The development of a semiotic framework to analyze teaching and learning processes: Examples in pre- and post-algebraic contexts

THE DEVELOPMENT OF A SEMIOTIC FRAMEWORK TO ANALYZE TEACHING AND LEARNING PROCESSES: EXAMPLES IN PRE- AND POST-ALGEBRAIC CONTEXTS

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DEVELOPPEMENT D'UN CADRE SÉMIOTIQUE POUR ANALYSER LES PROCESSUS D'ENSEIGNEMENT ET D'APPRENTISSAGE : EXEMPLES DANS DES CONTEXTES PRE ET POST-ALGEBRIQUES

Résumé – Cet article décrit un cadre théorique de sémiotique inspiré de Peirce pour l'analyse de processus d'enseignement et d'apprentissage mathématiques. Ce cadre est centré sur la notion de « paquet sémiotique », qui a déjà été présentée et discutés à un niveau international (Arzarello, Paola, Robutti, & Sabena 2009, Arzarello, Ferrara, & Robutti 2011). Dans ce numéro spécial, nous avons privilégié une présentation « historique », en mettant en avant les connexions et les différences avec les études cognitives et didactiques dont cette notion est issue. Le cadre est brièvement illustré par deux exemples dans des contextes pré- et post-algébriques. Ce choix reflète l'ouverture de SFIDA ces dernières années, par rapport à son orientation originelle strictement algébrique.

Mots clés: embodiment, fonctions, multimodalité, pensée pré-algébrique, paquet sémiotique, jeu sémiotique, variables.

EL DESSARROLLO DE UN MARCO SEMIOTICÓ PARA ANALIZAR PROCESOS DE ENSEÑANZA Y APRENDIZAJE : EJEMPLOS EN CONTEXTOS PRE Y POST-ALGEBRAICOS

Resumen – El presente documento esboza un marco semiótico inspirado en el trabajo de Peirce para el análisis de los procesos de enseñanza y aprendizaje de las matemáticas. Este marco se centra en la noción de "conjunto semiótico" que ya ha sido presentado y discutido a nivel internacional (por ejemplo, Arzarello, Paola, Robutti, y Sabena 2009; Arzarello, Ferrara, y Robutti 2011). En este número especial, hemos privilegiado une presentación "histórica", resaltando conexiones y diferencias con estudios cognitivos y didácticos de los cuales la noción emergió. El marco es brevemente ilustrado con dos ejemplos de contextos pre y post-algebraicos. Esta elección refleja la apertura

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de SFIDA en sus últimos años, con respecto a su original enfoque estrictamente algebraico.

Palabras-claves: embodiment, funciones, multimodalidad, pensamiento pre-algebraico, conjunto semiótico, juego semiótico, variables.

ABSTRACT

Abstract – This paper describes a Peircean-inspired semiotic framework for the analysis of mathematics teaching and learning processes. The framework is centered on the notion of "semiotic bundle," which has already been presented and discussed at the international level (e.g., Arzarello, Paola, Robutti, & Sabena 2009; Arzarello, Ferrara, & Robutti 2011). In this special issue, we opted for an "historical" presentation, highlighting connections with and differences from the cognitive and didactic studies from which the notion came. The framework is briefly illustrated by two examples from pre- and post-algebraic contexts. This choice reflects the openness of SFIDA in recent years with respect to its original strictly algebraic orientation.

Key words: embodiment, functions, multimodality, pre-algebraic thinking, semiotic bundle, semiotic game, variables.

EMBODIMENT AND MULTIMODALITY

The provocative book *Where Mathematics Comes From* by Lakoff and Núñez (2000) has prompted, increasingly in the last decade, an interest in embodied aspects in mathematics education. This work, criticizing Platonic idealism and the Cartesian mind-body dualism, advocates that mathematical ideas are founded on our bodily sensory-motor experiences and develop through metaphorical mechanisms. In our research, the theory of embodiment constitutes an important reference point; however, we think that it also has several limits, in particular, concerning the lack of attention to social, historical, and cultural aspects in the genesis of mathematical concepts (Radford, Bardini, Sabena, Diallo, & Simbagoye 2005; Schiralli & Sinclair 2003).

This paper presents a lens that tries to consider all these aspects and their integration: embodied and individual on the one side, historico-cultural and social on the other. In fact, as remarked by Radford et al. (2005):

> An account of the embodied nature of thinking must come to terms with the problem of the relationship between the body as a locus for the constitution of an individual's subjective meanings and the historically constituted cultural system of meanings and concepts that exists prior to that particular individual's actions. (p.114)

Radford et al. focus on languages, signs, artifacts, and so on, calling them "semiotic systems of cultural meanings" in order to indicate that they are rooted in culture and bearers of knowledge. In our work, we coordinate the perspective of embodiment with recent results on multimodality in communication (Kress, Jewitt, Ogborn, & Tsatsarelis 2001) and neuroscience (Ferrara & Nemirovsky 2005; Gallese & Lakoff 2005), and supplement them with cultural and semiotic dimensions (Arzarello & Robutti 2008).

On the basis of neuroscientific results, in the wake of the embodied cognition theory, Gallese and Lakoff (2005) use the notion of "multimodality" to highlight the role of the brain's sensory-motor system in conceptual knowledge. Their claim is, in particular, based on the discovery of mirror neurons that have a multimodal function; that is, they fire both when the subject performs an action and when he or she observes it, as well as when he or she imagines it. This firing entails that there is not any central "brain engine" responsible for sense making, controlling the different brain areas devoted to different sensorial modalities. Instead, there are multiple modalities that work together in an integrated way, up to overlapping with each other, like vision, touch, and hearing, but also motor control and planning. All of

them are active when we think, so all of them are important with respect to thinking processes involved in mathematics teaching and learning.

The term *multimodality* is also used in communication design to refer to the multiple modes we use to communicate and express meanings to our interlocutors: words, sounds, figures, and so forth (Kress et al. 2001). Today, also thanks to technological media, images are coming to the fore as important means for conveying messages. The image acquires a primary place, whether one considers interaction with the web, with software, and with a game, or with the pages of a magazine or a publicity billboard.

On the base of the discussion briefly summarized above, in our research we use multimodality for the coexistence and interplay of the diverse intertwined modalities. In classroom social interaction, not only words are present, but also gestures, gazes, postures, tones of voice, and so on. In this sense, a perspective on multimodality appears in the recent stream of mathematics education research that highlights the total involvement of the learner interacting with others, considering the individual as inseparable from the social and cultural aspects of learning (Arzarello, Paola, Robutti, & Sabena 2009; Ferrara & Savioli 2009; Nemirovsky & Ferrara 2009; Radford, Edwards, & Arzarello 2009; Roth 2009).

To face such complexity, our research, initially attentive to the embodied aspects involved in mathematics teaching and learning, has increasingly grown toward an integration of these aspects with a semiotic lens for looking at the students' and the teacher's behavior. This theoretical process has brought us to construct a model for an analysis of cognitive and didactic processes based on the signs intervening therein, drawing on Peirce's semiotic theory and including the role of artifacts. We present it in the next section.

THE INTRODUCTION OF THE SEMIOTIC BUNDLE

Our primary research goals deal with the cognitive dimension of teaching and learning processes. As a consequence, we looked for a cognitive model suitable to consider also the social, contextual, and historic-cultural situations in which learners interact with each other and with the teacher. A first step consisted in the introduction of the "Space of Action, Production and Communication" (in short, APC-space): a socio-cultural cognitive environment in which the multimodality of learning processes develops (Arzarello 2008). The APC-space is a metaphorical space where the students' cognitive processes mature through social interaction and is made of multiple

intertwined components: culture, sensory-motor experiences, embodied templates, semiotic resources, and artifacts, considered with their mutual relationships.

Investigating the APC-space, we observed a complex interplay of various modalities through which the students interact (with each other and with the teacher). Becoming sensitive to such complexity, we found that a semiotic lens was suitable to investigate the dynamism of thinking processes, as other scholars have pointed out (e.g., Bartolini-Bussi & Mariotti 2008; Duval 2006). Unlike in other semiotic perspectives, we chose to include, besides linguistic and mathematical systems, embodied and multimodal aspects. We adopted a Peircean approach (Peirce 1931/1958), according to which a sign is a triad constituted by:

- the sign or *representamen* (that represents);
- the *object* (that is represented); and
- the *interpretant*.

The triadic nature of signs had already been considered in studies about algebra in which Frege's model for sign, consisting in *Sinn*, *Zeichen* and *Bedeutung*, had been used (see the contribution in this special issue by Arzarello, Bazzini, and Chiappini).

In the Peircean description, the interpretant constitutes the most delicate and interesting part: It is another representation referred to the same object, like an equivalent significant—but Peirce would say "sign"—in another semiotic system (e.g., a drawing to explain the meaning of a word); an index to the single object, implying an element of universal quantification ("all the objects like this"); another definition in the same semiotic system (e.g., *salt* for sodium chloride); an emotive association (e.g., *dog* for fidelity); the use of synonyms; and so on. As a result, the sign is endowed with an intrinsic dynamic character. The interpretant is, in fact, a sign that translates and explains the previous one, and that other sign in turn requires another sign as interpretant, and so on, in a chain of infinite interpretations, establishing a process of *dynamic unlimited semiosis* (Peirce 1931/1958, vol. 2, par. 92).

Peirce's characterization of "signs" provides us with two features appropriate for our needs. The first concerns the generality of the definition of sign: Anything that can be interpreted by somebody in some respect can be considered a sign (Peirce 1931/1958, vol. 2, par. 228). The second concerns the dynamism of the semiotic processes, framed with the idea of the interpretant.

Assuming a Peircean approach, and considering both static and dynamic descriptions of signs, Arzarello, Bazzini, Ferrara, Robutti,

Sabena, and Villa (2006) have introduced the model of the *semiotic bundle* as:

> *a system of signs* ... that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher. (p. 100)

As an example, we can consider the set of gestures and the set of words that enter a certain problem-solving activity. The two sets are intertwined because they are used simultaneously during the activity: So they constitute elements of the semiotic bundle for that activity.

Unlike other semiotic approaches in mathematics education, the semiotic bundle includes all the bodily means of expression, such as gestures, gazes, sketches, and so on, as semiotic resources in teaching and learning. Such an approach widens the notion of semiotic system (Duval 2006; Ernest 2006), so that it is also possible to consider as signs nonrecursive and segmented forms of language (e.g., gestures), which we consider fundamental components of the multimodal activities in the classroom (Sabena 2008).

To summarize, from the triadic interpretation of sign based on Frege's triangle (which, at the beginning, allowed for the dynamism of the processes involved in treating algebraic formulas), we have moved in two subsequent steps (still grounded in a triadic idea of sign):

- the passage from Frege's to Peirce's account; and
- the enlargement of the notion of semiotic system to that of semiotic bundle.

At a theoretical level, this enlargement has been favored by a refinement of the tools used for observation: from the students' written productions, which constituted our data in the first research studies, to the videos, examined in detail, frame by frame, to analyze the ongoing processes more in depth. An a-posteriori tool for the investigation of multimodality in mathematics activities has been introduced with the help of teachers: the timeline (Arzarello et al. 2011; Arzarello, Ferrara, & Robutti 2011). The timeline is a table with many columns and some rows that are grouped in three main sections: speech, body, and inscriptions (of students and teacher), detailed on a time scale (see Fig. 1).

Two kinds of analyses may be carried out: a *diachronic analysis,* focused on the evolution of signs over time, and the transformation of their relationships (in periods with variable length, from few minutes to years); and a *synchronic analysis*, focused on the relations among

the signs used at a certain moment. The timeline per se is a static product composed of a series of static elements: images of gestures, sketches, drawings, words, and symbols. Scrolling it through time, however, gives one an idea of the complex multimodal nature of learning processes, and of the dynamics linking the various signs. Codes are introduced for representing these and other more specific aspects of the components of the semiotic bundle, like subjects' postures, gazes, tones of voice, and so forth. In this way, the focus is on the ongoing dynamic contextual teaching and learning processes in which the cognitive aspects intertwine with the didactic and communicative ones.

Figure 1. The timeline or semiotic line.

EXAMPLES OF ANALYSIS WITH THE SEMIOTIC BUNDLE

In this section, we present two examples of analysis through the use of the semiotic bundle, and we discuss some results obtained with this kind of analysis: the genetic aspect of gestures and the semiotic game.

1. Genetic aspects in the semiotic bundle

The first feature regards the genetic function of some signs with respect to others in the semiotic bundle, and with regard to the evolution of knowledge in students. An example is provided by a group of Grade 5 children solving a problem situation about the legend of Penelope (Arzarello et al. 2006). The problem is given through narration:

> ... On the island of Ithaca, Penelope had been waiting ten years for the return of her husband Ulysses from the war. On Ithaca, however, a lot

> of men wanted to take the place of Ulysses and marry Penelope. One day, the goddess Minerva told Penelope that Ulysses was returning and that his ship would arrive at Ithaca in 50 days. Penelope immediately summoned the suitors and told them: "I have decided that I will choose my bridegroom from among you, and the wedding will be celebrated when I have finished weaving a new piece of cloth for the nuptial bed. I will begin today, and I promise to weave every two days. When I have finished, the cloth will be my dowry." The suitors accepted. The cloth had to be 15 spans in length. Penelope immediately began to work, but one day she wove a span of cloth, while the following day, in secret, she undid half of it. …
> Will Penelope choose another bridegroom? Why?

The problem was presented at the beginning of an experiment in which the main didactical goals were as follows: (a) the social construction of mathematical knowledge; and (b) the consolidation of crucial concepts of Grade 5 (such as multiples, odd and even numbers, decimal numbers, relative integers), as well as two, more complex, topics; that is, recursivity and the covariance between variables (and its graphical representation). In fact, length and time constitute two crucial variables in the problem, and they have to be suitably coordinated for producing a mathematical model to solve the problem.

The students worked in small groups, and we videotaped one of the groups. The video, as well as the children's written productions, constituted the data for the analysis, which was carried out by means of the timeline.

Considering the timeline from a diachronic point of view, we could see that, while discussing the solution of the problem, the children produced signs of different kinds, such as words, gestures, and diagrams.

At first, to make sense of the story, they focused on the action of weaving and unraveling a span of cloth, and used many gestures to represent it. In particular, the group explored the situation given by the story through a specific gesture indicating the weaving and unraveling of one span (we call it the "basic gesture"; see Arzarello et al. 2006). This gesture, first introduced by a student (Fig. 2a), was soon imitated by all the other children (Fig. 2b). It became a shared reference in the APC-space within which the children made their knowledge evolve.

The use of gestures allowed the children to imitate Penelope's actions in an embodied and iconic way and to easily make those actions visible. Furthermore, the dynamic features of gestures (Fig. 2b–c) condensed the two crucial elements of the problem: the passing of *time* and the *length* of the cloth. Their existence as two entities did not appear at all explicit at that moment, but through gesturing the children made the problem more tangible and started coping with it.

Soon, using a semiotic bundle made of gestures and words, the students started to describe not only Penelope's actions but also their product, finally identifying regularity in her production. See the coordination of gestures and words (*synchronic analysis*) and the use of the adverb *always* to convey a certain level of generalization in the following excerpt:

> PAOLA: Look, because ... she made a span (*basic gesture*) and then, the day after, she undid a half (*shifting her left hand rightward*), and a half was left. ... Right? Then the day after ... (*moving her left hand leftward and repeating the gesture*).
> DAVIDE: (*interrupting*) A half was always left.

Figure 2. (a) Two hands gesture a span. (b–c) The shared gesture of weaving and unraveling a span.

After reaching a common understanding of what happened in the story, the children looked for a way to compute the days. To do that, they produced sketches that represented the work done by Penelope in four days; that is, a span (Fig. 3).

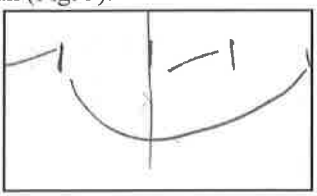

Figure 3. Written signs representing Penelope's 4-day work.

In their diagram, the students tried to represent on paper Penelope's work of weaving and unraveling: Weaving was expressed through vertical lines and unraveling with a sort of deleting mark. To indicate that the spans were linked to each other in the passage of time, a bow was added at the bottom. The final drawing (Fig. 3) shows the palpable need of representing the story using two different quantities, length and time. Time and cloth length are the two variables in play: They were in narrative form in the story, and the students needed to make them explicit and to find their covariance. Their covariance was expressed first through a gesture-speech bundle, and then through a written sign.

If we compare the diagram with the previous gestures, we observe that they share many iconic features: The vertical strokes resemble the slanted positions of arms and hands, and the deleting marks can be seen as referring to the unraveling gestures (see Fig. 2c). Indeed, the written sign appears to be related to the previous basic gestures by a *genetic relationship*.

Using the written signs as "gestures that have been fixed" (Vygotsky 1978, p. 107), the children made the crucial step of understanding that Penelope needed four days to make one span of cloth. In other words, the basic gesture has a generative function with respect to correctly managing the covariance of variables. Starting from this rule, the children built a numerical table through which they were able to reach the end of the story.

2. The emergence of semiotic games

The second example illustrates a didactic phenomenon identified by means of the semiotic bundle analysis: the so-called semiotic game between teacher and students (Arzarello & Paola 2007; Arzarello et al. 2009). A semiotic game may occur when the teacher is interacting with the students, as in classroom discussions or during group work. In a semiotic game, the teacher tunes into the students' semiotic resources (e.g., words and gestures), and uses them to make the mathematical knowledge evolve towards scientifically shared meanings. More specifically, the teacher uses one kind of sign to tune into the students' discourse (typically, a gesture), and another to support the evolution of meanings (typically, language). For instance, the teacher takes a gesture that one or more students have just performed, and repeats it, accompanying it with appropriate linguistic expressions and explanations. The semiotic games can develop if the students produce something meaningful with respect to the problem at hand, using some signs (words, gestures, drawings, etc.). It is up to the teacher to seize these moments to enact her or his semiotic game. Even a vague gesture can indicate a certain comprehension level on the part of a student who does not yet have the words to express it.[1] In a Vygotskian framework, it can indicate that the student is in the Zone

[1] It may not be easy for the teacher in the flow of action to identify the signs that are indexes of meaning construction by the students. For that reason, it is necessary that teachers become aware of the semiotic complexity at play, in order to avoid the risk of a sort of *effet Jourdain* (Brousseau 1986). In our experience, producing a timeline analysis works as a suitable activity for teachers themselves to reach that goal (Arzarello, Bazzini, Politano, & Sabena 2010).

of Proximal Development (Vygotsky 1978) for a certain concept, and the teacher may have the opportunity to intervene in the student's cognitive development. At the same time, the teacher is encouraging the student, signaling that her or his idea, although not fully expressed, is correct.

The teacher can interact with the students in different ways; namely, using different semiotic games. For instance: guiding a student's hand, to help him or her trace a sign in the air, on paper, or with technology; repeating a sign (gesture, or word) made by one of the students, to render it more incisive; adding a sign to one introduced by a student (e.g., a gesture to a word, or a word to a gesture); posing a question about a sign introduced by students; substituting another, more precise sign for a sign made by a student (e.g., a specific mathematical term to replace a generic term); acting on a tool; introducing a metaphor; and so on. These different ways can be partially overlapping because of the complexity of the interaction between students and teacher, and there can be other games not included in the list.

The following example shows a case of a semiotic game nested in the use of technology from a Grade 12 teaching experiment in a scientifically oriented Italian high school. In the experiment, the students had been introduced to approximate measures of areas under a curve in order to approach the concept of integral (for a complete account, see Robutti, Arzarello, & Bartolini Bussi 2004). Using programmable symbolic-graphic calculators, they had computed the areas using the classical rectangle method, but they were unable to bridge the gap between the approximation process and the exact value of the area; namely, between finite and infinite. Replacing n (the number of rectangles) by the symbol ∞ in the program for rectangles in the calculator, one student (Francesco) discovered that the calculator was unable to perform the computation. He brought up this issue in the class discussion:

> FRANCESCO: I put infinity instead of a number n, and the calculator answers "undef."

Instead of giving the calculator program a finite number of rectangles (under or over the curve), Francesco entered a symbol he knew, "∞," because he had the intuition that, as the process of area calculation was increasing in precision when the number of rectangles increased, the most precise (the exact) value should be the last one, and the last was infinity. He was expecting a numerical value as a result of the calculation, and when the response of the calculator was *undef*, he shared his surprise with his classmates and the teacher.

At this point, the teacher decided to use the sign *undef,* given by the calculator, to help the students bridge the gap between finite and infinite. Therefore, she introduced an ideal calculator that could do the same calculation and had the same program as the real calculator but without its limitations; namely, a calculator that did not give the answer *undef* but instead a numerical result. This metaphoric calculator could work with infinite values and do infinite computations.

> TEACHER: Now I am using an ideal calculator, which doesn't exist of course, and I imagine doing the calculation (Fig.4a).
> FRANCESCO: At the end we will have a root.
> TEACHER: A root?
> FRANCESCO: No, a number. … What is the name of those numbers?
> TEACHER: Real. And do the two sequences coincide? (Fig. 4b).

Figure 4a–b. The teacher's gestures in the semiotic game.

The ideal calculator—conceived of as an instrument that does the same calculations as those done by the real calculator but without limitations, neither in quantities nor in the number of operations—may constitute a suitable mediator to bridge the gap between finite and infinite.

What the teacher did in her semiotic game was to tune into the students' instrumented signs to introduce a metaphor taken from the previous activity of the students with the calculator: the metaphor of an ideal calculator, conceived of as having infinite potentiality, with the aim of supporting the link between the exact area and a real number, which are the same concept. While introducing the metaphor, she turned her arm (Fig. 4a) to show a process that goes on and on without limitation. Then she used another gesture, with her two hands very close, to shape two sequences (of areas) that coincide (Fig. 4b). The teacher's semiotic game is here very powerful because this concept of real number is decisive in approaching the exact value of the area under a curve with the use of the definite integral, which is usually a difficult concept for secondary school students (and not only them).

CONCLUDING REMARKS

In this paper, we presented the development that occurred in our research, starting from the consideration of the embodied aspects in mathematics learning, and looking for their integration with cultural and social ones. We presented the *semiotic bundle*, a theoretical notion elaborated in order to include both symbolic as well as embodied productions in teaching-learning processes. We illustrated it by means of two examples from pre-algebraic and post-algebraic contexts: such a choice reflects the openness of SFIDA in the last decade with respect to its original strict algebraic focus.

In the examples, we illustrated two results obtained with the semiotic bundle analysis that are related to the *genetic* character of gestures with respect to written signs, and to a new didactic phenomenon called a *semiotic game*.

The semiotic bundle appears to have great potential for looking at aspects that are often neglected in classroom analysis, such as those related to embodied aspects. Furthermore, the notion of bundle gives a flexible tool for the analysis: it may be used to analyze mathematics learning at any school grade and may easily be networked with other theoretical frameworks (an example is in Arzarello, Bosch, Gascón, & Sabena 2008).

Our current research is investigating the adaptability of the semantic bundle analysis with regard to teacher education. On the one hand, the semiotic frame (enlarged with respect to the Peircean viewpoint) helps shed light on relevant didactic phenomena concerning teaching and learning processes in mathematics, and not only on their products. Moreover, it offers the teacher new instruments for observing and reflecting on her or his didactic action, and for intervening (e.g., the semiotic game). On the other hand, fine-grained analyses require suitable formation and technical tools, and are time-consuming. These practical features make such analyses difficult to be implemented directly in teaching practice. Further research is needed to study how the theoretical framework is modified in different teacher education contexts, in order to meet teachers' needs and to make possible its applicability to more and larger contexts.

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