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Spaces of generalized splines over T-meshes

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Abstract

We consider a class of non-polynomial spaces, namely a noteworthy case of Extended Chebyshev spaces, and we generalize the concept of polynomial spline space over T-mesh to this non-polynomial setting: in other words, we focus on a class of spaces spanned, in each cell of the T-mesh, both by polynomial and by suitably-chosen non-polynomial functions, which we will refer to as generalized splines over T-meshes. For such spaces, we provide, under certain conditions on the regularity of the space, a study of the dimension and of the basis, based on the notion of minimal determining set, as well as some results about the dimension of refined and merged T-meshes. Finally, we study the approximation power of the just constructed spline spaces.

Keywords: T-mesh, Generalized splines, Dimension formula, Basis functions, Approximation power
2010 MSC: 41A15 (Spline approximation), 65D07 (Splines)

1. Introduction

The theory of Chebyshevian and Quasi-Chebyshevian spline spaces is a well-known tool which allows to generalize the classical concept of univariate polynomial spline spaces to a non-polynomial setting (see, e.g., [1] and [2]). Essentially, the elements of such spaces locally belong to Extended Chebyshev and Quasi-Extended Chebyshev spaces (see, e.g., [2]), respectively. Many papers considered particular cases of Chebyshevian and Quasi-Chebyshevian splines (see, e.g., [3], [4] and [5]).

This paper deals with the application of the concept of spline space over T-meshes to the noteworthy case of the Extended Chebyshev spaces considered in [6], in order to get a generalization of the polynomial spline spaces over T-meshes. The idea of spline spaces over T-meshes was first introduced for polynomial splines by Deng et al. in [7] and further studied by the same authors and several others (see, e.g., [8], [9], [10] and [1]). The basic idea consists of considering spline functions which are polynomials of a certain degree in each of the cells of the T-mesh, which, unlike the classical tensor-product meshes, allows T-junctions, that is, vertices where only three edges meet. This structure, unlike the one of tensor-product meshes, allows the use of local refinement techniques, and for this reason has gathered a lot of attention in the scientific community, which brought to the study not only of spline spaces over T-meshes, but also of the closely-related T-splines (see, e.g., [11], [12], [13] and [14]), the hierarchical splines (see, e.g., [15] and [16]), and the LR-splines (see, e.g., [17]). Our goal is then using the generalized splines of type [6] to define a class of spaces of non-polynomial splines over T-meshes. The relevance of this class of spline spaces and some of the basic concepts related to it have been recently discussed in some international conferences. The study of these non-polynomial spaces is justified by at least two reasons. First of all, the presence of non-polynomial functions allows to exactly reproduce certain shapes which can only be approximated by polynomial splines or NURBS (for example relevant curves like helices, cycloids, catenaries, or other transcendental curves). Moreover, as we will also point out in Section 4, choosing suitable non-polynomial functions also allows an easier computation of derivatives and integrals of certain surfaces with respect to using NURBS (see also [18], [4]). For these reasons, the same kind of non-polynomial functions have been recently used also to construct non-polynomial T-splines (see, e.g., [19]), and non-polynomial hierarchical splines spaces (see [20]). The goal of this work is to carry out a rigorous and deep study of this class of splines, which we will

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call Generalized splines over T-meshes, including results about the space dimension and the approximation power, which, as far as we know, are still missing in the literature.

Starting from some of the results obtained in [6] about a noteworthy class of univariate non-polynomial spaces, we define generalized spline spaces over T-meshes and construct a local representation in the Bernstein-Bézier fashion for their elements. For the above spaces we first provide the construction of a basis and a dimension formula by using the properties of the local Bernstein-Bézier representation and by generalizing to the non-polynomial case some of the techniques proposed for the polynomial one in [1]. We also analyze how the dimension of such spaces changes when we refine the T-mesh and when we merge two T-meshes.
Moreover, we also study the approximation power of the just constructed spline spaces. In particular, we do it by constructing a quasi-interpolant based on some new local approximants, whose construction is not trivial. In fact, the results about the univariate non-polynomial Hermite interpolants given in [6] cannot be directly extended to the bivariate case. On the other hand, also the bivariate averaged Taylor expansions used in [1] cannot be simply adapted to the non-polynomial case we consider here. Therefore, we instead defined a new local Hermite interpolant belonging to the non-polynomial spline space, whose existence is proved by using certain assumptions made about the non-polynomial functions spanning the space, as carefully explained in Section 4. This approach allows us to get, at least in certain cases, the same approximation order as in the polynomial case.

The paper is organized as follows. Section 2 includes several preliminary arguments about the non-polynomial spaces we will use to define the new spline spaces, including some important properties about the derivatives of the basis functions and the basic concepts about T-meshes. Section 3 presents the new generalized spline spaces over T-meshes, and includes a detailed proof of the dimension formula and of the construction of the basis; moreover, we also provide a study of how the spline space dimension changes when the T-mesh is refined, and of the dimension of a generalized spline space over two merged T-meshes. Section 3 also includes some examples of basis functions, with some remarks about their features. Finally, Section 4 is devoted to the study of the approximation power of the constructed generalized spline space.

2. Preliminaries

The spaces we will consider are of the type

$$\mathscr{P}^n_{u,v}([a,b]) := \langle 1, s, ..., s^{n-2}, u(s), v(s)\rangle, \qquad s \in [a,b], \quad 2 \le n \in \mathbb{N}, \tag{1}$$

where $u, v \in C^{n+1}([a,b])$; for $n = 1$ we set

$$\mathscr{P}^1_{u,v}([a,b]) := \langle u(s), v(s)\rangle, \qquad s \in [a,b].$$

We assume that $\dim\big(\mathscr{P}^n_{u,v}([a,b])\big) = n+1$; moreover, in order to prove some of the properties we are about to present, we will sometimes require the following additional conditions on $\mathscr{P}^n_{u,v}([a,b])$

$$\begin{aligned} \forall \psi \in \mathscr{P}^n_{u,v}([a,b]), \text{ if } \psi^{(n-1)}(s_1) = \psi^{(n-1)}(s_2) = 0, \quad s_1, s_2 \in [a,b],\ s_1 \neq s_2 \\ \text{then } \psi^{(n-1)}(s) = 0, \quad s \in [a,b]; \end{aligned} \tag{2}$$

$$\begin{aligned} \forall \psi \in \mathscr{P}^n_{u,v}([a,b]), \text{ if } \psi^{(n-1)}(s_1) = \psi^{(n)}(s_1) = 0, \quad s_1 \in (a,b), \\ \text{then } \psi^{(n-1)}(s) = 0, \quad s \in [a,b]. \end{aligned} \tag{3}$$

In the following, we will explicitly mention when such conditions are needed.

2.1. Normalized positive basis and its properties

In this subsection we consider a normalized positive basis for the space $\mathscr{P}^n_{u,v}([a,b])$. The procedure to obtain it and its fundamental properties are known and can be found in [6]. Therefore here we will just recall the main results obtained in [6], omitting the proofs. We will instead prove Property 2, which will be crucial in order to obtain some results later in the paper.
We will assume that the condition (2) holds.

The normalized positive basis can be constructed by using the following integral recurrence relation. By (2), there exist unique elements $U_{0,1,n}$ and $U_{1,1,n}$ belonging to $\langle u^{(n-1)}, v^{(n-1)}\rangle$ satisfying

$$\begin{aligned} U_{0,1,n}(a) = 1, \quad U_{0,1,n}(b) = 0, \\ U_{1,1,n}(a) = 0, \quad U_{1,1,n}(b) = 1, \end{aligned} \tag{4}$$

and

$$U_{0,1,n}(s), U_{1,1,n}(s) > 0, \qquad s \in (a,b). \tag{5}$$

Moreover, we define, for $k = 2, \ldots, n$ and $n \geq 2$

$$\begin{aligned} &U_{0,k,n}(s) := 1 - V_{0,k-1,n}(s) \\ &U_{i,k,n}(s) := V_{i-1,k-1,n}(s) - V_{i,k-1,n}(s), \qquad 1 \leq i \leq k-1 \\ &U_{k,k,n}(s) := V_{k-1,k-1,n}(s), \end{aligned} \tag{6}$$

where

$$V_{i,k,n}(s) := \int_a^s U_{i,k,n}/d_{i,k,n} dt, \tag{7}$$

and

$$d_{i,k,n}(s) := \int_a^b U_{i,k,n} dt,$$

for $i = 0, \ldots, k$, $k = 1, \ldots, n-1$. Note that (4) and (5) hold also in the particular case $n = 1$, and then $U_{0,1,1}$ and $U_{1,1,1}$ are a positive basis for $\mathscr{P}^1_{u,v}([a,b])$. The following results can be proved about the just defined functions.

Theorem 1. *For $k = 2, \ldots, n$ and $n \geq 2$, the set of functions $\{U_{0,k,n}, \ldots, U_{k,k,n}\}$ is a basis for the space*

$$\langle 1, s, \ldots, s^{k-2}, u^{(n-k)}(s), v^{(n-k)}(s)\rangle.$$

Moreover, it is a normalized positive basis, that is, satisfies the conditions $\sum_{i=0}^{k} U_{i,k,n}(s) = 1$ and $U_{i,k,n}(s) > 0$ for $s \in (a,b)$, $i = 0, \ldots, k$.

Corollary 1. *The set of functions $\{U_{0,n,n}, \ldots, U_{n,n,n}\}$ is a normalized positive basis for the space $\mathscr{P}^n_{u,v}([a,b])$, $n \geq 2$, $U_{i,n,n} = B_{i,n}$, where $\{B_{i,n}\}_{i=0}^n$ satisfy $\sum_{i=0}^n B_{i,n}(s) = 1$ and $B_{i,n}(s) > 0$ for $s \in (a,b)$, $i = 0, \ldots, n$. For $n = 1$, the set $\{U_{0,1,1}, U_{1,1,1}\}$ is a positive basis of $\mathscr{P}^1_{u,v}([a,b])$.*

Since in the case $n = 1$ we cannot, in general, guarantee the construction of a normalized positive basis, in the following we will assume $n \geq 2$. As a consequence of the results given in Sections 4 and 6 of [6], we get the following property.

Property 1. *For $i = 0, \ldots, k$, $k = 2, \ldots, n$ and $n \geq 2$, we have*

$$\begin{aligned} U^{(j)}_{i,k,n}(a) = 0, \qquad j = 0, \ldots, i-1, \\ U^{(j)}_{i,k,n}(b) = 0, \qquad j = 0, \ldots, k-i-1. \end{aligned}$$

In particular, if we consider $k = n$, we have

$$\begin{aligned} B^{(j)}_{i,n}(a) = 0, \qquad j = 0, \ldots, i-1, \\ B^{(j)}_{i,n}(b) = 0, \qquad j = 0, \ldots, n-i-1. \end{aligned}$$

Property 2. *For $k = 2, \ldots, n$ and $n \geq 2$, we have*

$$U^{(i)}_{i,k,n}(a) \neq 0, \qquad i = 0, \ldots, k-1, \tag{8}$$

$$U^{(k-i)}_{i,k,n}(b) \neq 0, \qquad i = 1, \ldots, k. \tag{9}$$

In particular, if we consider $k = n$, we have

$$\begin{aligned} B^{(i)}_{i,n}(a) \neq 0, \qquad i = 0, \ldots, n-1, \\ B^{(n-i)}_{i,n}(b) \neq 0, \qquad i = 1, \ldots, n. \end{aligned}$$

Proof. First, let us prove (8) by induction. For $k = 2$, (8) holds, since from (4), (6) and (7) we get

$$U_{0,2,n}(a) = 1 - V_{0,1,n}(a) = 1 - \int_a^a U_{0,1,n}(t)/d_{0,1,n}dt = 1 - 0 = 1,$$
$$U^{(1)}_{1,2,n}(a) = D[V_{0,1,n}(s) - V_{1,1,n}(s)]_{s=a} = \frac{U_{0,1,n}(a)}{d_{0,1,n}} - \frac{U_{1,1,n}(a)}{d_{1,1,n}} = \frac{1}{d_{0,1,n}} - 0 \neq 0.$$

Now, if (8) holds for k, it must be true for $k+1$ as well, since we have

$$U_{0,k+1,n}(a) = 1 - V_{0,k,n}(a) = 1 - \int_a^a U_{0,k,n}(t)/d_{0,k,n}dt = 1 - 0 = 1,$$
$$U^{(i)}_{i,k+1,n}(a) = \frac{U^{(i-1)}_{i-1,k,n}(a)}{d_{i-1,k,n}} - \frac{U^{(i-1)}_{i,k,n}(a)}{d_{i,k,n}} = \frac{U^{(i-1)}_{i-1,k,n}(a)}{d_{i-1,k,n}} \neq 0,$$

where we used (6), (7), Property 1 and the induction hypothesis. Analogously we can prove (9). □

Note that the above constructed basis is not only normalized positive, but it is also a Bernstein basis.

2.2. Some definitions on T-meshes

We will now recall the definition of T-mesh and of some related objects, using the notations of [1]. Note that the concept of T-mesh we will consider here may slightly differ from other ones in the literature, such as the more general used in [21], which allows the presence not only of *T-junctions*, but of *L-junctions* and *I-junctions* as well.

Definition 1. *A T-mesh is a collection of axis-aligned rectangles $\Delta = \{R_i\}_{i=1}^N$ such that the domain $\Omega \equiv \cup_i R_i$ is connected and any pair of rectangles (which we will call cells) $R_i, R_j \in \Delta$ intersect each other only at points on their edges.*

Note that this definition does not imply that the domain Ω is rectangular and allows the presence of holes in it. Tensor-product meshes are a particular case of T-meshes. If a vertex v of a cell belonging to Δ lies in the interior of an edge of another cell, then we call it a *T-junction.*

Definition 2. *Given a T-mesh Δ, a line segment e connecting the vertices w_1 and w_2 is called edge segment if there are no vertices lying in its interior. Instead, if all the vertices lying in its interior are T-junctions and if it cannot be extended to a longer segment with the same property, then we call it a composite edge.*

In the following, we will consider T-meshes which are *regular* and have no *cycles*, in the sense of the following definitions (see [1] for more details).

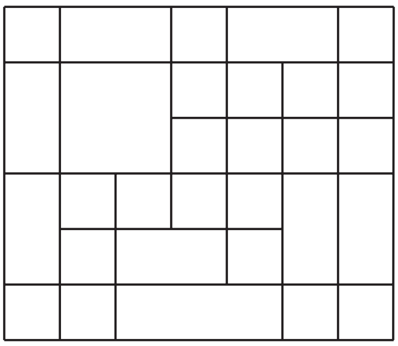


Figure 1: *An example of regular T-mesh.*

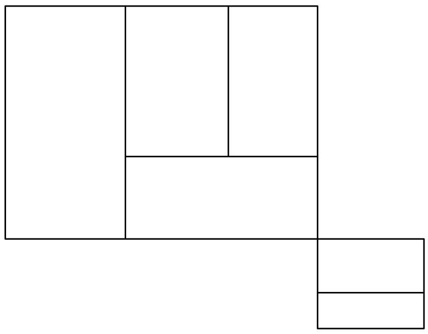


Figure 2: *An example of non-regular T-mesh.*

Definition 3. *A T-mesh Δ is regular if for each of its vertices w the set of all rectangles containing w has a connected interior.*

See Figures 1-2 for examples of regular and not regular T-meshes.

Definition 4. *Let $w_1, \ldots, w_n$ be a collection of T-junctions in a T-mesh Δ such that w_i lies in the interior of a composite edge having one of its endpoints at w_{i+1} (we assume $w_{n+1} = w_1$). Then $w_1, \ldots, w_n$ are said to form a cycle.*

See Figure 3 for an example of cycle in a T-mesh.

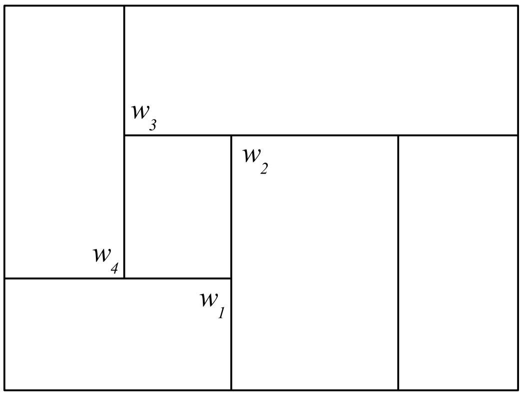


Figure 3: *The sequence* w_1, w_2, w_3, w_4 *is a cycle.*

3. Spaces of generalized splines on T-meshes

In this Section, we define the spaces of generalized splines over T-meshes, and we study their dimension by constructing a basis. The results obtained can be considered a generalization to non-polynomial splines spaces over T-meshes of the ones proved in [1] for the basic polynomial case.

3.1. Basics

Let Δ be a regular T-mesh without cycles, and let $0 \le r_1 < n_1$, $0 \le r_2 < n_2$, where r_1, r_2, n_1, n_2 are integers and $n_1, n_2 \ge 1$. We will use the notation $\mathbf{r} = (r_1, r_2)$ and $\mathbf{n} = (n_1, n_2)$.
We define the space of generalized splines over the T-mesh Δ of bi-degree $\mathbf{n}$ and smoothness $\mathbf{r}$, $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$, as

$$GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta) := \{p(s,t) \in C^{\mathbf{r}}(\Omega) : \, p|_R \in \mathscr{P}^{\mathbf{n}}_{\mathbf{u}^R,\mathbf{v}^R}(R) \; \forall R \in \Delta\}, \tag{10}$$

where $\Omega = \cup_{R \in \Delta} R$, $C^{\mathbf{r}}(\Omega)$ denotes the space of functions p such that their derivatives $D^i_s D^j_t p$ are continuous for all $0 \le i \le r_1$ and $0 \le j \le r_2$, and the space $\mathscr{P}^{\mathbf{n}}_{\mathbf{u}^R,\mathbf{v}^R}(R)$ is defined as

$$\mathscr{P}^{\mathbf{n}}_{\mathbf{u}^R,\mathbf{v}^R}(R) := \mathscr{P}^{n_1}_{u^R_1,v^R_1}([a_R, b_R]) \otimes \mathscr{P}^{n_2}_{u^R_2,v^R_2}([c_R, d_R]), \tag{11}$$

with $R := [a_R, b_R] \times [c_R, d_R]$, and $\mathbf{u}^R = (u^R_1, u^R_2)$ and $\mathbf{v}^R = (v^R_1, v^R_2)$ such that $u^R_1, v^R_1 \in C^{n_1+1}([a_R, b_R])$, $u_2, v_2 \in C^{n_2+1}([c_R, d_R])$, $\dim\big(\mathscr{P}^{n_1}_{u^R_1,v^R_1}([a_R, b_R])\big) = n_1 + 1$, $\dim\big(\mathscr{P}^{n_2}_{u^R_2,v^R_2}([c_R, d_R])\big) = n_2 + 1$, and satisfying both (2) and (3). In other words, $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$ is a space of spline functions which, restricted to each cell R, are products of functions belonging to spaces of type (1).

We introduce now on each cell R a Bernstein-Bézier representation for the elements of $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$ based on the Bernstein basis of $\mathscr{P}^{n_1}_{u^R_1,v^R_1}([a_R, b_R])$ and $\mathscr{P}^{n_2}_{u^R_2,v^R_2}([c_R, d_R])$ constructed in Theorem 1; therefore, we need to assume that (2) is satisfied both by $\mathscr{P}^{n_1}_{u^R_1,v^R_1}([a_R, b_R])$ and $\mathscr{P}^{n_2}_{u^R_2,v^R_2}([c_R, d_R])$. Let us denote by $\{B^R_{i,n_1}\}^{n_1}_{i=0}$ and $\{B^R_{j,n_2}\}^{n_2}_{j=0}$ the Bernstein basis of, respectively, $\mathscr{P}^{n_1}_{u^R_1,v^R_1}([a_R, b_R])$ and $\mathscr{P}^{n_2}_{u^R_2,v^R_2}([c_R, d_R])$, to stress the dependence of the basis on the coordinates a_R, b_R, c_R, d_R of the vertices of the cell R. For any $p \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$, we can then give on the cell R the following representation:

$$p|_R(s,t) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} c^R_{ij} B^R_{i,n_1}(s) B^R_{j,n_2}(t), \tag{12}$$

where $c^R_{ij} \in \mathbb{R}$ are suitable coefficients. Let us define the set of *domain points associated to R*:

$$\mathscr{D}_{\mathbf{n},R} := \{\xi^R_{ij}\}_{i=0,j=0}^{n_1,n_2},$$

with

$$\xi^R_{ij} := \Big(\frac{(n_1-i)a_R + ib_R}{n_1}, \frac{(n_2-j)c_R + jd_R}{n_2}\Big), \qquad i=0,\dots,n_1,\ j=0,\dots,n_2.$$

We can then define the *set of domain points* for a given T-mesh Δ as

$$\mathscr{D}_{\mathbf{n},\Delta} := \bigcup_{R\in\Delta} \mathscr{D}_{\mathbf{n},R},$$

where we assume that multiple appearances of the same point are allowed. If we set

$$B^R_\xi(s,t) := B^R_{i,n_1}(s) B^R_{j,n_2}(t), \quad \text{where } \xi^R_{ij} := \xi,$$

then, for each $R \in \Delta$, we can re-write (12) in the more compact form

$$p|_R(s,t) = \sum_{\xi\in\mathscr{D}_{\mathbf{n},R}} c_\xi B^R_\xi(s,t),$$

which we call *Bernstein-Bézier form*; we refer to the c^R_ξ as the *B-coefficients*. It is then clear that any element of the space $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$ is completely determined by a set of B-coefficients $\{c_\xi\}_{\xi\in\mathscr{D}_{\mathbf{n},\Delta}}$. Of course, not every choice of the B-coefficients corresponds to an element in the spline space, since smoothness conditions must be satisfied.

3.2. Smoothness conditions

In order to study the consequences of the smoothness conditions required for $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$ on the determination of the B-coefficients of an element of the space, first we need to recall some more concepts about domain points.

Let w be the bottom-left vertex of a cell R, and $\mu := (\mu_1, \mu_2)$ with $\mu_1 \le n_1$ and $\mu_2 \le n_2$. We call the set $\mathscr{D}^R_\mu(w) := \{\xi_{ij}\}_{i=0,j=0}^{\mu_1,\mu_2}$ the *disk of size μ around w*. The disks around the other vertices of R can be defined analogously. Moreover, we say that the points ξ^R_{ij} with $0 \le i \le \nu$ lie within a distance ν from the edge $e = \{a_R\} \times [c_R, d_R]$ and we use the notation $d(\xi^R_{ij}, e) \le \nu$. Analogous notations hold for the other edges of R.

Moreover, we can define the set of domain points

$$\mathscr{D}_\mu(w) := \bigcup_{R\in\Delta_w} \mathscr{D}^R_\mu(w),$$

where $\Delta_w \subset \Delta$ contains only the cells having w as one of their vertices and multiple appearances of a point are allowed in the union. Given a composite edge e, an edge $\tilde{e}$ lying on e and a domain point ξ of a cell which has $\tilde{e}$ as one of its edges, if $d(\xi, \tilde{e}) \le \nu$, then we write that $d(\xi, e) \le \nu$ as well.

The following lemma is a key step to be able to understand the influence of the smoothness conditions around a vertex, and it is analogous to Lemma 3.3 in [1].

Lemma 1. *Let $p \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$ and let w be a vertex of Δ. Let us consider two cells R and $\tilde{R}$ with vertices (in counter-clockwise order) w, w_2, w_3, w_4 and w, w_5, w_6, w_7, respectively. If the coefficients c_ξ, $\xi \in \mathscr{D}^R_{\mathbf{r}}(w)$ are given, then the coefficients c_η, $\eta \in \mathscr{D}^{\tilde{R}}_{\mathbf{r}}(w)$ are uniquely determined by the smoothness conditions at w.*

Proof. Let us assume that R and $\tilde{R}$ are like in Figure 4 (the proof for other configurations is analogous). Then, since we have regularity $\mathbf{r} = (r_1, r_2)$ at w, by using Property 1 we get

$$\sum_{i=0}^{h}\sum_{j=0}^{k} c^{\tilde{R}}_{ij} D^h_s B^{\tilde{R}}_{i,n_1}(a_{\tilde{R}}) D^k_t B^{\tilde{R}}_{j,n_2}(c_{\tilde{R}}) = \sum_{i=n_1-h}^{n_1}\sum_{j=n_2-k}^{n_2} c^R_{ij} D^h_s B^R_{i,n_1}(b_R) D^k_t B^R_{j,n_2}(d_R), \tag{13}$$

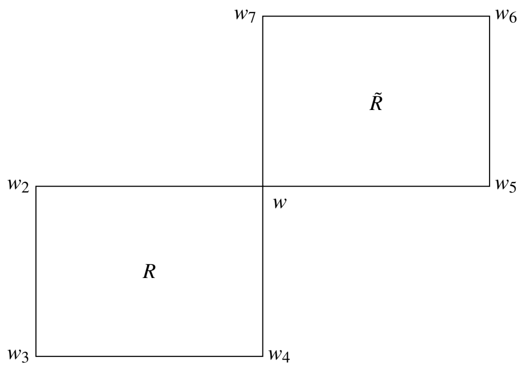


Figure 4: *The common vertex w shared by R and $\tilde{R}$, with the notation of Lemma 1.*

for $h = 0,...,r_1$, $k = 0,...,r_2$. By using Property 2 it can be shown that the system composed of equations (13), with a suitable re-ordering of the equations, is lower triangular, which proves the lemma. □

After having studied the influence of smoothness around a vertex, we now study the situation around edges. The two following lemmas can be considered generalizations of Lemma 3.5 and Lemma 3.6 in [1]. However, note that in our nonpolynomial setting Lemma 3.6 cannot be directly generalized, since having different (u_1^R, u_2^R), (v_1^R, v_2^R) in neighbouring cells leads to configurations which are significantly different from the polynomial case, as we will explain in the proof of Lemma 3. Given an edge e, we will use the following notation:

$$r_e := \begin{cases} r_1, & \text{if } e \text{ is vertical,} \\ r_2, & \text{if } e \text{ is horizontal,} \end{cases}$$

$$D_e := \begin{cases} D_s, & \text{if } e \text{ is vertical,} \\ D_t, & \text{if } e \text{ is horizontal,} \end{cases}$$

$$n_e := \begin{cases} n_2, & \text{if } e \text{ is vertical,} \\ n_1, & \text{if } e \text{ is horizontal,} \end{cases}$$

$$\{(a_e, c_e), (b_e, c_e)\} := \text{coordinates of the endpoints of } e,$$

$$\Delta_e = \{R \in \Delta : R \cap int(e) \neq \emptyset\}$$

$$u_e^R := \begin{cases} u_2^R, & \text{if } e \text{ is vertical,} \\ u_1^R, & \text{if } e \text{ is horizontal,} \end{cases}$$

$$v_e^R := \begin{cases} v_2^R, & \text{if } e \text{ is vertical,} \\ v_1^R, & \text{if } e \text{ is horizontal.} \end{cases}$$

Moreover, we will assume that for any $R \in \Delta$ and any edge e such that $R \in \Delta_e$, u_e^R, v_e^R are such that

$$\dim \mathscr{P}^{n_e}_{u_e^R, v_e^R}([a_e, b_e]) = n_e + 1. \tag{14}$$

Lemma 2. *Let e be a composite edge of Δ. Given $p \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$, for any $0 \leq j \leq r_e$, $D_e^j p|_e$ is a univariate function belonging to* $\bigcap_{R \in \Delta_e} \mathscr{P}^{n_e}_{u_e^R, v_e^R}([a_e, b_e])$.

Proof. Let us consider a horizontal composite edge e with endpoints $w_1 = (a_e, c_e)$ and $w_5 = (b_e, c_e)$ like the one showed in Figure 5, composed of the edges e_1, e_2, e_3. First, $p|_{R_1}(s, d_{R_1})$ gives the values of p both on e_1 and e_2, because both the edges belong to the same cell R_1; similarly, $p|_{R_2}(s, c_{R_2})$ gives the values of p both on e_2 and e_3, since they belong to R_2.
Since $p|_{e_2}$ belongs to $\mathscr{P}^{n_1}_{u_1^{R_1}, v_1^{R_1}}([a_{R_1}, b_{R_1}]) \cap \mathscr{P}^{n_1}_{u_1^{R_2}, v_1^{R_2}}([a_{R_2}, b_{R_2}])$, and by using assumption (14), we get that $p|_{e_1}$,

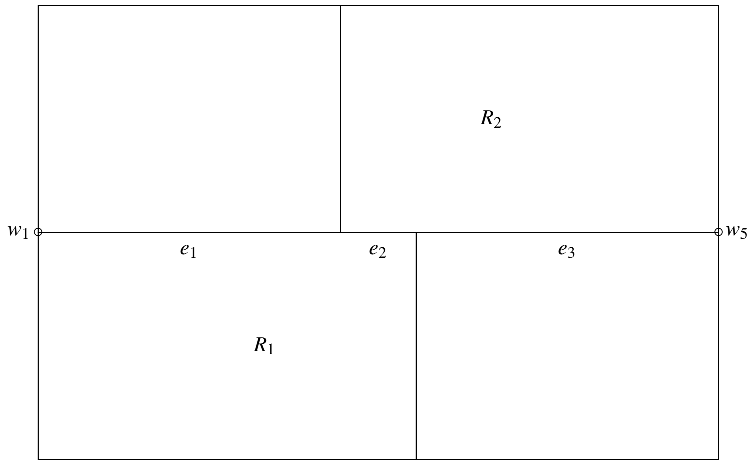


Figure 5: The cells considered in the proof of Lemma 2.

$p|_{e_2}$ and $p|_{e_3}$ coincide. These arguments can be extended to an arbitrary number of segments in a composite edge, to the case of vertical composite edges, and to derivatives of any order up to r_e. $\square$

Let us now consider a composite edge with endpoints w_1 and w_5, a cell R_e with vertices w_1, w_2, w_3, w_4, and another cell $\tilde{R}_e$ with vertices w_5, w_6, w_7, w_8. Moreover we assume that w_4 and w_6 lie on e as well (the other cases are analogous). Let us define

$$\mathscr{M}_e^k := \begin{cases} \left\{\xi_{ij}^{R_e}\right\}_{i=n_1-r_1,j=r_2+1}^{n_1,n_2-r_2-k}, & \text{if } e \text{ is vertical,} \\ \left\{\xi_{ij}^{R_e}\right\}_{i=r_1+1,j=n_2-r_2}^{n_1-r_1-k,n_2}, & \text{if } e \text{ is horizontal.} \end{cases}, \qquad k=1,2,3. \tag{15}$$

Moreover, we will use $\tilde{\mathbf{r}}_\mathbf{e}$ to denote $\mathbf{r}-(1,0)$ if e is horizontal, $\mathbf{r}-(0,1)$ if e is vertical, and $\hat{\mathbf{r}}_\mathbf{e}$ to denote $\mathbf{r}-(2,0)$ if e is horizontal, $\mathbf{r}-(0,2)$ if e is vertical. We also define, for every e:

$$d_e := \dim \bigcap_{R\in\Delta_e} \langle u_e^R, v_e^R \rangle.$$

Lemma 3. *Let e be a composite edge of the T-mesh Δ with endpoints $w_{e,a}$ and $w_{e,b}$. Let us assume that there exists a basis satisfying Properties 1 and 2 for the space $\bigcap_{R\in\Delta_e} \mathscr{P}^{n_e}_{u_e^R, v_e^R}([a_e, b_e])$. Then, the B-coefficients of a spline $p \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$ associated to domain points ξ such that $d(\xi, e) \le r_e$ are uniquely determined by the coefficients of p corresponding to the domain points belonging to one the following sets:*

- *if $d_e = 2$, $\tilde{\mathscr{M}}_e^{1,0}$;*
- *if $d_e = 1$, $\tilde{\mathscr{M}}_e^{1,1}$ or $\tilde{\mathscr{M}}_e^{2,0}$;*
- *if $d_e = 0$, $\tilde{\mathscr{M}}_e^{1,2}$, or $\tilde{\mathscr{M}}_e^{2,1}$, or $\tilde{\mathscr{M}}_e^{3,0}$;*

where

$$\begin{aligned}
\tilde{\mathscr{M}}_e^{1,0} &:= \mathscr{D}_\mathbf{r}^{R_e}(w_{e,a}) \cup \mathscr{D}_\mathbf{r}^{\tilde{R}_e}(w_{e,b}) \cup \mathscr{M}_e^1, \\
\tilde{\mathscr{M}}_e^{1,1} &:= \mathscr{D}_\mathbf{r}^{R_e}(w_{e,a}) \cup \mathscr{D}_{\tilde{\mathbf{r}}_\mathbf{e}}^{\tilde{R}_e}(w_{e,b}) \cup \mathscr{M}_e^1, \\
\tilde{\mathscr{M}}_e^{2,0} &:= \mathscr{D}_\mathbf{r}^{R_e}(w_{e,a}) \cup \mathscr{D}_\mathbf{r}^{\tilde{R}_e}(w_{e,b}) \cup \mathscr{M}_e^2, \\
\tilde{\mathscr{M}}_e^{1,2} &:= \mathscr{D}_\mathbf{r}^{R_e}(w_{e,a}) \cup \mathscr{D}_{\hat{\mathbf{r}}_\mathbf{e}}^{\tilde{R}_e}(w_{e,b}) \cup \mathscr{M}_e^1, \\
\tilde{\mathscr{M}}_e^{2,1} &:= \mathscr{D}_\mathbf{r}^{R_e}(w_{e,a}) \cup \mathscr{D}_{\tilde{\mathbf{r}}_\mathbf{e}}^{\tilde{R}_e}(w_{e,b}) \cup \mathscr{M}_e^2, \\
\tilde{\mathscr{M}}_e^{3,0} &:= \mathscr{D}_\mathbf{r}^{R_e}(w_{e,a}) \cup \mathscr{D}_\mathbf{r}^{\tilde{R}_e}(w_{e,b}) \cup \mathscr{M}_e^3.
\end{aligned}$$

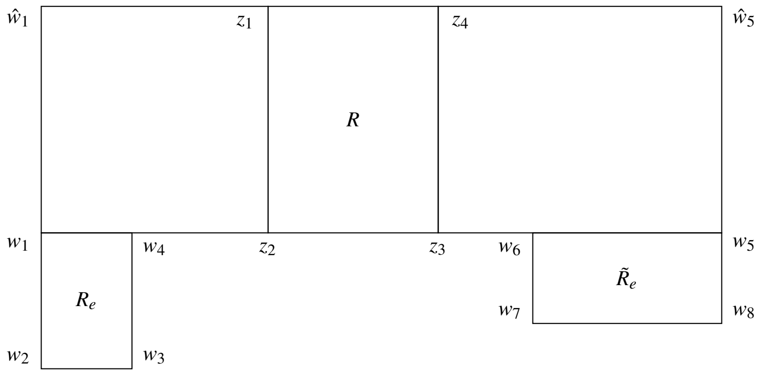


Figure 6: The cells considered in the proof of Lemma 3.

Proof. Let us consider a horizontal composite edge e as in Figure 6, with endpoints $w_{e,a} = w_1$ and $w_{e,b} = w_5$ (the proof is analogous for the vertical case). Let $R \in \Delta_e$, and let us denote its vertices of by z_1, z_2, z_3, z_4, with z_2 and z_3 lying on e. We will show that the B-coefficients corresponding to the domain points ξ belonging to $\mathscr{D}_{\mathbf{n},R}$ and such that $d(\xi, e) \leq r_2$ are uniquely determined.

Let $p \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$, and let us consider integers k and ℓ such that $1 \leq k \leq 3$, $0 \leq \ell \leq 2$, $k+\ell \leq 3$. First of all, assuming that the B-coefficients corresponding to the domain points in $\tilde{\mathscr{M}}_e^{k,\ell}$ are given, we can compute the derivatives

$$\left\{D_s^i D_t^j p(w_1)\right\}_{i=0,j=0}^{n_1-r_1-k,r_2}, \qquad \left\{D_s^i D_t^j p(w_5)\right\}_{i=0,j=0}^{r_1-\ell,r_2}. \tag{16}$$

In fact, by Property 1, the computation of these derivatives involves just the B-coefficients contained in $\tilde{\mathscr{M}}_e^{k,\ell}$. Note that, by Lemma 2, we know that $D_t^j p|_e$, $j = 0,...,r_2$, belongs to the univariate space $\bigcap_{R\in\Delta_e} \mathscr{P}^{n_1}_{u_1^R,v_1^R}([w_1,w_5])$.

If $d_e = 2$, then we set $(k,\ell) = (1,0)$, and the proof is analogous to the polynomial case.

If $d_e = 1$, by differentiating i times with respect to s, and by considering the basis of $\bigcap_{R\in\Delta_e} \mathscr{P}^{n_1}_{u_1^R,v_1^R}([w_1,w_5])$ on e satisfying Properties 1 and 2, denoted by $\{B_k\}_{k=0}^{n_1-1}$, we can write

$$D_s^i \sum_{k=0}^{n_1-1} a_{k,j} B_k(s) = D_s^i D_t^j p(s,t)|_e, \qquad j = 0,...,r_2, \tag{17}$$

If we assume to have the coefficients associated with the elements of $\tilde{\mathscr{M}}_e^{1,0}$, we can use the values (16) of derivatives in w_1 and w_5 to determine $a_{k,j}$'s from the $(n_1+1)(r_2+1)$ conditions

$$\begin{cases} D_s^i \displaystyle\sum_{k=0}^{n_1-1} a_{k,j} B_k(w_1) = D_s^i D_t^j p(w_1)|_e & i = 0,\dots,n_1-r_1-1, \\ D_s^i \displaystyle\sum_{k=0}^{n_1-1} a_{k,j} B_k(w_5) = D_s^i D_t^j p(w_5)|_e & i = 0,\dots,r_1, \end{cases} \qquad , \qquad j = 0,\dots,r_2.$$

For example, by considering $j = 0$, we obtain a linear system whose matrix is of the form

$$A = \begin{bmatrix} D_s^0 B_0(w_1) & D_s^0 B_1(w_1) & \dots & D_s^0 B_{n_1-r_1-1}(w_1) & D_s^0 B_{n_1-r_1}(w_1) & \dots & D_s^0 B_{n_1-1}(w_1) \\ D_s^1 B_0(w_1) & D_s^1 B_1(w_1) & \dots & D_s^1 B_{n_1-r_1-1}(w_1) & D_s^1 B_{n_1-r_1}(w_1) & \dots & D_s^1 B_{n_1-1}(w_1) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ D_s^{k_1} B_0(w_1) & D_s^{k_1} B_1(w_1) & \dots & D_s^{k_1} B_{n_1-r_1-1}(w_1) & D_s^{k_1} B_{n_1-r_1}(w_1) & \dots & D_s^{k_1} B_{n_1-1}(w_1) \\ D_s^{r_1} B_0(w_5) & D_s^{r_1} B_1(w_5) & \dots & D_s^{r_1} B_{n_1-r_1-1}(w_5) & D_s^{r_1} B_{n_1-r_1}(w_5) & \dots & D_s^{r_1} B_{n_1-1}(w_5) \\ D_s^{k_2} B_0(w_5) & D_s^{k_2} B_1(w_5) & \dots & D_s^{k_2} B_{n_1-r_1-1}(w_5) & D_s^{k_2} B_{n_1-r_1}(w_5) & \dots & D_s^{k_2} B_{n_1-1}(w_5) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ D_s^0 B_0(w_5) & D_s^0 B_1(w_5) & \dots & D_s^0 B_{n_1-r_1-1}(w_5) & D_s^0 B_{n_1-r_1}(w_5) & \dots & D_s^0 B_{n_1-1}(w_5) \end{bmatrix}$$

where $k_1 = n_1 - r_1 - 1$ and $k_2 = r_1 - 1$. By Properties 1 and 2, we know that the matrix A has the following sparsity structure (we mark with $\bullet$ non-zero entries, and with $\circ$ entries which could be either zero or nonzero).

$$\begin{bmatrix} \bullet & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \circ & \bullet & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \circ & \circ & \dots & \bullet & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \bullet & \circ & \circ & \dots & \circ \\ 0 & 0 & \dots & 0 & \bullet & \circ & \dots & \circ \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \bullet & \circ \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \bullet \end{bmatrix}$$

Note that there is a linear dependence, necessarily in the rows with the $\bullet$ in the same column, corresponding to derivative of order $n_1 - r_1 - 1$ w.r.t. w_1 and derivative of order r_1 w.r.t. w_5. So exactly one of them can be removed in order to obtain a square matrix, which is nonsingular because it is composed of a lower triangular upper part, and of an upper triangular lower part, with nonzero elements on the diagonal.
The same arguments hold for higher values of j, and so we do not need all the coefficients associated with the elements of $\tilde{\mathscr{M}}_e^{1,0}$: it is sufficient to know the coefficients of the elements of either $\tilde{\mathscr{M}}_e^{2,0}$ or $\tilde{\mathscr{M}}_e^{1,1}$.
If $d_e = 0$, then $D_t^j p|_e$, $j = 0,...,r_2$, belongs to the univariate space $\bigcap_{R\in\Delta_e} \mathscr{P}^{n_1}_{u_1^R, v_1^R}([w_1, w_5])$ whose basis we denote by $\{B_k\}_{k=0}^{n_1-2}$. In this case, assuming again to have all data about $\tilde{\mathscr{M}}_e^{1,0}$, the matrices of the systems determining the coefficients $a_{j,k}$'s have the following sparsity structure:

$$\begin{bmatrix} \bullet & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \circ & \bullet & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \circ & \circ & \dots & \bullet & 0 & 0 & \dots & 0 \\ \circ & \circ & \dots & \circ & \bullet & 0 & \dots & 0 \\ 0 & 0 & \dots & \bullet & \circ & \circ & \dots & \circ \\ 0 & 0 & \dots & 0 & \bullet & \circ & \dots & \circ \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \bullet & \circ \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \bullet \end{bmatrix}$$

The linear dependence is between the rows in which are considered derivatives of order $n_1 - r_1 - 2$ and $n_1 - r_1 - 1$ at w_1, and derivatives of order $r_1 - 1$ and r_1 at w_5. For every j, in order to get a square nonsingular matrix, we can delete the first two, or the last two, or the two central of these rows. These deletions corresponds respectively to considering only the sets $\tilde{\mathscr{M}}_e^{3,0}$, or $\tilde{\mathscr{M}}_e^{1,2}$, or $\tilde{\mathscr{M}}_e^{2,1}$.

By writing $p|_R$ in its Bernstein-Bézier form, we get the linear system:

$$\sum_{i=0}^{n_1} \sum_{j=0}^{r_2} c_{ij}^R D_s^h B_{i,n_1}^R(a_R) D_t^k B_{j,n_2}^R(c_R) = D_s^h D_t^k p|_R(a_R, c_R)$$

where $0 \le h \le n_1, 0 \le k \le r_2$, the unknowns are the c_{ij}^R, and the derivatives $D_s^h D_t^k p|_R(a_R, c_R)$ are known, since we have just determined $p|_e$ and its derivatives. By suitably re-ordering the indices (i, j) and (h,k) we obtain, by Property 1, a lower triangular system where the elements on the diagonal are nonzero due to Property 2. The Lemma is then proved. □

Remark 1. Note that, in order to prove Lemma 2, we do not require that $\bigcap_{R\in\Delta_e} \mathscr{P}^{n_1}_{u_1^R, v_1^R}([w_{1,e}, w_{5,e}])$ has a Bernstein-like basis (and then we do not require the conditions (2) and (3) for this space): we just need that the basis e satisfies Properties 1 and 2, which is sufficient to guarantee that the derivatives $D_s^i D_t^j p|_e$, for $i = 0,...,n_1$ and $j = 0,...,r_2$, are uniquely determined by the given data.

3.3. *Basis and dimension formula*

We will prove the construction of the basis, and a dimension formula for $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$, provided that $n_1 \geq 2r_1 + 3$ and $n_2 \geq 2r_2 + 3$. We recall the meaning of *determining set* and *minimal determining set*.

Definition 5. *Let $\mathscr{M} \subset \mathscr{D}_{\mathbf{n},\Delta}$. $\mathscr{M}$ is a* determining set *for $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$ if for any spline function p belonging to $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$ such that $c_\xi = 0, \forall \xi \in \mathscr{M} \Longrightarrow p \equiv 0$, where for any $\xi \in \mathscr{M}$, c_ξ is the corresponding B-coefficient of p. Furthermore, $\mathscr{M}$ is* minimal *if no (strict) subset of it satisfy this property.*

Let us denote by J_{NT} the set of vertices which are not T-junctions, and by C the set of composite edges of Δ. Moreover, let C_k be the subset of the composite edges $e \in C$ such that $d_e = k$, $k = 0,1,2$.
For any w in J_{NT}, let R_w be a cell with an edge e_w having an endpoint at w and such that it has maximum length among the edges with an endpoint at e_w. Moreover, let

$$\begin{aligned} \mathscr{M}_w &:= \mathscr{D}^{R_w}_{\mathbf{r}}(w), \qquad \text{for any } w \in J_{NT} \\ \mathscr{M}_R &:= \{\xi^R_{ij}\}^{n_1-r_1-1,n_2-r_2-1}_{i=r_1+1,r_2+1}, \qquad \text{for any } R \in \Delta \\ \mathscr{M} &:= \bigcup_{w\in J_{NT}} \mathscr{M}_w \cup \bigcup_{e\in C_2} \mathscr{M}^1_e \cup \bigcup_{e\in C_1} \mathscr{M}^2_e \cup \bigcup_{e\in C_0} \mathscr{M}^3_e \cup \bigcup_{R\in\Delta} \mathscr{M}_R \end{aligned} \tag{18}$$

where $\mathscr{M}^1_e, \mathscr{M}^2_e, \mathscr{M}^3_e$, are defined by (15).
The three following results of this subsection are essentially obtained by using arguments analogous to those used in [1] (they can be considered the generalization of Lemma 4.1, Lemma 4.2 and Theorem 4.3 in [1], respectively). However, we will briefly summarize their respective proofs in order to highlight the role played by some crucial assumptions about the absence of cycles in the T-mesh and about the regularity of the spline space.

Theorem 2. *The subset of domain points $\mathscr{M} \subset \mathscr{D}_{\mathbf{n},\Delta}$ is a determining set for $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$.*

Proof. In order to prove the lemma we need to show that if $p \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$, $p|_R = \sum_{\xi\in\mathscr{D}_{\mathbf{n},R}} c_\xi B^R_\xi$ for any $R \in \Delta$ with $c_\xi = 0\ \forall \xi \in \mathscr{M}$, then $p \equiv 0$. By hypothesis, for any $w \in J_{NT}$ $c_\xi = 0$ for all $\xi \in \mathscr{M}_w = D^{R_w}_{\mathbf{r}}(w)$, which implies, by Lemma 1, that $c_\xi = 0$ for all $\xi \in D_{\mathbf{r}}(w)$. Therefore, for any composite edge e with the endpoints in J_{NT}, by Lemma 3 we have $c_\xi = 0$ for all ξ such that $d(\xi, e) \leq r_e$, since by hypothesis $c_\xi = 0\ \forall \xi \in \bigcup_{e\in C_2} \mathscr{M}^1_e \cup \bigcup_{e\in C_1} \mathscr{M}^2_e \cup \bigcup_{e\in C_0} \mathscr{M}^3_e$. We determine the B-coefficients associated with the not yet considered domain points by using an iterative procedure consisting of two steps:

1. for each T-junction w on an already considered composite edge, Lemma 1 implies that $c_\xi = 0\ \forall \xi \in D_{\mathbf{r}}(w)$;
2. for each composite edge e whose endpoints have been already considered, Lemma 3 implies that $c_\xi = 0$ for all ξ such that $d(\xi, e) \leq r_e$.

Since the T-meshes has no cycles, this procedure stops after having considered all the vertices and edges. Then, all the B-coefficients corresponding to domain points within a distance r_e from any edge e are determined and are zero. The remaining coefficients are zeros as well, since they correspond to domain points ξ whose distance from any edge e is greater than r_e, that is, $\xi \in \cup_{R\in\Delta} \mathscr{M}_R$. □

Lemma 4. *For every $\xi \in \mathscr{M}$, there is one and only one $\psi_\xi \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$*

$$\gamma_\eta \psi_\xi = \delta_{\xi,\eta}, \qquad \eta \in \mathscr{M}, \tag{19}$$

where $\delta_{\xi,\eta}$ is the Kronecker delta and, for any $\eta \in \mathscr{D}_{\mathbf{n},\Delta}$, $\gamma_\eta : GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta) \to \mathbb{R}$ is the functional defined by

$$\gamma_\eta p = c_\eta, \qquad \text{with } c_\eta \text{ B-coefficient of } p \text{ associated to } \eta, \quad p \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta). \tag{20}$$

Proof. For any $\xi \in \mathscr{M}$, ψ_ξ can be constructed as follows: we set $c_\eta = \delta_{\xi,\eta}$, and then we determine the remaining coefficients by using the same procedure as in the proof of Theorem 2. Note that this way to determine coefficients does not lead to inconsistencies, since we assumed $n_1 \geq 2r_1 + 3$ and $n_2 \geq 2r_2 + 3$, which implies that the disks of size $\mathbf{r} = (r_1, r_2)$ centered at the vertices do not intersect. □

Theorem 3. *The subset of domain points $\mathscr{M} \subset \mathscr{D}_{\mathbf{n},\Delta}$ is a minimal determining set for $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$, the set $\{\psi_\xi\}_{\xi\in\mathscr{M}}$ is a basis for $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$, and*

$$\begin{aligned}\dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)) \;=\;& (r_1+1)(r_2+1)J_{NT}(\Delta)+(r_2+1)(n_1-2r_1-1)E^2_{hor}(\Delta)\\ &+(r_2+1)(n_1-2r_1-2)E^1_{hor}(\Delta)+(r_2+1)(n_1-2r_1-3)E^0_{hor}(\Delta)\\ &+(r_1+1)(n_2-2r_2-1)E^2_{ver}(\Delta)+(r_1+1)(n_2-2r_2-2)E^1_{ver}(\Delta)\\ &+(r_1+1)(n_2-2r_2-3)E^0_{ver}(\Delta)+(n_1-2r_1-1)(n_2-2r_2-1)N(\Delta)\end{aligned} \tag{21}$$

where

$J_{NT}(\Delta)$ *:= number of vertices of* Δ *which are not T-junctions,*
$E^i_{hor}(\Delta)$ *:= number of horizontal composite edges of* Δ *with* $d_e = i$*,*
$E^i_{ver}(\Delta)$ *:= number of vertical composite edges of* Δ *with* $d_e = i$*,*
$N(\Delta)$ *:= number of cells.*

Proof. The set of functions $\{\psi_\xi\}_{\xi\in\mathscr{M}}$ is a basis for $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$. In fact, (19) implies that they are linearly independent and therefore $\dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)) \geq |\mathscr{M}|$. On the other hand, since $\mathscr{M}$ is a determining set we have $\dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)) \leq |\mathscr{M}|$, and therefore we must conclude that $\dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)) = |\mathscr{M}|$ and that $\{\psi_\xi\}_{\xi\in\mathscr{M}}$ is a basis. Then, $\mathscr{M}$ is a minimal determining set and the formula for $\dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta))$ is obtained from (18). □

Remark 2. From the dimension formula of Theorem 3, it is clear that the dimension of the spline space depends on the dimensions d_e of the spaces $\cap_{R\in\Delta_e}\langle u^R_e, v^R_e\rangle$, $e \in C$. In particular, if for any composite edge e, $d_e \geq 1$, then we can relax the conditions on regularity and order, that is, it is sufficient to assume that $n_1 \geq 2r_1+2$, $n_2 \geq 2r_2+2$, instead of $n_1 \geq 2r_1+3$, $n_2 \geq 2r_2+3$. Similarly, if for any composite edge $d_e = 2$ holds, we can further relax the above conditions and replace them with $n_1 \geq 2r_1+1$, $n_2 \geq 2r_2+1$, which are exactly the same conditions required in the polynomial case.

Lemma 5. *The elements of the basis ψ_ξ, $\xi \in \mathscr{M}$, form a partition of the unity.*

Proof. For the spline $p = \sum_{\xi\in\mathscr{M}}\psi_\xi$ we have $\gamma_\eta p = 1$ for any $\eta \in \mathscr{M}$. Note that, since the local Bernstein-like basis $\{B^R_{i,n_1}(s)B^R_{j,n_2}(t)\}_{i=0,\dots,n_1,j=0,\dots,n_2}$ satisfy the partition of unity, setting all the B-coefficients c^R_ξ to 1, $\xi \in \mathscr{D}_{\mathbf{n},R}$, $R \in \Delta$, gives the constant function 1, which belongs to $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$. In other words, $\gamma_\eta 1 = 1$ for any $\eta \in \mathscr{D}_{\mathbf{n},\Delta}$. On the other hand, we know that the B-coefficient associated to the points of the minimal determining set uniquely determine an element of $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$. Then, we must have $p = \sum_{\xi\in\mathscr{M}}\psi_\xi = 1$. □

3.4. Examples

Let $\mathbf{n} = (5,5), \mathbf{r} = (1,1)$, an let us consider the T-mesh Δ in Figure 7 and the spline spaces over it

$$\begin{aligned}&S_1 := GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta) = \{p(s,t) \in C^{\mathbf{r}}(\Omega) : p|_R \in \mathscr{P}^{\mathbf{n}}_{\mathbf{u}^R,\mathbf{v}^R}(R)\ \forall R \in \Delta\},\\ &\mathbf{u}^{R_i} = (\cosh(3s),\cosh(3t)),\quad \mathbf{v}^{R_i} = (\sinh(3s),\sinh(3t)),\qquad i = 1,2,\dots,7,\end{aligned} \tag{22}$$

$$\begin{aligned}&S_2 := GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta) = \{p(s,t) \in C^{\mathbf{r}}(\Omega) : p|_R \in \mathscr{P}^{\mathbf{n}}_{\mathbf{u}^R,\mathbf{v}^R}(R)\ \forall R \in \Delta\},\\ &\mathbf{u}^{R_1} = (\cos(1.9s),\cos(1.9t)),\quad \mathbf{v}^{R_1} = (\sin(1.9s),\sin(1.9t)),\\ &\mathbf{u}^{R_4} = (\cosh(3s),\cosh(3t)),\quad \mathbf{v}^{R_4} = (\sinh(3s),\sinh(3t)),\\ &\mathbf{u}^{R_i} = (s^4,t^4),\quad \mathbf{v}^{R_i} = (s^5,t^5),\qquad i = 2,3,5,6,7,\end{aligned} \tag{23}$$

$$\begin{aligned}&S_3 := GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta) = \{p(s,t) \in C^{\mathbf{r}}(\Omega) : p|_R \in \mathscr{P}^{\mathbf{n}}_{\mathbf{u}^R,\mathbf{v}^R}(R)\ \forall R \in \Delta\},\\ &\mathbf{u}^{R_i} = (s^4,t^4),\quad \mathbf{v}^{R_i} = (s^5,t^5),\qquad i = 1,2,\dots,7.\end{aligned} \tag{24}$$

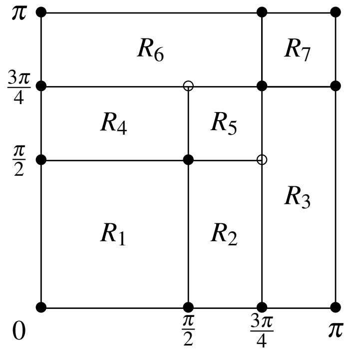


Figure 7: *T-mesh, where black circles represent vertices belonging to J_{NT} and empty circles are T-junctions.*

In other words, S_1 is a generalized spline space locally spanned by hyperbolic and polynomial functions, S_2 is locally spanned by trigonometric and polynomial, hyperbolic and polynomial or only polynomial functions, while S_3 is a polynomial spline space over the T-mesh Δ.

Note that in the three cases $E^1_{hor}(\Delta) = E^1_{ver}(\Delta) = 0$, but for S_1 and S_3 we have $E^2_{hor}(\Delta) + E^2_{ver}(\Delta) = 18$, $E^0_{hor}(\Delta) + E^0_{ver}(\Delta) = 0$, while for S_2 we have $E^2_{hor}(\Delta) + E^2_{ver}(\Delta) = 14$, $E^0_{hor}(\Delta) + E^0_{ver}(\Delta) = 4$. In fact, for S_3 there are 4 composite edges with $d_e = 0$, that is

$$\begin{aligned}
&\text{segment with endpoints}(0, 3/4\pi) \text{ end } (3/4\pi, 3/4\pi),\\
&\text{segment with endpoints}(0, \pi/2) \text{ end } (\pi/2, \pi/2),\\
&\text{segment with endpoints}(\pi/2, \pi/2) \text{ end } (\pi/2, 3/4\pi),\\
&\text{segment with endpoints}(\pi/2, 0) \text{ end } (\pi/2, \pi/2).
\end{aligned}$$

Then, by Theorem 3 we get $\dim(S_1) = \dim(S_3) = 148$ and $\dim(S_2) = 132$. In all the cases (included the polyonomial one, see [1]), the basis functions ψ_ξ, $\xi \in \mathscr{M}$, can be determined by setting to 1 the B-coefficient corresponding to one point of the respective minimal determining set $\mathscr{M}$, to 0 the B-coefficients of the other points of $\mathscr{M}$, and then computing the remaining coefficients by using the scheme described in the proof of Theorem 2.

It is worth stressing that, in spite of the different dimension, by (15) and (18) the minimal determining set for S_2 is a subset of the ones for S_1 and S_3 (which coincide). Therefore, for the three cases there are several basis functions which are associated to the same domain points and can be compared (see Figures 9 and 8). From the actual computation of their values, it is evident that the elements of the basis are not necessarily non-negative (see, for example, the basis function $\psi_{\xi^{R_1}_{14}}$ shown in Figure 8). Moreover, we observe that some elements of the global basis coincide with elements of a local basis. For example, in Figure 9 $\psi_{\xi^{R_4}_{22}}$ is both an element of the local basis in the cell R_4 and an element of the global basis.

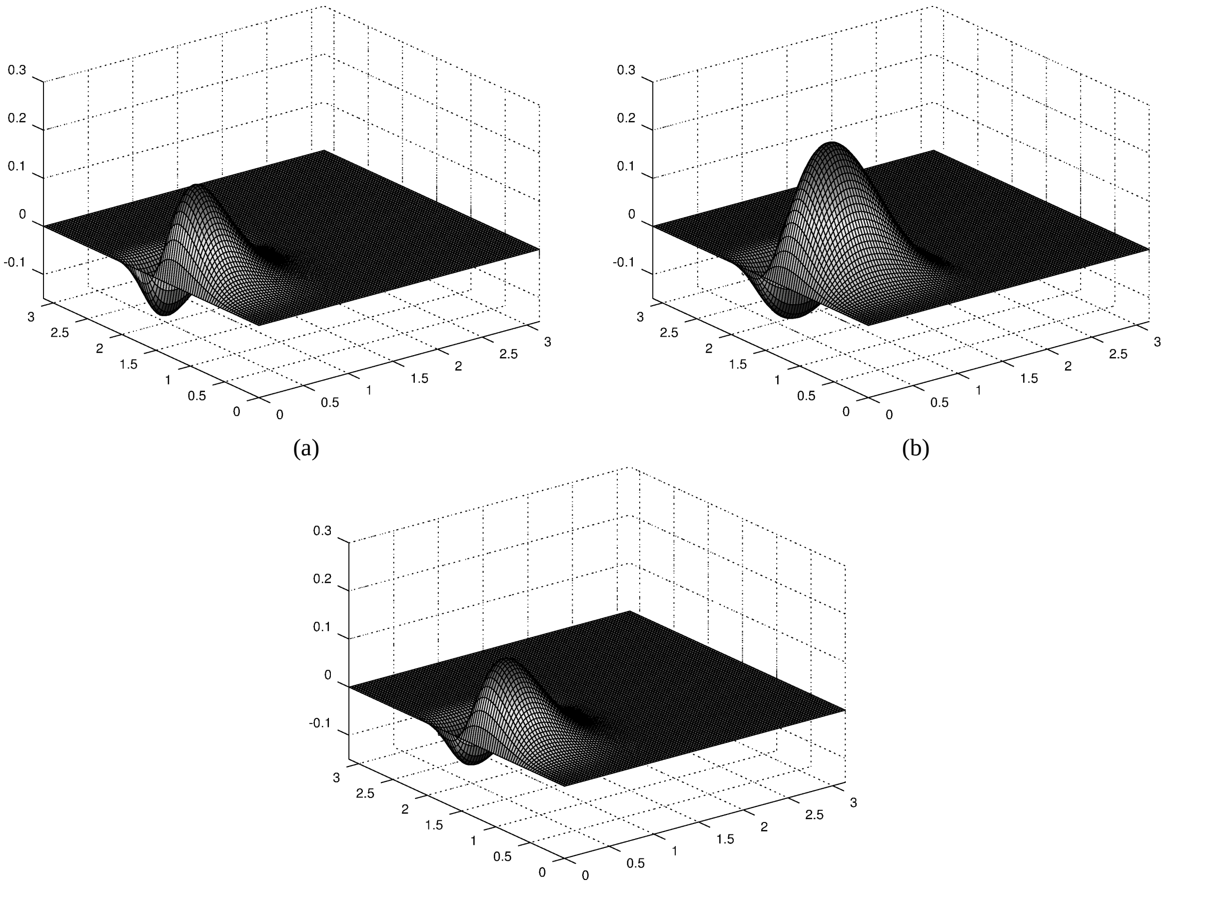
Finally, let us show another example: we consider the T-mesh Δ in Figure 10 and the corresponding spline space $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$, with $\mathbf{n} = (3,3), \mathbf{r} = (1,1), \mathbf{u}^R = (\cos(s), \cos(t))$, $\mathbf{v}^R = (\sin(s), \sin(t))$, for any $R \in \Delta$. This example allows us to show that the basis is not guaranteed to have a local support. In fact, we can observe that, for example, the basis function $\psi_{\xi^{R_2}_{00}}$ takes non-zero values in all the cells of the T-mesh (see Figure 10).

3.5. T-mesh refinement and merging

Two key features of T-meshes are the possibility of local refinement and the ability to easily merge two T-meshes (and the corresponding surfaces). We will then discuss how the space dimension changes when we refine a T-mesh and when we merge two T-meshes, using an approach analogous to [7], where such computations were done for the corresponding polynomial spaces.

3.5.1. Edge insertion

While for a tensor-product mesh inserting a new knot (in either direction) means inserting an entire row or columns of knots in the mesh, in T-meshes we can insert a single edge subdividing only one cell into two smaller



(a) (b)

(c)

Figure 8: *The global basis element associated to the domain point* $\xi_{14}^{R_1}$ *for (a) the space* (22)*, (b) the space* (23) *and (c) the polynomial space* (24)*. Note that for all the considered cases it is not a positive function.*

cells.

In the following, we assume that (1) and (2) always hold, both before and after a knot insertion. Moreover, at each refinement we will add a new edge splitting an existing cell into two parts, but we will not introduce any new non-polynomial functions: in the two new cells the considered non-polynomial functions are the same as in the original cell, so that, globally, the new spline space contains the previous one (the spaces are nested).

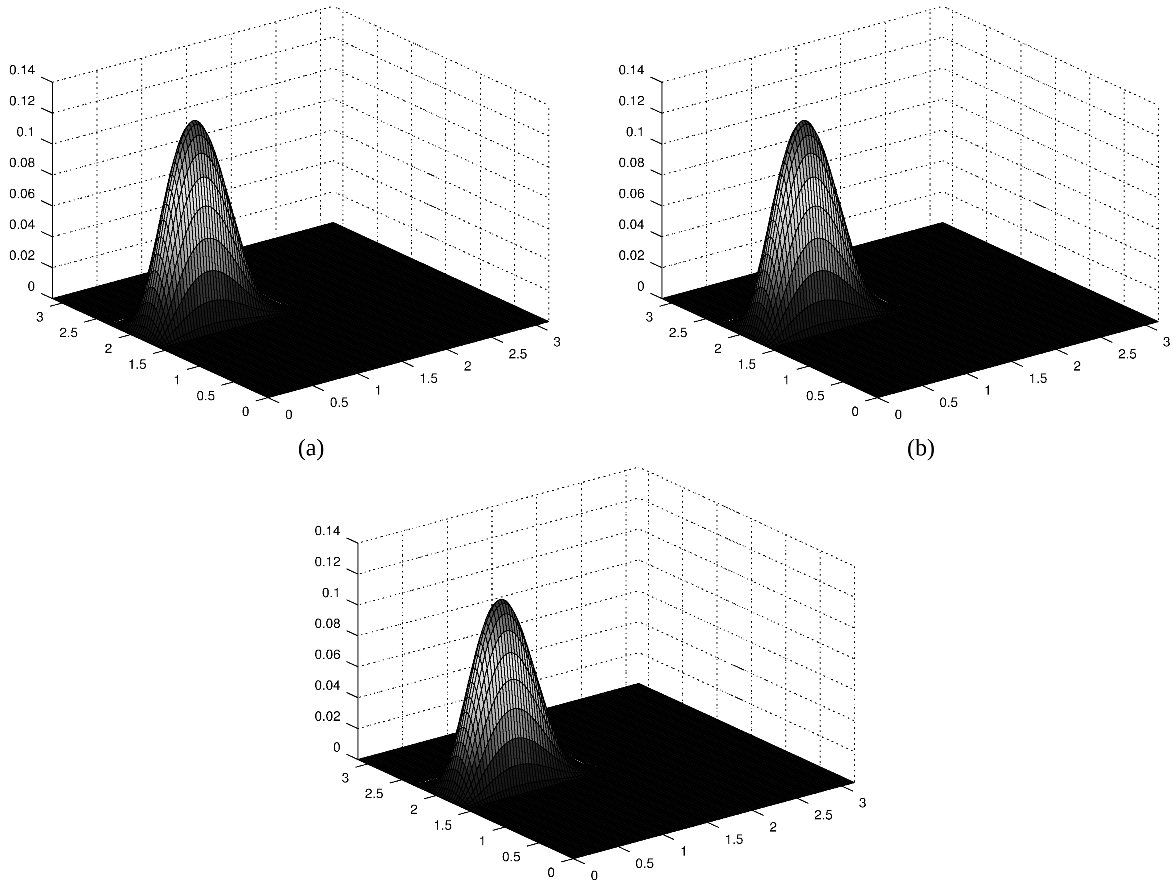
We consider three possible cases of edge insertion.

- Case (a) (see Figure 11(a)). The edge insertion adds two new T-junctions and one new composite edge (the inserted edge itself). Since in the dimension formula (21) only the number of vertices which are not T-junctions is used ($J_{NT}(\Delta)$), the new vertices do not produce any change in the dimension, while the new composite edge does. Note that for such composite edge $d_e = 2$, since we assumed that the refinement generates nested spaces. Then, if we denote by $\tilde{\Delta}$ the T-mesh obtained by inserting the edge in Δ, we have that, if the inserted edge is horizontal,

$$\begin{aligned}\dim(GS_{\mathbf{u},\mathbf{v}}^{\mathbf{n},\mathbf{r}}(\tilde{\Delta})) &= \dim(GS_{\mathbf{u},\mathbf{v}}^{\mathbf{n},\mathbf{r}}(\Delta)) \\ &\quad +(n_2-2r_2-1)(n_1-2r_1-1)+(r_2+1)(n_1-2r_1-1),\end{aligned}$$

  while, if the edge inserted is vertical,

$$\begin{aligned}\dim(GS_{\mathbf{u},\mathbf{v}}^{\mathbf{n},\mathbf{r}}(\tilde{\Delta})) &= \dim(GS_{\mathbf{u},\mathbf{v}}^{\mathbf{n},\mathbf{r}}(\Delta)) \\ &\quad +(n_1-2r_1-1)(n_2-2r_2-1)+(r_1+1)(n_2-2r_2-1).\end{aligned}$$



(a) (b) (c)

Figure 9: *In (a) the element* $\psi_{\xi_{22}^{R_4}}$ *of the basis of the space* (22) *coincides with an element of the local Bernstein-Bézier basis. The same behaviour holds for the corresponding element of the basis of (b) the space* (23) *(and therefore the basis function is exactly the same, since the two spaces in* R^4 *are spanned by the same functions) and of (c) the polynomial spline space* (24).

- Case (b) (see Figure 11(b)). The edge insertion adds one new T-junction, one new vertex which is not a T-junction and one new composite edge (the inserted edge itself). Moreover, note that the inserted edge splits into two new edges an edge in the opposite direction: the values of d_e for these two parts after splitting could be different from the value of d_e for the original edge (they could not be lower, but they could be higher; see the example of Figure 12(a)). Let $\Delta(\Sigma d)$ denote the difference between the sum of the values of d_e after splitting, and the value of d_e before. Then, if the inserted edge is horizontal, we have

$$\begin{aligned}\dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\tilde{\Delta})) &= \dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)) \\ &+(n_2-2r_2-1)(n_1-2r_1-1)+(r_2+1)(n_1-2r_1-1) \\ &+(r_1+1)(n_2-2r_2-3+\Delta(\Sigma d))+(r_1+1)(r_2+1)\end{aligned}$$

  while, if the edge inserted is vertical,

$$\begin{aligned}\dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\tilde{\Delta})) &= \dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)) \\ &+(n_1-2r_1-1)(n_2-2r_2-1)+(r_1+1)(n_2-2r_2-1) \\ &+(r_2+1)(n_1-2r_1-3+\Delta(\Sigma d))+(r_1+1)(r_2+1)\end{aligned}$$

- Case (c) (see Figure 11(c)). The edge insertion adds two new vertices which are not T-junctions. Moreover, in this case two edges in the opposite direction are split into two edges, each of them possibly with a different

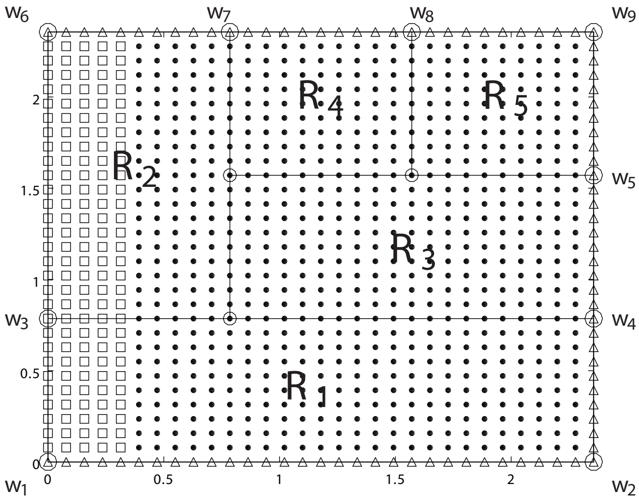


Figure 10: *Sign of $\psi_{\xi_{00}^{R_2}}$: at evaluation points marked with a square $\psi_{\xi_{00}^{R_2}}$ takes positive values, at evaluation points marked with a circle it takes negative values, and at evaluation points marked with a triangle it is zero.*

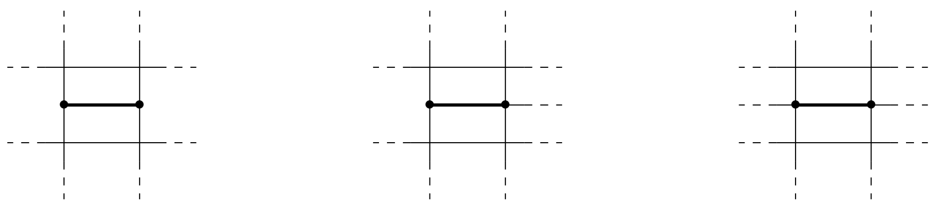
value of d_e (see Figure 12(b)). If the inserted edge is horizontal we have

$$\begin{aligned}\dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\tilde{\Delta})) &= \dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)) \\ &\quad +(n_2-2r_2-1)(n_1-2r_1-1)+(r_2+1)(n_1-2r_1-1) \\ &\quad +(r_1+1)(2n_2-4r_2-6+\Delta(\Sigma d))+2(r_1+1)(r_2+1)\end{aligned}$$

while, if the edge inserted is vertical

$$\begin{aligned}\dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\tilde{\Delta})) &= \dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)) \\ &\quad +(n_1-2r_1-1)(n_2-2r_2-1)+(r_1+1)(n_2-2r_2-1) \\ &\quad +(r_2+1)(2n_1-4r_1-6+\Delta(\Sigma d))+2(r_1+1)(r_2+1)\end{aligned}$$

where, in this case, $\Delta(\Sigma d)$ denotes the sum of the differences between the values of d_e for the two split edges and the values of d_e of the new edges after the split.



(a) (b) (c)

Figure 11: *Edge insertion where (a) J_{NT} remains the same, (b) J_{NT} increases by 1, (c) J_{NT} increases by 2.*

3.5.2. Merging two T-meshes

We consider two T-meshes Δ_1 and Δ_2 having a common boundary segment. The new T-mesh $\Delta_1 \cup \Delta_2$ is obtained by the union of the sets of cells of Δ_1 and Δ_2. In the following, we assume that (1) and (2) always hold, both before and after merging.

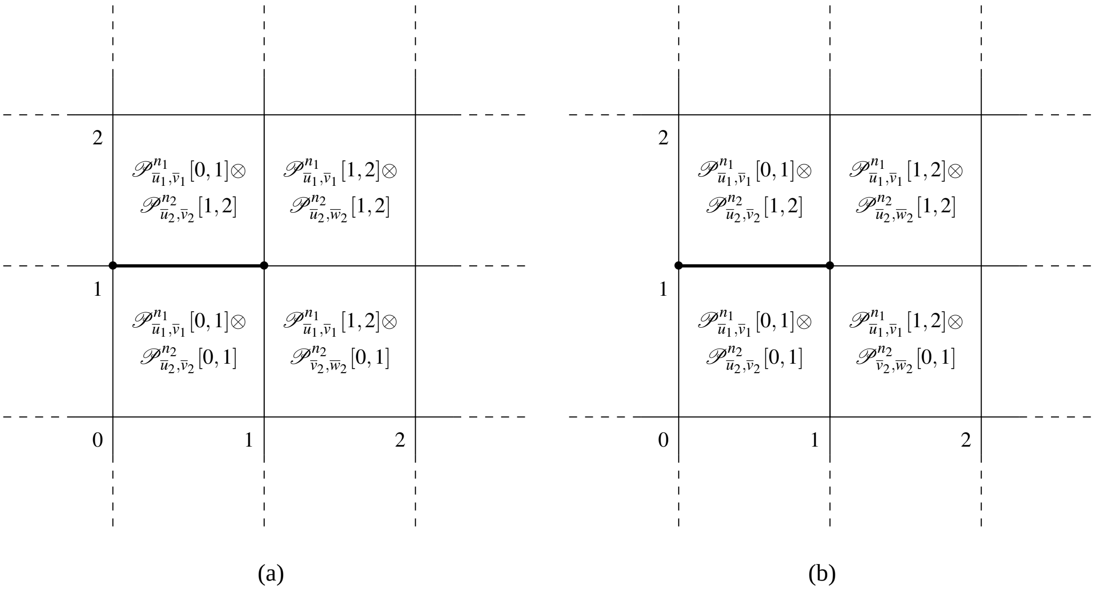


Figure 12: *Examples where a composite edge for which $d_e = 0$ is split, by the edge insertion (represented by the thick black segment), into two composite edges for which $d_e = 1$ (in (a) J_{NT} increases by 2, while in (b) J_{NT} increases by 1).*

First, we observe that

$$N(\Delta_1 \cup \Delta_2) = N(\Delta_1) + N(\Delta_2).$$

Let us denote by W_1^b the number of vertices of Δ_1 along the common boundary which are not corner vertices, and by W_2^b the same quantity for Δ_2 (such vertices are not T-junctions, respectively in Δ_1 and Δ_2, since they are on the boundary). We denote instead by W^I the number of the common boundary vertices which are not corner vertices. So there are $W_1^b + W_2^b + 2 - W^I$ vertices which, after the merging, become T-junctions and then

$$J_{NT}(\Delta_1 \cup \Delta_2) = J_{NT}(\Delta_1) + J_{NT}(\Delta_2) - (W_1^b + W_2^b + 2 - W^I).$$

There are $W_1^b + 1$ edges of Δ_1 on the boundary segment in common with Δ_2, and $W_2^b + 1$ edges of Δ_2 on the boundary segment in common with Δ_1. These edges are composite edges for which $d_e = 2$, since the vertices on the boundary of a T-mesh are not considered T-junctions. After having merged the two T-meshes, on the common boundary there are W^I vertices which are not T-junctions, which means that there are $W^I + 1$ composite edges, which can have different values of d_e: let us say that E_I^i of them are composite edges with $d_e = i$, for $i = 0, 1, 2$, and so $E_I^0 + E_I^1 + E_I^2 = W^I + 1$. If the common boundary segment is horizontal, then we have

$$E_{hor}^2(\Delta_1 \cup \Delta_2) = E_{hor}^2(\Delta_1) + E_{hor}^2(\Delta_2) - (W_1^b + W_2^b + 2 - E_I^2),$$
$$E_{hor}^i(\Delta_1 \cup \Delta_2) = E_{hor}^i(\Delta_1) + E_{hor}^i(\Delta_2) + E_I^i, \quad i = 0, 1,$$

and, as a consequence, the new dimension of is :

$$\begin{aligned} \dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta_1 \cup \Delta_2)) &= \dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta_1)) + \dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta_2)) \\ &- (r_1+1)(r_2+1)(W_1^b + W_2^b + 2 - W^I) \\ &- (r_2+1)(n_1 - 2r_1 - 1)(W_1^b + W_2^b + 2 - E_I^2) \\ &+ (r_2+1)(n_1 - 2r_1 - 2)E_I^1 + (r_2+1)(n_1 - 2r_1 - 3)E_I^0. \end{aligned}$$

If the common boundary segment is vertical, we have

$$E_{ver}^2(\Delta_1 \cup \Delta_2) = E_{ver}^2(\Delta_1) + E_{ver}^2(\Delta_2) - (W_1^b + W_2^b + 2 - E_I^2),$$
$$E_{ver}^i(\Delta_1 \cup \Delta_2) = E_{ver}^i(\Delta_1) + E_{ver}^i(\Delta_2) + E_I^i, \quad i = 0, 1,$$

and then

$$\begin{aligned}
\dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta_1 \cup \Delta_2)) &= \dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta_1)) + \dim(GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta_2)) \\
&- (r_1+1)(r_2+1)(W_1^b + W_2^b + 2 - W^I) \\
&- (r_1+1)(n_2-2r_2-1)(W_1^b + W_2^b + 2 - E_I^2) \\
&+ (r_1+1)(n_2-2r_2-2)E_I^1 + (r_1+1)(n_2-2r_2-3)E_I^0
\end{aligned}$$

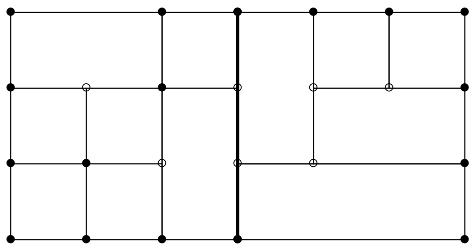


Figure 13: Example of two merged T-meshes (common boundary represented by the thick black segment)

4. Approximation power

This section is devoted to the study of the approximation properties of the generalized spline spaces over T-meshes. We will prove these properties for the case where the couples of nonpolynomial functions $\mathbf{u}^R$ and $\mathbf{v}^R$ are the same in each cell R, that is, $\mathbf{u}^R = (u_1^R, u_2^R) = (u_1, u_2) = \mathbf{u}$ and $\mathbf{v}^R = (v_1^R, v_2^R) = (v_1, v_2) = \mathbf{v}$ for any $R \in \Delta$. Moreover, we will assume that (u_1, u_2) and (v_1, v_2) give a space $\mathscr{P}^{\mathbf{n}}_{\mathbf{u},\mathbf{v}}([\min_{R\in\Delta} a_R, \max_{R\in\Delta} b_R] \times [\min_{R\in\Delta} c_R, \max_{R\in\Delta} d_R])$ invariant under translations. More precisely, we assume that, for any $(s_0, t_0) \in \mathbb{R}^2$,

$$\begin{aligned}
&\psi(s,t) \in \mathscr{P}^{\mathbf{n}}_{\mathbf{u},\mathbf{v}}([\min_{R\in\Delta} a_R, \max_{R\in\Delta} b_R] \times [\min_{R\in\Delta} c_R, \max_{R\in\Delta} d_R]) \\
\Longrightarrow\ &\psi(s-s_0, t-t_0) \in \mathscr{P}^{\mathbf{n}}_{\mathbf{u},\mathbf{v}}([\min_{R\in\Delta} a_R, \max_{R\in\Delta} b_R] \times [\min_{R\in\Delta} c_R, \max_{R\in\Delta} d_R]),
\end{aligned} \tag{25}$$

or, equivalently,

$$\begin{aligned}
\psi(s) \in \mathscr{P}^{n_1}_{u_1,v_1}([\min_{R\in\Delta} a_R, \max_{R\in\Delta} b_R]) &\implies \psi(s-s_0) \in \mathscr{P}^{n_1}_{u_1,v_1}([\min_{R\in\Delta} a_R, \max_{R\in\Delta} b_R]), \\
\psi(t) \in \mathscr{P}^{n_2}_{u_2,v_2}([\min_{R\in\Delta} c_R, \max_{R\in\Delta} d_R]) &\implies \psi(t-t_0) \in \mathscr{P}^{n_2}_{u_2,v_2}([\min_{R\in\Delta} c_R, \max_{R\in\Delta} d_R]).
\end{aligned}$$

In order to better understand what this assumption actually means, we observe that the results in [22] (see Section 3) imply that a space of type $\mathscr{P}^n_{u,v}([a,b])$, $n \geq 2$, invariant under translations must satisfy

$$\psi(s) \in \mathscr{P}^n_{u,v}([a,b]) \Longrightarrow \psi'(s) \in \mathscr{P}^{n-1}_{u,v}([a,b]). \tag{26}$$

By using elementary arguments of the theory of ordinary differential equations, we obtain that, in order to satisfy (26) (and (2)-(3) as well), both u_1, v_1 and u_2, v_2 must be chosen in one of the following ways:

- $u(s) = e^{\lambda s}$, $v(s) = e^{\mu s}$, with $\lambda, \mu \in \mathbb{R}$, $\lambda \neq \mu$;
- $u(s) = e^{\lambda s}$, $v(s) = s e^{\lambda s}$;
- $u(s) = e^{\alpha s}\cos(\beta s)$, $v(s) = e^{\alpha s}\sin(\beta s)$, with $\alpha, \beta \in \mathbb{R}$ and $\beta(b-a) < \pi$.

It can be easily verified that with any of the above choices the corresponding space is invariant under translations. As a consequence, the assumption (25) is equivalent to choosing (u_1, u_2) and (v_1, v_2) as mentioned above. Moreover, note that in this case the condition (14) is satisfied for any T-mesh.

Remark 3. It is easy to verify that all the possible choices of u and v reported above satisfy not only (26) but also

$$\psi(s) \in \mathscr{P}^n_{u,v}([a,b]) \Longrightarrow \int \psi(s)ds \in \mathscr{P}^{n+1}_{u,v}([a,b]). \tag{27}$$

This leads to generalized spline spaces $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$ satisfying

$$\begin{aligned}
\psi(s,t) \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta) \quad &\Longrightarrow \quad D_s\psi(s,t) \in GS^{\tilde{\mathbf{n}}_\mathbf{s},\tilde{\mathbf{r}}_\mathbf{s}}_{\mathbf{u},\mathbf{v}}(\Delta)\\
\psi(s,t) \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta) \quad &\Longrightarrow \quad \int \psi(s,t)ds \in GS^{\hat{\mathbf{n}}_\mathbf{s},\hat{\mathbf{r}}_\mathbf{s}}_{\mathbf{u},\mathbf{v}}(\Delta)\\
\psi(s,t) \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta) \quad &\Longrightarrow \quad D_t\psi(s,t) \in GS^{\tilde{\mathbf{n}}_\mathbf{t},\tilde{\mathbf{r}}_\mathbf{t}}_{\mathbf{u},\mathbf{v}}(\Delta)\\
\psi(s,t) \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta) \quad &\Longrightarrow \quad \int \psi(s,t)dt \in GS^{\hat{\mathbf{n}}_\mathbf{t},\hat{\mathbf{r}}_\mathbf{t}}_{\mathbf{u},\mathbf{v}}(\Delta),
\end{aligned}$$

where $\tilde{\mathbf{n}}_\mathbf{s} = (n_1-1,n_2)$, $\hat{\mathbf{n}}_\mathbf{s} = (n_1+1,n_2)$, $\tilde{\mathbf{n}}_\mathbf{t} = (n_1,n_2-1)$, $\hat{\mathbf{n}}_\mathbf{t} = (n_1,n_2+1)$, and $\tilde{\mathbf{r}}_\mathbf{s} = (r_1-1,r_2)$, $\hat{\mathbf{r}}_\mathbf{s} = (r_1+1,r_2)$, $\tilde{\mathbf{r}}_\mathbf{t} = (r_1,r_2-1)$, $\hat{\mathbf{r}}_\mathbf{t} = (r_1,r_2+1)$. In other words, we get spaces whose elements have derivatives and integrals belonging to spaces of the same type. Such nice behaviour with respect to the fundamental derivation and integration operators is of a certain interest in some applications, in particular in isogeometric analysis (see, e.g., [13], [14], [18]). Moreover, we observe that noteworthy cases of generalized spline spaces allowing to exactly reproduce certain shapes (conic sections, helices, cycloids, catenaries; see also [18]), such as $u(s) = \cos(\beta s)$, $v(s) = \sin(\beta s)$ and $u(s) = \cosh(\lambda s)$, $v(s) = \sinh(\lambda s)$, satisfy the invariance under translations.

We will obtain the approximation order by using similar arguments to the ones used in [1], and introducing a new suitable quasi-interpolant operator. In fact, the local approximants used in [1], that is, the averaged Taylor expansions, cannot be simply generalized to our non-polynomial case. Moreover, also the results on the approximation power obtained in [6] for the univariate case, by using Hermite interpolation in spaces of type $\mathscr{P}^n_{u,v}([a,b])$, cannot be directly extended to the bivariate case, due to the difficulty to find a suitable differential operator and the corresponding Green's function needed to construct a non-polynomial Taylor expansion. For these reasons, we adopt an alternative approach: we construct a bivariate Hermite interpolant belonging to the spline space, whose existence is rigorously proved by using the assumption (2) and (3). This also allows us to obtain an approximation order, which is essentially the same as in polynomial case.

Given a function $f \in C^{\mathbf{n+1}}(\Omega)$ and $(s_0,t_0) \in (a,b)\times(c,d)$, we define the interpolant $Q_L(f;s_0,t_0)(s,t)$ as the function satisfying the two following conditions

1. it belongs to $\mathscr{P}^{\mathbf{n}}_{\mathbf{u},\mathbf{v}}([a,b]\times[c,d])$,
2. its polynomial expansion of coordinate bi-degree (n_1,n_2) coincides with the polynomial expansion of f of the same bi-degree, that is, $Q_L(f;s_0,t_0)(s,t)$ is a Hermite interpolant of coordinate bi-degree (n_1,n_2).

Since $Q_L(f;s_0,t_0)$ is a Hermite interpolant, the Taylor expansion of the difference $f - Q_L(f;s_0,t_0)$ does not contain any term of degree smaller than or equal to k, where $k := \min\{n_1,n_2\}$, and then $\|f - Q_L(f;s_0,t_0)\| = O(h^{k+1})$, where $h := \mathrm{diam}([a,b]\times[c,d])$.

In order to show that $Q_L(f;s_0,t_0)(s,t)$ exists and is unique for any $f \in C^{\mathbf{n+1}}(\Omega)$ and $(s_0,t_0) \in (a,b)\times(c,d)$, let us write the explicit expressions of a generic element belonging to $\mathscr{P}^{\mathbf{n}}_{\mathbf{u},\mathbf{v}}([a,b]\times[c,d])$

$$\begin{aligned}
&\sum_{i=0}^{n_1-2}\sum_{j=0}^{n_2-2} a_{ij}\frac{(s-s_0)^i}{i!}\frac{(t-t_0)^j}{j!} + \sum_{i=0}^{n_1-2} b_i\frac{(s-s_0)^i}{i!}u_2(t) + \sum_{i=0}^{n_1-2} c_i\frac{(s-s_0)^i}{i!}v_2(t)\\
&+\sum_{j=0}^{n_2-2} d_j u_1(s)\frac{(t-t_0)^j}{j!} + \sum_{j=0}^{n_2-2} e_j v_1(s)\frac{(t-t_0)^j}{j!}\\
&+\nu_1 u_1(s)u_2(t) + \nu_2 u_1(s)v_2(t) + \nu_3 v_1(s)u_2(t) + \nu_4 v_1(s)v_2(t)
\end{aligned}$$

and of its Taylor expansion of coordinate bi-degree (n_1,n_2)

$$\begin{gathered}
\sum_{i=0}^{n_1-2}\sum_{j=0}^{n_2-2} a_{ij}\frac{(s-s_0)^i(t-t_0)^j}{i!j!}+\sum_{i=0}^{n_1-2}\sum_{j=0}^{n_2}\frac{b_iD_t^ju_2(t_0)}{i!j!}(s-s_0)^i(t-t_0)^j\\
+\sum_{i=0}^{n_1-2}\sum_{j=0}^{n_2}\frac{c_iD_t^jv_2(t_0)}{i!j!}(s-s_0)^i(t-t_0)^j+\sum_{i=0}^{n_1}\sum_{j=0}^{n_2-2}\frac{d_jD_s^iu_1(s_0)}{i!j!}(s-s_0)^i(t-t_0)^j\\
+\sum_{i=0}^{n_1}\sum_{j=0}^{n_2-2}\frac{e_jD_s^iv_1(s_0)}{i!j!}(s-s_0)^i(t-t_0)^j\\
+v_1\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\frac{D_s^iu_1(s_0)D_t^ju_2(t_0)}{i!j!}(s-s_0)^i(t-t_0)^j\\
+v_2\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\frac{D_s^iu_1(s_0)D_t^jv_2(t_0)}{i!j!}(s-s_0)^i(t-t_0)^j\\
+v_3\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\frac{D_s^iv_1(s_0)D_t^ju_2(t_0)}{i!j!}(s-s_0)^i(t-t_0)^j\\
+v_4\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\frac{D_s^iv_1(s_0)D_t^jv_2(t_0)}{i!j!}(s-s_0)^i(t-t_0)^j.
\end{gathered}$$

Then, the condition requiring that $Q_L(f;s_0,t_0)$ is a Hermite interpolant of coordinate bi-degree (n_1,n_2) corresponds to the following equations:

$$\begin{aligned}
&a_{ij}+b_iD_t^ju_2(t_0)+c_iD_t^jv_2(t_0)+d_jD_s^iu_1(s_0)+e_jD_s^iv_1(s_0)+v_1D_s^iu_1(s_0)D_t^ju_2(t_0)\\
&\quad+v_2D_s^iu_1(s_0)D_t^jv_2(t_0)+v_3D_s^iv_1(s_0)D_t^ju_2(t_0)+v_4D_s^iv_1(s_0)D_t^jv_2(t_0)=D_s^iD_t^jf(s_0,t_0),
\end{aligned}$$

for $0\le i\le n_1-2$, $0\le j\le n_2-2$,

$$\begin{aligned}
&b_iD_t^ju_2(t_0)+c_iD_t^jv_2(t_0)+v_1D_s^iu_1(s_0)D_t^ju_2(t_0)\\
&\quad+v_2D_s^iu_1(s_0)D_t^jv_2(t_0)+v_3D_s^iv_1(s_0)D_t^ju_2(t_0)+v_4D_s^iv_1(s_0)D_t^jv_2(t_0)=D_s^iD_t^jf(s_0,t_0),
\end{aligned}$$

for $0\le i\le n_1-2$, and $j=n_2-1,n_2$,

$$\begin{aligned}
&d_jD_s^iu_1(s_0)+e_jD_s^iv_1(s_0)+v_1D_s^iu_1(s_0)D_t^ju_2(t_0)\\
&\quad+v_2D_s^iu_1(s_0)D_t^jv_2(t_0)+v_3D_s^iv_1(s_0)D_t^ju_2(t_0)+v_4D_s^iv_1(s_0)D_t^jv_2(t_0)=D_s^iD_t^jf(s_0,t_0),
\end{aligned}$$

for $i=n_1-1,n_1$, $0\le j\le n_2-2$, and

$$\begin{aligned}
&v_1D_s^iu_1(s_0)D_t^ju_2(t_0)+v_2D_s^iu_1(s_0)D_t^jv_2(t_0)\\
&\quad+v_3D_s^iv_1(s_0)D_t^ju_2(t_0)+v_4D_s^iv_1(s_0)D_t^jv_2(t_0)=D_s^iD_t^jf(s_0,t_0),
\end{aligned}$$

for $i=n_1-1,n_1,\ j=n_2-1,n_2$. By using a suitable reordering of the unknowns $a_{ij},b_i,c_i,d_j,e_j,v_k$, we obtain a linear system whose matrix is

$$A=\begin{bmatrix} I & \star & \star & \star\\ 0 & A_1 & 0 & \star\\ 0 & 0 & A_2 & \star\\ 0 & 0 & 0 & A_3\end{bmatrix}$$

where I is the identity matrix of size $(n_1-1)(n_2-1)\times(n_1-1)(n_2-1)$, $\star$ stands for blocks of suitable size, 0 stand for null matrices of suitable size, and

$$A_1=\begin{bmatrix}
D_t^{n_2-1}u_2(t_0) & 0 & \dots & 0 & D_t^{n_2-1}v_2(t_0) & 0 & \dots & 0\\
D_t^{n_2}u_2(t_0) & 0 & \dots & 0 & D_t^{n_2}v_2(t_0) & 0 & \dots & 0\\
0 & D_t^{n_2-1}u_2(t_0) & \dots & 0 & 0 & D_t^{n_2-1}v_2(t_0) & \dots & 0\\
0 & D_t^{n_2}u_2(t_0) & \dots & 0 & 0 & D_t^{n_2}v_2(t_0) & \dots & 0\\
0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0\\
0 & 0 & \dots & D_t^{n_2-1}u_2(t_0) & 0 & 0 & \dots & D_t^{n_2-1}v_2(t_0)\\
0 & 0 & \dots & D_t^{n_2}u_2(t_0) & 0 & 0 & \dots & D_t^{n_2}v_2(t_0)
\end{bmatrix}$$

$$A_2 = \begin{bmatrix} D_s^{n_1-1}u_1(s_0) & 0 & \dots & 0 & D_s^{n_1-1}v_1(s_0) & 0 & \dots & 0 \\ D_s^{n_1}u_1(s_0) & 0 & \dots & 0 & D_s^{n_1}v_1(s_0) & 0 & \dots & 0 \\ 0 & D_s^{n_1-1}u_1(s_0) & \dots & 0 & 0 & D_s^{n_1-1}v_1(s_0) & \dots & 0 \\ 0 & D_s^{n_1}u_1(s_0) & \dots & 0 & 0 & D_s^{n_1}v_1(s_0) & \dots & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & D_s^{n_1-1}u_1(s_0) & 0 & 0 & \dots & D_s^{n_1-1}v_1(s_0) \\ 0 & 0 & \dots & D_s^{n_1}u_1(s_0) & 0 & 0 & \dots & D_s^{n_1}v_1(s_0) \end{bmatrix}$$

$$A_3 = \begin{bmatrix} D_s^{n_1-1}u_1(s_0)D_t^{n_2-1}u_2(t_0) & D_s^{n_1-1}u_1(s_0)D_t^{n_2-1}v_2(t_0) & D_s^{n_1-1}v_1(s_0)D_t^{n_2-1}u_2(t_0) & D_s^{n_1-1}v_1(s_0)D_t^{n_2-1}v_2(t_0) \\ D_s^{n_1}u_1(s_0)D_t^{n_2-1}u_2(t_0) & D_s^{n_1}u_1(s_0)D_t^{n_2-1}v_2(t_0) & D_s^{n_1}v_1(s_0)D_t^{n_2-1}u_2(t_0) & D_s^{n_1}v_1(s_0)D_t^{n_2-1}v_2(t_0) \\ D_s^{n_1-1}u_1(s_0)D_t^{n_2}u_2(t_0) & D_s^{n_1-1}u_1(s_0)D_t^{n_2}v_2(t_0) & D_s^{n_1-1}v_1(s_0)D_t^{n_2}u_2(t_0) & D_s^{n_1-1}v_1(s_0)D_t^{n_2}v_2(t_0) \\ D_s^{n_1}u_1(s_0)D_t^{n_2}u_2(t_0) & D_s^{n_1}u_1(s_0)D_t^{n_2}v_2(t_0) & D_s^{n_1}v_1(s_0)D_t^{n_2}u_2(t_0) & D_s^{n_1}v_1(s_0)D_t^{n_2}v_2(t_0) \end{bmatrix}$$

The matrix A_1 has size $2(n_1-1)\times 2(n_1-1)$, A_2 has size $2(n_2-1)\times 2(n_2-1)$, A_3 has size 4×4. The existence and uniqueness of the interpolation operator Q_L is then equivalent to the non-singularity of this matrix. Since A is an upper triangular block matrix, its non-singularity can be proved by studying A_1, A_2, A_3 (I is obviously non-singular). The matrices A_1 and A_2 are not singular, due to their structure and to the fact that (2) and (3) hold. In fact, we have

$$\begin{aligned} |\det(A_1)| &= |D_t^{n_2-1}u_2(t_0)D_t^{n_2}v_2(t_0) - D_t^{n_2}u_2(t_0)D_t^{n_2-1}v_2(t_0)|^{n_1-1} \\ |\det(A_2)| &= |D_s^{n_1-1}u_1(s_0)D_s^{n_1}v_1(s_0) - D_s^{n_1}u_1(s_0)D_s^{n_1-1}v_1(s_0)|^{n_2-1}. \end{aligned}$$

Moreover, it can be easily verified that determinant of A_3 is $-[\det(D_1)]^2[\det(D_2)]^2$, where

$$D_1 \quad := \quad \begin{bmatrix} D_s^{n_1-1}u_1(s_0) & D_s^{n_1-1}v_1(s_0) \\ D_s^{n_1}u_1(s_0) & D_s^{n_1}v_1(s_0) \end{bmatrix}$$

and

$$D_2 \quad := \quad \begin{bmatrix} D_t^{n_2-1}u_2(t_0) & D_t^{n_2-1}v_2(t_0) \\ D_t^{n_2}u_2(t_0) & D_t^{n_2}v_2(t_0) \end{bmatrix} \quad .$$

If we assume that (3) holds, it can be shown (see [6]) that $\det(D_1)\neq 0$ and $\det(D_2)\neq 0$, and so $\det(A_3)\neq 0$.
In general, $|\det(A)|$ depends on (s_0,t_0), which in the following will be chosen as a point in the interior of the cells of the T-mesh, and then partly depending on the T-mesh itself. In order to prove the approximation properties, we will assume that there is a lower bound for $|\det(A)|$ which does not depend on the refinement of the T-mesh. Note that this true in the cases where the nonpolynomial functions are $e^{\lambda s}$ and $e^{\mu s}$, $e^{\lambda s}$ and $se^{\lambda s}$, and $e^{\alpha s}\cos(\beta s)$ and $e^{\alpha s}\sin(\beta s)$.

Given a function $f\in C^{\mathbf{n}+\mathbf{1}}(\Omega)$, we now define the following quasi-interpolant belonging to the generalized spline space $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$

$$Qf := \sum_{\xi\in\mathscr{M}} \gamma_\xi(Q_L(f;s_\xi,t_\xi))\psi_\xi \tag{28}$$

where

- $\mathscr{M}$ is the minimal determining set constructed in Section 3.3;
- ψ_ξ are the elements of the basis of the spline space on the T-mesh Δ associated to $\mathscr{M}$;
- γ_ξ are the linear functionals defined in (20) that associate to a spline $p\in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$ the correspondent B-coefficients, needed to express p as a linear combination of the basis ψ_ξ:

$$p = \sum_{\zeta\in\mathscr{M}} \gamma_\zeta p\,\psi_\zeta, \qquad \forall p\in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$$

- (s_ξ,t_ξ) is the center of the biggest circle included in the rectangle R_ξ, which is a cell containing ξ. Note that such a point lies in the interior of R_ξ, allowing the construction of $Q_L(f;s_\xi,t_\xi)$.

Note that Q is a linear operator, being the functionals γ_ξ linear, and it is a projection onto $GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$, that is, $Qp = p$ for every $p \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$. In order to study the approximation properties of Q, we need to prove the generalization to our non-polynomial setting of Lemmas 3.1, Lemma 3.2 and Theorem 6.1 in [1].

Lemma 6. *Let $p \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$. Let $R \in \Delta$, and let $p|_R = \sum_{\eta\in\mathscr{D}_{\mathbf{n},R}} c^R_\eta B^R_\eta(s,t)$. We denote by c the vector containing all the coefficients c^R_η, $\eta \in \mathscr{D}_{\mathbf{n},R}$. Then, there exists a constant K_1, depending only on n_1 and n_2, such that*

$$\frac{||c||_\infty}{K_1} \le ||p||_R \le ||c||_\infty,$$

where $||c||_\infty$ stands for the max*-norm of c and $||\cdot||_R$ for the* sup*-norm of a function restricted to R.*

Proof. This is a straightforward generalization of the polynomial case: the upper bound follows from the fact that the basis functions are nonnegative and sum to one, while the lower bound can be proved with the following argument: the matrix $M := [B^R_\eta(\zeta)]_{\zeta,\eta\in\mathscr{D}_{\mathbf{n},R}}$ is non-singular by well-known results on tensor-product interpolation. Then $Mc = r$, where r is the vector $\{p(\zeta)\}_{\zeta\in\mathscr{D}_{\mathbf{n},R}}$. As a consequence, we have

$$||c||_\infty \le ||M^{-1}r||_\infty \le ||M^{-1}||_\infty ||r||_\infty \le ||M^{-1}||_\infty ||p||_R = K_1||p||_R.$$

The result is then achieved by setting $K_1 = ||M^{-1}||_\infty$. □

Lemma 7. *Given a rectangle R, let A_R be its area. Then there exists a constant K_2, depending only on n_1 and n_2, such that*

$$\frac{A_R^{1/q}}{K_2}||c||_q \le ||p||_{q,R} \le A_R^{1/q}||c||_q,$$

where $||c||_q$ stands for the q-norm of the vector c and $||\cdot||_{q,R}$ for the q-norm of a function restricted to R.

Proof. It is sufficient to use equivalence of norms on finite dimensional spaces, considering that both a classical polynomial space and the more general space in which we work have finite dimension. Then, the result is obtained, for any $1 \le q < \infty$, by generalizing Theorem 2.7 in [23]. □

To prove the approximation property of the quasi-interpolant, we will need the following result about the minimal determining set and the B-coefficients.

Definition 6. *Let e be a composite edge of Δ, and let $e_1,\dots,e_m$ be a maximal sequence of composite edges such that for each $i = 1,\dots,m$, one endpoint of e_i lies in the interior of e_{i+1}, where we assume $e_{m+1} = e$. We call $e_1,\dots,e_m$ a chain ending at e. We call m the length of the chain.*

Theorem 4. *Let (u_1,u_2) and (v_1,v_2) be such that $\mathscr{P}^{\mathbf{n}}_{\mathbf{u},\mathbf{v}}([\min_{R\in\Delta} a_R, \max_{R\in\Delta} b_R]\times[\min_{R\in\Delta} c_R, \max_{R\in\Delta} d_R])$ is invariant under translations in the sense of* (25)*. For every composite edge e consisting of m edge segments $e_1,\dots,e_m$ with $m \ge 1$, let $\alpha_e := \max\{|e|/|e_1|, |e|/|e_m|\}$, and let β_e be the length of the longest chain ending at e. For each rectangle R in Δ, let κ_R be the ratio of the lengths of its longest and of its shortest edges. Recalling that C is the set of all composite edges of Δ, we set $\alpha_\Delta := \max_{e\in C}\alpha_e, \beta_\Delta := \max_{e\in C}\beta_e, \kappa_\Delta := \max_{R\in\Delta}\kappa_R$. Moreover, let $L := \max\{b_\Delta - a_\Delta, d_\Delta - c_\Delta\}$, where $a_\Delta := \min_{R\in\Delta} a_R$, $b_\Delta := \max_{R\in\Delta} b_R$, $c_\Delta := \min_{R\in\Delta} c_R$, $d_\Delta := \max_{R\in\Delta} d_R$. Then, for any $p \in GS^{\mathbf{n},\mathbf{r}}_{\mathbf{u},\mathbf{v}}(\Delta)$, its associated B-coefficients satisfy*

$$|c_\eta| \le K_3 \max_{\xi\in\mathscr{M}} |c_\xi|, \qquad \eta \in \mathscr{D}_{\mathbf{n},\Delta}$$

where K_3 is a constant depending only on $\mathbf{n}, \alpha_\Delta, \beta_\Delta, \kappa_\Delta, L$.

Proof. In order to prove the bound, it is enough to show that it holds for any $\eta \in \mathscr{D}^{\tilde{R}}_{\mathbf{r}}(w)$, with $w \in J_{NT}$ and $\tilde{R} \in \Delta$ having w as one of its vertices, and for any η such that $d(\eta,e) \le r_e$, with $e \in C$, since the remaining part of the proof coincides with the one of Theorem 6.1 in [1].

For any $\tilde{R}$ having $w \in J_{NT}$ as one of its vertices there exists R sharing w as one of its vertices and such that $\mathscr{D}_{\mathbf{r}}^{R}(w) \subset \mathscr{M}$ (see Figure 4). Since the regularity is $\mathbf{r} = (r_1, r_2)$, we have

$$\sum_{i=0}^{h}\sum_{j=0}^{k} c_{ij}^{\tilde{R}} D_s^h B_{i,n_1}^{\tilde{R}}(a_{\tilde{R}}) D_t^k B_{j,n_2}^{\tilde{R}}(c_{\tilde{R}}) = \sum_{i=n_1-h}^{n_1}\sum_{j=n_2-k}^{n_2} c_{ij}^{R} D_s^h B_{i,n_1}^{R}(b_R) D_t^k B_{j,n_2}^{R}(d_R), \tag{29}$$

for $h = 0,\dots,r_1$, $k = 0,\dots,r_2$. Starting from (29), we can obtain a similar linear system by replacing the partial derivatives with the directional derivatives multiplied for suitable powers of edges' lengths. In other words, we obtain a system of the form $\mathbf{c}_{\mathbf{r}}^{\tilde{R}} = (M^{\tilde{R}})^{-1}\Lambda M^{R}\mathbf{c}_{\mathbf{r}}^{R}$, where $\mathbf{c}_{\mathbf{r}}^{R}$ and $\mathbf{c}_{\mathbf{r}}^{\tilde{R}}$ are the vectors of B-coefficients associated, respectively, with the sets $\mathscr{D}_{\mathbf{r}}^{R}(w)$ and $\mathscr{D}_{\mathbf{r}}^{\tilde{R}}(w)$, M^R and $M^{\tilde{R}}$ are matrices of directional derivatives of Bernstein basis functions, and Λ is a matrix dependent only on topological quantities of the elements of the mesh around w, whose entries, in turn, can be bounded by a constant dependent only on κ_Δ and $\mathbf{n}$.
Note that the norms of M^R and $(M^{\tilde{R}})^{-1}$ are bounded by constants depending only on L. In fact, let $L_{h,R} := b_R - a_R$ and $L_{v,R} := d_R - c_R$ for any cell $R \in \Delta$, and let also $l_\Delta := \min_{R\in\Delta}\min\{L_{h,R}, L_{v,R}\}$ and $L_\Delta := \max_{R\in\Delta}\max\{L_{h,R}, L_{v,R}\}$. Note that if we consider another cell with the same sizes $L_{h,R}$ and $L_{v,R}$, but with the bottom-left corner at $(0,0)$, the assumption of invariance under translations (25) implies that the local Bernstein-Bézier basis on such a cell is obtained by translation from the one on R. Then, since $[l_\Delta, L_\Delta] \subset (0,L]$, we have $\|M^R\| \le \sup_{l_\Delta \le L_{h,\bar{R}}, L_{v,\bar{R}} \le L_\Delta} \|M^{\bar{R}}\| \le \sup_{0<L_{h,\bar{R}}, L_{v,\bar{R}} \le L} \|M^{\bar{R}}\|$, where $\bar{R}$ is a rectangle whose bottom-left corner is $(0,0)$, and with width $L_{h,\bar{R}}$ and height $L_{v,\bar{R}}$.
Note that $\sup_{0<L_{h,\bar{R}}, L_{v,\bar{R}} \le L} \|M^{\bar{R}}\|$ is bounded, since $\|M^{\bar{R}}\|$ is a continuous function of the variables $L_{h,\bar{R}}, L_{v,\bar{R}}$, and the limits of $\|M^{\bar{R}}\|$ for $(L_{h,\bar{R}}, L_{v,\bar{R}}) \to (0,0)$, for $L_{h,\bar{R}} \to 0$ and for $L_{v,\bar{R}} \to 0$ are bounded (such cases correspond to replacing in (13) suitable elements of the Bernstein-like basis with the corresponding ones of the polynomial Bernstein basis, which leads to bounded norms, thanks to the invariance for affine transformations of the polynomial case). As a consequence, $\sup_{0<L_{h,\bar{R}}, L_{v,\bar{R}} \le L} \|M^{\bar{R}}\|$ is finite and dependent only on L. Similar remarks apply for the matrix $(M^{\tilde{R}})^{-1}$.
Analogous observations about the linear systems describing the smoothness conditions on the composite edges (see Lemma 3) give similar inequalities where the norms can be bounded by constants depending only on the global extrema of the mesh and by constants depending only on $\alpha_\Delta, \kappa_\Delta$ and $\mathbf{n}$. ☐

Remark 4. Let us define, for any cell R in Δ:

$$\begin{aligned}\Gamma_R &:= \{\xi \in \mathscr{M} : \mathrm{supp}(\psi_\xi) \cap R \neq \emptyset\},\\ \Omega_R &:= \cup_{\xi\in\Gamma_R}\mathrm{supp}(\psi_\xi),\end{aligned}$$

Note that, if $\eta \in \mathscr{D}_{\mathbf{n},R}$, then

$$|c_\eta| \le K_3 \max_{\xi\in\Gamma_R} |c_\xi| \tag{30}$$

since it can be shown that the coefficients corresponding to the domain points $\xi \in \mathscr{M}\setminus\Gamma_R$ do not have influence on the coefficients of $\mathscr{D}_{\mathbf{n},R}$.

Let $\xi \in \mathscr{M}$ and $F \in C^{\mathbf{n}}(\Omega)$. By applying Lemma 7 with $p = Q_L(F; s_\xi, t_\xi)$, we obtain that $c_\xi = \gamma_\xi(Q_L(F; s_\xi, t_\xi))$ and

$$|\gamma_\xi(Q_L(F; s_\xi, t_\xi))| \le \frac{K_2}{A_{R_\xi}^{1/q}} ||Q_L(F; s_\xi, t_\xi)||_{q,R_\xi}.$$

If we denote by $T^{(n_1,n_2)}Q_L(F; s_\xi, t_\xi)$ the Taylor expansion of $Q_L(F; s_\xi, t_\xi)$ at (s_ξ, t_ξ) of bi-degree (n_1, n_2), for $1 \le q < \infty$, we get

$$\begin{aligned}|\gamma_\xi(Q_L(F; s_\xi, t_\xi))| &\le \frac{K_2}{A_{R_\xi}^{1/q}} ||Q_L(F; s_\xi, t_\xi)||_{q,R_\xi} \le \frac{K_2}{A_{R_\xi}^{1/q}} A_{R_\xi}^{1/q} ||Q_L(F; s_\xi, t_\xi)||_{\infty,R_\xi}\\ &\le K_2 \max_{(s,t)\in R_\xi} |T^{(n_1,n_2)}Q_L(F, s_\xi, t_\xi)(s,t)| + O((\mathrm{diam}(R_\xi))^{k+1}),\end{aligned}$$

where $k := \min\{n_1, n_2\}$. An analogous bound can be obtained for $q = \infty$ by using Lemma 6.
For $\eta \in \mathscr{D}_{\mathbf{n},R}$, by using Theorem 4 and (30), we have

$$|c_\eta| \le K_3 \max_{\xi\in\Gamma_R} |c_\xi| \le K_2 K_3 \max_{\xi\in\Gamma_R} \max_{(s,t)\in R_\xi} |T^{(n_1,n_2)} Q_L(F; s_\xi, t_\xi)(s,t)| + O((\max_{\xi\in\Gamma_R} \mathrm{diam}(R_\xi))^{k+1})$$

That allows us to obtain a bound for $\|QF\|$

$$\begin{aligned} ||QF||_{q,R} &\le A_R^{1/q} ||QF||_{\infty,R} = A_R^{1/q} \left\| \sum_{\eta\in\mathscr{D}_{\mathbf{n},R}} \gamma_\eta B_\eta^R \right\|_{\infty,R} \\ &\le A_R^{1/q} K_2 K_3 \max_{\xi\in\Gamma_R} \max_{(s,t)\in R_\xi} |T^{(n_1,n_2)} Q_L(F; s_\xi, t_\xi)(s,t)| + O((\max_{\xi\in\Gamma_R} \mathrm{diam}(R_\xi))^{k+1}). \end{aligned} \tag{31}$$

Now, we can finally get an approximation result for the quasi-interpolant Q. Given a cell $R_\zeta \in \Delta$, we have

$$\begin{aligned} ||f - Qf||_{q,R_\zeta} &\le ||f - Q_L(f; s_\zeta, t_\zeta)||_{q,R_\zeta} + ||Q_L(f; s_\zeta, t_\zeta) - Qf||_{q,R_\zeta} \\ &= ||f - Q_L(f; s_\zeta, t_\zeta)||_{q,R_\zeta} + ||Q(f - Q_L(f; s_\zeta, t_\zeta))||_{q,R_\zeta} \\ &\le O((\mathrm{diam}(R_\zeta))^{k+1}) + A_{R_\zeta}^{1/q} K_2 K_3 \max_{\xi\in\Gamma_{R_\zeta}} \max_{(s,t)\in R_\xi} |T^{(n_1,n_2)} Q_L(f - Q_L(f; s_\zeta, t_\zeta); s_\xi, t_\xi)(s,t)| \\ &\quad + O((\max_{\xi\in\Gamma_{R_\zeta}} \mathrm{diam}(R_\xi))^{k+1}) \end{aligned}$$

where we used the fact that Q is linear and it is a projection on $GS_{\mathbf{u},\mathbf{v}}^{\mathbf{n},\mathbf{r}}(\Delta)$, and we applied inequality (31) to $F = f - Q_L(f; s_\zeta, t_\zeta)$. Since $|D_s^i D_t^j (f - Q_L(f; s_\zeta, t_\zeta))| = O(\|(s,t) - (s_\zeta, t_\zeta)\|^{\max\{0, k+1-i-j\}})$ (for $0 \le i \le n_1$ and $0 \le j \le n_2$), and $\|(s_\xi, t_\xi) - (s_\zeta, t_\zeta)\| \le \mathrm{diam}(\Omega_{R_\zeta})$, we have

$$\begin{aligned} &|T^{(n_1,n_2)} Q_L(f - Q_L(f; s_\zeta, t_\zeta); s_\xi, t_\xi)| \\ &\le \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} |D_s^i D_t^j (f - Q_L(f; s_\zeta, t_\zeta))|_{(s_\xi, t_\xi)} (s - s_\xi)^i (t - t_\xi)^j = O((\mathrm{diam}(\Omega_{R_\zeta}))^{k+1}). \end{aligned}$$

Moreover, it can be proved that there exists a constant K_4, depending only on $\alpha_\Delta, \beta_\Delta, \kappa_\Delta$, such that $\mathrm{diam}(\Omega_R) \le K_4\, \mathrm{diam}(R)$ for any $R \in \Delta$. Then, we get

$$||f - Qf||_{q,R_\zeta} \le O((\mathrm{diam}(R_\zeta))^{k+1}) + O((\mathrm{diam}(\Omega_{R_\zeta}))^{k+1}) = O((\mathrm{diam}(R_\zeta))^{k+1})$$

Then, we can state the following result.

Theorem 5. *Let the* mesh size *of* Δ *be* $H = \max_{R\in\Delta} \mathit{diam}(R)$. *Then, for any* $f \in C^{\mathbf{n}+\mathbf{1}}(\Omega)$ *and for any cell* $R_\zeta \in \Delta$ *the quasi-interpolation operator* Q *defined in* (28) *satisfies*

$$\|f - Qf\|_{q,R_\zeta} = O(H^{k+1})$$

5. Conclusions

In this paper we provided a deep study of the generalized spline spaces over T-meshes, which extend the concept of spline spaces over T-mesh to a noteworthy case of Chebyshevian spline spaces. We showed that, in spite of the different functions locally considered, the overall behaviour of the new spline spaces is analogous to the classical polynomial case. In fact, thanks to the properties of the chosen non-polynomial functions we can use a local Bernstein-Bézier representation and generalize the arguments used in [1] to the considered non-polynomial case, to get a basis (associated to a minimal determining set) and a dimension formula. Moreover, we also studied the change of the spline space dimension when the T-mesh is refined, as well as the dimension of a generalized spline space over two merged T-meshes. We showed that the analogy with the polynomial case extends to the approximation order, which we obtained by considering a quasi-interpolant based on newly defined local Hermite interpolants.

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