Cubic fourfolds containing a plane and K3 surfaces of Picard rank two

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Cubic fourfolds containing a plane and $K3$ surfaces of Picard rank two

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Abstract

We present some new examples of families of cubic hypersurfaces in $\mathbb{P}^5(\mathbb{C})$ containing a plane whose associated quadric bundle does not have a rational section.

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1 Introduction

Let X be a smooth cubic hypersurface in $\mathbb{P}^5(\mathbb{C})$. Investigating the rationality of X is a classical problem in algebraic geometry. The general X is conjectured to be not rational but not a single example of non rational cubic fourfold is known.

Cubic fourfolds containing a quartic scroll or a quintic del Pezzo surface are rational (see [F], [Mo]). Idem for those fourfolds containing a plane and a Veronese surface (see [Tr]). Beauville and Donagi showed in [BD] that also pfaffian cubic fourfolds are rational.

The closure of the locus of pfaffian cubic fourfolds is a divisor $\mathcal{C}_{14}$ in the moduli space $\mathcal{C}$ of all cubic fourfolds, while the fourfolds containing a plane form a divisor $\mathcal{C}_8$ (see [H2]). The general fourfold containing a plane is also expected to be non rational. Nevertheless, Hassett showed in [H1] that there exists a countable infinite collection of divisors in $\mathcal{C}_8$ which parameterize rational cubic fourfolds. The fourfolds containing a plane are birational to the total space of a quadric surface bundle by projecting from the plane: Hassett's examples are rational since the associated quadric bundle has a rational section. We call these hypersurfaces *trivially rational*.

Auel et al. (see [ABBV]) have described a divisor in $\mathcal{C}_8$ whose very general member parameterizes rational but not trivially rational cubic fourfolds. They are all pfaffian, so rational. In a recent paper, Bolognesi and Russo proved that every cubic hypersurface belonging to $\mathcal{C}_{14}$ is rational [BR].

Using results on the Hodge structure of cubic fourfolds and $K3$ surfaces, we present a family of cubic fourfolds containing a plane which are not trivially

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rational. We don't know if these fourfolds are rational. The rational example in [ABBV] is in our family.

The paper is organized as follows. In Sections 2 and 3 we recall some basic notions on lattices and $K3$ surfaces. We focus on $K3$ surfaces of Picard rank two recalling the fundamental work of Nikulin in [N]. Then in 3.1 we present the $K3$ surfaces of Picard rank two which are double covers of the plane ramified over a sextic curve. In 3.1.1 we construct a family $S_{(b,c)}$ of double planes with Picard rank two. In Section 4 we recall how these surfaces are related to cubic 4−folds containing a plane. Such a cubic X is birational to a quadric bundle $Y \xrightarrow{\pi} \mathbb{P}^2$ which, in the general case, ramifies over a smooth sextic curve C. The Hodge structure of X is strictly related to the Hodge structure of the $K3$ surface S obtained as a double cover of the plane ramified over C and parameterizing the rulings of the quadrics in the fibration $Y \xrightarrow{\pi} \mathbb{P}^2$ (see [V, §1]). We use the following fact: the lattice $A(X)$ of 2−cycles modulo numerical equivalence on X has rank three and even discriminant if S has Picard rank two and even Néron-Severi discriminant (see [V, §1 Proposition 2]). In case of $\mathrm{rk}(A(X)) = 3$ it is known that the quadric bundle $Y \xrightarrow{\pi} \mathbb{P}^2$ does not have a rational section if and only if the discriminant of $A(X)$ is even (see Proposition 4.0.4).

We prove that if X is not trivially rational, the discriminant $d(A(X))$ is even, without restrictions on the rank of $A(X)$ (see Proposition 4.0.6).

In 4.1 we recover the cubic hypersurfaces associated to the double planes $S_{(b,c)}$ using the additional datum of an odd theta characteristic on the discriminant sextic (see [B, V]).

In Theorem 4.1.2 we prove that the fourfolds corresponding to $S_{(b,c)}$ with d even are not trivially rational. The rational example in [ABBV, Theorem 11] correspond to fourfolds associated to $S_{(2,-1)}$.
Theorem 4.1.2 gives only a sufficient condition for the existence of not trivially rational 4−folds: there are cubic fourfolds containing a plane associated to double planes $S_{(b,c)}$ with b odd which are not trivially rational (see Proposition 4.1.4).

2 Lattices

A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a $\mathbb{Z}$-valued symmetric bilinear form $b_L(x, y)$. A lattice is called *even* if the quadratic form q_L associated to the bilinear form has only even values, *odd* otherwise. The *discriminant* $d(L)$ of a lattice is the determinant of the matrix of its bilinear form. A lattice is called *non-degenerate* if the discriminant is non-zero and *unimodular* if the discriminant is ± 1. If the lattice L is non-degenerate, the pair (s_+, s_-), where $s_\pm$ denotes the multiplicity of the eigenvalue ± 1 for the quadratic form associated to $L \otimes \mathbb{R}$, is called *signature* of L. Finally, we call $s_+ + s_-$ the *rank* of L and L is said *indefinite* if the associate quadratic form has both positive and negative values.

Given a lattice L, the lattice $L(m)$ is the $\mathbb{Z}$-module L with bilinear form $b_{L(m)}(x, y) = m b_L(x, y)$. An *isometry* of lattices is an isomorphism preserving the bilinear form. Given a sublattice $L' \subset L$, the embedding is *primitive* if $\frac{L}{L'}$ is free.

Let $L^* = Hom_{\mathbb{Z}}(L, \mathbb{Z}) = \{x \in L \otimes \mathbb{Q} \;:\; b_L(x, l) \in \mathbb{Z}, \forall l \in L\}$ be the *dual* of the lattice L. There is a natural embedding $L \hookrightarrow L^*$ given by $l \mapsto b_L(l, -)$. There is the following

Lemma 2.0.1. *[BPV, I,Lemma 2.1.] Let L be a non-degenerate lattice. Then*

1. $[L^* : L] = |d(L)|$
2. $[L : L']^2 = \dfrac{d(L')}{d(L)}$, *where $L' \subset L$ is a sublattice with* $\mathrm{rk}(L') = \mathrm{rk}(L)$.

Denote by L a non-degenerate even lattice. The bilinear form b_L induces a $\mathbb{Q}$-valued bilinear form on L^* and so a finite quadratic form

$$q_{A_L} : L^*/L \longrightarrow \mathbb{Q}/2\mathbb{Z}$$

called the *discriminant form* of L. The group $L^*/L := A_L$ is the *discriminant group* of L.

2.1 Examples.

i) The lattice $\langle n \rangle$ is a free $\mathbb{Z}$-module of rank one, $\mathbb{Z}\langle e \rangle$, with bilinear form $b(e, e) = n$.

ii) The *hyperbolic lattice* is the even, unimodular, indefinite lattice with $\mathbb{Z}$-module $\mathbb{Z}\langle e_1, e_2 \rangle$ and bilinear form given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We write

$$U = \left\{ \mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

iii) The lattice E_8 has $\mathbb{Z}^8$ as $\mathbb{Z}$-module and the matrix of the bilinear form is the Cartan matrix of the root system of E_8. It is an even, unimodular and positive definite lattice.

3 K3 surfaces of rank two

A $K3$ surface is a smooth projective surface S with trivial canonical class and $H^1(S, \mathcal{O}_S) = 0$.

It is well known that $H^2(S, \mathbb{Z})$ is an even, unimodular, indefinite lattice, with respect to the intersection form on S. It has rank 22, signature $(3, 19)$ and it is isomorphic to

$$\Lambda := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

The lattice Λ will be called the $K3$ *lattice.* The Hodge numbers are $(1, 20, 1)$, (see [BPV, VIII]). Denote by

$$NS(S) \cong H^2(S,\mathbb{Z}) \cap H^{1,1}(S)$$

the *Néron-Severi lattice* of S, it is a primitive sublattice of $H^2(S,\mathbb{Z})$. Rational, algebraic and homological equivalence coincide on a $K3$ surface.

The orthogonal complement $T(S)$ of $NS(S)$ in $H^2(S,\mathbb{Z})$ is the *transcendental lattice* of S.

The *rank* of S, $\rho(S)$, is the rank of $NS(S)$. The Hodge Index Theorem implies that $NS(S)$ has signature $(1, \rho(S) - 1)$ and that $T(S)$ has signature $(2, 20 - \rho(S))$. Let $l \in NS(S)$ be a class with $l^2 > 0$. The *primitive cohomology* $H^2(S,\mathbb{Z})^0$ is the orthogonal complement of the lattice $< l >$.

Main tools for the study of K3 surfaces are the Torelli Theorem (see [LP] and [PSS]) and the Surjectivity of the Period Map (see [T]). The *period* of S is given by $[\omega_S] = \mathbb{P}(H^{2,0}(S))$ in the period domain

$$\Omega = \{x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | x \cdot x = 0,\, x \cdot \bar{x} > 0\} \subset \mathbb{P}(\Lambda \otimes \mathbb{C}).$$

By the Torelli Theorem and the Surjectivity of the Period Map, an element ω in the period domain determines the $K3$ surface: given $\omega \in \Omega$ there exists a $K3$ surface S_ω (unique up to isomorphism) with period ω such that $H^2(S_\omega,\mathbb{Z})$ is isometric to Λ.

Nikulin in [N] made a deep study of lattice theory and integral quadratic forms with applications to the study of $K3$ surfaces. We recall the following which is crucial for our purposes

Theorem 3.0.1. *[N, Theorem 1.14.4] [M, Corollary 2.9] If $\rho(S) \leq 10$, then every even lattice M of signature $(1, \rho - 1)$ occurs as the Néron-Severi group of some $K3$ surface and the primitive embedding $M \hookrightarrow \Lambda$ is unique.*

Corollary 3.0.2. *All even lattices of rank 2 and signature $(1, 1)$ occur as the Néron-Severi lattice $NS(S)$ of some K3 surface S of rank two and the primitive embedding $NS(S) \hookrightarrow \Lambda$ is unique. Any such lattice has the form*

$$M = \left\{ \mathbb{Z}^2,\ \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \right\}$$

with $a \geq 0$ and $b^2 - 4ac > 0$.

3.1 $K3$ surfaces double planes of rank two

A double covering of the projective plane $\varphi : S \longrightarrow \mathbb{P}^2$ branched along a smooth sextic C is a $K3$ surface: $\varphi_*(\mathcal{O}_S) \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}(3)$, so $H^1(S, \mathcal{O}_S) = 0$ and $\omega_S \cong \varphi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{O}(3)) \cong \mathcal{O}_S$. The $K3$ surface S in this case is called a *double plane.* For general references on double planes, see [En] and [S]. An ample class $l \in NS(S)$

with $l^2 = 2$ is the pull-back of the class of a line in $\mathbb{P}^2$. If S has rank two the Néron-Severi lattice has the form

$$L_{(b,c)} = \left\{ \mathbb{Z}^2, \begin{pmatrix} 2 & b \\ b & 2c \end{pmatrix} \right\}.$$

3.1.1 Examples.

i) Consider S a $K3$ surface double plane ramified over a smooth sextic with Néron-Severi lattice of the form

$$L_{(1,-1)} = \left\{ \mathbb{Z}^2, \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \right\}.$$

This can be realized by taking a double cover of the plane ramified over a sextic curve having a tritangent line l. The pull-back of l to S is a divisor splitting into two irreducible components l_1, l_2 . The corresponding divisor classes are linearly independent. Both curves are isomorphic to l and $l_1^2 = l_2^2 = -2$.

ii) Analogously, if the Néron-Severi lattice has the form

$$L_{(2,-1)} = \left\{ \mathbb{Z}^2, \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} \right\}$$

the corresponding double plane S can be realized with a ramification sextic C which is tangent to a conic D in 6 points with multiplicity two. As before, $\varphi^*(D) = D_1 + D_2$, with D_1, D_2 isomorphic to D and $D_1^2 = D_2^2 = -2$.

The previous examples can be generalized as follows.

Lemma 3.1.1. *If $b > 0$ and $b^2 - 4c > 0$, then the lattice*

$$L_{(b,c)} = \left\{ \mathbb{Z}^2, \begin{pmatrix} 2 & b \\ b & 2c \end{pmatrix} \right\}$$

is the Néron-Severi lattice of a double plane $S_{(b,c)}$ with a smooth ramification sextic.

Proof. The lattice $L_{(b,c)}$ is even and it has signature $(1,1)$. By Theorem 3.0.1 and Corollary 3.0.2, $L_{(b,c)}$ occurs as the Picard group of a $K3$ surface: denote by $S_{(b,c)} = S_\alpha$ the $K3$ surface defined by $\alpha \in \Omega$ with $\alpha^\perp = L_{(b,c)}$ and, moreover, generic with this property, hence $L_{(b,c)} = NS(S_{(b,c)})$. Let H, A be the classes $(1,0)$ and $(0,1)$ in $NS(S_{(b,c)})$, respectively. For each divisor Γ with $\Gamma^2 = -2$ we have the Picard–Lefschetz reflection π_Γ of $NS(S_{(b,c)})$ defined by $D \mapsto D + (D\Gamma)\Gamma$. If D' is another divisor on $S_{(b,c)}$, then $\pi_\Gamma(D)\pi_\Gamma(D') = DD'$, because $\Gamma^2 = -2$. The cone of big and nef divisors is a fundamental domain for the group generated by the above reflections (see for example [Huy1, Chapter 8, Corollary 2.11]). In particular, we can find divisors Γ_i with $\Gamma_i\Gamma_j = -2\delta_{i,j}$, $i = 1, \ldots, l$, such that

$$H' := H + \sum_{i=1}^{l}(H\Gamma_i)\Gamma_i$$

is nef. Let

$$A' := A + \sum_{i=1}^{l}(A\Gamma_i)\Gamma_i.$$

Thus $NS(S_{(b,c)}) =< H, A >=< H', A' >$. Omitting the prime in the superscript we can thus assume that H is nef.

Let $H = F + M$ be its decomposition in the fixed part F and the mobile part M, then M is nef too. Observe that $M^2 = H^2 = 2$ (see for example [Huy1, Chapter 2, Remark 3.3.]). Since, moreover, M is without fixed part by definition, it defines a double cover $\varphi : S_{(b,c)} \longrightarrow \mathbb{P}^2$. The ramification curve C is smooth since a point $x \in S$ is singular iff $\varphi(x)$ is a singular point of C (see for example [S, p.8]). $\square$

4 Cubic 4−folds containing a plane

Let X be a smooth cubic hypersurface in $\mathbb{P}^5(\mathbb{C})$. Consider the cohomology group $H^4(X, \mathbb{Z})$ and denote with

$$A(X) = H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$$

the lattice of the middle integral cohomology Hodge classes. Those classes are algebraic since X verifies the integral Hodge conjecture (see [Mu] and [Zu]). The *transcendental lattice* $T(X)$ is the orthogonal complement of $A(X)$ (with respect to the intersection form on X).

From now on X will indicate a cubic hypersurface in $\mathbb{P}^5$ containing a plane. Consider the projection from the plane P onto a plane in $\mathbb{P}^5$ disjoint from P. Blowing up X along P one obtains a quadric bundle $\pi : Y \longrightarrow \mathbb{P}^2$ branched over C, the discriminant sextic. If X does not contain a second plane intersecting P, the curve C is smooth and this means that the quadrics of the bundle have rank ≥ 3 (see [V, §1 Lemme 2]).

Denote by Q the class of such a quadric. One has $P + Q = H^2$, where H is the hyperplane class associated to the embedding $X \hookrightarrow \mathbb{P}^5(\mathbb{C})$. The hypersurface X is said to be *very general* if $A(X) = < H^2, P >$ $(= < H^2, Q >)$. Denote $L :=< H^2, P >^{\perp}$.

X is rational iff Y is rational and a sufficient condition for the rationality of Y is the existence of a rational section.

Definition 4.0.2. We call a cubic hypersurface $X \subset \mathbb{P}^5$ containing a plane *trivially rational* if the associated quadric bundle has a rational section.

This fact may be translated in a condition on the parity of the intersection of some $2-$cycles on X. More precisely, for a $2-$cycle T in X consider the intersection index

$$\delta(T) = T \cdot Q.$$

Note that $\delta(P) = -2$ and $\delta(H^2) = 2$ So, if X is very general the index δ takes only even values. There is the following result (see [H2, Theorem 3.1.], [ABBV, Proposition 2], [H1, Lemma 4.4.]).

Theorem 4.0.3. *A cubic fourfold X containing a plane is trivially rational if and only if there exists a cycle T in $A(X)$ with $\delta(T)$ odd.*

Using this Theorem it is easy to give (lattice-theoretic) hints to construct cubic fourfolds with $rk(A(X)) > 2$ and not trivially rational (see [H1, Lemma 4.4.] and [ABBV, Proposition 2]).

Proposition 4.0.4. *Let X be a cubic fourfold containing a plane with* $\mathrm{rk}(A(X)) = 3$. *Thus X is trivially rational if and only if $d(A(X))$ is odd.*

Proof. The quadric bundle $\pi : Y \longrightarrow \mathbb{P}^2$ has a rational section if and only if there exists a cycle $T \in A(X)$ such that $\delta(T)$ is odd (by Theorem 4.0.3). Since $A(X)$ has rank 3, the sublattice $< H^2, Q, T >$ has finite index, hence Lemma 2.0.1 implies that, if $< H^2, Q, T >$ has odd discriminant, then $d(A(X))$ is odd as well. □

Our aim now is to build some geometric examples. To do this, we need to better understand the links between Hodge theory and the geometry on a cubic $4-$fold containing a plane. Here we follow Voisin [V, §1].

Let $\varphi : S \longrightarrow \mathbb{P}^2$ be the double cover branched over C, the discriminant sextic of the quadric bundle $Y \longrightarrow \mathbb{P}^2$. The surface S parameterizes the rulings of the quadrics of the fibration. Let F be the Fano variety of lines in X, the subvariety of the Grassmannian $Gr(1,5)$ parameterizing lines contained in X. The divisor $D \subset F$ consisting of lines meeting P is identified with

$D = \{(l, s) \in F \times S : l$ is in the ruling of the quadric parameterized by $\varphi(s)\}$.

giving a $\mathbb{P}^1-$bundle

$$f : D \longrightarrow S. \tag{1}$$

The incidence graph restricted to D

$$\begin{tikzcd} D \times X \supset Z_D \arrow[r, "p"] \arrow[d, "q"] & D \\ X & \end{tikzcd}$$

defines the Abel-Jacobi map:

$$\alpha_D = p_* q^* : H^4(X, \mathbb{Q}) \longrightarrow H^2(D, \mathbb{Q})$$

which induces an isomorphism of Hodge structures, see [V, §1 Proposition 1]. Before stating the next result, we recall that we denote by L the orthogonal complement of the lattice $< H^2, P >$ in $H^4(X,\mathbb{Z})$, where H is the hyperplane class and P is the class of a plane contained in X.

Proposition 4.0.5. *([V, §1 Proposition 2],[ABBV, Proposition 1]) Let X be a smooth cubic fourfold containing a plane. Then $\alpha_D(L) \subset f^*(H^2(S,\mathbb{Z})_0(-1))$ is a polarized Hodge substructure of index 2. Moreover, $\alpha_D(T(X)) \subset f^*T(S)(-1)$ is a sublattice of index ϵ dividing 2. In particular,* $\mathrm{rk}\, A(X) = \mathrm{rk}\,(NS(S)) + 1$ *and* $d(A(X)) = (-1)^{\rho(S)-1}2^{2(\epsilon-1)}d(NS(S))$.

We can also derive the following result, which amplifies Proposition 4.0.4.

Proposition 4.0.6. *Let X be a cubic fourfold containing a plane. If X is not trivially rational, then $\alpha_D(T(X)) \subset f^*T(S)(-1)$ is a sublattice of index 2 and $d(A(X))$ is even.*

Proof. The $\mathbb{P}^1$-bundle $f : D \longrightarrow S$ in (1) produces an element of order two in the Brauer group $Br(S)$ of S. The quadric bundle associated to X does not have a rational section if and only if this element is not trivial in $Br(S)$ (see [Ku, Proposition 4.7.]). Recall that, if S is a $K3$ surface, then

$$Br(S) \cong T(S)^* \otimes \mathbb{Q}/\mathbb{Z} \cong \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z})$$

(see for example [vG, §2.1.]). An element of order 2 in $Br(S)$ defines a surjective homomorphism

$$\alpha : T(S) \longrightarrow \mathbb{Z}/2\mathbb{Z} \tag{2}$$

and thus a sublattice T_α of index 2 in $T(S)$. Voisin [V, §1] and van Geemen [vG, §9] give a geometric realization for this element α (see also [HVV11, §2]). More precisely, there exists $k \in H^2(S,\mathbb{Z})$ such that

$$\alpha_D(L) \cong \{v \in H^2(S,\mathbb{Z})^0 \ : < v, k >_S \equiv 0 (\mathrm{mod} 2)\}$$

and k induces an element φ in $\mathrm{Hom}(H^2(S,\mathbb{Z})^0, \mathbb{Z}/2\mathbb{Z})$ which restricts to α in $T(S)$. By definition, $\ker\varphi \cong \alpha_D(L)$ and, since $\alpha_D(T(X)) \subseteq \alpha_D(L)$, we have $\alpha_D(T(X)) \subseteq f^*(T_\alpha)(-1)$. Thus $\alpha_D(T(X)) \subset f^*T(S)(-1)$ is a sublattice of index 2 and $d(A(X))$ is even by Proposition 4.0.5. □

Remark 4.0.7. The lattice T_α is isometric to the transcendental lattice $T(S,\alpha)$ of the α-twisted Hodge structure of S (see [Huy3, Proposition 4.7] and [Huy2, Lemma 2.15]). If $u, v \in L$ one has that $< u, v >_X = - < \alpha_D(u), \alpha_D(v) >_S$ (see [V, Proposition 2 ii)]). Thus Proposition 4.0.6 implies that, if X is not trivially rational, then $\alpha_D(T(X))$ is isometric to $T(S,\alpha)(-1)$.

4.1 Theta-characteristics on the ramification curve C

A theta-characteristic on a smooth curve C is a line bundle κ such that $\kappa^{\otimes 2} = K_C$. We write $h^0(\kappa) := \dim H^0(C, \kappa)$.

Denote with Q_x a quadric of the bundle $Y \longrightarrow \mathbb{P}^2$. The map $x \mapsto Q_x \cap P$ gives a net of conics whose discriminant curve is a plane cubic C_1. The curve C_1 cuts a divisor $2D$ on the sextic C and thus it determines an effective theta-characteristic on C (see [V, §1 Lemme 7]). Conversely, the cubic hypersurface X is determined by the curve C plus an odd theta-characteristic (see [V, §1 Proposition 4]). The same result is implied by the following

Proposition 4.1.1. *([B, Proposition 4.2.]) Let C be a smooth plane curve of degree d, defined by an equation $F = 0$ and κ an odd theta-characteristic on C with $h^0(\kappa) = 1$. Thus, κ admits a minimal resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^{d-3} \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^{d-3} \oplus \mathcal{O}_{\mathbb{P}^2} \longrightarrow \kappa \longrightarrow 0$$

with a symmetric matrix $M \in M_{(d-2)\times(d-2)}(\mathbb{C}[X_0, X_1, X_2])$ satisfying $\det M = F$, and of the form

$$M = \begin{pmatrix} L_{1,1} & \dots & L_{1,d-3} & Q_1 \\ \vdots & & \vdots & \vdots \\ L_{1,d-3} & \dots & L_{d-3,d-3} & Q_{d-3} \\ Q_1 & \dots & Q_{d-3} & H \end{pmatrix} \tag{3}$$

where the forms $L_{i,j}$, Q_i, H are linear, quadratic and cubic respectively.

Conversely, the cokernel of a symmetric matrix M as above is an odd theta-characteristic κ on C with $h^0(\kappa) = 1$.

We can now prove our main result.

Theorem 4.1.2. *Consider the couple $(S_{(b,c)}, \kappa)$ where $S_{(b,c)}$ is a double plane defined as in Lemma 3.1.1 and κ is a theta characteristic on the ramification curve C with $h^0(\kappa) = 1$. If b is even, then $(S_{(b,c)}, \kappa)$ determines a cubic fourfold which is not trivially rational.*

Proof. Let C be the ramification curve of $S := S_{(b,c)}$ and take a theta characteristic κ on C with $h^0(\kappa) = 1$. Proposition 4.1.1 says that the curve C has a determinantal representation $F = \det M = 0$ with

$$M = \begin{pmatrix} L_{1,1} & L_{1,2} & L_{1,3} & Q_1 \\ L_{1,2} & L_{2,2} & L_{2,3} & Q_2 \\ L_{1,3} & L_{2,3} & L_{3,3} & Q_3 \\ Q_1 & Q_2 & Q_3 & H \end{pmatrix}.$$

The geometric interpretation is the following. Choose projective coordinates $[Z_1, Z_2, Z_3, X_0, X_1, X_2]$ in $\mathbb{P}^5(\mathbb{C})$ and define the cubic fourfold $X = X(S, \kappa)$ as the zero set

$$\sum_{i,j=1}^{3} Z_i Z_j L_{i,j}(X_0, X_1, X_2) + \sum_{i=1}^{3} 2Z_i Q_i(X_0, X_1, X_2) + H(X_0, X_1, X_2) = 0.$$

The cubic X is smooth and it contains the plane $P := \{X_0 = X_1 = X_2 = 0\}$. The curve C is the discriminant of the quadric bundle obtained by projecting the hypersurface X from P.

The $K3$ surface S has rank two and b is even, so the discriminant of $NS(S)$ is even. This means that $A(X)$ has rank three and even discriminant by Proposition 4.0.5. That X is not trivially rational follows now from Proposition 4.0.4. □

Remark 4.1.3. Auel et al. in [ABBV] (see Theorem 11) show an explicit example of a pfaffian (hence rational) cubic fourfold associated to a $K3$ surface of type $S_{(2,-1)}$.

Theorem 4.1.2 gives only a sufficient condition for the existence of not trivially rational $4-$folds.

Proposition 4.1.4. *There exist double planes $S_{(b,c)}$ with b odd determining cubic fourfolds containing a plane which are not trivially rational.*

Proof. In [ABBV, Theorem 4] it is proved that the general fourfold X in one of the irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{14}$ has $A(X)$ with intersection matrix given by

	H^2	P	T
H^2	3	1	4
P	1	3	2
T	4	2	10

(4)

The discriminant sextic C of the quadric bundle associated to X is smooth and let $S = S_{(b,c)}$ the double plane branched on C. Since $d(A(X) = 36$, X is not trivially rational by Proposition 4.0.4. Thus, $d(NS(S))) = -9$ by Proposition 4.0.5 and Proposition 4.0.6. We conclude that b is odd . □

Remark 4.1.5. Actually, the cubic in the previous example is already known to be rational since it is a pfaffian.

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