^2}{g_2} - g_1\right)a_{1,\varepsilon} + g\left(\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon} - \frac{g}{g_2}a_{1,\varepsilon}\right) - C\varepsilon.
\end{aligned}$$

In particular, if $R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}} \le |x| \le R_\varepsilon$, where $\check{\eta}_{2,\varepsilon} = \tilde{\eta}_{2,\varepsilon}$, via (4.21), which implies that the second term in the right-hand side is $-g^2g_2^{-1}\check{\eta}_{1,\varepsilon}^2$, and the exponential decay of the ground state $\hat{\eta}_{1,\varepsilon}$ for $r = |x| > R_{1,\varepsilon}$, we obtain that

$$\begin{aligned}
Q(r) &\ge \left(\frac{g^2}{g_2} - g_1\right)a_{1,\varepsilon} - Ce^{-cD}\varepsilon^{\frac{2}{3}}\\
&\overset{(1.3),(1.30),(1.34),(3.13)}{\ge} c(r - R_{1,\varepsilon})^2 + cD\varepsilon^{\frac{2}{3}} - Ce^{-cD}\varepsilon^{\frac{2}{3}},\\
&\ge c(r - R_{1,\varepsilon})^2 + cD\varepsilon^{\frac{2}{3}},
\end{aligned}$$

increasing the value of D if needed, provided that $\varepsilon > 0$ is sufficiently small. Clearly, in view of (4.23), the same lower bound holds if $R_\varepsilon \le |x| \le R_\varepsilon + \delta$. On the other side, if $R_\varepsilon + \delta \le |x| \le R_{2,\varepsilon}$, where $a_{1,\varepsilon} = -c(r^2 - R_{1,\varepsilon}^2)$, $\check{\eta}_{2,\varepsilon} = \hat{\eta}_{2,\varepsilon}$ and $\left|\hat{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon} - \frac{g}{g_2} a_{1,\varepsilon}\right| \le C\varepsilon^{\frac{2}{3}}$, we deduce that $Q(r) \ge c(r - R_{1,\varepsilon})^2$. The latter lower bound also holds if $|x| \ge R_{2,\varepsilon}$. So far, we have shown that

$$Q(x) \ge c(r - R_{1,\varepsilon})^2 + cD\varepsilon^{\frac{2}{3}}, \quad |x| \ge R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}, \tag{4.89}$$

provided that $\varepsilon > 0$ is sufficiently small. By (4.79), (4.87), (4.88), (4.89), and the maximum principle, we deduce that

$$\eta_{1,\varepsilon} \ge 0 \ \text{ if } |x| \ge R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}.$$

The desired strict positivity of $\eta_{1,\varepsilon}$ follows immediately from (4.87), (4.88), the above relation, and the strong maximum principle. The corresponding property for $\eta_{2,\varepsilon}$ can be proven analogously. □

The following lemma is motivated from Lemma 2.2, and will be used in the next section.

Lemma 4.13. *Given $D > 1$ sufficiently large, we have*

$$\eta_{i,\varepsilon}(s) \le \eta_{i,\varepsilon}(r) \exp\left\{-\frac{D^{\frac{1}{3}}}{\varepsilon^{\frac{2}{3}}}(s^2 - r^2)\right\} \ \textit{for} \ s \ge r \ge R_{i,\varepsilon} + D\varepsilon^{\frac{2}{3}}, \quad i = 1, 2,$$

provided that $\varepsilon > 0$ is sufficiently small.

Proof. Throughout this proof, the generic constants $c, C > 0$ are independent of both small $\varepsilon > 0$ and large $D > 1$. Abusing notation slightly, let

$$u(x) = u(s) = \exp\left\{-\frac{D^{\frac{1}{3}}}{\varepsilon^{\frac{2}{3}}} s^2\right\}, \quad x \in \mathbb{R}^2, \quad s = |x|.$$

It is easy to see that

$$-\varepsilon^2 \Delta u + c\left[(s - R_{1,\varepsilon})^2 + D\varepsilon^{\frac{2}{3}}\right] u \ge \left[-4D^{\frac{2}{3}}\varepsilon^{\frac{2}{3}} s^2 + c(s - R_{1,\varepsilon})^2 + cD\varepsilon^{\frac{2}{3}}\right] u \ge 0, \tag{4.90}$$

if $s = |x| \ge R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}$, for sufficiently large $D > 1$, provided that $\varepsilon > 0$ is sufficiently small. Here $c > 0$ is as in (4.89). For such D, ε, and any $r \ge R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}$, it follows that the function

$$v(s) = \eta_{1,\varepsilon}(r) \exp\left\{-\frac{D^{\frac{1}{3}}}{\varepsilon^{\frac{2}{3}}}(s^2 - r^2)\right\}, \quad s = |x| \ge r,$$

is an upper solution of the linear elliptic equation that is defined by the left-hand side of (4.90). On the other side, by virtue of (4.88) and (4.89), the function $\eta_{1,\varepsilon}(s)$ is a lower solution of the same equation for $|x| > r$, which clearly coincides with the upper solution v on ∂B_r. Hence, by the maximum principle, and (4.79), we deduce that

$$\eta_{1,\varepsilon}(s) \le v(s), \quad \forall\, s = |x| \ge r \ge R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}},$$

for any large $D > 1$, provided that $\varepsilon > 0$ is sufficiently small. The validity of the asserted estimate for $\eta_{1,\varepsilon}$ now follows immediately, while that for $\eta_{2,\varepsilon}$ follows analogously. □

4.6. Improved uniform estimates away from $R_{1,\varepsilon}$ and $R_{2,\varepsilon}$. In this subsection we show that the uniform estimates in (4.79), for the difference $\eta_{i,\varepsilon}-\check{\eta}_{i,\varepsilon}$, can be improved outside of an $\mathcal{O}(\varepsilon^{\frac{2}{3}})$-neighborhood of $R_{i,\varepsilon}$, $i=1,2$.

The results of this subsection, as well as those of the following one, are not essential for the proof of Theorem 1.5, and, depending on the reader's preference, can be skipped on a first reading.

Proposition 4.14. *If $\varepsilon>0$ is sufficiently small, there exist $C,\delta>0$, with $\delta<\min\left\{\frac{R_{1,0}}{4},\frac{R_{2,0}-R_{1,0}}{4}\right\}$, such that*

$$\left|\eta_{i,\varepsilon}(r)-\check{\eta}_{i,\varepsilon}(r)\right|\leq C\varepsilon^2 \quad \text{if } |r-R_{1,\varepsilon}|\geq\delta \text{ and } |r-R_{2,\varepsilon}|\geq\delta,\quad i=1,2.$$

Proof. We prove the assertion in the case where $r\in[0,R_{1,\varepsilon}-\delta]$, which reduces to show that

$$\left|\eta_{i,\varepsilon}(r)-\sqrt{a_{i,\varepsilon}(r)}\right|\leq C\varepsilon^2 \ \text{ if } \ r\in[0,R_{1,\varepsilon}-\delta],\ i=1,2, \tag{4.91}$$

[recall the construction of $\check{\eta}_{i,\varepsilon}$ and also see (4.12)]. In the remaining intervals the proof carries over analogously.

We first require a rough uniform bound for the radial derivatives of the functions

$$u\equiv\eta_{1,\varepsilon}-\sqrt{a_{1,\varepsilon}} \ \text{ and } \ v\equiv\eta_{2,\varepsilon}-\sqrt{a_{2,\varepsilon}}, \tag{4.92}$$

say over the interval $\left[\delta,R_{1,\varepsilon}-\frac{\delta}{50}\right]$. It follows from (4.63), (4.68), and (4.79) (with a smaller constant $\delta>0$), that

$$\|\Delta\varphi\|_{L^2\left(\frac{\delta}{2},R_{1,\varepsilon}-\frac{\delta}{100}\right)}=\|\Delta(\eta_{1,\varepsilon}-\hat{\eta}_{1,\varepsilon})\|_{L^2\left(\frac{\delta}{2},R_{1,\varepsilon}-\frac{\delta}{100}\right)}\leq C\varepsilon^{-\frac{2}{3}}.$$

In turn, by interior elliptic regularity theory and (4.77), we deduce that

$$\|\varphi\|_{H^2\left(\delta,R_{1,\varepsilon}-\frac{\delta}{50}\right)}\leq C\varepsilon^{-\frac{2}{3}}.$$

Hence, from (4.80), via (4.81) with φ' in place of v, we get that

$$|\varphi'(r)|\leq C,\quad r\in\left[\delta,R_{1,\varepsilon}-\frac{\delta}{50}\right].$$

Thanks to (3.13) and (4.12), we infer that

$$\|u'\|_{L^\infty\left(\delta,R_{1,\varepsilon}-\frac{\delta}{50}\right)}\leq C. \tag{4.93}$$

Similarly we have

$$\|v'\|_{L^\infty\left(\delta,R_{1,\varepsilon}-\frac{\delta}{50}\right)}\leq C. \tag{4.94}$$

Observe that u, v satisfy

$$\mathcal{M}(u,v)\equiv\begin{pmatrix}-\varepsilon^2\Delta u+g_1\eta_1(\eta_1+\sqrt{a_{1,\varepsilon}})u+g\eta_1(\eta_2+\sqrt{a_{2,\varepsilon}})v\\ -\varepsilon^2\Delta v+g_2\eta_2(\eta_2+\sqrt{a_{2,\varepsilon}})v+g\eta_2(\eta_1+\sqrt{a_{1,\varepsilon}})u\end{pmatrix}=\begin{pmatrix}\varepsilon^2\Delta\sqrt{a_{1,\varepsilon}}\\ \varepsilon^2\Delta\sqrt{a_{2,\varepsilon}}\end{pmatrix}, \tag{4.95}$$

$x \in B_{\left(R_{1,\varepsilon}-\frac{\delta}{50}\right)}$, having dropped some ε subscripts for convenience. By virtue of (1.3), there exists a unique solution (u_0, v_0) to the linear algebraic system

$$\begin{cases} 2g_1 a_{1,\varepsilon} u + 2g\sqrt{a_{1,\varepsilon}}\sqrt{a_{2,\varepsilon}} v = \varepsilon^2 \Delta \sqrt{a_{1,\varepsilon}}, \\ 2g\sqrt{a_{1,\varepsilon}}\sqrt{a_{2,\varepsilon}} u + 2g_2 a_{2,\varepsilon} v = \varepsilon^2 \Delta \sqrt{a_{2,\varepsilon}}, \end{cases}$$

$x \in B_{\left(R_{1,\varepsilon}-\frac{\delta}{50}\right)}$. It follows readily that

$$\|u_0\|_{C^2\left(\bar{B}_{\left(R_{1,\varepsilon}-\frac{\delta}{50}\right)}\right)} \le C\varepsilon^2 \quad \text{and} \quad \|v_0\|_{C^2\left(\bar{B}_{\left(R_{1,\varepsilon}-\frac{\delta}{50}\right)}\right)} \le C\varepsilon^2, \tag{4.96}$$

(keep in mind that $a_{i,\varepsilon} \ge c$ in $B_{\left(R_{1,\varepsilon}-\frac{\delta}{50}\right)}$, $i = 1, 2$). We can write

$$(u, v) = (u_0, v_0) + (\tilde{u}, \tilde{v}), \tag{4.97}$$

where $\tilde{u}, \tilde{v}$ satisfy

$$\mathcal{M}(\tilde{u}, \tilde{v}) = \begin{pmatrix} \varepsilon^2 \Delta u_0 \\ \varepsilon^2 \Delta v_0 \end{pmatrix} + \begin{pmatrix} g_1(2a_{1,\varepsilon} - \eta_1^2 - \sqrt{a}_{1,\varepsilon}\eta_1)u_0 + g(2\sqrt{a_{1,\varepsilon}}\sqrt{a_{2,\varepsilon}} - \eta_1\eta_2 - \sqrt{a_{2,\varepsilon}}\eta_1)v_0 \\ g_2(2a_{2,\varepsilon} - \eta_2^2 - \sqrt{a}_{2,\varepsilon}\eta_2)v_0 + g(2\sqrt{a_{1,\varepsilon}}\sqrt{a_{2,\varepsilon}} - \eta_1\eta_2 - \sqrt{a_{1,\varepsilon}}\eta_2)u_0 \end{pmatrix}, \tag{4.98}$$

$x \in B_{\left(R_{1,\varepsilon}-\frac{\delta}{50}\right)}$.

Consider any $\rho \in \left(0, R_{1,\varepsilon} - \frac{\delta}{50}\right]$. By testing the above equation by $(\tilde{u}, \tilde{v})$ in the $L^2(B_\rho) \times L^2(B_\rho)$ sense, making use of (2.4), Proposition 4.8, (4.96), and Young's inequality, it follows readily that

$$\int_{B_\rho} \left(\varepsilon^2|\nabla\tilde{u}|^2 + \varepsilon^2|\nabla\tilde{v}|^2 + \tilde{u}^2 + \tilde{v}^2\right) dx \le C\varepsilon^{\frac{20}{3}} + C\varepsilon^2|\tilde{u}'(\rho)\tilde{u}(\rho)| + C\varepsilon^2|\tilde{v}'(\rho)\tilde{v}(\rho)|, \tag{4.99}$$

provided that $\varepsilon > 0$ is sufficiently small. Setting in this relation $\rho = R_{1,\varepsilon} - \frac{\delta}{50}$, using (4.79), (4.93), (4.94), and (4.96), we obtain that

$$\int_{B_{\left(R_{1,\varepsilon}-\frac{\delta}{50}\right)}} \left(\varepsilon^2|\nabla\tilde{u}|^2 + \varepsilon^2|\nabla\tilde{v}|^2 + \tilde{u}^2 + \tilde{v}^2\right) dx \le C\varepsilon^{\frac{20}{3}} + C\varepsilon^3.$$

Thus, there exists $r_1 \in \left(R_{1,\varepsilon} - \frac{\delta}{49}, R_{1,\varepsilon} - \frac{\delta}{50}\right)$ such that

$$|\tilde{u}(r_1)| + |\tilde{v}(r_1)| \le C\varepsilon^{\frac{10}{3}} + C\varepsilon^{\frac{3}{2}} \quad \text{and} \quad |\tilde{u}'(r_1)| + |\tilde{v}'(r_1)| \le C\varepsilon^{\frac{1}{2}}.$$

Doing the same procedure with $\rho = r_1$, using the above estimates instead of (4.79) when estimating the boundary terms in (4.99), yields that

$$\int_{B_{r_1}} \left(\varepsilon^2|\nabla\tilde{u}|^2 + \varepsilon^2|\nabla\tilde{v}|^2 + \tilde{u}^2 + \tilde{v}^2\right) dx \le C\varepsilon^{\frac{20}{3}} + C\varepsilon^4.$$

Thus, there exists $r_2 \in \left(R_{1,\varepsilon} - \frac{\delta}{48}, r_1\right)$ such that

$$|\tilde{u}(r_2)| + |\tilde{v}(r_2)| \leq C\varepsilon^{\frac{10}{3}} + C\varepsilon^2 \text{ and } |\tilde{u}'(r_2)| + |\tilde{v}'(r_2)| \leq C\varepsilon.$$

Iterating this scheme a finite number of times provides us with an $r_* \in \left(R_{1,\varepsilon} - \frac{\delta}{2}, R_{1,\varepsilon} - \frac{\delta}{3}\right)$ such that

$$\int_{B_{r_*}} (\tilde{u}^2 + \tilde{v}^2)\, dx \leq C\varepsilon^{\frac{20}{3}}. \tag{4.100}$$

Now, via (4.96), (4.98), and interior elliptic regularity theory, we find that

$$\|\tilde{u}\|_{H^2\left(B_{(R_{1,\varepsilon}-\delta)}\right)} \leq C\varepsilon^2 \quad \text{and} \quad \|\tilde{v}\|_{H^2\left(B_{(R_{1,\varepsilon}-\delta)}\right)} \leq C\varepsilon^2. \tag{4.101}$$

By the two-dimensional Agmon inequality [7, Lem. 13.2], we infer that

$$\|\tilde{u}\|_{L^\infty\left(B_{(R_{1,\varepsilon}-\delta)}\right)} \leq C\|\tilde{u}\|^{\frac{1}{2}}_{H^2\left(B_{(R_{1,\varepsilon}-\delta)}\right)}\|\tilde{u}\|^{\frac{1}{2}}_{L^2\left(B_{(R_{1,\varepsilon}-\delta)}\right)} \overset{(4.100),(4.101)}{\leq} C\varepsilon^3.$$

Analogously, we have

$$\|\tilde{v}\|_{L^\infty\left(B_{(R_{1,\varepsilon}-\delta)}\right)} \leq C\varepsilon^3.$$

The desired estimate (4.91) follows at once from (4.92), (4.96), (4.97), and the above two relations. □

In the following proposition, we prove estimates in the intermediate zones, bridging the estimates (4.79) and those provided by Proposition 4.14, on the left side of $R_{i,\varepsilon}$, $i = 1, 2$.

Proposition 4.15. *The following estimates hold:*

$$\left|\eta_{1,\varepsilon}(r) - \check{\eta}_{1,\varepsilon}(r)\right| \leq C\varepsilon^2|r - R_{1,\varepsilon}|^{-\frac{3}{2}}, \quad \left|\eta_{2,\varepsilon}(r) - \check{\eta}_{2,\varepsilon}(r)\right| \leq C\varepsilon^2|r - R_{1,\varepsilon}|^{-1},$$
$$r \in [R_{1,\varepsilon} - \delta, R_{1,\varepsilon} - D\varepsilon^{\frac{2}{3}}],$$

and

$$\left|\eta_{2,\varepsilon}(r) - \check{\eta}_{2,\varepsilon}(r)\right| \leq C\varepsilon^2|r - R_{2,\varepsilon}|^{-\frac{3}{2}} \quad \text{if } r \in [R_{2,\varepsilon} - \delta, R_{2,\varepsilon} - D\varepsilon^{\frac{2}{3}}],$$

for some constants $C, \delta, D > 0$ (δ as in Proposition 4.14), provided that $\varepsilon > 0$ is sufficiently small.

Proof. We only prove the assertions of the proposition that are related to $R_{1,\varepsilon}$, since those related to $R_{2,\varepsilon}$ follow analogously and are in fact considerably simpler to verify because $\eta_{1,\varepsilon}$ is small beyond all orders for $r \geq R_{1,\varepsilon} + \delta$.

From (4.57) and (4.77), there exists $C > 0$ and a sequence $D_j \to \infty$ such that

$$\left|(\eta_{2,\varepsilon} - \check{\eta}_{2,\varepsilon})(R_{1,\varepsilon} - D_j\varepsilon^{\frac{2}{3}})\right| \leq C\varepsilon^{\frac{4}{3}}, \tag{4.102}$$

for sufficiently small $\varepsilon > 0$, $j \geq 1$.

For $r \in [R_{1,\varepsilon} - \delta, R_{1,\varepsilon} - D_j \varepsilon^{\frac{2}{3}}]$, we can write

$$\varphi = \eta_{1,\varepsilon} - \check{\eta}_{1,\varepsilon} = \eta_{1,\varepsilon} - \hat{\eta}_{1,\varepsilon}, \quad \psi = \eta_{2,\varepsilon} - \check{\eta}_{2,\varepsilon} = \eta_{2,\varepsilon} - \tilde{\eta}_{2,\varepsilon}, \tag{4.103}$$

where φ, ψ satisfy (4.63).

Let $\varphi_0, \ \psi_0$ be determined from the problems

$$\begin{cases} -\varepsilon^2 \Delta\varphi_0 + \left[\left(g_1 - \frac{g^2}{g_2}\right)(3\hat{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon}) + 2\frac{g^2}{g_2}\hat{\eta}_{1,\varepsilon}^2\right]\varphi_0 = E_1, \\ \qquad\qquad r \in (R_{1,\varepsilon} - \delta, R_{1,\varepsilon} - D_j \varepsilon^{\frac{2}{3}}), \\ \varphi_0(R_{1,\varepsilon} - \delta) = \varphi(R_{1,\varepsilon} - \delta), \ \ \varphi_0(R_{1,\varepsilon} - D_j\varepsilon^{\frac{2}{3}}) = \varphi(R_{1,\varepsilon} - D_j\varepsilon^{\frac{2}{3}}), \end{cases}$$

and

$$\begin{cases} -\varepsilon^2 \Delta\psi_0 + 2g_2\tilde{\eta}_{2,\varepsilon}^2\psi_0 = N_2(\varphi, \psi) + E_2, \ \ r \in (R_{1,\varepsilon} - \delta, R_{1,\varepsilon} - D_j\varepsilon^{\frac{2}{3}}), \\ \psi_0(R_{1,\varepsilon} - \delta) = \psi(R_{1,\varepsilon} - \delta), \ \ \psi_0(R_{1,\varepsilon} - D_j\varepsilon^{\frac{2}{3}}) = \psi(R_{1,\varepsilon} - D_j\varepsilon^{\frac{2}{3}}), \end{cases}$$

where $E_i, \ N_i(\cdot, \cdot), i = 1, 2$, are as in (4.25) and (4.64) respectively. By virtue of (4.27), (4.28), (4.30), (4.33), (4.34), (4.45), (4.48), (4.79), Proposition 4.14, and (4.102), via a standard barrier argument, we deduce that

$$|\varphi_0(r)| \le C\varepsilon^2 + C\varepsilon \exp\left\{c\frac{r - R_{1,\varepsilon} + D_j\varepsilon^{\frac{2}{3}}}{\varepsilon^{\frac{2}{3}}}\right\}, \tag{4.104}$$

$$|\psi_0(r)| \le C\varepsilon^2 + C\varepsilon^4|r - R_{1,\varepsilon}|^{-4} + C\varepsilon^{\frac{4}{3}} \exp\left\{c\frac{r - R_{1,\varepsilon} + D_j\varepsilon^{\frac{2}{3}}}{\varepsilon}\right\}, \tag{4.105}$$

if $r \in [R_{1,\varepsilon} - \delta, R_{1,\varepsilon} - D_j\varepsilon^{\frac{2}{3}}]$.

We can write

$$\varphi = \varphi_0 + \tilde{\varphi}, \quad \psi = \psi_0 + \tilde{\psi},$$

where $\tilde{\varphi}, \ \tilde{\psi}$ satisfy

$$\begin{cases} \mathcal{L}(\tilde{\varphi}, \tilde{\psi}) = \begin{pmatrix} \tilde{E}_1 \\ \tilde{E}_2 \end{pmatrix}, \ \ r \in (R_{1,\varepsilon} - \delta, R_{1,\varepsilon} - D_j\varepsilon^{\frac{2}{3}}), \\ \tilde{\varphi}(R_{1,\varepsilon} - \delta) = \tilde{\psi}(R_{1,\varepsilon} - \delta) = 0, \ \ \tilde{\varphi}(R_{1,\varepsilon} - D_j\varepsilon^{\frac{2}{3}}) = \tilde{\psi}(R_{1,\varepsilon} - D_j\varepsilon^{\frac{2}{3}}) = 0, \end{cases} \tag{4.106}$$

with $\mathcal{L}$ as in (4.42), for some functions $\tilde{E}_i, i = 1, 2$, satisfying the following pointwise estimates:

$$\begin{aligned} |\tilde{E}_1| &\le C|\hat{\eta}_{1,\varepsilon}\psi_0| + |N_1(\varphi, \psi)| \text{via (4.46), (4.64), (4.79), (4.105)} \\ &\le C\varepsilon^2 + C\varepsilon^4|r - R_{1,\varepsilon}|^{-\frac{7}{2}} \\ &\quad + C_j\varepsilon^{\frac{5}{3}} \exp\left\{c\frac{r - R_{1,\varepsilon} + D_j\varepsilon^{\frac{2}{3}}}{\varepsilon}\right\}, \end{aligned} \tag{4.107}$$

$$|\tilde{E}_2| \le C|\hat{\eta}_{1,\varepsilon}\varphi_0| \overset{(4.46),(4.104)}{\le} C\varepsilon^2 + C_j\varepsilon^{\frac{4}{3}} \exp\left\{c\frac{r - R_{1,\varepsilon} + D_j\varepsilon^{\frac{2}{3}}}{\varepsilon^{\frac{2}{3}}}\right\}, \tag{4.108}$$

for $r \in [R_{1,\varepsilon} - \delta, R_{1,\varepsilon} - D_j\varepsilon^{\frac{2}{3}}]$.

Our plan is to solve the second equation in (4.106) for $\tilde{\psi}$ and substitute into the first, thus reducing the system to one scalar equation for $\tilde{\varphi}$. Then, we derive estimates for $\tilde{\varphi}$ in some carefully chosen weighted norms that we define afterwards. Let $I = (R_{1,\varepsilon} - \delta, R_{1,\varepsilon} - D_j \varepsilon^{\frac{2}{3}})$ and

$$\rho(r) = R_{1,\varepsilon} - r, \quad r \in I,$$

for $\phi \in C^2(I)$, we define

$$\|\phi\|_* = \varepsilon^2 \|\rho^{\frac{1}{2}} \phi_{rr}\|_{L^\infty(I)} + \varepsilon^2 \|\rho^{-\frac{1}{2}} \phi_r\|_{L^\infty(I)} + \|\rho^{\frac{3}{2}} \phi\|_{L^\infty(I)}, \tag{4.109}$$

(for related weighted norms we refer the interested reader to the monograph [31] and the references therein). In particular, we rely on the following a-priori estimate: there exist $\varepsilon_0, j_0, K > 0$ such that if $h \in C(\bar{I})$ and $\phi \in C_r^2\left(|x| \in \bar{I}\right)$ satisfy

$$-\varepsilon^2 \Delta\phi + \left(g_1 - \frac{g^2}{g_2}\right)(3\hat{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon})\phi = h \ \text{ in } I; \ \phi = 0 \ \text{ on } \partial I, \tag{4.110}$$

with $0 < \varepsilon < \varepsilon_0$, $j \geq j_0$, then

$$\|\phi\|_* \leq K \|\rho^{\frac{1}{2}} h\|_{L^\infty(I)}. \tag{4.111}$$

We stress that the above constant K is also independent of δ [i.e. $\varepsilon_0 = \varepsilon_0(\delta)$]. In the remainder of this proof, we denote by k/K a small/large generic constant that is independent of large j and small δ, ε. The proof of this estimate proceeds in two steps. Firstly, similarly to [27, Prop. 3.5], using the following consequence of (4.17)

$$k|r - R_{1,\varepsilon}| \leq 3\hat{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon} \leq K|r - R_{1,\varepsilon}|, \quad r \in I, \tag{4.112}$$

and the maximum principle (in the equation for $\rho^{\frac{3}{2}}\phi$), one obtains the partial estimate

$$\|\rho^{\frac{3}{2}} \phi\|_{L^\infty(I)} \leq K \|\rho^{\frac{1}{2}} h\|_{L^\infty(I)}. \tag{4.113}$$

Then, the full a-priori estimate follows by going back to the equation for ϕ and using the upper bound in (4.112). The details are given in Appendix B. From now on, we fix such a large j and drop the subscript from D_j.

In view of the second row in (4.42) and (4.106), we can write

$$\tilde{\psi} = -\frac{g}{g_2}\frac{\hat{\eta}_{1,\varepsilon}}{\tilde{\eta}_{2,\varepsilon}}\tilde{\varphi} + w + z, \quad r \in I, \tag{4.114}$$

where

$$-\varepsilon^2 \Delta w + 2g_2 \tilde{\eta}_{2,\varepsilon}^2 w = -\varepsilon^2 \frac{g}{g_2} \Delta\left(\frac{\hat{\eta}_{1,\varepsilon}}{\tilde{\eta}_{2,\varepsilon}}\tilde{\varphi}\right) \ \text{ in } I; \ \ w = 0 \ \text{ on } \ \partial I, \tag{4.115}$$

and

$$-\varepsilon^2 \Delta z + 2g_2 \tilde{\eta}_{2,\varepsilon}^2 z = \tilde{E}_2 \ \text{ in } I; \ \ z = 0 \ \text{ on } \ \partial I. \tag{4.116}$$

Using the pointwise estimates

$$\begin{cases} 0 < \hat{\eta}_{1,\varepsilon} \leq K\rho^{\frac{1}{2}}, \ |\nabla\hat{\eta}_{1,\varepsilon}| \leq K\rho^{-\frac{1}{2}}, \ |\Delta\hat{\eta}_{1,\varepsilon}| \leq K\rho^{-\frac{3}{2}}, \\ k \leq \tilde{\eta}_{2,\varepsilon} \leq K, \quad |\nabla\tilde{\eta}_{2,\varepsilon}| \leq K, \quad |\Delta\tilde{\eta}_{2,\varepsilon}| \leq K + K\varepsilon^2\rho^{-4}, \end{cases}$$

for $r \in I$, which follow readily from (1.33a), (4.17) (having increased j if needed), (4.21), (4.30), (4.33), (4.34), and (4.46), we can bound pointwise the right-hand side of (4.115) as

$$\begin{aligned}\varepsilon^2 \frac{g}{g_2}\left|\Delta\left(\frac{\hat{\eta}_{1,\varepsilon}}{\tilde{\eta}_{2,\varepsilon}}\tilde{\varphi}\right)\right| &\leq K\varepsilon^2\left(\rho^{\frac{1}{2}}|\Delta\tilde{\varphi}| + \rho^{-\frac{1}{2}}|\nabla\tilde{\varphi}| + \rho^{-\frac{3}{2}}|\tilde{\varphi}|\right)\\ &\leq K\left(\varepsilon^2\rho^{\frac{1}{2}}|\tilde{\varphi}_{rr}| + \varepsilon^2\rho^{-\frac{1}{2}}|\tilde{\varphi}_r| + \rho^{\frac{3}{2}}|\tilde{\varphi}|\right)\\ &= K\|\tilde{\varphi}\|_*.\end{aligned}$$

Hence, by the maximum principle, we deduce that

$$\|w\|_{L^\infty(I)} \leq K\|\tilde{\varphi}\|_*. \tag{4.117}$$

On the other side, from (4.108), (4.116) and a standard comparison argument, it follows that

$$|z(r)| \leq C\varepsilon^2 + C\varepsilon^{\frac{4}{3}}\exp\left\{c\frac{r - R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}}{\varepsilon^{\frac{2}{3}}}\right\}, \quad r \in I. \tag{4.118}$$

Substituting (4.114) into the first equation of (4.106), recalling (4.42), we arrive at

$$-\varepsilon^2\Delta\tilde{\varphi} + \left(g_1 - \frac{g^2}{g_2}\right)(3\hat{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon})\tilde{\varphi} = \tilde{E}_1 - 2g\hat{\eta}_{1,\varepsilon}\tilde{\eta}_{2,\varepsilon}(w+z) \text{ in } I; \quad \tilde{\varphi} = 0 \text{ on } \partial I.$$

Making use of the a-priori estimate (4.111), bound (4.117), and the easy estimates

$$\begin{aligned}\|\rho^{\frac{1}{2}}\tilde{E}_1\|_{L^\infty(I)} &\leq C\varepsilon^2 \ \text{[recall (4.107)]},\\ \|\rho^{\frac{1}{2}}\hat{\eta}_{1,\varepsilon}z\|_{L^\infty(I)} &\leq C\varepsilon^2 \ \text{[recall (4.46), (4.118)]},\end{aligned}$$

we obtain that

$$\|\tilde{\varphi}\|_* \leq C\varepsilon^2 + \delta K\|\tilde{\varphi}\|_*,$$

where we also exploited that $0 < \hat{\eta}_{1,\varepsilon} \leq K\rho^{\frac{1}{2}} \leq K\delta^{\frac{1}{2}}$ in I. Consequently, choosing a sufficiently small δ, and fixing it from now on, we infer that

$$\|\tilde{\varphi}\|_* \leq C\varepsilon^2.$$

In particular, for small ε, we have that

$$|\tilde{\varphi}(r)| \leq C\varepsilon^2|r - R_{1,\varepsilon}|^{-\frac{3}{2}}, \quad r \in [R_{1,\varepsilon} - \delta, R_{1,\varepsilon} - D\varepsilon^{\frac{2}{3}}].$$

In turn, from the second equation in (4.106), recalling (4.46) and (4.108), via a standard barrier argument, we find that

$$\left|\tilde{\psi}(r)\right| \leq C\varepsilon^2|r - R_{1,\varepsilon}|^{-1}, \quad r \in [R_{1,\varepsilon} - \delta, R_{1,\varepsilon} - D\varepsilon^{\frac{2}{3}}].$$

The desired assertion of the proposition now follows at once from (4.103), (4.104), (4.105), and the above two relations. □

Lemma 4.16. *Given $D > 0$, we have that*

$$\left|\eta_{2,\varepsilon}(r) - \check{\eta}_{2,\varepsilon}(r)\right| \leq C\varepsilon^{\frac{4}{3}}, \quad |r - R_{1,\varepsilon}| \leq D\varepsilon^{\frac{2}{3}},$$

provided that $\varepsilon > 0$ is sufficiently small.

Proof. As in the beginning of the proof of Proposition 4.15, given $D > 0$, if $\varepsilon > 0$ is sufficiently small, there exist $r_- \in \left(R_{1,\varepsilon} - (D+1)\varepsilon^{\frac{2}{3}}, R_{1,\varepsilon} - D\varepsilon^{\frac{2}{3}}\right)$ and $r_+ \in \left(R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}, R_{1,\varepsilon} + (D+1)\varepsilon^{\frac{2}{3}}\right)$ such that

$$|\psi(r_\pm)| \leq C\varepsilon^{\frac{4}{3}},$$

where $\psi = \eta_{2,\varepsilon} - \check{\eta}_{2,\varepsilon}$. Keeping in mind the proof of Proposition 4.3, (4.42), (4.46) and (4.79)), it follows readily from the second equation of (4.63) that ψ satisfies

$$-\varepsilon^2\Delta\psi + 2g_2\tilde{\eta}_{2,\varepsilon}^2\psi = \mathcal{O}(\varepsilon^{\frac{4}{3}}), \quad \text{uniformly on } [r_-, r_+], \ \text{as } \varepsilon \to 0.$$

The assertion of the lemma follows directly from the above two relations and the maximum principle, since $\tilde{\eta}_{2,\varepsilon} \geq c$ in this region. □

Lemma 4.17. *If $\varepsilon > 0$ is sufficiently small, we have*

$$\left|\eta_{1,\varepsilon}(r) - \check{\eta}_{1,\varepsilon}(r)\right| \leq C\varepsilon \exp\left\{c\frac{R_{1,\varepsilon} - r}{\varepsilon^{\frac{2}{3}}}\right\}, \quad R_{1,\varepsilon} \leq r \leq R_{1,\varepsilon} + \delta,$$
$$\left|\eta_{2,\varepsilon}(r) - \check{\eta}_{2,\varepsilon}(r)\right| \leq C\varepsilon^2 + C\varepsilon^{\frac{4}{3}} \exp\left\{c\frac{R_{1,\varepsilon} - r}{\varepsilon^{\frac{2}{3}}}\right\}, \quad R_{1,\varepsilon} \leq r \leq R_{1,\varepsilon} + \delta,$$

and

$$\left|\eta_{2,\varepsilon}(r) - \check{\eta}_{2,\varepsilon}(r)\right| \leq C\varepsilon \exp\left\{c\frac{R_{2,\varepsilon} - r}{\varepsilon^{\frac{2}{3}}}\right\}, \quad R_{2,\varepsilon} \leq r \leq R_{2,\varepsilon} + \delta.$$

Proof. We only prove the estimates that concern $R_{1,\varepsilon}$ because those concerning $R_{2,\varepsilon}$ follow analogously. As in the proof of Proposition 4.15, let $\varphi = \eta_{1,\varepsilon} - \check{\eta}_{1,\varepsilon}$ and $\psi = \eta_{2,\varepsilon} - \check{\eta}_{2,\varepsilon}$. In view of (4.42), (4.63), (4.64), the relations

$$E_1 = 0, \quad |E_2| \leq C\varepsilon^{\frac{4}{3}} \exp\left\{c\frac{R_{1,\varepsilon} - r}{\varepsilon^{\frac{2}{3}}}\right\} + C\varepsilon^2,$$
$$r \in [R_{1,\varepsilon}, R_{1,\varepsilon} + 3\delta] \ [\text{from } (4.27),\ (4.28),\ (4.30)],$$

(4.79), and Proposition 4.15, we infer that

$$\begin{cases} -\varepsilon^2\Delta\varphi + p(r)\varphi = \mathcal{O}(\hat{\eta}_{1,\varepsilon}\psi), \\ -\varepsilon^2\Delta\psi + q(r)\psi = \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon^{\frac{4}{3}}) \exp\left\{c\frac{R_{1,\varepsilon} - r}{\varepsilon^{\frac{2}{3}}}\right\}, \end{cases} \tag{4.119}$$

uniformly on $[R_{1,\varepsilon}, R_{1,\varepsilon} + 3\delta]$, as $\varepsilon \to 0$, for some smooth functions p, q satisfying

$$p \geq c\varepsilon^{\frac{2}{3}} \ \text{ and } \ q \geq c \ [\text{recall } (4.45) \text{ and } (4.48)]. \tag{4.120}$$

Note that, from (4.79) and Lemma 4.16, we have

$$\begin{cases} \varphi(R_{1,\varepsilon}) = \mathcal{O}(\varepsilon), \quad \varphi(R_{1,\varepsilon} + 2\delta) = \mathcal{O}(\varepsilon), \\ \psi(R_{1,\varepsilon}) = \mathcal{O}(\varepsilon^{\frac{4}{3}}), \ \psi(R_{1,\varepsilon} + 3\delta) = \mathcal{O}(\varepsilon), \end{cases} \tag{4.121}$$

as $\varepsilon \to 0$. A standard barrier argument yields that

$$|\psi(r)| \le C\varepsilon^{\frac{4}{3}} \exp\left\{c\frac{R_{1,\varepsilon} - r}{\varepsilon^{\frac{2}{3}}}\right\} + C\varepsilon^2 + C\varepsilon \exp\left\{c\frac{r - R_{1,\varepsilon} - 3\delta}{\varepsilon}\right\}, \quad r \in [R_{1,\varepsilon}, R_{1,\varepsilon} + 3\delta],$$

which implies that

$$|\psi(r)| \le C\varepsilon^{\frac{4}{3}} \exp\left\{c\frac{R_{1,\varepsilon} - r}{\varepsilon^{\frac{2}{3}}}\right\} + C\varepsilon^2, \quad r \in [R_{1,\varepsilon}, R_{1,\varepsilon} + 2\delta],$$

provided that $\varepsilon > 0$ is sufficiently small, as asserted. Now, via (4.119) and (4.13), we arrive at

$$-\varepsilon^2 \Delta\varphi + p(r)\varphi = \mathcal{O}(\varepsilon^{\frac{5}{3}}) \exp\left\{c\frac{R_{1,\varepsilon} - r}{\varepsilon^{\frac{2}{3}}}\right\},$$

uniformly on $[R_{1,\varepsilon}, R_{1,\varepsilon}+2\delta]$, as $\varepsilon \to 0$. Keeping in mind (4.120) and (4.121), a standard barrier argument yields that

$$|\varphi(r)| \le C\varepsilon \exp\left\{c\frac{R_{1,\varepsilon} - r}{\varepsilon^{\frac{2}{3}}}\right\} + C\varepsilon \exp\left\{c\frac{r - R_{1,\varepsilon} - 2\delta}{\varepsilon^{\frac{2}{3}}}\right\}, \quad r \in [R_{1,\varepsilon}, R_{1,\varepsilon} + 2\delta],$$

which implies that

$$|\varphi(r)| \le 2C\varepsilon \exp\left\{c\frac{R_{1,\varepsilon} - r}{\varepsilon^{\frac{2}{3}}}\right\}, \quad r \in [R_{1,\varepsilon}, R_{1,\varepsilon} + \delta],$$

as asserted. □

4.7. Improved estimate for the Lagrange multipliers. In the sequel, building on Proposition 3.3, via the results of the previous subsection, we are able to considerably improve the estimate for $\lambda_{i,\varepsilon} - \lambda_{i,0}$ of the aforementioned proposition.

Proposition 4.18. *If $\varepsilon > 0$ is sufficiently small, we have*

$$|\lambda_{i,\varepsilon} - \lambda_{i,0}| \le C|\log\varepsilon|\varepsilon^2, \quad i = 1, 2.$$

Proof. Motivated by the proof of the corresponding estimate for the scalar equation, as given in [27, Thm. 1.1], we first show that

$$\int_{\mathbb{R}^2} (\eta_{1,\varepsilon}^2 - a_{1,\varepsilon}^+)\,dx = \mathcal{O}(|\log\varepsilon|\varepsilon^2), \tag{4.122}$$

$$\int_{B_{R_{1,\varepsilon}}} (\eta_{2,\varepsilon}^2 - a_{2,\varepsilon})\,dx + \int_{\mathbb{R}^2 \setminus B_{R_{1,\varepsilon}}} \left[\eta_{2,\varepsilon}^2 - \left(a_{2,\varepsilon} + \frac{g}{g_2} a_{1,\varepsilon}\right)^+\right] dx = \mathcal{O}(|\log\varepsilon|\varepsilon^2), \tag{4.123}$$

as $\varepsilon \to 0$, and then exploit that

$$\int_{\mathbb{R}^2} \eta_{i,\varepsilon}^2 \, dx = \int_{\mathbb{R}^2} a_i \, dx = 1, \quad i = 1, 2. \tag{4.124}$$

It suffices to establish only the validity of estimate (4.123) because that of (4.122) follows verbatim. By (4.21), (4.23), Proposition 4.15 and Lemma 4.16, we obtain that

$$\begin{aligned} \eta_{2,\varepsilon}^2 - a_{2,\varepsilon} &= \tilde{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon} + \mathcal{O}\left(\varepsilon^2 |r - R_{1,\varepsilon}|^{-1}\right) \\ &= \tfrac{g}{g_2}(a_{1,\varepsilon} - \hat{\eta}_{1,\varepsilon}^2) + \mathcal{O}\left(\varepsilon^2 |r - R_{1,\varepsilon}|^{-1}\right), \end{aligned}$$

uniformly on $[R_{1,\varepsilon} - \delta, R_{1,\varepsilon} - D\varepsilon^{\frac{2}{3}}]$, as $\varepsilon \to 0$. Analogously, making use of Lemma 4.16, we see that

$$\eta_{2,\varepsilon}^2 - a_{2,\varepsilon} = \frac{g}{g_2}(a_{1,\varepsilon} - \hat{\eta}_{1,\varepsilon}^2) + \mathcal{O}(\varepsilon^{\frac{4}{3}}),$$

uniformly on $[R_{1,\varepsilon} - D\varepsilon^{\frac{2}{3}}, R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}]$, as $\varepsilon \to 0$. Hence, via Proposition 4.14, we find that

$$\begin{aligned} \int_{B_{R_{1,\varepsilon}}} (\eta_{2,\varepsilon}^2 - a_{2,\varepsilon}) \, dx &= \frac{g}{g_2} \int_{B_{R_{1,\varepsilon}}} (a_{1,\varepsilon} - \hat{\eta}_{1,\varepsilon}^2) \, dx + \mathcal{O}(|\log \varepsilon| \varepsilon^2) \\ &\overset{(4.12)}{=} \frac{g}{g_2} \int_{R_{1,\varepsilon} - \delta < |x| < R_{1,\varepsilon}} (a_{1,\varepsilon} - \hat{\eta}_{1,\varepsilon}^2) \, dx + \mathcal{O}(|\log \varepsilon| \varepsilon^2), \end{aligned} \tag{4.125}$$

as $\varepsilon \to 0$. Similarly, keeping in mind (4.13), we have

$$\int_{R_{1,\varepsilon} < |x| < R_{2,\varepsilon} - \delta} \left(\eta_{2,\varepsilon}^2 - a_{2,\varepsilon} - \frac{g}{g_2} a_{1,\varepsilon} \right) dx = -\frac{g}{g_2} \int_{R_{1,\varepsilon} < |x| < R_{1,\varepsilon} + \delta} \hat{\eta}_{1,\varepsilon}^2 \, dx + \mathcal{O}(\varepsilon^2), \tag{4.126}$$

as $\varepsilon \to 0$, where we use Lemmas 4.16–4.17 instead of Proposition 4.15. On the other side, thanks to (4.12) and Proposition 4.15, for $r \in [R_{2,\varepsilon} - \delta, R_{2,\varepsilon} - D\varepsilon^{\frac{2}{3}}]$, we find that

$$\begin{aligned} \eta_{2,\varepsilon}^2 - \left(a_{2,\varepsilon} + \tfrac{g}{g_2} a_{1,\varepsilon}\right) &= \hat{\eta}_{2,\varepsilon}^2 - \left(a_{2,\varepsilon} + \tfrac{g}{g_2} a_{1,\varepsilon}\right) + 2\hat{\eta}_{2,\varepsilon}(\eta_{2,\varepsilon} - \hat{\eta}_{2,\varepsilon}) + (\eta_{2,\varepsilon} - \hat{\eta}_{2,\varepsilon})^2 \\ &= \hat{\eta}_{2,\varepsilon}^2 - \left(a_{2,\varepsilon} + \tfrac{g}{g_2} a_{1,\varepsilon}\right) + \mathcal{O}\left(\varepsilon^2 |r - R_{2,\varepsilon}|^{-1}\right), \end{aligned}$$

uniformly, as $\varepsilon \to 0$. Analogously, making use of (4.12) and (4.79), we see that

$$\eta_{2,\varepsilon}^2 - \left(a_{2,\varepsilon} + \frac{g}{g_2} a_{1,\varepsilon}\right) = \hat{\eta}_{2,\varepsilon}^2 - \left(a_{2,\varepsilon} + \frac{g}{g_2} a_{1,\varepsilon}\right) + \mathcal{O}(\varepsilon^{\frac{4}{3}}),$$

uniformly on $[R_{2,\varepsilon} - D\varepsilon^{\frac{2}{3}}, R_{2,\varepsilon} + D\varepsilon^{\frac{2}{3}}]$, as $\varepsilon \to 0$. Thus, we get that

$$\begin{aligned} &\int_{R_{2,\varepsilon} - \delta < |x| < R_{2,\varepsilon}} \left(\eta_{2,\varepsilon}^2 - a_{2,\varepsilon} - \frac{g}{g_2} a_{1,\varepsilon} \right) dx \\ &\quad = \int_{R_{2,\varepsilon} - \delta < |x| < R_{2,\varepsilon}} \left(\hat{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon} - \frac{g}{g_2} a_{1,\varepsilon} \right) dx + \mathcal{O}(|\log \varepsilon| \varepsilon^2), \end{aligned} \tag{4.127}$$

as $\varepsilon \to 0$. Similarly, using Lemma 4.17 instead of Proposition 4.15, keeping in mind (4.13), we obtain that

$$\int_{|x|>R_{2,\varepsilon}} \eta_{2,\varepsilon}^2\, dx = \int_{R_{2,\varepsilon}<|x|<R_{2,\varepsilon}+\delta} \hat{\eta}_{2,\varepsilon}^2\, dx + \mathcal{O}(\varepsilon^2) \text{ as } \varepsilon \to 0. \tag{4.128}$$

Now, estimate (4.123) follows readily by adding relations (4.125), (4.126), (4.127), (4.128), and using the estimates

$$\int_{||x|-R_{1,\varepsilon}|<\delta} (\hat{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon}^+)\, dx = \mathcal{O}(|\log\varepsilon|\varepsilon^2),$$
$$\int_{||x|-R_{2,\varepsilon}|<\delta} \left[\hat{\eta}_{2,\varepsilon}^2 - \left(a_{2,\varepsilon} + \frac{g}{g_2}a_{1,\varepsilon}\right)^+\right] dx = \mathcal{O}(|\log\varepsilon|\varepsilon^2),$$

as $\varepsilon \to 0$, which follow from the proof of Theorem 1.1 in [27]. The proof of relation (4.123) is complete.

By virtue of (3.14), increasing the value of D, if needed, we may assume that $|R_{i,\varepsilon} - R_{i,0}| \le D|\log\varepsilon|^{\frac{1}{2}}\varepsilon$, $i = 1, 2$, for small $\varepsilon > 0$. It follows from (4.122), (4.123) and (4.124), recalling (1.30) and (3.13), that

$$\int_{|x|<R_{1,0}-D|\log\varepsilon|^{\frac{1}{2}}\varepsilon} (a_{1,\varepsilon} - a_{1,0})\, dx = \mathcal{O}(|\log\varepsilon|\varepsilon^2),$$
$$\int_{|x|<R_{1,0}-D|\log\varepsilon|^{\frac{1}{2}}\varepsilon} (a_{2,\varepsilon} - a_{2,0})\, dx$$
$$+ \int_{R_{1,0}+D|\log\varepsilon|^{\frac{1}{2}}\varepsilon<|x|<R_{2,0}-D|\log\varepsilon|^{\frac{1}{2}}\varepsilon} \left[(a_{2,\varepsilon} - a_{2,0}) + \frac{g}{g_2}(a_{1,\varepsilon} - a_{1,0})\right] dx$$
$$= \mathcal{O}(|\log\varepsilon|\varepsilon^2),$$

as $\varepsilon \to 0$. In view of (1.30), (1.31), and (3.13), this leads to the following system:

$$(\lambda_{1,\varepsilon} - \lambda_{1,0}) - \frac{g}{g_2}(\lambda_{2,\varepsilon} - \lambda_{2,0}) = \mathcal{O}(|\log\varepsilon|\varepsilon^2),$$
$$\frac{1}{\Gamma}\left[(\lambda_{2,\varepsilon} - \lambda_{2,0}) - \frac{g}{g_1}(\lambda_{1,\varepsilon} - \lambda_{1,0})\right]\left(\lambda_{1,\varepsilon} - \frac{g}{g_2}\lambda_{2,\varepsilon}\right)$$
$$+ (\lambda_{2,\varepsilon} - \lambda_{1,\varepsilon})(\lambda_{2,\varepsilon} - \lambda_{2,0}) = \mathcal{O}(|\log\varepsilon|\varepsilon^2),$$

as $\varepsilon \to 0$. Now, recalling that $g < g_2$, the assertion of the proposition follows straightforwardly. □

4.8. *Proof of Theorem 4.1.*

Proof. Let (η_1, η_2) be the unique positive minimizer of E_ε^0 in $\mathcal{H}$ provided by Theorem 1.3 (2). We saw in Proposition 3.3 that the associated Lagrange multipliers $\lambda_{i,\varepsilon}$ satisfy $|\lambda_{i,\varepsilon} - \lambda_{i,0}| \le \varepsilon|\log\varepsilon|^{1/2}$, $i = 1, 2$. In view of (4.79) and Proposition 4.12, the solution $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$ that is provided by Proposition 4.8 also fashions a positive radial solution of the system (1.5), with *the same* Lagrange multipliers $\lambda_{i,\varepsilon}$. Therefore, by Theorem 1.3 (1), it coincides with (η_1, η_2).

Estimate (4.1) is proven in Proposition 4.18. Estimates (4.2)-(4.3) follow from Proposition 4.14, the definition of $\check{\eta}_{i,\varepsilon}$, and the second estimate in (4.12). Estimates (4.4)-(4.5) follow readily from Proposition 4.15, the definition of $\check{\eta}_{i,\varepsilon}$ [especially recall (4.21) for the second estimate in (4.4)], and (4.17). Estimate (4.6) follows readily from Lemma 4.13, (4.12) and (4.79); estimate (4.7) follows from (4.13), (4.21) and Lemma 4.17. Finally, relations (4.8), (4.9) and (4.10) are consequences of (4.15), (4.21), (4.79) and Lemma 4.17. □

4.9. Proof of Theorem 1.4. The desired minimizer $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$ is that of Theorem 4.1. Clearly, estimate (1.19) is the same as (4.1). Estimate (1.20) follows readily by combining (1.30), (4.4), (4.5), (4.6), (4.7), (4.9), and (4.10). Estimate (1.21) follows readily from (4.2), (4.3), (4.4) and (4.5). In view of (1.12), (4.2) and (4.3), we infer that (1.22) holds. Finally, the decay estimate (1.23) follows immediately from (4.6).

5. Estimates for the Annulus Case

In this section, we explain how to extend the previous section to prove Theorem 1.6.

5.1. Construction of an approximate solution.

5.1.1. Outer approximations As before, we work with the equivalent problem (1.29), where $a_{1,\varepsilon}$, $a_{2,\varepsilon}$ are the same as in (1.30), and $\lambda_{1,\varepsilon}$, $\lambda_{2,\varepsilon}$ are provided by Theorem 1.3 in the case of (1.9). This time, the problem with both diffusion terms neglected has a unique continuous, nonnegative solution given by

$$\begin{array}{lll}
\eta_1 = \left(a_{1,\varepsilon} + \frac{g}{g_1} a_{2,\varepsilon}\right)^{\frac{1}{2}}, & \eta_2 = 0, & 0 \le r \le R^-_{2,\varepsilon}, \\
\eta_1 = a^{\frac{1}{2}}_{1,\varepsilon}, & \eta_2 = a^{\frac{1}{2}}_{2,\varepsilon}, & R^-_{2,\varepsilon} \le r \le R_{1,\varepsilon}, \\
\eta_1 = 0, & \eta_2 = \left(a_{2,\varepsilon} + \frac{g}{g_2} a_{1,\varepsilon}\right)^{\frac{1}{2}}, & R_{1,\varepsilon} \le r \le R^+_{2,\varepsilon}, \\
\eta_1 = 0, & \eta_2 = 0, & r \ge R^+_{2,\varepsilon},
\end{array}$$

where

$$(R^-_{2,\varepsilon})^2 = \frac{1}{\Gamma_1}\left(\lambda_{2,\varepsilon} - \frac{g}{g_1}\lambda_{1,\varepsilon}\right), \quad (R^+_{2,\varepsilon})^2 = \lambda_{2,\varepsilon},$$

and

$$R^2_{1,\varepsilon} = \frac{1}{\Gamma_2}\left(\lambda_{1,\varepsilon} - \frac{g}{g_2}\lambda_{2,\varepsilon}\right).$$

In view of (3.13), and Remark 3.4, we have that

$$|R_{1,\varepsilon} - R_{1,0}| + |R^{\pm}_{2,\varepsilon} - R^{\pm}_{2,0}| \le C|\log\varepsilon|^{\frac{1}{2}}\varepsilon. \tag{5.1}$$

5.1.2. Inner approximations. Here, we define approximate solutions of the problem in overlapping intervals around each point $R^-_{2,0} < R_{1,0} < R^+_{2,0}$.

On $[0, R_{1,0}-\delta]$, where $\sqrt{a_1}$ is away from zero and has bounded gradient, we neglect only the term $\varepsilon^2\Delta\eta_{1,\varepsilon}$ from (1.29), and get the following problem:

$$\begin{cases} g_1\eta_1\left(\eta_1^2 - a_{1,\varepsilon}(r)\right) + g\eta_1\left(\eta_2^2 - a_{2,\varepsilon}(r)\right) = 0, \\ -\varepsilon^2\Delta\eta_2 + g_2\eta_2\left(\eta_2^2 - a_{2,\varepsilon}(r)\right) + g\eta_2\left(\eta_1^2 - a_{1,\varepsilon}(r)\right) = 0. \end{cases}$$

From the first equation, we find that

$$\eta_1^2 = a_{1,\varepsilon} + \frac{g}{g_1}(a_{2,\varepsilon} - \eta_2^2). \tag{5.2}$$

Then, from the second equation, we obtain that

$$-\varepsilon^2\Delta\eta_2 + \left(g_2 - \frac{g^2}{g_1}\right)\eta_2(\eta_2^2 - a_{2,\varepsilon}) = 0.$$

The function $a_{2,\varepsilon}$ is negative in $[0, R^-_{2,\varepsilon})$ and positive in $(R^-_{2,\varepsilon}, \infty)$. We consider a function $A_{2,\varepsilon}$ which coincides with $a_{2,\varepsilon}$ on $[0, R_{1,0}+\delta]$, changes sign once in $(R_{1,0}+\delta, \infty)$, and diverges to $-\infty$ as $r \to \infty$. We then take as an approximation for η_2 on $[0, R_{1,0}-\delta]$ the restriction of the unique positive solution $\hat{\eta}^-_{2,\varepsilon}$ of the problem

$$\varepsilon^2\Delta\eta = \left(g_2 - \frac{g^2}{g_1}\right)\eta\left(\eta^2 - A_{2,\varepsilon}(r)\right) \text{ in } \mathbb{R}^2, \quad \eta \to 0 \text{ as } r \to \infty.$$

The properties of $\hat{\eta}^-_{2,\varepsilon}$ which we require are contained in Appendix A. Accordingly, we take as an approximation for η_1 on $[0, R_{1,0}-\delta]$ the one given by (5.2) with $\hat{\eta}^-_{2,\varepsilon}$ in place of η_2. The approximations for $\eta_{1,\varepsilon}$ and $\eta_{2,\varepsilon}$ on $[R_{1,0}+\delta, R^+_{2,0}-\delta]$ are the same ones as in the case of two disks, namely those given by (1.33a) and (4.21) respectively. Analogously, if $r \geq R_{1,0}+\delta$, we take as an approximation for η_1 the trivial solution, while for η_2 the unique positive solution of the problem (1.33b) which we now call $\hat{\eta}^+_{2,\varepsilon}$.

5.1.3. Gluing approximate solutions. Let

$$r_\varepsilon = \frac{R^-_{2,\varepsilon} + R_{1,\varepsilon}}{2}, \quad R_\varepsilon = \frac{R_{1,\varepsilon} + R^+_{2,\varepsilon}}{2}.$$

Analogously to Sect. 4.2, we can define a smooth global approximate solution $(\check{\eta}_{1,\varepsilon}, \check{\eta}_{2,\varepsilon})$ such that

$$\check{\eta}_{1,\varepsilon} = \begin{cases} \left(a_{1,\varepsilon} + \frac{g}{g_1}a_{2,\varepsilon} - \frac{g}{g_1}(\hat{\eta}^-_{2,\varepsilon})^2\right)^{\frac{1}{2}}, & 0 \leq r \leq r_\varepsilon, \\ a^{\frac{1}{2}}_{1,\varepsilon} + \mathcal{O}_{C^2}(\varepsilon^2), & r_\varepsilon \leq r \leq r_\varepsilon + \delta, \\ \hat{\eta}_{1,\varepsilon}, & r_\varepsilon + \delta \leq r, \end{cases} \tag{5.3}$$

$$\check{\eta}_{2,\varepsilon} = \begin{cases} \hat{\eta}_{2,\varepsilon}^{-}, & 0 \le r \le r_\varepsilon, \\ a_{2,\varepsilon}^{\frac{1}{2}} + \mathcal{O}_{C^2}(\varepsilon^2), & r_\varepsilon \le r \le r_\varepsilon + \delta, \\ \left(a_{2,\varepsilon} + \frac{g}{g_2} a_{1,\varepsilon} - \frac{g}{g_2} \hat{\eta}_{1,\varepsilon}^2\right)^{\frac{1}{2}}, & r_\varepsilon + \delta \le r \le R_\varepsilon, \\ \left(a_{2,\varepsilon} + \frac{g}{g_2} a_{1,\varepsilon}\right)^{\frac{1}{2}} + \mathcal{O}_{C^2}(\varepsilon^2), & R_\varepsilon \le r \le R_\varepsilon + \delta, \\ \hat{\eta}_{2,\varepsilon}^{+}, & R_\varepsilon + \delta \le r. \end{cases} \tag{5.4}$$

5.2. Estimates for the error on the approximate solution. The remainder $\mathcal{E}(\check{\eta}_{1,\varepsilon}, \check{\eta}_{2,\varepsilon})$ that is left when substituting the approximate solution $(\check{\eta}_{1,\varepsilon}, \check{\eta}_{2,\varepsilon})$ to the system (1.29) is as in (4.25).

For convenience, we set

$$A_\varepsilon = \{x \in \mathbb{R}^2 \ : \ r_\varepsilon < |x| < R_\varepsilon\}. \tag{5.5}$$

Analogously to Proposition 4.3 for the case of two disks, we have

Proposition 5.1. *The following estimates hold for small $\varepsilon > 0$:*

$$\|E_1\|_{L^2(A_\varepsilon)} \le C\varepsilon^2, \quad \|E_2\|_{L^2(A_\varepsilon)} \le C\varepsilon^{\frac{5}{3}},$$

and

$$\|E_1\|_{L^2(\mathbb{R}^2 \setminus A_\varepsilon)} \le C\varepsilon^{\frac{5}{3}}, \quad |E_2\|_{L^2(\mathbb{R}^2 \setminus A_\varepsilon)} \le C\varepsilon^2.$$

5.3. Linear analysis. In the sequel, we consider the linearization of (1.29) about the approximate solution $(\check{\eta}_{1,\varepsilon}, \check{\eta}_{2,\varepsilon})$, namely the linear operator that is given by (4.41) for this choice of $(\check{\eta}_{1,\varepsilon}, \check{\eta}_{2,\varepsilon})$.

As in the case of two disks, using that

$$3\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon} \ge \begin{cases} c \max\{\varepsilon^{\frac{2}{3}}, \check{\eta}_{2,\varepsilon}^2\}, & |r - R_{2,\varepsilon}^-| \le \delta, \\ c, & r \in [0, R_{2,\varepsilon}^- - \delta] \cup [R_{2,\varepsilon}^- + \delta, r_\varepsilon], \end{cases}$$

$$3\check{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon} \ge \begin{cases} c \max\{\varepsilon^{\frac{2}{3}}, \check{\eta}_{1,\varepsilon}^2\}, & |r - R_{1,\varepsilon}| \le \delta, \\ c, & r \in [r_\varepsilon, R_{1,\varepsilon} - \delta] \cup [R_{1,\varepsilon} + \delta, R_\varepsilon], \end{cases}$$

and

$$3\check{\eta}_{2,\varepsilon}^2 - a_{2,\varepsilon} - \frac{g}{g_2} a_{1,\varepsilon} \ge \begin{cases} c \max\{\varepsilon^{\frac{2}{3}}, \check{\eta}_{2,\varepsilon}^2\}, & |r - R_{2,\varepsilon}^+| \le \delta, \\ c, & r \in [R_\varepsilon, R_{2,\varepsilon}^+ - \delta] \cup [R_{2,\varepsilon}^+ + \delta, \infty), \end{cases}$$

we can establish an analog of Proposition 4.6.

Proposition 5.2. *The assertions of Proposition 4.6 are valid, provided that in (4.56) and in the definition of the $|||\cdot|||$-norm in (4.57), B_{R_ε} is replaced by A_ε defined in (5.5).*

5.4. Existence and properties of a positive solution to the system (1.29). As in Sect. 4.5, using the properties of the linearized operator that we discussed above, we construct a positive, radial solution $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$ to (1.29), near the approximate one $(\check{\eta}_{1,\varepsilon}, \check{\eta}_{2,\varepsilon})$, for small $\varepsilon > 0$. As before, the first part of the uniqueness Theorem 1.3 guarantees that this solution is the desired minimizer.

Using the $|||\cdot|||$-norm, as redefined in Proposition 5.2, we can show that Propositions 4.7, 4.8 and Corollary 4.9 remain unchanged. We still denote the corresponding solution by $(\eta_{1,\varepsilon}, \eta_{2,\varepsilon})$. The assertion of the Lemma 4.10 also remains the same. The only difference in the proof is that, say in the equation for v_n, we rearrange the terms differently, namely write

$$-\Delta v_n + g_1 v_n^3 - \left[g_1 a_{1,\varepsilon_n}(x_n + \varepsilon_n \mu_n y) + g a_{2,\varepsilon_n}(x_n + \varepsilon_n \mu_n y)\right] \mu_n^2 v_n$$
$$+ g\mu_n^2 \eta_{2,\varepsilon_n}^2 (x_n + \varepsilon_n \mu_n y) v_n = 0,$$

with

$$\|\eta_{2,\varepsilon}\|_{L^p(B_\delta)} \le C_p \varepsilon^{\frac{2}{3}+\frac{4}{3p}}, \quad p \ge 2.$$

Then, the analog of Corollary 4.11 is

$$\|\eta_{1,\varepsilon} - \sqrt{a_{1,\varepsilon} + \frac{g}{g_1} a_{2,\varepsilon}}\|_{L^\infty(B_\delta)} + \|\eta_{2,\varepsilon}\|_{L^\infty(B_\delta)} \le C\varepsilon^{\frac{1}{3}}.$$

The positivity of the constructed solution, namely the analog of Proposition 4.12, requires some additional considerations, since $\eta_{2,\varepsilon}$ is also small in the disk $|x| < R_{2,\varepsilon}^-$:

Proposition 5.3. *If $\varepsilon > 0$ is sufficiently small, the constructed solutions satisfy*

$$\eta_{i,\varepsilon} > 0 \ \text{ in } \mathbb{R}^2, \quad i = 1, 2.$$

Proof. The main difference with the previous case is in the domain $|x| < R_{2,\varepsilon}^-$, which we describe below.

We know that

$$\eta_{1,\varepsilon} = \sqrt{\frac{\lambda_{1,\varepsilon} - r^2}{g_1}} + \mathcal{O}(\varepsilon^{\frac{2}{3}}), \quad r \in [0, R_{2,\varepsilon}^- - D\varepsilon^{\frac{2}{3}}], \tag{5.6}$$

where $\mathcal{O}(\varepsilon^{\frac{2}{3}})$ in dependent of $D > 1$ [this follows directly from the analog of Proposition 4.14 or from the analogs of (4.22) and Corollary 4.9], and

$$\eta_{2,\varepsilon}(R_{2,\varepsilon}^- - D\varepsilon^{\frac{2}{3}}) \ge c\varepsilon^{\frac{1}{3}} > 0.$$

The function $\eta_{2,\varepsilon}$ satisfies the elliptic equation

$$-\varepsilon^2 \Delta\eta_{2,\varepsilon} + (r^2 + g_2\eta_{2,\varepsilon}^2 + g\eta_{1,\varepsilon}^2 - \lambda_{2,\varepsilon})\eta_{2,\varepsilon} = 0.$$

In view of the above, the desired positivity of $\eta_{2,\varepsilon}$ follows directly from the maximum principle once we show that

$$r^2 + g_2\eta_{2,\varepsilon}^2 + g\eta_{1,\varepsilon}^2 - \lambda_{2,\varepsilon} > 0, \quad r \in [0, R_{2,\varepsilon}^- - D\varepsilon^{\frac{2}{3}}].$$

Note that, thanks to (5.6), the left-hand side equals

$$\Gamma_1 r^2 + \frac{g}{g_1}\lambda_{1,\varepsilon} - \lambda_{2,\varepsilon} + g_2\eta_{2,\varepsilon}^2 + \mathcal{O}(\varepsilon^{\frac{2}{3}}),$$

where $\mathcal{O}(\varepsilon^{\frac{2}{3}})$ in dependent of $D > 1$. In view of (5.1), it suffices to show that

$$\Gamma_1 r^2 + \frac{g}{g_1}\lambda_{1,0} - \lambda_{2,0} \geq cD\varepsilon^{\frac{2}{3}}, \quad r \in [0, R_{2,0}^- - D\varepsilon^{\frac{2}{3}}], \tag{5.7}$$

for some constant $c > 0$ that is independent of D, ε, provided that D is sufficiently large and ε sufficiently small. Observe that, since $r \leq R_{2,0}^- - D\varepsilon^{\frac{2}{3}}$, we have

$$r^2 \leq (R_{2,\varepsilon}^-)^2 + D^2\varepsilon^{\frac{4}{3}} - 2DR_{2,0}^-\varepsilon^{\frac{2}{3}} \leq (R_{2,\varepsilon}^-)^2 - DR_{2,0}^-\varepsilon^{\frac{2}{3}},$$

provided that $\varepsilon \leq \varepsilon(D)$. Now, recalling that

$$\Gamma_1 < 0,$$

we can bound the left-hand side of (5.7) from below by

$$\Gamma_1(R_{2,0}^-)^2 + cD\varepsilon^{\frac{2}{3}} + \frac{g}{g_1}\lambda_{1,0} - \lambda_{2,0}. \tag{5.8}$$

We use (1.26) to find that the quantity (5.8) equals $cD\varepsilon^{\frac{2}{3}}$ with $c = -\Gamma_1 R_{2,0}^- > 0$. □

5.5. Proof of Theorem 1.6. The proof for the case where (1.9) holds, instead of (1.8), proceeds along the same lines as the proof of Theorem 1.4. This time, we have to decompose $[0, \infty)$ into four intervals with boundary points $R_{2,0}^- < R_{1,0} < R_{2,0}^+$. We point out that the reduced problem near $R_{2,0}^-$ is a scalar equation of the form (A.1) where $a(r) < 0$ for $r \in [0, R_{2,0}^-)$, $a(R_{2,0}^-) = 0$, and $a(r_2) = 0$ for some $r_2 > R_{2,0}^-$, which is covered in Theorem A.1.

6. The Auxiliary Functions $F_{1,\varepsilon}$, $F_{2,\varepsilon}$

Assume that (1.3), (1.8) and (1.18) hold. In this section, we consider the auxiliary functions

$$F_{i,\varepsilon}(r) = \frac{\xi_{i,\varepsilon}(r)}{\eta_{i,\varepsilon}^2(r)}, \quad \text{with } \xi_{i,\varepsilon}(r) = \int_r^\infty s\eta_{i,\varepsilon}^2(s)ds, \quad r \geq 0, \quad i = 1, 2, \tag{6.1}$$

which will play an important role when analyzing the energy with rotation. In particular, we will link them to the limiting functions

$$F_{i,0}(r) = \begin{cases} \frac{\xi_{i,0}(r)}{a_i(r)}, & 0 \leq r < R_{i,0}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{with } \xi_{i,0}(r) = \int_r^\infty s a_i(s)\, ds, \tag{6.2}$$

where a_i is as in (1.12). Note that $F_{i,0}$ is bounded in $\mathbb{R}^2$ since $a_i > 0$ for $r < R_{i,0}$, as observed in (3.6), (3.7), and $F_{i,0}(R_{i,0}) = 0$. Note also that $F_{i,0}$ is merely continuous at $R_{i,0}$, as $F_{i,0}'$ has a finite jump discontinuity across that point.

This section is devoted to proving the following.

Proposition 6.1. *Assume that* (1.3) *and* (1.8) *hold. Let* $F_{i,\varepsilon}$ *be given by* (6.1) *and* $F_{i,0}$ *by* (6.2). *Then*

$$F_{i,\varepsilon}(r) \le \begin{cases} C(R_{i,0}-r)+C\varepsilon^{\frac{2}{3}}, & \text{if } 0 \le r \le R_{i,0}, \\ C\varepsilon^{\frac{2}{3}}, & \text{if } r \ge R_{i,0}, \end{cases}$$

and $\|F_{i,\varepsilon}-F_{i,0}\|_{L^\infty(\mathbb{R}^2)} \le C\varepsilon^{\frac{1}{3}}$, $i = 1, 2$, *provided that* $\varepsilon > 0$ *is sufficiently small.*

This proposition follows from Corollary 6.4 and Lemma 6.5. The proof is made under the additional assumption (1.18). If $g_1 = g_2$, a simpler proof holds since $F_{1,\varepsilon} = F_{2,\varepsilon}$ and the property is that for a single equation [3]. The scalar counterparts

$$f_{i,\varepsilon}(r) = \frac{1}{\hat{\eta}^2_{i,\varepsilon}(r)} \int_r^\infty s\hat{\eta}^2_{i,\varepsilon}(s)ds, \quad r \ge 0, \quad i = 1, 2, \tag{6.3}$$

and their convergence to the corresponding limiting functions

$$f_{1,0}(r) = \frac{\int_r^\infty s a^+_{1,0}(s)ds}{a^+_{1,0}(r)} \quad \text{and} \quad f_{2,0}(r) = \frac{\int_r^\infty s\left(a_{2,0}+\frac{g}{g_2}a_{1,0}\right)^+(s)ds}{\left(a_{2,0}+\frac{g}{g_2}a_{1,0}\right)^+(r)}, \tag{6.4}$$

have been studied in [3, Lem. 2.2]. We have

$$F_{1,0} \equiv f_{1,0}, \quad \text{and} \quad F_{2,0} \equiv f_{2,0} \ \textit{only} \text{ on } r \ge R_{1,0}. \tag{6.5}$$

Actually, the ground states in the latter lemma had unit L^2-norm but its proof carries over to the above case, yielding the following lemma.

Lemma 6.2. *Suppose that* u_ε *is as in Proposition* 4.2 *with* $|\lambda_\varepsilon - \lambda_0| \le C|\log\varepsilon|^{\frac{1}{2}}\varepsilon$. *The auxiliary functions*

$$\xi_\varepsilon(r) = \int_r^\infty s u^2_\varepsilon(s)ds \ \text{ and } \ f_\varepsilon(r) = \frac{\xi_\varepsilon(r)}{u^2_\varepsilon(r)}, \quad r \ge 0.$$

satisfy

$$f_\varepsilon(r) \le \begin{cases} C(r_0-r)+C\varepsilon^{\frac{2}{3}}, & \text{if } 0 \le r \le r_0, \\ C\varepsilon^{\frac{2}{3}}, & \text{if } r \ge r_0, \end{cases}$$

and $\|f_\varepsilon - f_0\|_{L^\infty(\mathbb{R}^2)} \le C\varepsilon^{\frac{1}{3}}$, *where*

$$f_0(r) = \begin{cases} \frac{1}{A_0(r)}\int_r^{r_0} sA_0(s)ds, & \text{if } r < r_0, \\ 0, & \text{if } r \ge r_0, \end{cases}$$

provided that $\varepsilon > 0$ *is sufficiently small.*

The main task in this section is to show the following proposition.

Proposition 6.3. *If $\varepsilon > 0$ is sufficiently small, then*

$$\left|F_{1,\varepsilon}(r) - f_{1,\varepsilon}(r)\right| \leq C\varepsilon^{\frac{2}{3}}, \quad r \geq 0, \tag{6.6}$$

and

$$\left|F_{2,\varepsilon}(r) - f_{2,\varepsilon}(r)\right| \leq C\varepsilon^{\frac{2}{3}}, \quad r \geq R_{1,\varepsilon}. \tag{6.7}$$

Proof. From now on, let us fix a large $D > 1$ such that Lemma 4.13 is valid. The latter lemma, similarly to [3, Lem. 2.2], implies that

$$0 < F_{1,\varepsilon}(r) \leq C\varepsilon^{\frac{2}{3}}, \quad r \geq R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}. \tag{6.8}$$

Since the above estimate also holds for $f_{1,\varepsilon}$, by virtue of Lemma 6.2, we infer that (6.6) is valid for $r \geq R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}$.

If $r \leq R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}$, via (4.12), (4.15), and Corollaries 4.9, 4.11, we have

$$\frac{\eta_{1,\varepsilon}^2(R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}})}{\eta_{1,\varepsilon}^2(r)} \leq C.$$

Therefore, we can write

$$\begin{aligned} F_{1,\varepsilon}(r) &= \frac{1}{\eta_{1,\varepsilon}^2(r)} \int_r^{R_{1,\varepsilon}+D\varepsilon^{\frac{2}{3}}} s\eta_{1,\varepsilon}^2(s)ds + \frac{\eta_{1,\varepsilon}^2(R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}})}{\eta_{1,\varepsilon}^2(r)} F_{1,\varepsilon}(R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}) \\ &\overset{(6.8)}{=} \frac{1}{\eta_{1,\varepsilon}^2(r)} \int_r^{R_{1,\varepsilon}+D\varepsilon^{\frac{2}{3}}} s\eta_{1,\varepsilon}^2(s)ds + \mathcal{O}(\varepsilon^{\frac{2}{3}}), \end{aligned} \tag{6.9}$$

uniformly in $r \geq 0$, as $\varepsilon \to 0$. After rearranging terms, we find that

$$\begin{aligned} F_{1,\varepsilon}(r) - f_{1,\varepsilon}(r) = &\frac{1}{\eta_{1,\varepsilon}^2(r)} \int_r^{R_{1,\varepsilon}+D\varepsilon^{\frac{2}{3}}} s\left[\eta_{1,\varepsilon}^2(s) - \hat{\eta}_{1,\varepsilon}^2(s)\right] ds \\ &+ \frac{\hat{\eta}_{1,\varepsilon}^2(r) - \eta_{1,\varepsilon}^2(r)}{\eta_{1,\varepsilon}^2(r)\hat{\eta}_{1,\varepsilon}^2(r)} \int_r^{R_{1,\varepsilon}+D\varepsilon^{\frac{2}{3}}} s\hat{\eta}_{1,\varepsilon}^2(s)ds + \mathcal{O}(\varepsilon^{\frac{2}{3}}), \end{aligned}$$

uniformly in $r \leq R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}$, as $\varepsilon \to 0$. Since on this interval we can set

$$\varphi = \eta_{1,\varepsilon} - \check{\eta}_{1,\varepsilon} = \eta_{1,\varepsilon} - \hat{\eta}_{1,\varepsilon},$$

we obtain that

$$\begin{aligned} F_{1,\varepsilon}(r) - f_{1,\varepsilon}(r) = &\frac{1}{\eta_{1,\varepsilon}^2(r)} \int_r^{R_{1,\varepsilon}+D\varepsilon^{\frac{2}{3}}} s(\varphi^2 + 2\hat{\eta}_{1,\varepsilon}\varphi)ds \\ &- \left[\frac{\varphi^2(r)}{\eta_{1,\varepsilon}^2(r)\hat{\eta}_{1,\varepsilon}^2(r)} + \frac{2\varphi(r)}{\eta_{1,\varepsilon}^2(r)\hat{\eta}_{1,\varepsilon}(r)}\right] \int_r^{R_{1,\varepsilon}+D\varepsilon^{\frac{2}{3}}} s\hat{\eta}_{1,\varepsilon}^2(s)ds + \mathcal{O}(\varepsilon^{\frac{2}{3}}), \end{aligned} \tag{6.10}$$

uniformly in $r \leq R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}$, as $\varepsilon \to 0$. The above terms can be estimated by first decomposing the interval $[0, R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}]$ as $[0, R_{1,\varepsilon} - \delta] \cup [R_{1,\varepsilon} - \delta, R_{1,\varepsilon} - \varepsilon^{\frac{1}{3}}] \cup [R_{1,\varepsilon} - \varepsilon^{\frac{1}{3}}, R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}]$ (with $\delta > 0$ fixed small), then making use of the uniform estimates in (4.79) and Proposition 4.14 for φ, and those in (4.15) and (4.17) for $\hat{\eta}_{1,\varepsilon}$. To illustrate the procedure, let us estimate in detail the term

$$\frac{1}{\eta^2_{1,\varepsilon}(r)} \int_r^{R_{1,\varepsilon}+D\varepsilon^{\frac{2}{3}}} s\hat{\eta}_{1,\varepsilon}(s)\varphi(s)ds.$$

If $R_{1,\varepsilon} - \varepsilon^{\frac{1}{3}} \leq r \leq s \leq R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}}$, since (4.15) and (4.79) imply that $\eta_{1,\varepsilon}(r) \geq c\varepsilon^{\frac{1}{3}}$ and $\hat{\eta}_{1,\varepsilon}(s) \leq C\varepsilon^{\frac{1}{6}}$, using (4.79) to bound φ, we deduce that

$$\begin{aligned}\left|\frac{1}{\eta^2_{1,\varepsilon}(r)} \int_r^{R_{1,\varepsilon}+D\varepsilon^{\frac{2}{3}}} s\hat{\eta}_{1,\varepsilon}(s)\varphi(s)ds\right| &\leq C\varepsilon^{-\frac{2}{3}}(R_{1,\varepsilon} + D\varepsilon^{\frac{2}{3}} - r)\varepsilon^{\frac{1}{6}}\varepsilon \\ &\leq C\varepsilon^{-\frac{2}{3}}\varepsilon^{\frac{1}{3}}\varepsilon^{\frac{1}{6}}\varepsilon = C\varepsilon^{\frac{2}{3}+\frac{1}{6}}.\end{aligned}$$

If $R_{1,\varepsilon} - \delta \leq r \leq s \leq R_{1,\varepsilon} - \varepsilon^{\frac{1}{3}}$, arguing similarly, this time noting that $\eta_{1,\varepsilon}(r) \geq c\varepsilon^{\frac{1}{6}}$, we find that

$$\left|\frac{1}{\eta^2_{1,\varepsilon}(r)} \int_r^{R_{1,\varepsilon}+D\varepsilon^{\frac{2}{3}}} s\hat{\eta}_{1,\varepsilon}(s)\varphi(s)ds\right| \leq C\varepsilon^{-\frac{1}{3}}\varepsilon = C\varepsilon^{\frac{2}{3}}.$$

Lastly, if $0 \leq r \leq s \leq R_{1,\varepsilon} - \delta$, where $\eta_{1,\varepsilon} \geq c$, via Proposition 4.14, we get that

$$\left|\frac{1}{\eta^2_{1,\varepsilon}(r)} \int_r^{R_{1,\varepsilon}+D\varepsilon^{\frac{2}{3}}} s\hat{\eta}_{1,\varepsilon}(s)\varphi(s)ds\right| \leq C\varepsilon^{\frac{2}{3}}.$$

The remaining terms in (6.10) can be estimated analogously to complete the proof of (6.6). We point out that a rather delicate term is

$$\frac{\varphi(r)}{\eta^2_{1,\varepsilon}(r)\hat{\eta}_{1,\varepsilon}(r)} \int_r^{R_{1,\varepsilon}+D\varepsilon^{\frac{2}{3}}} s\hat{\eta}^2_{1,\varepsilon}(s)\, ds$$

when $r \in [R_{1,\varepsilon} - \delta, R_{1,\varepsilon} - \varepsilon^{\frac{1}{3}}]$, which can be estimated as follows: Since in this interval we have $\hat{\eta}_{1,\varepsilon}(r) \geq c(R_{1,\varepsilon} - r)^{\frac{1}{2}} \geq C\varepsilon^{1/6}$ [from (4.17) and (1.34)], and the same holds for $\eta_{1,\varepsilon}$ via (4.79), it follows that

$$\left|\frac{\varphi(r)}{\eta^2_{1,\varepsilon}(r)\hat{\eta}_{1,\varepsilon}(r)} \int_r^{R_{1,\varepsilon}+D\varepsilon^{\frac{2}{3}}} s\hat{\eta}^2_{1,\varepsilon}(s)ds\right| \leq C\frac{\varepsilon\varepsilon^{-\frac{1}{6}}}{\eta^2_{1,\varepsilon}(r)}\left(|r - R_{1,\varepsilon}| + \varepsilon^{\frac{2}{3}}\right) \leq C\varepsilon^{\frac{5}{6}}.$$

The validity of estimate (6.7) can be verified analogously, using (4.21) to show that $|\eta_{2,\varepsilon} - \hat{\eta}_{2,\varepsilon}| \leq C\varepsilon^{\frac{2}{3}}$ in $[R_{1,\varepsilon}, R_{2,\varepsilon} - \delta]$. □

The assertion of the following corollary is analogous to the first assertion of Lemma 6.2 for the scalar case.

Corollary 6.4. *If $\varepsilon > 0$ is sufficiently small, we have*

$$0 < F_{i,\varepsilon}(r) \leq \begin{cases} C(R_{i,0} - r) & \text{if } 0 \leq r \leq R_{i,0} - \varepsilon^{\frac{2}{3}}, \\ C\varepsilon^{\frac{2}{3}} & \text{if } r \geq R_{i,0} - \varepsilon^{\frac{2}{3}}, \end{cases} \tag{6.11}$$

$i = 1, 2$*, where* $\delta > 0$ *is independent of* ε *such that* $R_{2,0} - R_{1,0} > 4\delta$ *and* $R_{1,0} > 4\delta$.

Proof. The desired estimate (6.11) follows readily from the fact that it holds with $f_{i,\varepsilon}$ in place of $F_{i,\varepsilon}$ (see Lemma 6.2), via Theorem 4.1 and Proposition 6.3. □

The next lemma is a natural extension of the second assertion of Lemma 6.2.

Lemma 6.5. *If $\varepsilon > 0$ is sufficiently small, we have*

$$\|F_{i,\varepsilon} - F_{i,0}\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^{\frac{1}{3}}, \quad i = 1, 2,$$

where $F_{i,0}$ are as in (6.2).

Proof. The proof is based on the fact that

$$\|f_{i,\varepsilon} - f_{i,0}\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^{\frac{1}{3}}, \quad i = 1, 2, \quad \text{(see Lemma 6.2)},$$

where $f_{i,0}$ are as in (6.4). In view of Proposition 6.3 and (6.5), we infer that the assertion of the lemma is valid for $i = 1$ and that there exists some $C > 0$ such that

$$\left|F_{2,\varepsilon}(r) - F_{2,0}(r)\right| \leq C\varepsilon^{\frac{1}{3}}, \quad r \geq R_{1,\varepsilon} + \varepsilon^{\frac{2}{3}}, \tag{6.12}$$

[recall also (3.14)]. So, for the proof to be completed, it remains to show that there exists some $C > 0$ such that

$$\left|F_{2,\varepsilon}(r) - F_{2,0}(r)\right| \leq C\varepsilon^{\frac{1}{3}}, \quad 0 \leq r \leq R_{1,\varepsilon} + \varepsilon^{\frac{2}{3}}. \tag{6.13}$$

To this end, for $0 \leq r \leq R_{1,\varepsilon} + \varepsilon^{\frac{2}{3}}$, we write

$$F_{2,\varepsilon}(r) = \frac{1}{\eta^2_{2,\varepsilon}(r)} \int_r^{R_{1,\varepsilon}+\varepsilon^{\frac{2}{3}}} s\eta^2_{2,\varepsilon}(s)ds + \frac{\eta^2_{2,\varepsilon}(R_{1,\varepsilon} + \varepsilon^{\frac{2}{3}})}{\eta^2_{2,\varepsilon}(r)} F_{2,\varepsilon}(R_{1,\varepsilon} + \varepsilon^{\frac{2}{3}}),$$

and

$$F_{2,0}(r) = \frac{1}{\eta^2_{2,0}(r)} \int_r^{R_{1,\varepsilon}+\varepsilon^{\frac{2}{3}}} s\eta^2_{2,0}(s)ds + \frac{\eta^2_{2,0}(R_{1,\varepsilon} + \varepsilon^{\frac{2}{3}})}{\eta^2_{2,0}(r)} F_{2,0}(R_{1,\varepsilon} + \varepsilon^{\frac{2}{3}}).$$

Now, estimate (6.13) follows readily from (6.12) and the property that

$$\left|\eta_{2,\varepsilon}(r) - \eta_{2,0}(r)\right| \leq C\varepsilon^{\frac{2}{3}}, \quad 0 \leq r \leq R_{1,\varepsilon} + \varepsilon^{\frac{2}{3}}.$$

The latter estimate is a consequence of (4.20)-(4.21), the fact that $\|\hat{\eta}^2_{1,\varepsilon} - a^+_{1,\varepsilon}\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^{\frac{2}{3}}$ (see Proposition 4.2), Corollary 4.9 and Proposition 4.14. □

Finally, we have another estimate which will be used later.

Lemma 6.6. *There exists $C > 0$ such that $\|\nabla\xi_{i,\varepsilon}\|_{L^\infty(\mathbb{R}^2)} \leq C$, $i = 1, 2$.*

Proof. We have

$$\xi'_{i,\varepsilon}(r) = -r\eta^2_{i,\varepsilon}(r) \tag{6.14}$$

so that the result follows from Lemma 2.2. □

7. The Energy Minimizer with Rotation

In this section, we study the behavior of the minimizers of the energy functional E_ε^Ω in the space $\mathcal{H}$, defined in (1.1) and (1.2) respectively, as $\varepsilon \to 0$. In the following we assume

$$\Omega \leq C|\log \varepsilon|, \tag{7.1}$$

for some constant C independent of ε. Any minimizer $(u_1, u_2) = (u_{1,\varepsilon}, u_{2,\varepsilon})$ of E_ε^Ω in $\mathcal{H}$ solves the following system

$$\begin{cases} -\varepsilon^2\Delta u_1 + u_1(|x|^2 + g_1|u_1|^2 + g|u_2|^2) + 2\varepsilon^2 i\Omega x^\perp \cdot \nabla u_1 = \mu_{1,\varepsilon} u_1 & \text{in } \mathbb{R}^2, \\ -\varepsilon^2\Delta u_2 + u_2(|x|^2 + g_2|u_2|^2 + g|u_1|^2) + 2\varepsilon^2 i\Omega x^\perp \cdot \nabla u_2 = \mu_{2,\varepsilon} u_2 & \text{in } \mathbb{R}^2, \end{cases}$$

for some Lagrange multipliers $\mu_{1,\varepsilon}, \mu_{2,\varepsilon}$. The existence of a minimizer when Ω satisfies (7.1) is a consequence of Lemma 7.1 below and of the compactness induced by the fact that the harmonic potential $|x|^2$ diverges as $|x| \to \infty$.

7.1. Energy estimates. The following proof uses some ideas from [24, Lem. 3.1].

Lemma 7.1. *We have*

$$\Omega \sum_{j=1}^{2} \left| \int_{\mathbb{R}^2} x^\perp \cdot (iu_j, \nabla u_j)\, dx \right| \leq \sum_{j=1}^{2} \int_{\mathbb{R}^2} \frac{|\nabla u_j|^2}{4}\, dx + 2\Omega^2(R_{1,0}^2 + R_{2,0}^2)$$
$$+ \frac{2\Omega^2 g_1 \Gamma}{\Gamma_2} \int_{\mathbb{R}^2 \setminus D_1} a_{1,0}^- |u_1|^2\, dx + 2\Omega^2 g_2 \int_{\mathbb{R}^2 \setminus D_2} \left(a_{2,0} + \frac{g}{g_2} a_{1,0}\right)^- |u_2|^2\, dx.$$

Proof. We have

$$\left| \Omega \int_{\mathbb{R}^2} x^\perp \cdot (iu_i, \nabla u_i)\, dx \right| \leq \int_{\mathbb{R}^2} \left(\frac{|\nabla u_i|^2}{4} + \Omega^2 |x|^2 |u_i|^2 \right) dx.$$

We need to estimate the second term in the right hand side. Let us start with $i = 1$. Notice that

$$|x|^2 \leq -\frac{2g_1\Gamma}{\Gamma_2} a_{1,0}(x) = \frac{2g_1\Gamma}{\Gamma_2} a_{1,0}^-(x) \qquad \text{for } |x| \geq \sqrt{2} R_{1,0}.$$

This implies

$$\int_{\mathbb{R}^2} |x|^2|u_1|^2\, dx = \int_{\{|x|\leq\sqrt{2}R_{1,0}\}} |x|^2|u_1|^2\, dx + \int_{\{|x|>\sqrt{2}R_{1,0}\}} |x|^2|u_1|^2\, dx$$
$$\leq 2R_{1,0}^2 + \frac{2g_1\Gamma}{\Gamma_2} \int_{\mathbb{R}^2\setminus D_1} a_{1,0}^- |u_1|^2\, dx.$$

Similarly, for $i = 2$, we have

$$|x|^2 \leq 2g_2 \left(a_{2,0} + \frac{g}{g_2} a_{1,0}\right)^- (x) \quad \text{for } |x| \geq \sqrt{2} R_{2,0},$$

so that

$$\int_{\mathbb{R}^2} |x|^2|u_2|^2\, dx \leq 2R_{2,0}^2 + 2g_2 \int_{\mathbb{R}^2\setminus D_2} \left(a_{2,0} + \frac{g}{g_2} a_{1,0}\right)^- |u_2|^2\, dx,$$

which concludes the proof of the lemma. □

In analogy to Proposition 3.2, we have the following lemma.

Lemma 7.2. *Let* (u_1, u_2) *be a minimizer of* E_ε^Ω *in* $\mathcal{H}$. *There exists* $C > 0$ *independent of* ε *such that, for* $i = 1, 2$, *we have*

$$\int_{\mathbb{R}^2} |\nabla u_i|^2 \, dx \le C|\log \varepsilon|^2,$$

$$\int_{\mathbb{R}^2} \left(|u_i|^2 - a_i\right)^2 \, dx \le C\varepsilon^2 |\log \varepsilon|^2,$$

$$\int_{\mathbb{R}^2 \setminus D_1} |u_1|^2 a_{1,0}^- \, dx + \int_{\mathbb{R}^2 \setminus D_2} (|u_1|^2 + |u_2|^2) \left(a_{2,0} + \frac{g}{g_2} a_{1,0}\right)^- \, dx \le C\varepsilon^2 |\log \varepsilon|^2.$$

Proof. On the one hand, by the definition of minimizers, by Lemma 3.1 and Proposition 3.2, we have

$$E_\varepsilon^\Omega(u_1, u_2) \le E_\varepsilon^\Omega(\eta_1, \eta_2) = E_\varepsilon^0(\eta_1, \eta_2) \le C|\log \varepsilon| + K, \tag{7.2}$$

with K as in Lemma 3.1. On the other hand, we have

$$E_\varepsilon^\Omega(u_1, u_2) = \tilde{E}_\varepsilon^0(u_1, u_2) + K - \Omega \sum_{j=1}^{2} \int_{\mathbb{R}^2} x^\perp \cdot (iu_j, \nabla u_j) \, dx.$$

The right hand side can be bounded from below by means of (2.4) and of Lemma 7.1 as follows:

$$\begin{aligned} E_\varepsilon^\Omega(u_1, u_2) \ge & \sum_{i=1}^{2} \int_{\mathbb{R}^2} \left\{ \frac{|\nabla u_i|^2}{4} + \frac{\gamma}{4\varepsilon^2} (|u_i|^2 - a_i)^2 \right\} dx \\ & + g_1 \Gamma \left(\frac{1}{2\varepsilon^2} - \frac{2\Omega^2}{\Gamma_2} \right) \int_{\mathbb{R}^2 \setminus D_1} a_{1,0}^- |u_1|^2 \, dx \\ & + \frac{g}{2\varepsilon^2} \int_{\mathbb{R}^2 \setminus D_2} |u_1|^2 \left(a_{2,0} + \frac{g}{g_2} a_{1,0} \right)^- dx \\ & + g_2 \left(\frac{1}{2\varepsilon^2} - 2\Omega^2 \right) \int_{\mathbb{R}^2 \setminus D_2} \left(a_{2,0} + \frac{g}{g_2} a_{1,0} \right)^- |u_2|^2 \, dx \\ & - 2\Omega^2 (R_{1,0}^2 + R_{2,0}^2) + K. \end{aligned}$$

The result follows by combining the last inequality with (7.2) and by using (7.1). □

In analogy to Lemma 2.1, we have the following

Lemma 7.3. *Let* (u_1, u_2) *be a minimizer of* E_ε^Ω *in* $\mathcal{H}$. *For* ε *sufficiently small, we have*

$$|u_i|^2 \le \mu_{i,\varepsilon}/g_i, \qquad \|\nabla u_i\|_{L^\infty(\mathbb{R}^2)} \le \frac{C\sqrt{\mu_{i,\varepsilon}}(\mu_{i,\varepsilon} + \mu_{j,\varepsilon} + 1) + C}{\varepsilon}$$

for some $C > 0$, $i = 1, 2$, $j \ne i$.

Proof. For ε sufficiently small, the following holds

$$\Delta(|u_j|^2) = 2|\nabla u_j|^2 + 2(u_j, \Delta u_j) \geq \frac{2}{\varepsilon^2}|u_j|^2(g_j|u_j|^2 - |\mu_{j,\varepsilon}|),$$

where we use $-2\Omega x^\perp \cdot (iu_j, \nabla u_j) \geq -\Omega^2|x|^2|u_j|^2 - |\nabla u_j|^2$ and condition (7.1). We can proceed very similarly to Lemma 2.1: let

$$w_i = \frac{g_i|u_i|^2 - |\mu_{i,\varepsilon}|}{\varepsilon^2} \quad \text{we have} \quad \Delta w_i^+ \geq 2(w_i^+)^2$$

so that we conclude again with the non-existence result by Brezis [14]. Note that by testing the equation of u_i by u_i itself, and working as above, yields that $\mu_{i,\varepsilon} > 0$.

To prove the second part, fix $x \in \mathbb{R}^2$, $L > 0$ and for $y \in B_{2L}(x)$, let $z_i(y) = u_i(\varepsilon(y - x))$. Then

$$-\Delta z_i = -z_i(\varepsilon^2|y - x|^2 + g_i|z_i|^2 + g|z_j|^2 - \mu_{i,\varepsilon}) - 2\varepsilon^2 i\Omega(y - x)^\perp \cdot \nabla z_i =: h_{i,\varepsilon}(y).$$

We have, by Lemma 7.2 and by (7.1),

$$\varepsilon^2\Omega\|(y - x)^\perp \cdot \nabla z_i\|_{L^2(B_{2L}(x))} = \varepsilon\Omega\|x^\perp \cdot \nabla u_i\|_{L^2(B_{2\varepsilon L}(0))} \leq C$$

for a constant C independent of x. Therefore, using also the L^∞-bound above, we have $\|h_{i,\varepsilon}\|_{L^2(B_{2L}(x))} \leq C\sqrt{\mu_{i,\varepsilon}}(\mu_{i,\varepsilon} + \mu_{j,\varepsilon} + 1) + C$. We deduce that $\|z_i\|_{H^2(B_L(x))} \leq C\sqrt{\mu_{i,\varepsilon}}(\mu_{i,\varepsilon} + \mu_{j,\varepsilon} + 1) + C$ and we conclude by a bootstrap argument. □

Lemma 7.4. *Let (u_1, u_2) be a minimizer of E_ε^Ω in $\mathcal{H}$ and denote by $\mu_{i,\varepsilon}$ the associated Lagrange multipliers. There exists $C > 0$ independent of ε such that, for $i = 1, 2$,*

$$|\mu_{i,\varepsilon}| \leq C.$$

Proof. We test the equation for u_i by u_i itself and integrate by parts, which is possible since $u_i \in H^1(\mathbb{R}^2, \mathbb{C})$. The term containing Ω can be bounded by means of Lemma 7.1, whereas the other terms can be rewritten as in Proposition 3.3. Finally, the desired bound follows from the energy estimates of Lemma 7.2. □

7.2. Non-existence of vortices. The proof presented here is an adaptation of the proof of the main theorem in [3]. Let us start with the following splitting of the energy, which is introduced in [5].

Lemma 7.5. *Let (u_1, u_2) be any minimizer of E_ε^Ω in $\mathcal{H}$ and let (η_1, η_2) be the unique positive minimizer of E_ε^0 in $\mathcal{H}$ provided by Theorem 1.3. Let*

$$v_i = \frac{u_i}{\eta_i}, \quad \textit{for } i = 1, 2.$$

Then

$$E_\varepsilon^\Omega(u_1, u_2) = E_\varepsilon^0(\eta_1, \eta_2) + F_\varepsilon^\Omega(v_1, v_2), \quad \textit{where}$$

$$F_\varepsilon^\Omega(v_1, v_2) = \sum_{j=1}^2 \int_{\mathbb{R}^2} \left\{ \frac{\eta_j^2}{2}|\nabla v_j|^2 + \frac{g_j}{4\varepsilon^2}\eta_j^4(|v_j|^2 - 1)^2 - \eta_j^2\Omega x^\perp \cdot (iv_j, \nabla v_j) \right\} dx$$
$$+ \frac{g}{2\varepsilon^2}\int_{\mathbb{R}^2} \eta_1^2\eta_2^2(1 - |v_1|^2)(1 - |v_2|^2)\, dx.$$

We skip the proof since it is similar to the one of Proposition 2.9. An integration by parts and assumption (1.3) yield

Lemma 7.6. *Let $F_{i,\varepsilon}$ be the auxiliary functions introduced in* (6.1) *and let γ be as in* (2.4). *Then we have*

$$\tilde{F}_\varepsilon^\Omega(v_1, v_2) = \sum_{i=1}^2 \int_{\mathbb{R}^2} \left\{ \frac{\eta_i^2}{2} \left(|\nabla v_i|^2 - 4\Omega F_{i,\varepsilon} J v_i \right) + \frac{\gamma}{4\varepsilon^2} \eta_i^4 (|v_i|^2 - 1)^2 \right\} dx \le 0,$$

where $J v_j = (i\partial_{x_1} v_j, \partial_{x_2} v_j)$ stands for the Jacobian of v_j.

Proof. First we prove that we can rewrite F_ε^Ω in terms of $F_{i,\varepsilon}$ as follows

$$F_\varepsilon^\Omega(v_1, v_2) = \sum_{i=1}^2 \int_{\mathbb{R}^2} \left\{ \frac{\eta_i^2}{2} \left(|\nabla v_i|^2 - 4\Omega F_{i,\varepsilon} J v_i \right) + \frac{g_i}{4\varepsilon^2} \eta_i^4 (|v_i|^2 - 1)^2 \right\} dx + \frac{g}{2\varepsilon^2} \int_{\mathbb{R}^2} \eta_1^2 \eta_2^2 (1 - |v_1|^2)(1 - |v_2|^2)\, dx. \tag{7.3}$$

Indeed, by (6.14), the following holds

$$\nabla^\perp \xi_i = (-\partial_{x_2} \xi_i, \partial_{x_1} \xi_i) = -\eta_i^2 x^\perp,$$

and Stokes theorem yields

$$\int_{\partial B_R} \xi_j (i v_j, \nabla v_j)^\perp \cdot \nu \, d\sigma = \int_{B_R} \{ -\xi_j \nabla \times (i v_j, \nabla v_j) + \eta_j^2 x^\perp \cdot (i v_j, \nabla v_j) \}\, dx,$$

where $\nabla \times (i v_j, \nabla v_j) = \partial_{x_1} (i v_j, \partial_{x_2} v_j) - \partial_{x_2} (i v_j, \partial_{x_1} v_j) = 2 J v_j$. The boundary term vanishes because, by Corollary 6.4, for R large, we have

$$\left| \int_{\partial B_R} \xi_j (i v_j, \nabla v_j)^\perp \cdot \nu \, d\sigma \right| = \left| \int_{\partial B_R} F_{j,\varepsilon} (i u_j, \nabla u_j)\, d\sigma \right| \le C \varepsilon^{2/3} \int_{\partial B_R} (|\nabla u_j|^2 + |u_j|^2)\, d\sigma$$

which vanishes along a sequence $R_k \to \infty$. Hence we have obtained

$$\int_{\mathbb{R}^2} \eta_j^2 x^\perp \cdot (i v_j, \nabla v_j)\, dx = 2 \int_{\mathbb{R}^2} \eta_j^2 F_{j,\varepsilon} J v_j \, dx,$$

and (7.3) is proved. Then, reasoning as in (2.3), we deduce that $F_\varepsilon^\Omega(v_1, v_2) \ge \tilde{F}_\varepsilon^\Omega(v_1, v_2)$. On the other hand, since (u_1, u_2) is a minimizer and (η_1, η_2) is real valued, we have

$$E_\varepsilon^\Omega(u_1, u_2) \le E_\varepsilon^\Omega(\eta_1, \eta_2) = E_\varepsilon^0(\eta_1, \eta_2),$$

which, by Lemma 7.5, implies that $F_\varepsilon^\Omega(v_1, v_2) \le 0$. □

The rest of the section is devoted to proving that $v_i = 1$.

Let $0 \le \chi_i \le 1$ be regular cut-off functions with the property that

$$\chi_i(r) = 1 \text{ for } r \leq R_{i,0} - 2|\log\varepsilon|^{-3/2} \quad \text{and} \quad \chi_i(r) = 0 \text{ for } r \geq R_{i,0} - |\log\varepsilon|^{-3/2},$$

and moreover $\|\nabla\chi_i\|_{L^\infty(\mathbb{R}^2)} \leq 2|\log\varepsilon|^{3/2}$. We estimate $\tilde{F}_\varepsilon^\Omega(v_1, v_2)$ according to the following splitting

$$\tilde{F}_\varepsilon^\Omega(v_1, v_2) = A_1 + B_1 - C_1 + A_2 + B_2 - C_2,$$

where

$$A_i = \int_{\mathbb{R}^2} \chi_i \left\{ \frac{\eta_i^2}{2} |\nabla v_i|^2 + \frac{\gamma}{4\varepsilon^2} \eta_i^4 (|v_i|^2 - 1)^2 \right\} dx$$

$$B_i = \int_{\mathbb{R}^2} (1 - \chi_i) \left\{ \frac{\eta_i^2}{2} \left(|\nabla v_i|^2 - 4\Omega F_{i,\varepsilon} J v_i \right) + \frac{\gamma}{4\varepsilon^2} \eta_i^4 (|v_i|^2 - 1)^2 \right\} dx$$

$$C_i = 2\Omega \int_{\mathbb{R}^2} \chi_i \xi_i J v_i \, dx.$$

Lemma 7.6 immediately provides

$$A_1 + B_1 + A_2 + B_2 \leq C_1 + C_2. \tag{7.4}$$

Proposition 7.7. *With the notation above, for ε small, we have $B_i \geq 0$ for $i = 1, 2$ so that $A_1 + A_2 \leq C_1 + C_2$.*

Proof. Due to the definition of B_i, we can restrict our attention to the set

$$\text{supp}(1 - \chi_i) = \{x : |x| > R_{i,0} - 2|\log\varepsilon|^{-3/2}\}.$$

Corollary 6.4 implies that in such a set we have $F_{i,\varepsilon} \leq C|\log\varepsilon|^{-3/2}$. Hence assumption (7.1) implies that $\Omega F_{i,\varepsilon} \leq 1/4$, for ε sufficiently small. Recalling that $|Jv_i| \leq |\nabla v_i|^2/2$, we deduce that

$$|\nabla v_i|^2 - 4\Omega F_{i,\varepsilon} J v_i \geq \frac{1}{2} |\nabla v_i|^2,$$

and as a consequence,

$$B_i \geq \int_{\mathbb{R}^2} (1 - \chi_i) \left\{ \frac{\eta_i^2}{4} |\nabla v_i|^2 + \frac{\gamma}{4\varepsilon^2} \eta_i^4 (|v_i|^2 - 1)^2 \right\} dx. \tag{7.5}$$

The second part of the statement is obtained by combining with (7.4). □

Lemma 7.8. *Let*

$$\tilde{\varepsilon}_i = \varepsilon\gamma^{-1/2} \left(\inf_{\{supp\chi_i\}} \eta_i \right)^{-1}.$$

There exists $C > 0$ such that $\tilde{\varepsilon}_i \leq C\varepsilon|\log\varepsilon|^{3/4}$.

Proof. Clearly $a_i \geq c|\log\varepsilon|^{-3/2}$ in $\{\text{supp}\chi_i\}$. Hence property (1.20) implies that

$$\eta_i \geq \sqrt{a_i} - C\varepsilon^{1/3} \geq c|\log\varepsilon|^{-3/4} \quad \text{in } \{\text{supp}\chi_i\}, \tag{7.6}$$

which provides the statement. □

Lemma 7.9. *There exists C independent of ε such that, for small ε,*

$$\sum_{i=1}^{2}\int_{\{supp\chi_i\}}\left\{\frac{|\nabla v_i|^2}{2}+\frac{1}{4\tilde{\varepsilon}_i^2}(|v_i|^2-1)^2\right\}dx \leq C|\log\varepsilon|^{3/2}(C_1+C_2).$$

Proof. Recalling that $\tilde{\varepsilon}_i^2 \geq \varepsilon^2/(\gamma\eta_i^2)$ in $\{\mathrm{supp}\chi_i\}$ and relation (7.6), we deduce

$$\int_{\{\mathrm{supp}\chi_i\}}\left\{\frac{|\nabla v_i|^2}{2}+\frac{1}{4\tilde{\varepsilon}_i^2}(|v_i|^2-1)^2\right\}dx$$
$$\leq C|\log\varepsilon|^{3/2}\int_{\{\mathrm{supp}\chi_i\}}\left\{\frac{\eta_i^2}{2}|\nabla v_i|^2+\frac{\gamma}{4\varepsilon^2}\eta_i^4(|v_i|^2-1)^2\right\}dx.$$

On the other side, estimate (7.5) implies that

$$\int_{\mathbb{R}^2}\left\{\frac{\eta_i^2}{2}|\nabla v_i|^2+\frac{\gamma}{4\varepsilon^2}\eta_i^4(|v_i|^2-1)^2\right\}dx \leq A_i+2B_i.$$

The result follows by summing the above for $i = 1, 2$ and combining with (7.4). □

Proposition 7.10. *Suppose that*

$$2\Omega\max_{i=1,2}\{\|F_{i,0}\|_{L^\infty(\mathbb{R}^2)}\} \leq |\log\varepsilon|-(\alpha+1)\log|\log\varepsilon|, \tag{7.7}$$

for a suitable $\alpha > 0$, where $F_{i,0}$ are as in (6.2). There exists $C > 0$ independent of ε such that

$$A_i+B_i+|C_i| \leq C|\log\varepsilon|^{-11}, \quad \textit{for } i = 1, 2.$$

Proof. We use a result by Jerrard [25], as it is stated in [3]. Following the last mentioned paper, we let

$$\alpha = 1300, \quad k = 1+\alpha\frac{\log|\log\varepsilon|}{|\log\varepsilon|}, \quad \beta = \frac{k-1}{100}. \tag{7.8}$$

Notice that

$$\varepsilon^\beta = |\log\varepsilon|^{-\alpha/100} = |\log\varepsilon|^{-13}. \tag{7.9}$$

As in [3], we can write [25, Lemma 8] as

$$\sum_{i=1}^{2}|C_i| \leq 2\Omega k\sum_{i=1}^{2}\int_{\mathbb{R}^2}\frac{\chi_i\xi_i}{|\log\tilde{\varepsilon}_i|}\left\{\frac{|\nabla v_i|^2}{2}+\frac{1}{4\tilde{\varepsilon}_i^2}(|v_i|^2-1)^2\right\}dx+C\varepsilon^\beta(1+\sum_{i=1}^{2}|C_i|),$$

where $\tilde{\varepsilon}_i$ is defined in Lemma 7.8. This formulation only makes use of the estimates in Lemmas 6.6 and 7.9, so that it holds also in our case. Now, recalling that $\xi_i = F_{i,\varepsilon}\eta_i^2$ and that $\tilde{\varepsilon}_i^2 \geq \varepsilon^2/(\gamma\eta_i^2)$ in $\{\mathrm{supp}\chi_i\}$, we deduce

$$\sum_{i=1}^{2}|C_i| \leq 2\Omega k\sum_{i=1}^{2}\frac{\|F_{i,\varepsilon}\|_{L^\infty(\mathbb{R}^2)}}{|\log\tilde{\varepsilon}_i|}A_i+C\varepsilon^\beta(1+\sum_{i=1}^{2}|C_i|),$$

so that

$$(1 - C\varepsilon^{\beta}) \sum_{i=1}^{2} |C_i| \leq 2\Omega k \sum_{i=1}^{2} \frac{\|F_{i,\varepsilon}\|_{L^\infty(\mathbb{R}^2)}}{|\log \tilde{\varepsilon}_i|} A_i + C\varepsilon^{\beta}.$$

We estimated $F_{i,\varepsilon}$ in Lemma 6.5, which provides

$$\|F_{i,\varepsilon}\|_{L^\infty(\mathbb{R}^2)} \leq (1 + C\varepsilon^{1/3})\|F_{i,0}\|_{L^\infty(\mathbb{R}^2)} \leq (1 + C\varepsilon^{\beta})\|F_{i,0}\|_{L^\infty(\mathbb{R}^2)},$$

where the last inequality holds for ε sufficiently small by virtue of (7.9). Also, Lemma 7.8 implies that, for every $K > 0$, we have

$$|\log \tilde{\varepsilon}_i| \geq (|\log \varepsilon| - \log|\log \varepsilon|)(1 + K\varepsilon^{\beta})$$

for ε sufficiently small with respect to K. By combining these facts with assumption (7.7) and with our choice of k in (7.8) we obtain

$$\begin{aligned} |C_1| + |C_2| &\leq \left(1 - \alpha \frac{\log|\log \varepsilon|}{|\log \varepsilon| - \log|\log \varepsilon|}\right) k(A_1 + A_2) + C\varepsilon^{\beta} \\ &\leq \left(1 - \alpha^2 \frac{\log^2|\log \varepsilon|}{|\log \varepsilon|^2}\right)(A_1 + A_2) + C\varepsilon^{\beta}. \end{aligned} \tag{7.10}$$

Recalling Proposition 7.7 we deduce

$$A_1 + A_2 \leq |C_1| + |C_2| \leq \left(1 - \alpha^2 \frac{\log^2|\log \varepsilon|}{|\log \varepsilon|^2}\right)(A_1 + A_2) + C\varepsilon^{\beta},$$

so that

$$A_1 + A_2 \leq \frac{C\varepsilon^{\beta}}{\alpha^2} \frac{|\log \varepsilon|^2}{\log^2|\log \varepsilon|} \leq C|\log \varepsilon|^{-11},$$

where in the last step we replaced relation (7.9). Being A_i non-negative quantities, the last estimate holds for both terms. In turn we deduce from (7.10) that $|C_i| \leq C|\log \varepsilon|^{-11}$, and from (7.4) that

$$B_1 + B_2 \leq A_1 + B_1 + A_2 + B_2 \leq C_1 + C_2 \leq C|\log \varepsilon|^{-11}.$$

Being B_i non-negative by Proposition 7.7, the estimate holds for both terms. □

We can now derive a "clearing-out" property (see also [10]).

Proposition 7.11. *Suppose that* (7.7) *holds. For ε sufficiently small, we have*

$$|v_i| \geq \frac{1}{2} \quad \text{in } \{supp \chi_i\}.$$

Proof. We shall prove that

$$|v_i| > 1 - |\log \varepsilon|^{-1} \quad \text{in } \{\operatorname{supp} \chi_i\}, \tag{7.11}$$

for $i = 1, 2$, which implies the statement. By combining Lemma 7.9 with Proposition 7.10 we obtain

$$\sum_{i=1}^{2} \int_{\{\operatorname{supp} \chi_i\}} \left\{ \frac{|\nabla v_i|^2}{2} + \frac{1}{4\tilde{\varepsilon}_i^2}(|v_i|^2 - 1)^2 \right\} dx \le C |\log \varepsilon|^{3/2} |\log \varepsilon|^{-11}.$$

Then Lemma 7.8 provides

$$\frac{1}{\varepsilon^2} \int_{\{\operatorname{supp} \chi_i\}} (|v_i|^2 - 1)^2 \, dx \le C |\log \varepsilon|^{-8} \tag{7.12}$$

for $i = 1, 2$. Next we observe that

$$\|\nabla v_i\|_{L^\infty(\{\operatorname{supp} \chi_i\})} \le C \frac{|\log \varepsilon|^{3/2}}{\varepsilon}. \tag{7.13}$$

This comes from the fact that $\|\nabla u_i\|_{L^\infty(\mathbb{R}^2)} \le C/\varepsilon$, as can be seen by combining Lemmas 7.3 and 7.4, and that $\nabla v_i = \nabla u_i/\eta_i - u_i \nabla \eta_i/\eta_i^2$, together with estimate (7.6). Suppose by contradiction that (7.11) does not hold, i.e. there exists $x_0 \in \{\operatorname{supp} \chi_i\}$ such that

$$|v_i(x_0)| \le 1 - |\log \varepsilon|^{-1} \quad \text{as } \varepsilon \to 0.$$

Then (7.13) implies

$$|v_i(x)| \le 1 - C |\log \varepsilon|^{-1} \quad \text{in } B_{r_0}(x_0) \quad \text{with } r_0 = \varepsilon |\log \varepsilon|^{-5/2},$$

so that

$$\frac{1}{\varepsilon^2} \int_{B_{r_0}(x_0) \cap \{\operatorname{supp} \chi_i\}} (|v_i|^2 - 1)^2 \, dx \ge C |\log \varepsilon|^{-7},$$

which contradicts (7.12) for ε sufficiently small. Therefore (7.11) is proved. □

7.3. Proof of Theorem 1.5. We are now in position to give the proof of Theorem 1.5.

Proof of Theorem 1.5. We take $\Omega \le \omega_0 |\log \varepsilon| - \omega_1 \log |\log \varepsilon|$ with ω_0, ω_1 such that (7.7) holds (recall that $F_{i,0}$ is bounded in $\mathbb{R}^2$). Thanks to the previous proposition the quantity $w_i = v_i/|v_i|$ is well defined in $\{\operatorname{supp} \chi_i\}$ and satisfies $J w_i = 0$ (see [3]). Hence, we find that

$$\begin{aligned} C_j &= 2\Omega \int_{\{\operatorname{supp} \chi_j\}} \chi_j \xi_j (J v_j - J w_j) \, dx \\ &= 2\Omega \int_{\{\operatorname{supp} \chi_j\}} \nabla^\perp (\chi_j \xi_j) [(i v_j, \nabla v_j) - (i w_j, \nabla w_j)] \, dx. \end{aligned}$$

Writing $v_j = \rho_j e^{i\phi_j}$ in $\{\text{supp}\chi_j\}$ we see that $(iv_j, \nabla v_j) = \rho_j^2 \nabla\phi_j$ and $(iw_j, \nabla w_j) = \nabla\phi_j$, so that Proposition 7.11 implies

$$|(iv_j, \nabla v_j) - (iw_j, \nabla w_j)| = \frac{|\rho_j^2 - 1|}{\rho_j} |\rho_j \nabla\phi_j| \le 2|\rho_j^2 - 1||\rho_j \nabla\phi_j|$$
$$\le 2\left||v_j|^2 - 1\right| |\nabla v_j|.$$

We insert it in the previous estimate to obtain

$$C_j \le 2\Omega \|\nabla(\chi_j \xi_j)\|_{L^\infty(\{\text{supp}\chi_j\})} \int_{\{\text{supp}\chi_j\}} 2\left||v_j|^2 - 1\right| \cdot |\nabla v_j|\, dx$$
$$\le 4\sqrt{2}\Omega \|\nabla(\chi_j \xi_j)\|_{L^\infty(\mathbb{R}^2)} \int_{\{\text{supp}\chi_j\}} \left\{ \frac{\tilde\varepsilon_j}{2}|\nabla v_j|^2 + \frac{1}{4\tilde\varepsilon_j}\left(|v_j|^2 - 1\right)^2 \right\} dx$$
$$\le C\Omega\varepsilon|\log\varepsilon|^{9/4} \int_{\{\text{supp}\chi_j\}} \left\{ \frac{|\nabla v_j|^2}{2} + \frac{1}{4\tilde\varepsilon_j^2}\left(|v_j|^2 - 1\right)^2 \right\} dx,$$

where we used Lemma 7.8 and the estimate $\|\nabla(\chi_i\xi_i)\|_{L^\infty(\mathbb{R}^2)} \le C|\log\varepsilon|^{3/2}$. We sum for $i = 1, 2$ and then we use the assumption (7.1), and Lemma 7.9, to obtain $C_1 + C_2 \le C\varepsilon|\log\varepsilon|^{19/4}(C_1 + C_2)$, so that $C_1 + C_2 \le (C_1 + C_2)/2$ for ε sufficiently small. Since $C_1 + C_2$ is non-negative by Proposition 7.7, we conclude that $C_1 + C_2 = 0$. In turn, relation (7.4) and Proposition 7.7 imply also $A_i = B_i = 0$ for $i = 1, 2$, that is

$$A_i = \int_{\mathbb{R}^2} \chi_i \left\{ \frac{\eta_i^2}{2}|\nabla v_i|^2 + \frac{\gamma}{4\varepsilon^2}\eta_i^4(|v_i|^2 - 1)^2 \right\} dx = 0$$

and [see (7.5)]

$$\int_{\mathbb{R}^2} (1 - \chi_i) \left\{ \frac{\eta_i^2}{4}|\nabla v_i|^2 + \frac{\gamma}{4\varepsilon^2}\eta_i^4(|v_i|^2 - 1)^2 \right\} dx = 0.$$

Therefore, we infer that v_i are both constants of modulus 1 as we wanted to prove. □

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Appendix A. The Scalar Ground State

Throughout Sects. 4.2–4.3, we have referred to the following

Theorem A.1. *Assume that $a \in C^1[0, \infty)$ satisfies $a'(0) = 0$, there exist positive numbers $r_1 < r_2 < \cdots < r_n$ such that $a(r_i) = 0$, $a(r) \neq 0$ if $r \neq r_i$, and $(-1)^i a'(r_i) > 0$, $i = 1, \ldots, n$, and $a(r) \to -\infty$ as $r \to \infty$. Assume also that $\mu_\varepsilon \in \mathbb{R}$ satisfy $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$.*

Let $A_\varepsilon = a + \mu_\varepsilon$. For sufficiently small $\varepsilon > 0$, by the implicit function theorem, there exist $0 < r_{1,\varepsilon} < r_{2,\varepsilon} < \cdots < r_{n,\varepsilon}$ such that $r_{i,\varepsilon} \to r_i$ as $\varepsilon \to 0$, satisfying $A_\varepsilon(r_{i,\varepsilon}) = 0$, $A_\varepsilon(r) \neq 0$ if $r \neq r_{i,\varepsilon}$, and $(-1)^i A'_\varepsilon(r_{i,\varepsilon}) > 0$, $i = 1, \ldots, n$.

If $\varepsilon > 0$ is sufficiently small, there exists a positive radially symmetric solution $\eta_\varepsilon \in C^2(\mathbb{R}^2)$ to the problem

$$\varepsilon^2 \Delta\eta = \eta\left(\eta^2 - A_\varepsilon(x)\right), \quad x \in \mathbb{R}^2, \quad \eta(x) \to 0 \text{ as } |x| \to \infty, \tag{A.1}$$

such that

$$\|\eta_\varepsilon - \sqrt{A_\varepsilon^+}\|_{L^\infty(\mathbb{R}^2)} \le C\varepsilon^{\frac{1}{3}}, \tag{A.2}$$

and

$$3\eta_\varepsilon^2 - A_\varepsilon \ge \begin{cases} c|r - r_{i,\varepsilon}| + c\varepsilon^{\frac{2}{3}}, & \text{if } |r - r_{i,\varepsilon}| \le \delta, \\ c, & \text{otherwise}, \end{cases} \tag{A.3}$$

for some $\delta \in \left(0, \frac{1}{4}\min_{i=1,\cdots,n-1}\{r_{i+1} - r_i\}\right)$. More precisely, we have

$$\begin{aligned} \eta_\varepsilon(r) &= \varepsilon^{\frac{1}{3}}(-1)^{i+1}\beta_{i,\varepsilon} V\left(\beta_{i,\varepsilon}\frac{r - r_{i,\varepsilon}}{\varepsilon^{\frac{2}{3}}}\right) \\ &+ \begin{cases} \mathcal{O}\left(\varepsilon + |r - r_{i,\varepsilon}|^{\frac{3}{2}}\right) & \text{if } 0 \le (-1)^i(r - r_{i,\varepsilon}) \le \delta, \\ \mathcal{O}(\varepsilon)\exp\left\{-c\frac{|r - r_{i,\varepsilon}|}{\varepsilon^{\frac{2}{3}}}\right\} & \text{if } -\delta \le (-1)^i(r - r_{i,\varepsilon}) \le 0, \end{cases} \end{aligned} \tag{A.4}$$

where

$$\beta_{i,\varepsilon}^3 = -a'(r_{i,\varepsilon}), \quad i = 1, \ldots, n,$$

and V is the Hastings–McLeod solution, as described in (4.11). Estimate (A.4) can be differentiated once to give

$$\eta_\varepsilon'(r) = \varepsilon^{-\frac{1}{3}}(-1)^{i+1}\beta_{i,\varepsilon}^2 V'\left(\beta_{i,\varepsilon}\frac{r - r_{i,\varepsilon}}{\varepsilon^{\frac{2}{3}}}\right) + \mathcal{O}\left(\varepsilon^{\frac{1}{3}} + |r - r_{i,\varepsilon}|^{\frac{1}{2}}\right) \quad \text{if } |r - r_{i,\varepsilon}| \le \delta, \tag{A.5}$$

uniformly, as $\varepsilon \to 0$. On the other side, we have

$$\eta_\varepsilon(r) - \sqrt{A_\varepsilon(r)} = \varepsilon^2\mathcal{O}(|r - r_{i,\varepsilon}|^{-\frac{5}{2}}) \quad \text{if } C\varepsilon^{\frac{2}{3}} \le (-1)^i(r - r_{i,\varepsilon}) \le \delta, \tag{A.6}$$

uniformly, as $\varepsilon \to 0$. Furthermore,

$$\left|\eta_\varepsilon - \sqrt{A_\varepsilon^+}\right| \le C\varepsilon^2 \text{ in } I_\delta \equiv [0, r_{1,\varepsilon} - \delta] \cup [r_{1,\varepsilon} + \delta, r_{2,\varepsilon} - \delta] \cup \cdots \cup [r_{n,\varepsilon} + \delta, \infty), \tag{A.7}$$

and

$$\eta_\varepsilon(r) \le C\varepsilon^{\frac{1}{3}}\exp\left\{-c\varepsilon^{-\frac{2}{3}}\min_{i=1,\ldots,n}|r - r_{i,\varepsilon}|\right\} \text{ if } A_\varepsilon^+(r) = 0. \tag{A.8}$$

Moreover, if $a(x) = a(|x|) \in C^4(\mathbb{R}^2)$,

$$\eta_\varepsilon' - \left(\sqrt{A_\varepsilon}\right)' = \varepsilon^2\mathcal{O}(|r - r_{i,\varepsilon}|^{-\frac{7}{2}}), \tag{A.9}$$

and

$$\Delta\eta_\varepsilon - \Delta\left(\sqrt{A_\varepsilon}\right) = \varepsilon^2\mathcal{O}(|r - r_{i,\varepsilon}|^{-\frac{9}{2}}), \text{ if } C\varepsilon^{\frac{2}{3}} \le (-1)^i(r - r_{i,\varepsilon}) \le \delta, \tag{A.10}$$

uniformly, as $\varepsilon \to 0$, and

$$\|\eta_\varepsilon - \sqrt{A_\varepsilon^+}\|_{C^2(I_\delta)} \le C\varepsilon^2. \tag{A.11}$$

Proof. All the assertions up to (A.8) are essentially contained in [27, Thm. 1.1], where in fact no radial symmetry is imposed on $a(\cdot)$. Actually, relation (A.5) can be proven by combining the proof of Corollary 4.1 in [27] with relation (3.40) therein. In passing, we note that $V'(s) < 0$, $s \in \mathbb{R}$.

Let us further assume that $a(x) = a(|x|) \in C^4(\mathbb{R}^2)$. In order to establish relations (A.9)-(A.10), we need a refinement of (A.6). Motivated from the identity

$$\varepsilon^2 \Delta \left(\eta_\varepsilon - \sqrt{A_\varepsilon}\right) - \eta_\varepsilon \left(\eta_\varepsilon + \sqrt{A_\varepsilon}\right)\left(\eta_\varepsilon - \sqrt{A_\varepsilon}\right) = -\varepsilon^2 \Delta \left(\sqrt{A_\varepsilon}\right)$$

if $C\varepsilon^{\frac{2}{3}} \le (-1)^i (r - r_{i,\varepsilon}) \le \delta$ (see also [24, Prop. 2.1]), we let

$$\eta_\varepsilon - \sqrt{A_\varepsilon} = \varepsilon^2 \frac{\Delta\left(\sqrt{A_\varepsilon}\right)}{2A_\varepsilon} + \phi \quad \text{if } C\varepsilon^{\frac{2}{3}} \le (-1)^i (r - r_{i,\varepsilon}) \le \delta, \tag{A.12}$$

for some fluctuation function ϕ. Pushing further the analysis in [17, Thm. 2.1] or [29, Thm. 1.1], it can be shown that

$$|\phi(r)| \le C\varepsilon^4 \quad \text{if } \delta \le (-1)^i (r - r_{i,\varepsilon}) \le 2\delta. \tag{A.13}$$

Making use of (A.6), and recalling that

$$A'_\varepsilon(r_{i,\varepsilon}) = a'(r_{i,\varepsilon}) \to -c_i < 0 \quad \text{as } \varepsilon \to 0, \tag{A.14}$$

it follows readily that

$$\varepsilon^2 \Delta\phi - \eta_\varepsilon \left(\eta_\varepsilon + \sqrt{A_\varepsilon}\right)\phi = \varepsilon^4 \mathcal{O}\left(|r - r_{i,\varepsilon}|^{-\frac{9}{2}}\right) \quad \text{if } C\varepsilon^{\frac{2}{3}} \le (-1)^i (r - r_{i,\varepsilon}) \le \delta,$$

uniformly, as $\varepsilon \to 0$. Since

$$\eta_\varepsilon \left(\eta_\varepsilon + \sqrt{A_\varepsilon}\right) \ge c|r - r_{i,\varepsilon}| \quad \text{if } C\varepsilon^{\frac{2}{3}} \le (-1)^i (r - r_{i,\varepsilon}) \le \delta,$$

[from (A.6) and (A.14)], a standard comparison argument yields that

$$|\phi(r)| \le C\varepsilon^4 |r - r_{i,\varepsilon}|^{-\frac{11}{2}}, \quad C\varepsilon^{\frac{2}{3}} \le (-1)^i (r - r_{i,\varepsilon}) \le \delta,$$

where we have also used that

$$\left|\phi\left(r_{i,\varepsilon} + (-1)^i \delta\right)\right| \le C\varepsilon^4 \quad \text{and} \quad \left|\phi\left(r_{i,\varepsilon} + (-1)^i C\varepsilon^{\frac{2}{3}}\right)\right| \le C\varepsilon^{\frac{1}{3}},$$

which follow from (A.13) and (A.6) respectively; one plainly uses barriers of the form $\pm M\varepsilon^4 |r - r_{i,\varepsilon}|^{-\frac{11}{2}}$ with M chosen sufficiently large, see also [21, Lem. 2.1] or [26, Lem. 3.10] for related arguments when the problem is independent of ε (for the present argument to work it is crucial that $|r - r_{i,\varepsilon}| \ge \varepsilon^{\frac{2}{3}}$). Consequently, recalling (A.12), we have shown the following refinement of (A.6):

$$\eta_\varepsilon - \sqrt{A_\varepsilon} = \varepsilon^2 \frac{\Delta\left(\sqrt{A_\varepsilon}\right)}{2A_\varepsilon} + \varepsilon^4 \mathcal{O}\left(|r - r_{i,\varepsilon}|^{-\frac{11}{2}}\right),$$

uniformly if $C\varepsilon^{\frac{2}{3}} \le (-1)^i(r - r_{i,\varepsilon}) \le \delta$, as $\varepsilon \to 0$, which complements (A.13). In turn, via (A.1) and some straightforward calculations, this can be shown to imply (A.10). Equivalently, we have that

$$\left(r(\eta_\varepsilon - \sqrt{A_\varepsilon})'\right)' = \varepsilon^2 \mathcal{O}\left(|r - r_{i,\varepsilon}|^{-\frac{9}{2}}\right),$$

if $C\varepsilon^{\frac{2}{3}} \le (-1)^i(r - r_{i,\varepsilon}) \le \delta$. Integrating the above identity from $r_{i,\varepsilon} + (-1)^i\delta$ to $r_{i,\varepsilon} + (-1)^i C\varepsilon^{\frac{2}{3}}$, and using that $\left(\eta_\varepsilon - \sqrt{A_\varepsilon}\right)'\left(r_{i,\varepsilon} + (-1)^i\delta\right) = \mathcal{O}(\varepsilon^2)$ as $\varepsilon \to 0$ (from [24, Prop. 2.1]), we arrive at (A.9). Finally, relation (A.11) is shown in [24, Prop. 2.1] to hold in the C^1-topology but their proof carries over to yield the same estimate in C^m, $m \ge 2$, via a standard bootstrap argument (as in [9, Thm. 1]), provided that the coefficients in the equation are sufficiently smooth. □

Appendix B. Proof of the Technical Estimate (4.111) in Proposition 4.15

Here we present the

Proof of (4.111). Suppose that ϕ satisfies (4.110) for some $h \in C(\bar{I})$.

Firstly, we establish (4.113). Let

$$\Phi = \rho^{\frac{3}{2}}\phi.$$

It is easy to see that Φ satisfies

$$-\varepsilon^2\Phi_{rr} - \varepsilon^2\left(3\rho^{-1} + \frac{1}{r}\right)\Phi_r + Q(r)\Phi = \rho^{\frac{3}{2}}h, \quad r \in I; \quad \Phi = 0 \text{ on } \partial I, \tag{B.1}$$

where

$$Q(r) = \left(g_1 - \frac{g^2}{g_2}\right)(3\hat{\eta}_{1,\varepsilon}^2 - a_{1,\varepsilon}) - \frac{15}{4}\varepsilon^2\rho^{-2} - \frac{3}{2r}\varepsilon^2\rho^{-1}.$$

Observe that, thanks to the lower bound in (4.112), we have

$$Q(r) \ge \rho\left(k - \frac{15}{4}\varepsilon^2\rho^{-3} - K\varepsilon^2\rho^{-2}\right) \ge \rho(k - KD_j^{-3}) \ge k\rho, \tag{B.2}$$

provided that D_j is sufficiently large. We may assume, without loss of generality, that $\Phi, h \ge 0$ (by writing $h = h^+ - h^-$ if necessary). If Φ attains its maximum value at a point $r_0 \in I$, then $\Phi_{rr}(r_0) \le 0$ and $\Phi_r(r_0) = 0$. So, letting $\rho_0 = R_{1,\varepsilon} - r_0$, via (B.1) and (B.2), we obtain that

$$k\rho_0\Phi(r_0) \le \rho_0^{\frac{3}{2}}h(r_0),$$

i.e., $\Phi(r_0) \le K\|\rho^{\frac{1}{2}}h\|_{L^\infty(I)}$ which clearly implies the validity of (4.113).

By (4.110), the upper bound in (4.112), and (4.113), we find that

$$\varepsilon^2\|\rho^{\frac{1}{2}}\Delta\phi\|_{L^\infty(I)} \le K\|\rho^{\frac{1}{2}}h\|_{L^\infty(I)}. \tag{B.3}$$

From this, we derive a pointwise estimate for ϕ_r by making use of the identity

$$r\phi_r(r) - r_0\phi_r(r_0) = \int_{r_0}^{r} s\Delta\phi ds, \quad \forall\, r_0, r \in I. \tag{B.4}$$

We can choose $r_0 \in (R_{1,\varepsilon} - 2D_j\varepsilon^{\frac{2}{3}}, R_{1,\varepsilon} - D_j\varepsilon^{\frac{2}{3}})$ such that

$$\phi_r(r_0) = \frac{\phi(R_{1,\varepsilon} - D_j\varepsilon^{\frac{2}{3}}) - \phi(R_{1,\varepsilon} - 2D_j\varepsilon^{\frac{2}{3}})}{D_j\varepsilon^{\frac{2}{3}}}.$$

It follows from (4.113) that

$$|\phi_r(r_0)| \le K\varepsilon^{-\frac{5}{3}}\|\rho^{\frac{1}{2}}h\|_{L^\infty(I)}.$$

In turn, via (B.3) and (B.4), we get that

$$|\phi_r(r)| \le K\varepsilon^{-\frac{5}{3}}\|\rho^{\frac{1}{2}}h\|_{L^\infty(I)} + \|\rho^{\frac{1}{2}}\Delta\phi\|_{L^\infty(I)}\left|\int_{r_0}^{r}(R_{1,\varepsilon} - s)^{-\frac{1}{2}}ds\right|$$

$$\le K\varepsilon^{-\frac{5}{3}}\|\rho^{\frac{1}{2}}h\|_{L^\infty(I)} + K\varepsilon^{-2}\|\rho^{\frac{1}{2}}h\|_{L^\infty(I)}\left|\rho_0^{\frac{1}{2}} - \rho^{\frac{1}{2}}\right|,$$

$r \in I$. Hence, since $\rho \ge D_j\varepsilon^{\frac{2}{3}}$ and $D_j\varepsilon^{\frac{2}{3}} \le \rho_0 \le 2D_j\varepsilon^{\frac{2}{3}}$, we infer that

$$\rho^{-\frac{1}{2}}|\phi_r(r)| \le K\varepsilon^{-2}\|\rho^{\frac{1}{2}}h\|_{L^\infty(I)}, \quad r \in I.$$

Now, the desired estimate (4.111) follows by combining (4.113), (B.3) and the above relation. □

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