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On the dimension of Tchebycheffian spline spaces over planar T-meshes

Cesare Bracco[a], Tom Lyche[b], Carla Manni[c], Fabio Roman[d], Hendrik Speleers[c]

[a] *Department of Mathematics and Computer Science, University of Florence, Italy*
[b] *Department of Mathematics, University of Oslo, Norway*
[c] *Department of Mathematics, University of Rome 'Tor Vergata', Italy*
[d] *Department of Mathematics, University of Turin, Italy*

Abstract

In this paper we define Tchebycheffian spline spaces over planar T-meshes and we address the problem of determining their dimension. We extend to the Tchebycheffian spline context the homological approach previously used to characterize polynomial spline spaces over T-meshes, and we exploit this characterization in the study of the dimension. In particular, we give combinatorial lower and upper bounds for the dimension, and we show that these bounds coincide if the dimensions of the underlying extended Tchebycheff section spaces are large enough with respect to the smoothness, under some mild conditions on the T-mesh. Finally, we illustrate that the dimension of Tchebycheffian spline spaces over T-meshes can be unstable, which means that it can depend on the exact geometry of the T-mesh. These results are extensions of those known in the literature for polynomial spline spaces over T-meshes.

Keywords: Tchebycheffian splines; T-meshes; dimension formula; dimension bounds; instability

1. Introduction

Tchebycheff spaces, or more precisely extended Tchebycheff spaces, are natural generalizations of algebraic polynomial spaces [18, 31]. They are a popular tool in approximation theory, in particular because they form a very flexible substitute for algebraic polynomial spaces to solve Hermite interpolation problems. Besides algebraic polynomial spaces, important examples of extended Tchebycheff spaces are the null spaces of differential operators with real constant coefficients.

Univariate Tchebycheffian splines are smooth piecewise functions with sections in extended Tchebycheff spaces. They have several advantages over classical (algebraic) polynomial splines, mainly due to the wide variety that extended Tchebycheff spaces offer. Despite this flexibility, many results of the polynomial framework extend in a natural way to the larger Tchebycheffian spline framework, ranging from approximation theory to geometric modelling, see [24, 28, 31]. In particular, Tchebycheffian splines admit a representation in terms of basis functions with similar properties as polynomial B-splines. Moreover, the elegant blossoming approach and classical algorithms (like degree elevation, knot insertion, differentiation formulas, etc.) can be rephrased for them [15, 22, 28].

Multivariate extensions of Tchebycheffian splines can be easily obtained via the tensor-product approach and have been applied in different contexts. For example, tensor-products of so-called generalized splines (which are a special class of Tchebycheffian splines) are a promising problem-dependent tool in isogeometric analysis, a recent paradigm for the numerical treatment of partial differential equations [26].

Adaptive local refinement is important for both geometric modelling and numerical simulation. However, a simple tensor-product spline structure lacks adequate local refinement. This triggered the interest in alternative spline structures supporting local refinement. Confining the discussion to local tensor-product structures, we mention (analysis-suitable) T-splines [21, 33], hierarchical splines [13, 14],

Email addresses: cesare.bracco@unifi.it (Cesare Bracco), tom@math.uio.no (Tom Lyche), manni@mat.uniroma2.it (Carla Manni), fabio.roman@unito.it (Fabio Roman), speleers@mat.uniroma2.it (Hendrik Speleers)

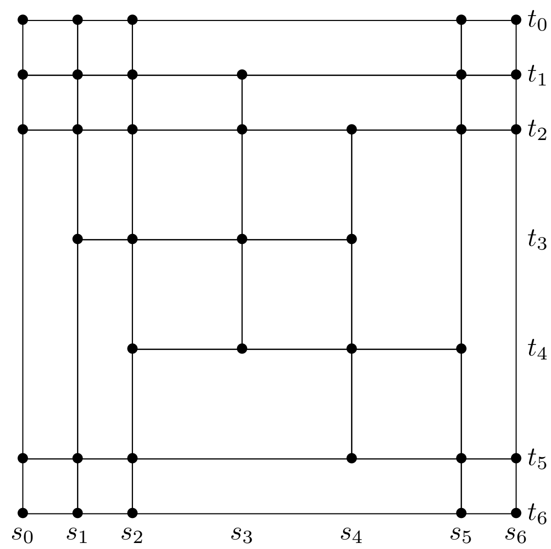


Figure 1: An unstable T-mesh.

and locally refined (LR-) splines [12]. All of them can be seen as special cases of polynomial splines over T-meshes [9, 10, 32]. In the more recent literature we also find some specific generalizations to the Tchebycheffian spline setting. For example, generalized T-splines [3, 4], hierarchical generalized splines [27] and generalized splines on T-meshes [5, 6] have been addressed. A multiresolution approach based on specific tensor-product Tchebycheffian splines has been considered in [23]. However, Tchebycheffian splines in their wide generality over T-meshes have not been previously investigated.

In this paper we consider Tchebycheffian spline spaces over T-meshes. As in the polynomial case, a complete understanding of these spline spaces requires the knowledge of the dimension of the spline space defined on a prescribed T-mesh for a given smoothness. Of course, it is of particular interest to understand when the dimension only depends on combinatorial quantities of the T-mesh (such as number of vertices, edges and faces), on the given smoothness, and on the componentwise dimensions (say $p_i + 1$) of the underlying extended Tchebycheff section spaces. The dimension of the spline space is said to be unstable if it depends on the exact geometry of the T-mesh. Spline spaces with unstable dimensions are not robust for practical use. Hence, it is important to detect whether or not there are instabilities in the dimension and to identify stable families of spaces. This instability phenomenon can be illustrated with the T-mesh in Figure 1: the dimension of the C^1 quadratic spline space over the depicted T-mesh is 37 but reduces to 36 if, for example, the value s_3 is slightly perturbed.

The dimension of polynomial spline spaces on a prescribed T-mesh for a given componentwise degree p_i and smoothness r_i has been addressed by several authors using different techniques, see [9, 12, 20, 29, 32, 35] and references therein, and it turns out to be a very challenging problem. Lower and upper bounds for the dimension are known, and an explicit expression has been determined in some special cases. In particular, the dimension is known for spline spaces over so-called quasi-cross-cut T-meshes [29] – these are meshes where each edge extends to the boundary – and for spline spaces with $p_i \geq 2r_i+1$ under some mild conditions on the T-mesh [9, 29, 32]. On the other hand, instability in the dimension can occur if the degree is not large enough with respect to the smoothness, see [20] for the case $p_i = r_i + 1$ and [1] for some specific examples with a larger gap between degree and smoothness.

The dimension problem of spline spaces over T-meshes faces the same difficulties as the dimension problem of polynomial spline spaces of total degree p over triangulations, see [19] and references therein. In this case, the dimension is known for spline spaces over quasi-cross-cut partitions [7, 30], and for spline spaces with $p \geq 3r+2$ [16, 17]. Instability in the dimension has been illustrated for $p = 2r$ in [11]. Some similar results are known for spline spaces of total degree p over general rectilinear partitions, see [7, 25] and references therein.

Among the various techniques to tackle the dimension problem, one can use the homological approach proposed in [29], where the technique introduced in [2] and developed in [30] for polynomial splines over triangulations has been fine-tuned for polynomial splines over planar T-meshes. In this paper we address the problem of finding the dimension of Tchebycheffian spline spaces over planar T-meshes. To this end, we generalize the techniques and the results presented in [29]. More precisely, besides characterizing the Tchebycheffian spline space as a suitable homology space,

- we provide a dimension formula in terms of combinatorial quantities of the T-mesh, the smoothness, the dimensions of the underlying extended Tchebycheff spaces, and homology quantities;
- we derive lower and upper bounds for the dimension (under a specific assumption on the underlying extended Tchebycheff spaces);
- we provide an explicit expression for the (stable) dimension of spline spaces over quasi-cross-cut T-meshes, and of spline spaces with $p_i \geq 2r_i + 1$ under some mild conditions on the T-mesh; the latter conditions are usually satisfied by T-meshes of interest in applications and identify a family of T-meshes larger than the one considered in [29];
- we illustrate that the dimension of Tchebycheffian spline spaces over T-meshes can be unstable, by generalizing the examples given in [20].

As mentioned above, this paper is a generalization of [29] from the polynomial spline setting to the Tchebycheffian spline setting, and the reading of this paper should go hand in hand with the reading of [29]. Nevertheless, it is worth pointing out that the extension is not so straightforward because the ring structure of algebraic polynomials cannot be used anymore in this general setting. Besides the stated results on the dimension, this new interpretation of the approach in [29] is an additional contribution of the paper and it strengthens the structural similarity between algebraic polynomial and general Tchebycheffian spline spaces.

We recall that generalized spline spaces are a special class of Tchebycheffian spline spaces. Results on the dimension of generalized spline spaces over T-meshes have been provided in [6] by extending the approach based on so-called minimal determining sets, see [32]. The homological approach for the dimension problem has been considered in [5] in the case of generalized spline spaces over T-meshes. More precisely, only the characterization of the dimension in terms of homology quantities has been given in [5]. To the best of our knowledge, there are no other results about the dimension of Tchebycheffian spline spaces over T-meshes in the literature.

The remainder of the paper is divided into five sections. In Section 2 we give the definition of a Tchebycheffian spline space over T-meshes, and we provide their characterization in terms of the homology of a suitable complex. Section 3 collects several technical results to be used in Section 4 for determining the dimension of the considered Tchebycheffian spline spaces. Examples of instability in the dimension of Tchebycheffian spline spaces over T-meshes are provided in Section 5. Finally, we end in Section 6 with some concluding remarks.

2. Tchebycheffian spline spaces over T-meshes

In this section we formulate the definitions of the meshes and of the spaces we are dealing with.

2.1. T-meshes and smoothness

Let us consider a region $\Omega \subset \mathbb{R}^2$ which is a finite union of closed axis-aligned rectangles, called *cells*, with pairwise disjoint interiors. We assume that Ω is simply connected and its interior Ω^o is connected; see Figure 2 for an illustration. The smallest rectangle containing Ω is denoted by $[a_h, b_h] \times [a_v, b_v]$.

We now define a T-mesh on Ω using the notation and definition given in [5, 29].

Definition 2.1 (T-mesh)**.** *A T-mesh $\mathcal{T} := (\mathcal{T}_2, \mathcal{T}_1, \mathcal{T}_0)$ on Ω is defined as:*

- *$\mathcal{T}_2$ is the collection of cells in Ω;*

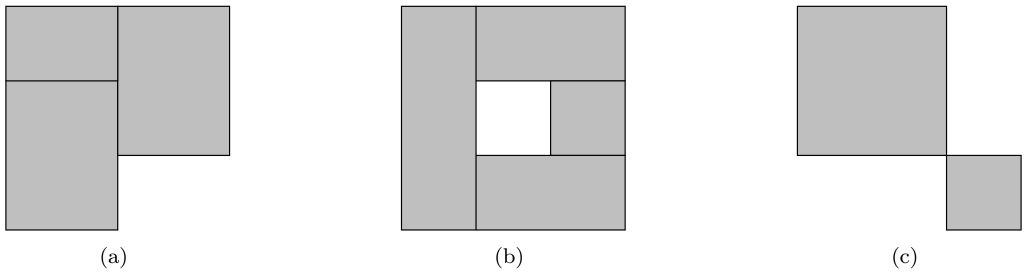


Figure 2: (a) A simply connected region with connected interior. (b) A region which is not simply connected. (c) A region where the interior is not connected.

- $\mathcal{T}_1 = \mathcal{T}_1^h \cup \mathcal{T}_1^v$ *is a finite set of closed axis-aligned horizontal and vertical segments in* $\bigcup_{\sigma\in\mathcal{T}_2} \partial\sigma$*, called* edges*;*
- $\mathcal{T}_0 := \bigcup_{\tau\in\mathcal{T}_1} \partial\tau$ *is a finite set of points, called* vertices*;*

such that

- *for each* $\sigma \in \mathcal{T}_2$*,* $\partial\sigma$ *is a finite union of elements of* $\mathcal{T}_1$*;*
- *for* $\sigma, \sigma' \in \mathcal{T}_2$ *with* $\sigma \neq \sigma'$*,* $\sigma \cap \sigma' = \partial\sigma \cap \partial\sigma'$ *is a finite union of elements of* $\mathcal{T}_1 \cup \mathcal{T}_0$*;*
- *for* $\tau, \tau' \in \mathcal{T}_1$ *with* $\tau \neq \tau'$*,* $\tau \cap \tau' = \partial\tau \cap \partial\tau' \subset \mathcal{T}_0$*;*
- *for each* $\gamma \in \mathcal{T}_0$*,* $\gamma = \tau_h \cap \tau_v$ *where* τ_h *is a horizontal edge and* τ_v *is a vertical edge.*

We denote by $\mathcal{T}_1^o$ the set of *interior edges*, i.e., the edges intersecting the interior of Ω. Analogously, $\mathcal{T}_0^o$ represents the set of vertices in Ω^o, called *interior vertices*. The elements of the sets $\mathcal{T}_1 \setminus \mathcal{T}_1^o$ and $\mathcal{T}_0 \setminus \mathcal{T}_0^o$ are the *boundary edges* and the *boundary vertices*, respectively. We say that an interior vertex is a *crossing vertex* if it belongs to 4 distinct edges; it is a *T-vertex* if it belongs to exactly 3 edges. Moreover, $\mathcal{T}_1^{o,h}$ and $\mathcal{T}_1^{o,v}$ indicate the sets of the horizontal and vertical interior edges of $\mathcal{T}$, respectively, and we set $\mathcal{T}_1^o := \mathcal{T}_1^{o,h} \cup \mathcal{T}_1^{o,v}$. Then, the interior T-mesh is given by $\mathcal{T}^o := (\mathcal{T}_2, \mathcal{T}_1^o, \mathcal{T}_0^o)$.

A *segment* of $\mathcal{T}$ is a connected union of edges of $\mathcal{T}$ belonging to the same straight line. Given any $\tau \in \mathcal{T}_1^o$, we denote by $\rho(\tau)$ the maximal segment composed of edges of $\mathcal{T}_1^o$ containing τ. Moreover, we denote by $\text{MS}(\mathcal{T})$ the set of all such maximal segments. If $\rho \in \text{MS}(\mathcal{T})$ does not intersect the boundary of the T-mesh, we say that ρ is a *maximal interior segment*. The set of all horizontal (respectively vertical) maximal interior segments is denoted by $\text{MIS}_h(\mathcal{T})$ (respectively $\text{MIS}_v(\mathcal{T})$), and we set $\text{MIS}(\mathcal{T}) := \text{MIS}_h(\mathcal{T}) \cup \text{MIS}_v(\mathcal{T})$. Given any $\gamma \in \mathcal{T}_0^o$, we define $\rho_h(\gamma) := \rho(\tau_h)$ and $\rho_v(\gamma) := \rho(\tau_v)$, such that $\gamma = \tau_h \cap \tau_v$ and $\tau_h \in \mathcal{T}_1^{o,h}$, $\tau_v \in \mathcal{T}_1^{o,v}$.

Finally, we denote by f_2 the number of rectangles, by f_1^h and f_1^v the number of horizontal and vertical interior edges, respectively, and by f_0 the number of interior vertices of $\mathcal{T}$.

Example 2.1. *Consider the T-mesh $\mathcal{T}$ depicted in Figure 3. In this case, we have*

- $\mathcal{T}_2 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\}$*,* $f_2 = 7$*;*
- $\mathcal{T}_1^{o,h} = \{\tau_1^h, \tau_2^h, \tau_3^h, \tau_4^h, \tau_5^h, \tau_6^h, \tau_7^h\}$*,* $f_1^h = 7$*;*
- $\mathcal{T}_1^{o,v} = \{\tau_1^v, \tau_2^v, \tau_3^v, \tau_4^v, \tau_5^v, \tau_6^v, \tau_7^v\}$*,* $f_1^v = 7$*;*
- $\mathcal{T}_0^o = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8\}$*,* $f_0 = 8$*.*
- $\text{MIS}(\mathcal{T}) = \{\rho_1 := \tau_4^v \cup \tau_5^v, \rho_2 := \tau_6^v \cup \tau_7^v, \rho_3 := \tau_4^h\}$*.*

For some other examples, we refer to [5, Section 2]. Since we are interested in (non-polynomial) spline spaces over T-meshes where the smoothness of the elements of the space across the edges of the T-mesh is given, we also need to define what we mean by smoothness.

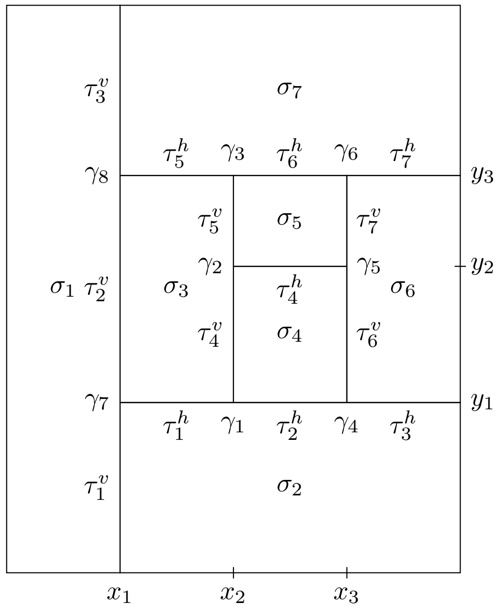


Figure 3: Example of a T-mesh.

Definition 2.2 (Smoothness). *With each edge $\tau \in \mathcal{T}_1^o$, we associate an integer $r(\tau) \geq -1$. We say that $f \in C^{r(\tau)}(\tau)$ if the partial derivatives of f up to order $r(\tau)$ are continuous across the edge τ. We assume that $r(\tau) = r(\tau')$ for all τ, τ' lying on the same straight line, and we refer to this as the constant smoothness (along lines) assumption. Letting*

$$\boldsymbol{r} := \{\, r(\tau),\ \forall \tau \in \mathcal{T}_1^o \,\},$$

we call $\boldsymbol{r}$ a smoothness distribution on $\mathcal{T}$. We define the following class of smooth functions on Ω:

$$C^{\boldsymbol{r}}(\mathcal{T}) := \{\, f : \Omega \to \mathbb{R} : f \in C^{r(\tau)}(\tau),\ \forall \tau \in \mathcal{T}_1^o \,\}.$$

Given a smoothness distribution $\boldsymbol{r}$ on $\mathcal{T}$, with each vertex $\gamma \in \mathcal{T}_0^o$, we associate two integers $r_h(\gamma), r_v(\gamma)$, where $r_h(\gamma) := r(\tau_v)$ and $r_v(\gamma) := r(\tau_h)$ such that $\gamma = \tau_h \cap \tau_v$ and $\tau_h \in \mathcal{T}_1^{o,h}$, $\tau_v \in \mathcal{T}_1^{o,v}$. Moreover, with each maximal segment $\rho \in \mathrm{MS}(\mathcal{T})$, we associate an integer $r(\rho) := r(\tau)$, where τ is any interior edge belonging to ρ. Note that the integers $r_h(\gamma), r_v(\gamma)$ and $r(\rho)$ are well defined by the constant smoothness (along lines) assumption.

2.2. Tchebycheffian spline spaces

We start by defining extended Tchebycheff spaces on a certain interval [18].

Definition 2.3 (Extended Tchebycheff space). *Given an integer $p \geq 0$ and an interval J, a space $\mathbb{T}_p(J) \subset C^p(J)$ of dimension $p+1$ is an extended Tchebycheff space on J if any Hermite interpolation problem with $p+1$ data on J has a unique solution in $\mathbb{T}_p(J)$. In other words, for any integer $m \geq 1$, let $\bar{x}_1, \dots, \bar{x}_m$ be distinct points in J and let $d_1, \dots, d_m$ be integers such that $p + 1 = \sum_{i=1}^{m}(d_i + 1)$, then for any set $\{f_{i,k} \in \mathbb{R}\}_{k=0,\dots,d_i,\, i=1,\dots,m}$ there exists a unique $q \in \mathbb{T}_p(J)$ such that*

$$D_x^k q(\bar{x}_i) = f_{i,k}, \qquad k = 0, \dots, d_i, \quad i = 1, \dots, m.$$

From the definition it follows that any non-trivial element in $\mathbb{T}_p(J)$ has at most p roots in J counting multiplicities. It is clear that if $\mathbb{T}_p(J)$ is an extended Tchebycheff space, then $\mathbb{T}_p(J')$ is also an extended Tchebycheff space for any $J' \subset J$.

Example 2.2. *The space $\mathbb{P}_p := \langle 1, x, \ldots, x^p \rangle$ of algebraic polynomials of degree less than or equal to p is an extended Tchebycheff space on the real line.*

Example 2.3. *The space of trigonometric functions $\mathbb{E}_{2n} := \langle 1, \sin x, \cos x, \ldots, \sin(nx), \cos(nx) \rangle$ is an extended Tchebycheff space on any interval $[a, b]$ with $0 < n(b - a) < \pi$.*

Example 2.4. *The kernel (null space) $\mathbb{L}_p$ of a differential operator $L_p := \sum_{i=0}^{p+1} c_i D_x^i$ with real coefficients and $c_{p+1} := 1$ is an extended Tchebycheff space of dimension $p + 1$. If the characteristic polynomial has only real roots then $\mathbb{L}_p$ is an extended Tchebycheff space on the real line. On the other hand, if some of the roots are not real then $\mathbb{L}_p$ is an extended Tchebycheff space on a suitable interval J. Note that the space $\mathbb{P}_p$ (see Example 2.2) is a special case with $L_p = D_x^{p+1}$, and also the space $\mathbb{E}_{2n}$ (see Example 2.3) with $L_{2n} = D_x \prod_{j=1}^{n} (D_x^2 + j^2 I)$.*

Example 2.5. *The space $\mathbb{G}_p^{U,V} := \langle 1, x, \ldots, x^{p-2}, U(x), V(x) \rangle$ is an extended Tchebycheff space on J, under the assumption that $\langle D_x^{p-1} U(x), D_x^{p-1} V(x) \rangle$ is an extended Tchebycheff space on J, see [5, 6, 8]. Noteworthy cases are the spaces*

- $\mathbb{G}_{p,\alpha}^{\exp} := \langle 1, x, \ldots, x^{p-2}, \mathrm{e}^{\alpha x}, \mathrm{e}^{-\alpha x} \rangle$, $0 < \alpha \in \mathbb{R}$, $J = \mathbb{R}$;
- $\mathbb{G}_{p,\alpha}^{\mathrm{trig}} := \langle 1, x, \ldots, x^{p-2}, \sin(\alpha x), \cos(\alpha x) \rangle$, $0 < \alpha(b - a) < \pi$, $J = [a, b]$;
- $\mathbb{G}_{p,n}^{\mathrm{vdp}} := \left\langle 1, x, \ldots, x^{p-2}, \left(\frac{b-x}{b-a}\right)^n, \left(\frac{x-a}{b-a}\right)^n \right\rangle$, $p \le n$, $J = (a, b)$.

Extended Tchebycheff spaces are a natural extension of the space of algebraic polynomials, because they enjoy the same structural properties as polynomial spaces. In particular, from Definition 2.3 it follows that for any $\bar{x} \in J$ the extended Tchebycheff space $\mathbb{T}_p(J)$ admits a Taylor-like basis $\{\psi_{\bar{x},i}\}_{i=0}^p$ such that

$$D_x^k \psi_{\bar{x},i}(\bar{x}) = \delta_{ik}, \qquad k = 0, \ldots, p, \qquad i = 0, \ldots, p,$$

where δ_{ik} stands for the classical Kronecker delta.

In the following, we denote by ℓ either h or v. Let $p_\ell \in \mathbb{N}$ with $p_\ell \ge 0$, and let $\mathbb{T}_{p_\ell}^\ell([a_\ell, b_\ell])$ be an extended Tchebycheff space of dimension $p_\ell + 1$ on $[a_\ell, b_\ell]$. Then, we define the tensor-product space

$$\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}} := \mathbb{T}_{p_h}^h([a_h, b_h]) \otimes \mathbb{T}_{p_v}^v([a_v, b_v]), \tag{2.1}$$

where $\boldsymbol{p} := (p_h, p_v)$ and $\boldsymbol{T} := (T_h, T_v) := (\mathbb{T}_{p_h}^h, \mathbb{T}_{p_v}^v)$. If the space (2.1) is the space of bivariate algebraic polynomials of bi-degree $\boldsymbol{p}$, then it will be denoted by $\mathbb{P}_{\boldsymbol{p}}$. We are now ready to define the Tchebycheffian spline space over a T-mesh.

Definition 2.4 (Tchebycheffian spline space over a T-mesh)**.** *Let $\mathcal{T}$ be a T-mesh with a smoothness distribution $\boldsymbol{r}$, and let $p_h, p_v \in \mathbb{N}$ with $p_h, p_v \ge 0$. We define the space of Tchebycheffian splines over the T-mesh $\mathcal{T}$, denoted by $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})$, as the space of functions in $C^{\boldsymbol{r}}(\mathcal{T})$ such that, restricted to any cell $\sigma \in \mathcal{T}_2$, they belong to $\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}$, i.e.,*

$$\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}) := \left\{ s \in C^{\boldsymbol{r}}(\mathcal{T}) : s_{|\sigma} \in \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}, \, \sigma \in \mathcal{T}_2 \right\}.$$

In particular, in the case of bivariate algebraic polynomials,

$$\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{r}}(\mathcal{T}) := \left\{ s \in C^{\boldsymbol{r}}(\mathcal{T}) : s_{|\sigma} \in \mathbb{P}_{\boldsymbol{p}}, \, \sigma \in \mathcal{T}_2 \right\}.$$

Note that, since $\mathbb{T}_{p_h}^h$ is an extended Tchebycheff space, if the smoothness $r(\tau_v) \ge p_h$ associated with a vertical edge $\tau_v \in \mathcal{T}_1^{o,v}$ then for any two cells σ, σ' adjacent to τ_v we have

$$s_{|\sigma \cup \sigma'} \in \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}, \quad s \in \mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}).$$

A similar property holds for horizontal edges. Therefore, in the following we assume

$$r(\tau_v) < p_h, \quad \forall \tau_v \in \mathcal{T}_1^{o,v}, \qquad r(\tau_h) < p_v, \quad \forall \tau_h \in \mathcal{T}_1^{o,h}.$$

2.3. A homological characterization of Tchebycheffian spline spaces over T-meshes

In this section we describe an alternative characterization of the spline space $\mathbb{S}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T})$ which will play a fundamental role in our analysis of the dimension of $\mathbb{S}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T})$. This characterization is based on a homological approach similar to the one used in [5, 29]. To this end, we first recall the definition of complex and homology.

Definition 2.5 (i-homology)**.** *A complex is a sequence of objects and morphisms*

$$\mathcal{A}:\ \cdots \to A_{i+1} \overset{\delta_{i+1}}{\to} A_i \overset{\delta_i}{\to} A_{i-1}\cdots$$

where $\operatorname{im}\delta_{i+1} \subseteq \ker\delta_i$*. The i-homology of $\mathcal{A}$ is defined as $H_i(\mathcal{A}) := \ker\delta_i/\operatorname{im}\delta_{i+1}$. The complex is exact at position i if $H_i(\mathcal{A}) = 0$.*

We now define some subspaces of $\mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}}$ that will be used in the alternative characterization of $\mathbb{S}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T})$ as the kernel of a suitable linear map. For each vertical edge τ of $\mathcal{T}$ we consider the following subspace of $\mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}}$:

$$\mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\tau) := \big\{\, q \in \mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}} :\ D^k_x q(\bar{x}, y) \equiv 0,\ \forall y \in [a_v, b_v],\ k = 0,\dots,r(\tau)\,\big\}, \tag{2.2}$$

where $\bar{x}$ is the abscissa of any point of τ. Analogously, for each horizontal edge τ we set

$$\mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\tau) := \big\{\, q \in \mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}} :\ D^l_y q(x, \bar{y}) \equiv 0,\ \forall x \in [a_h, b_h],\ l = 0,\dots,r(\tau)\,\big\}, \tag{2.3}$$

where $\bar{y}$ is the ordinate of any point of τ. Moreover, for each vertex $\gamma := (\bar{x}, \bar{y})$ we define the subspace

$$\mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\gamma) := \big\{\, q \in \mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}} : D^k_x D^l_y q(\bar{x}, \bar{y}) \equiv 0,\ k = 0,\dots,r_h(\gamma),\ \ l = 0,\dots,r_v(\gamma)\,\big\}. \tag{2.4}$$

As done in [29] in the algebraic polynomial case, we define the following complexes, see also [2, 5, 34]:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
\mathfrak{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T}^o): & & 0 \arrow[r, "\hat{\partial}_2"] \arrow[d] & \bigoplus\limits_{\tau\in\mathcal{T}^o_1} \mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\tau) \arrow[r, "\hat{\partial}_1"] \arrow[d] & \bigoplus\limits_{\gamma\in\mathcal{T}^o_0} \mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\gamma) \arrow[r, "\hat{\partial}_0"] \arrow[d] & 0 \\
\mathfrak{P}^{\boldsymbol{T}}_{\boldsymbol{p}}(\mathcal{T}^o): & 0 \arrow[r, "\partial_3"] & \bigoplus\limits_{\sigma\in\mathcal{T}_2} \mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}} \arrow[r, "\partial_2"] \arrow[d] & \bigoplus\limits_{\tau\in\mathcal{T}^o_1} \mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}} \arrow[r, "\partial_1"] \arrow[d] & \bigoplus\limits_{\gamma\in\mathcal{T}^o_0} \mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}} \arrow[r, "\partial_0"] \arrow[d] & 0 \\
\mathfrak{S}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T}^o): & 0 \arrow[r, "\bar{\partial}_3"] & \bigoplus\limits_{\sigma\in\mathcal{T}_2} \mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}} \arrow[r, "\bar{\partial}_2"] \arrow[d] & \bigoplus\limits_{\tau\in\mathcal{T}^o_1} \mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}}/\mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\tau) \arrow[r, "\bar{\partial}_1"] \arrow[d] & \bigoplus\limits_{\gamma\in\mathcal{T}^o_0} \mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}}/\mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\gamma) \arrow[r, "\bar{\partial}_0"] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd} \tag{2.5}$$

The maps of the complex $\mathfrak{P}^{\boldsymbol{T}}_{\boldsymbol{p}}(\mathcal{T}^o)$ are induced by the usual boundary maps, so they are defined as follows. We consider all the edges $\tau \in \mathcal{T}_1$ oriented, and we use the notation $\tau = [\gamma_1\gamma_2]$, where $\gamma_1, \gamma_2 \in \mathcal{T}_0$. The opposite edge is denoted by $[\gamma_2\gamma_1]$, and by convention we set $[\gamma_1\gamma_2] = -[\gamma_2\gamma_1]$.

- The map ∂_3 is the identity map.
- The map $\partial_2 : \bigoplus_{\sigma\in\mathcal{T}_2} \mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}} \to \bigoplus_{\tau\in\mathcal{T}^o_1} \mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}}$ is given by

  $$\partial_2(q) := \bigoplus_{\tau\in\mathcal{T}^o_1} \sum_{\sigma\in S(\tau)} q_\sigma, \qquad q \in \bigoplus_{\sigma\in\mathcal{T}_2} \mathbb{P}^{\boldsymbol{T}}_{\boldsymbol{p}},$$

  where, for each $\tau \in \mathcal{T}^o_1$, $S(\tau)$ is the set of cells in $\mathcal{T}_2$ which contain τ, and for each cell $\sigma \in \mathcal{T}_2$, whose counter-clockwise boundary is formed by the edges $\tau_1 = [\gamma_1\gamma_2], \dots, \tau_l = [\gamma_l\gamma_1]$, q_σ is the component of q associated with the cell σ if the boundary of σ contains τ and its opposite if the boundary of σ contains the opposite of τ.

- The map $\partial_1 : \bigoplus_{\tau\in\mathcal{T}_1^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}} \to \bigoplus_{\gamma\in\mathcal{T}_0^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}$ is given by

$$\partial_1(q) := \bigoplus_{\gamma\in\mathcal{T}_0^o} \sum_{\tau\in E(\gamma)} q_\tau, \qquad q \in \bigoplus_{\tau\in\mathcal{T}_1^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}},$$

  where, for each $\gamma \in \mathcal{T}_0^o$, $E(\gamma)$ is the set of edges in $\mathcal{T}_1^o$ which have γ as one of the endpoints, and, for each oriented edge $\tau = [\gamma_1\gamma_2] \in \mathcal{T}_1^o$, q_τ is the component of q associated with τ if $\gamma = \gamma_2$ and its opposite if $\gamma = \gamma_1$.

- For each $q \in \bigoplus_{\gamma\in\mathcal{T}_0^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}$, $\partial_0(q) := 0$.

The maps of the complex $\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o)$, denoted by $\hat{\partial}_2$, $\hat{\partial}_1$ and $\hat{\partial}_0$, are obtained from ∂_2, ∂_1 and ∂_0 by restriction. Indeed, for each $\tau \in \mathcal{T}_1^o$ and $\gamma \in \mathcal{T}_0^o$, an element q_τ of $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau)$ also belongs to $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\gamma)$, provided that the edge τ has an endpoint in γ, and therefore $\sum_{\tau\in E(\gamma)} q_\tau$ belongs to $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\gamma)$ as well. As a consequence, the image of the restriction of ∂_1 to $\bigoplus_{\tau\in\mathcal{T}_1^o} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau)$ is included in $\bigoplus_{\gamma\in\mathcal{T}_0^o} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\gamma)$.

The maps of $\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o)$, denoted by $\bar{\partial}_2$, $\bar{\partial}_1$ and $\bar{\partial}_0$, are naturally induced since the considered vector spaces are quotients of the ones of $\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{T}}(\mathcal{T}^o)$.

Note that, by construction, we have $\hat{\partial}_i \circ \hat{\partial}_{i+1} = 0$, $\partial_i \circ \partial_{i+1} = 0$, $\bar{\partial}_i \circ \bar{\partial}_{i+1} = 0$, $i = 0, 1$.

The vertical maps in each column of the diagram in (2.5) are the inclusion and the quotient map, respectively.

We will now study the homology of the complexes in (2.5). Our interest is motivated by the fact that the homology of the cells in $\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o)$ is related to the space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})$. More precisely, we have the following proposition. Its proof is completely analogous to the proof of [5, Proposition 1] which extends the result in [29, Proposition 2.9].

Proposition 2.1. *For the complex $\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o)$ in (2.5), we have*

$$H_2(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o)) = \ker \bar{\partial}_2 = \mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}). \tag{2.6}$$

In order to determine the dimension of the space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})$, we can consider the Euler characteristic of the complex

$$\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o):\ 0 \overset{\bar{\partial}_3}{\to} \bigoplus_{\sigma\in\mathcal{T}_2} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}} \overset{\bar{\partial}_2}{\to} \bigoplus_{\tau\in\mathcal{T}_1^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau) \overset{\bar{\partial}_1}{\to} \bigoplus_{\gamma\in\mathcal{T}_0^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\gamma) \overset{\bar{\partial}_0}{\to} 0,$$

namely

$$\begin{aligned}\dim\Big(\bigoplus_{\sigma\in\mathcal{T}_2} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}\Big) &- \dim\Big(\bigoplus_{\tau\in\mathcal{T}_1^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau)\Big) + \dim\Big(\bigoplus_{\gamma\in\mathcal{T}_0^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\gamma)\Big)\\ &= \dim\big(H_2(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big) - \dim\big(H_1(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big) + \dim\big(H_0(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big).\end{aligned}$$

Taking into account the equality (2.6) we get the relation

$$\begin{aligned}\dim\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})\big) = \dim\Big(\bigoplus_{\sigma\in\mathcal{T}_2} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}\Big) &- \dim\Big(\bigoplus_{\tau\in\mathcal{T}_1^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau)\Big) + \dim\Big(\bigoplus_{\gamma\in\mathcal{T}_0^o} \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\gamma)\Big)\\ &+ \dim\big(H_1(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big) - \dim\big(H_0(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big).\end{aligned} \tag{2.7}$$

In the next section we will investigate the different terms in (2.7). Afterwards, in Section 4 we will give combinatorial lower and upper bounds for the dimension of $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})$, and we show that these bounds coincide in some special cases.

3. A deeper look at the terms in (2.7)

In this section we study in more detail the terms in the formula (2.7). Our arguments are based on homological techniques, extending the ones used in [29] for investigating the dimension of the space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{r}}(\mathcal{T})$. First, we address the three terms in (2.7) related to the section space $\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}$, and then we investigate the two homology terms in (2.7).

3.1. Properties of the section space $\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}$

We now analyze the subspaces of $\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}$ appearing in the dimension formula (2.7). The next proposition extends the results in [29, Lemma 1.5] and [5, Lemma 1] to the Tchebycheffian setting.

Proposition 3.1. *The following dimension formulas hold:*

1. $\dim\big(\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}\big) = (p_h+1)(p_v+1)$;
2. $\dim\big(\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau)\big) = \begin{cases} (p_h+1)(r(\tau)+1), & \tau \text{ horizontal} \\ (r(\tau)+1)(p_v+1), & \tau \text{ vertical} \end{cases}$;
3. $\dim\big(\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\gamma)\big) = (r_h(\gamma)+1)(r_v(\gamma)+1)$.

Proof. Proving the first formula is trivial. In order to prove the second formula, we note that a general element of $\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}$ can be written as

$$q(x,y) = \sum_{i=0}^{p_h}\sum_{j=0}^{p_v} a_{i,j}\psi_{\bar{x},i}^{h}(x)\psi_{\bar{y},j}^{v}(y), \tag{3.1}$$

where $\{\psi_{\bar{x},i}^{h}\}_{i=0}^{p_h}$ and $\{\psi_{\bar{y},j}^{v}\}_{j=0}^{p_v}$ are Taylor-like bases of the spaces $\mathbb{T}_{p_h}^{h}$ and $\mathbb{T}_{p_v}^{v}$, respectively, such that

$$D_x^k\psi_{\bar{x},i}^{h}(\bar{x}) = \delta_{ik}, \qquad k=0,\dots,p_h, \qquad i=0,\dots,p_h,$$
$$D_y^l\psi_{\bar{y},j}^{v}(\bar{y}) = \delta_{jl}, \qquad l=0,\dots,p_v, \qquad j=0,\dots,p_v.$$

Let us assume that τ is vertical where $\bar{x}$ is its abscissa (the proof for τ horizontal is analogous). An element belonging to $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau)$ must then satisfy the conditions

$$0 = D_x^k q(\bar{x},y) = \sum_{i=0}^{p_h}\sum_{j=0}^{p_v} a_{i,j}D_x^k\psi_{\bar{x},i}^{h}(\bar{x})\psi_{\bar{y},j}^{v}(y) = \sum_{j=0}^{p_v} a_{k,j}\psi_{\bar{y},j}^{v}(y), \qquad k=0,\dots,r(\tau),$$

for some $\bar{y}\in[a_v,b_v]$ and for all $y\in[a_v,b_v]$. This implies that $a_{k,j}=0$ for $k=0,\dots,r(\tau)$ and $j=0,\dots,p_v$. As a consequence,

$$\dim\big(\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau)\big) = (p_h+1)(p_v+1) - (r(\tau)+1)(p_v+1). \tag{3.2}$$

Then, $\dim\big(\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau)\big) = \dim\big(\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}\big) - \dim\big(\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau)\big) = (r(\tau)+1)(p_v+1)$. Finally, we prove the third item of the lemma. Let us assume that $(\bar{x},\bar{y})$ are the coordinates of γ. An element belonging to $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\gamma)$ can be expressed in the form (3.1) and must satisfy the following conditions

$$0 = D_x^kD_y^l q(\bar{x},\bar{y}) = \sum_{i=0}^{p_h}\sum_{j=0}^{p_v} a_{i,j}D_x^k\psi_{\bar{x},i}^{h}(\bar{x})D_y^l\psi_{\bar{y},j}^{v}(\bar{y}) = a_{k,l}, \qquad k=0,\dots,r_h(\gamma), \quad l=0,\dots,r_v(\gamma),$$

which means that

$$\dim\big(\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\gamma)\big) = (p_h+1)(p_v+1) - (r_h(\gamma)+1)(r_v(\gamma)+1).$$

Then, since $\dim\big(\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}/\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\gamma)\big) = \dim\big(\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}\big) - \dim\big(\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\gamma)\big)$, the proof is complete. □

Consider an extended Tchebycheff space $\mathbb{T}_p(J)$ of dimension $p+1$ on J. Given $d\in\mathbb{N}$ and $\bar{x}\in J$, we define

$$\mathbb{I}_p^{T,d}(\bar{x}) := \big\{\, q\in\mathbb{T}_p(J) : D_x^k q(\bar{x}) = 0,\ k=0,\dots,d \,\big\}, \tag{3.3}$$

where $T := \mathbb{T}_p$. We note that

- $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau_h) = \mathbb{T}_{p_h}^{h}\otimes\mathbb{I}_{p_v}^{T_v,r(\tau_h)}(\bar{y})$, if τ_h is a horizontal edge and $\bar{y}$ is the ordinate of any point of τ_h;
- $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau_v) = \mathbb{I}_{p_h}^{T_h,r(\tau_v)}(\bar{x})\otimes\mathbb{T}_{p_v}^{v}$, if τ_v is a vertical edge and $\bar{x}$ is the abscissa of any point of τ_v;

- $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\gamma) = \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau_h) + \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau_v)$ if γ is a vertex such that $\gamma = \tau_h \cap \tau_v$.

The following property is an important ingredient for the dimension results later on in Section 3.2 and Section 4.

Definition 3.1 (**$\boldsymbol{d}$**-sum property)**.** *Consider an extended Tchebycheff space $\mathbb{T}_p(J)$ of dimension $p+1$ on J. Let $\boldsymbol{d} := (d_1, \ldots, d_m)$ with $0 \le d_i \le p$, $d_i \in \mathbb{N}$, $i = 1, \ldots, m$. We say that $\mathbb{T}_p(J)$ has the **$\boldsymbol{d}$**-sum property if for any set of m distinct points $\bar{x}_1, \ldots, \bar{x}_m \in J$ we have*

$$\dim\Bigg(\sum_{i=1}^{m} \mathbb{I}_p^{T,d_i}(\bar{x}_i)\Bigg) = \min\Bigg(p+1, \sum_{i=1}^{m} p - d_i\Bigg), \tag{3.4}$$

where $\mathbb{I}_p^{T,d_i}(\bar{x}_i)$ is defined in (3.3).

Example 3.1. *The space of algebraic polynomials $\mathbb{P}_p$ has the **$\boldsymbol{d}$**-sum property for any $\boldsymbol{d} := (d_1, \ldots, d_m)$ with $0 \le d_i \le p$, $d_i \in \mathbb{N}$, $i = 1, \ldots, m$ and for any $m \in \mathbb{N}$; see [29, Proposition 1.8].*

Proposition 3.2. *Consider an extended Tchebycheff space $\mathbb{T}_p(J)$ of dimension $p+1$ on J. Let $\boldsymbol{d} := (d_1, \ldots, d_m)$ with $0 \le d_i \le p$, $d_i \in \mathbb{N}$, $i = 1, \ldots, m$. It holds*

- *if $m = 1$, then $\mathbb{T}_p(J)$ has the **$\boldsymbol{d}$**-sum property;*
- *if $p \ge d_k + d_l + 1$ for at least a pair $k, l \in \{1, \ldots, m\}$, then $\mathbb{T}_p(J)$ has the **$\boldsymbol{d}$**-sum property.*

Proof. The first item immediately follows from Definition 2.3. Let us now prove the second item, for which it suffices to consider $m \ge 2$. From the Grassmann formula for the dimension of vector spaces we deduce

$$\begin{aligned}\dim\Bigg(\sum_{i=1}^{m} \mathbb{I}_p^{T,d_i}(\bar{x}_i)\Bigg) &\ge \dim\big(\mathbb{I}_p^{T,d_k}(\bar{x}_k) + \mathbb{I}_p^{T,d_l}(\bar{x}_l)\big)\\ &= \dim\big(\mathbb{I}_p^{T,d_k}(\bar{x}_k)\big) + \dim\big(\mathbb{I}_p^{T,d_l}(\bar{x}_l)\big) - \dim\big(\mathbb{I}_p^{T,d_k}(\bar{x}_k) \cap \mathbb{I}_p^{T,d_l}(\bar{x}_l)\big).\end{aligned}$$

Moreover, from Definition 2.3 we get

$$\dim\big(\mathbb{I}_p^{T,d_i}(\bar{x}_i)\big) = p - d_i,$$

$$\dim\big(\mathbb{I}_p^{T,d_k}(\bar{x}_k) \cap \mathbb{I}_p^{T,d_l}(\bar{x}_l)\big) = (p + 1 - (d_k + 1) - (d_l + 1))_+ = (p - d_k - d_l - 1)_+,$$

where $(x)_+ := \max(x, 0)$. Since $p \ge d_k + d_l + 1$, we deduce

$$p + 1 \ge \dim\Bigg(\sum_{i=1}^{m} \mathbb{I}_p^{T,d_i}(\bar{x}_i)\Bigg) \ge (p - d_k) + (p - d_l) - (p - d_k - d_l - 1) = p + 1.$$

This gives the value in (3.4) for $\dim\big(\sum_{i=1}^{m} \mathbb{I}_p^{T,d_i}(\bar{x}_i)\big)$. $\square$

Inspired by the polynomial case (see Example 3.1), we make the following conjecture.

Conjecture 3.1. *Any extended complete Tchebycheff space $\mathbb{T}_p(J)$ has the **$\boldsymbol{d}$**-sum property for any $\boldsymbol{d} := (d_1, \ldots, d_m)$ with $0 \le d_i \le p$, $d_i \in \mathbb{N}$, $i = 1, \ldots, m$ and for any $m \in \mathbb{N}$. For the formal definition of an extended complete Tchebycheff space we refer to [31, Chapter 9].*

3.2. Properties of the homology spaces

In Proposition 3.3 we address the exactness of $\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{T}}(\mathcal{T}^o)$. These results can be proved with the same line of arguments as considered in [29] to prove Propositions D.1–D.3 for the algebraic polynomial case. Indeed, their proofs are just based on general properties of complexes and on the topological features of the T-mesh. For this reason, we omit the corresponding technical proofs.

Proposition 3.3. *For the complex* $\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o)$ *in (2.5), we have*

$$H_0(\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{T}}(\mathcal{T}^o)) = 0, \qquad H_1(\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{T}}(\mathcal{T}^o)) = 0, \qquad H_2(\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{T}}(\mathcal{T}^o)) = \mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}.$$

The next proposition extends [29, Lemma 2.2 and Proposition 2.7] to the case of Tchebycheffian splines. Their proofs are again based on general properties of complexes and on the topological features of the T-mesh, just like in [29]; therefore we omit them.

Proposition 3.4. *For the complexes* $\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o)$, $\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o)$ *and* $\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o)$ *in (2.5), we have*

$$H_0(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o)) = H_0(\mathfrak{P}_{\boldsymbol{p}}^{\boldsymbol{T}}(\mathcal{T}^o)) = 0;$$

$$H_1(\mathfrak{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o)) = H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o)).$$

From Proposition 3.4 and the formula in (2.7) we see that is important to study more in detail $H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))$ and its dimension. In the remainder of this section we will address this point.

For any $\gamma := (\bar{x}, \bar{y}) \in \mathcal{T}_0^o$, let $E_h(\gamma)$ and $E_v(\gamma)$ be the sets of horizontal and vertical interior edges containing γ. We also set $E(\gamma) := E_h(\gamma) \cup E_v(\gamma)$. Moreover, let $P_h(\gamma)$ and $P_v(\gamma)$ be the sets of pairs (τ, τ') of horizontal and vertical interior edges containing γ, and $P(\gamma) := P_h(\gamma) \cup P_v(\gamma)$. We now define the map

$$\phi_\gamma^{\boldsymbol{T}} : \bigoplus_{\tau \in E(\gamma)} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau) \longrightarrow \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\gamma), \tag{3.5}$$

which relates an element $q \in \bigoplus_{\tau \in E(\gamma)} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tau)$ with the element $\phi_\gamma^{\boldsymbol{T}}(q) \in \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\gamma)$ identified by

$$\phi_\gamma^{\boldsymbol{T}}(q) := \sum_{\tau \in E(\gamma)} q_\tau,$$

where q_τ stands for the component of q associated with the edge τ. In the polynomial case we denote this map by

$$\phi_\gamma : \bigoplus_{\tau \in E(\gamma)} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{r}}(\tau) \longrightarrow \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{r}}(\gamma), \tag{3.6}$$

for which the following proposition has been proved in [29, Proposition 2.3]. Note that in [29] the corresponding map (denoted by φ_γ) is defined on a set isomorphic to $\bigoplus_{\tau \in E(\gamma)} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{r}}(\tau)$.

Proposition 3.5. *For the map* ϕ_γ *in (3.6), we have*

$$\ker \phi_\gamma = \sum_{(\tau,\tau') \in P(\gamma)} K_{\tau,\tau'} + \sum_{\tau \in E_h(\gamma), \tau' \in E_v(\gamma)} K_{\tau,\tau'},$$

where

$$K_{\tau,\tau'} := \left\{ q \in \bigoplus_{\tilde{\tau} \in E(\gamma)} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{r}}(\tilde{\tau}) : q_\tau = -q_{\tau'} \ \textit{and} \ q_{\tilde{\tau}} = 0 \textit{ for each } \tilde{\tau} \notin \{\tau, \tau'\} \right\}.$$

We will now prove an analogous proposition for the case of extended Tchebycheff spaces.

Proposition 3.6. *For the map* $\phi_\gamma^{\boldsymbol{T}}$ *in (3.5), we have*

$$\ker \phi_\gamma^{\boldsymbol{T}} = \sum_{(\tau,\tau') \in P(\gamma)} K_{\tau,\tau'}^{\boldsymbol{T}} + \sum_{\tau \in E_h(\gamma), \tau' \in E_v(\gamma)} K_{\tau,\tau'}^{\boldsymbol{T}},$$

where

$$K_{\tau,\tau'}^{\boldsymbol{T}} := \left\{ q \in \bigoplus_{\tilde{\tau} \in E(\gamma)} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\tilde{\tau}) : q_\tau = -q_{\tau'} \ \textit{and} \ q_{\tilde{\tau}} = 0 \textit{ for each } \tilde{\tau} \notin \{\tau, \tau'\} \right\}.$$

Proof. We first recall that $\gamma := (\bar{x}, \bar{y})$. Let $\{\psi^h_{\bar{x},i}\}_{i=0}^{p_h}$ and $\{\psi^v_{\bar{y},j}\}_{j=0}^{p_v}$ be Taylor-like bases of the spaces $\mathbb{T}^h_{p_h}$ and $\mathbb{T}^v_{p_v}$, respectively. Then, we can write every component q_τ of any $q \in \bigoplus_{\tau\in E(\gamma)} \mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\tau)$ in the form (3.1). More precisely,

$$q_\tau = \sum_{j=r(\tau)+1}^{p_v} \psi^v_{\bar{y},j}(y) \sum_{i=0}^{p_h} \alpha^\tau_{i,j} \psi^h_{\bar{x},i}(x) \qquad \text{if } \tau \in E_h(\gamma),$$

and

$$q_\tau = \sum_{i=r(\tau)+1}^{p_h} \psi^h_{\bar{x},i}(x) \sum_{j=0}^{p_v} \alpha^\tau_{i,j} \psi^v_{\bar{y},j}(y) \qquad \text{if } \tau \in E_v(\gamma).$$

As a consequence, we have

$$\phi^{\boldsymbol{T}}_\gamma(q) = \sum_{i=0}^{p_h}\sum_{j=0}^{p_v} \sum_{\tau\in E(\gamma)} \alpha^\tau_{i,j} \psi^h_{\bar{x},i}(x) \psi^v_{\bar{y},j}(y),$$

taking $\alpha^\tau_{i,j} := 0$ for $j = 0, \ldots, r(\tau)$, $\tau \in E_h(\gamma)$, and $\alpha^\tau_{i,j} := 0$ for $i = 0, \ldots, r(\tau)$, $\tau \in E_v(\gamma)$. Therefore, an element belongs to $\ker \phi^{\boldsymbol{T}}_\gamma$ if and only if it satisfies

$$D^k_x D^l_y\big(\phi^{\boldsymbol{T}}_\gamma(q)\big)(\bar{x}, \bar{y}) = \sum_{\tau\in E_v(\gamma)} \alpha^\tau_{k,l} = 0, \qquad k > r_h(\gamma), \quad l \le r_v(\gamma), \tag{3.7}$$

$$D^k_x D^l_y\big(\phi^{\boldsymbol{T}}_\gamma(q)\big)(\bar{x}, \bar{y}) = \sum_{\tau\in E_h(\gamma)} \alpha^\tau_{k,l} = 0, \qquad k \le r_h(\gamma), \quad l > r_v(\gamma), \tag{3.8}$$

$$D^k_x D^l_y\big(\phi^{\boldsymbol{T}}_\gamma(q)\big)(\bar{x}, \bar{y}) = \sum_{\tau\in E(\gamma)} \alpha^\tau_{k,l} = 0, \qquad k > r_h(\gamma), \quad l > r_v(\gamma). \tag{3.9}$$

From its definition it is clear that any element of $K^{\boldsymbol{T}}_{\tau,\tau'}$ satisfies the conditions in (3.7)–(3.9). It follows that any element of $\sum_{(\tau,\tau')\in P(\gamma)} K^{\boldsymbol{T}}_{\tau,\tau'} + \sum_{\tau\in E_h(\gamma),\tau'\in E_v(\gamma)} K^{\boldsymbol{T}}_{\tau,\tau'}$ satisfies these conditions as well, and that this set is included in $\ker \phi^{\boldsymbol{T}}_\gamma$. Hence, it is sufficient to prove that

$$\ker \phi^{\boldsymbol{T}}_\gamma \subseteq \sum_{(\tau,\tau')\in P(\gamma)} K^{\boldsymbol{T}}_{\tau,\tau'} + \sum_{\tau\in E_h(\gamma),\tau'\in E_v(\gamma)} K^{\boldsymbol{T}}_{\tau,\tau'}. \tag{3.10}$$

Suppose now $q \in \bigoplus_{\tau\in E(\gamma)} \mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\tau)$ satisfying (3.7)–(3.9). Moreover, suppose γ is a crossing vertex with $E_h(\gamma) = \{\tau_1, \tau_2\}$ and $E_v(\gamma) = \{\tau_3, \tau_4\}$. Then, we can decompose q as

$$q = q^P + q^E,$$

where the coefficients of q^P are given by

$$\begin{aligned}
&\alpha^{P,\tau_k}_{i,j} := \alpha^{\tau_k}_{i,j}, \qquad k = 1,2,3,4, \quad i \le r_h(\gamma) \text{ or } j \le r_v(\gamma),\\
&\alpha^{P,\tau_1}_{i,j} := -\alpha^{\tau_2}_{i,j}, \quad \alpha^{P,\tau_2}_{i,j} := \alpha^{\tau_2}_{i,j}, \quad \alpha^{P,\tau_3}_{i,j} := -\alpha^{\tau_4}_{i,j}, \quad \alpha^{P,\tau_4}_{i,j} := \alpha^{\tau_4}_{i,j}, \qquad i > r_h(\gamma) \text{ and } j > r_v(\gamma),
\end{aligned}$$

and the coefficients of q^E are given by

$$\begin{aligned}
&\alpha^{E,\tau_k}_{i,j} := 0, \qquad k = 1,2,3,4, \quad i \le r_h(\gamma) \text{ or } j \le r_v(\gamma),\\
&\alpha^{E,\tau_1}_{i,j} := \alpha^{\tau_1}_{i,j} + \alpha^{\tau_2}_{i,j}, \quad \alpha^{E,\tau_2}_{i,j} := 0, \quad \alpha^{E,\tau_3}_{i,j} := \alpha^{\tau_3}_{i,j} + \alpha^{\tau_4}_{i,j}, \quad \alpha^{E,\tau_4}_{i,j} := 0, \qquad i > r_h(\gamma) \text{ and } j > r_v(\gamma).
\end{aligned}$$

We see that $q^P \in K^{\boldsymbol{T}}_{\tau_1,\tau_2} + K^{\boldsymbol{T}}_{\tau_3,\tau_4}$ because (3.7)–(3.8) hold and so $\alpha^{\tau_1}_{i,j} = -\alpha^{\tau_2}_{i,j}$ and $\alpha^{\tau_3}_{i,j} = -\alpha^{\tau_4}_{i,j}$. Moreover, $q^E \in K^{\boldsymbol{T}}_{\tau_1,\tau_3}$ because (3.9) holds and so $\alpha^{\tau_1}_{i,j} + \alpha^{\tau_2}_{i,j} = -\alpha^{\tau_3}_{i,j} - \alpha^{\tau_4}_{i,j}$. This means that $q \in K^{\boldsymbol{T}}_{\tau_1,\tau_2} + K^{\boldsymbol{T}}_{\tau_3,\tau_4} + K^{\boldsymbol{T}}_{\tau_1,\tau_3}$, and we may conclude that (3.10) holds for a crossing vertex γ. A similar argument can be applied when γ is a T-vertex. $\square$

Note that, for each element $q \in K^{\boldsymbol{T}}_{\tau,\tau'}$ and $\gamma := (\bar{x}, \bar{y})$, we have

$$
\begin{aligned}
&D_x^k D_y^l q_{\tilde{\tau}}(\bar{x},\bar{y}) = 0, \qquad && l \le r_v(\gamma) \qquad && \text{if } \tilde{\tau} \in \{\tau,\tau'\} = E_h(\gamma),\\
&D_x^k D_y^l q_{\tilde{\tau}}(\bar{x},\bar{y}) = 0, \qquad && k \le r_h(\gamma) \qquad && \text{if } \tilde{\tau} \in \{\tau,\tau'\} = E_v(\gamma),\\
&D_x^k D_y^l q_{\tilde{\tau}}(\bar{x},\bar{y}) = 0, \qquad && k \le r_h(\gamma) \text{ or } l \le r_v(\gamma) \qquad && \text{if } \tilde{\tau} \in \{\tau,\tau'\},\ \tau \in E_h(\gamma),\ \tau' \in E_v(\gamma).
\end{aligned}
$$

By exploiting the analogy between Propositions 3.5 and 3.6, the following result can be obtained as a direct generalization of [29, Proposition 2.4].

Proposition 3.7. *For the complex $\mathfrak{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T}^o)$ in (2.5), we have*

$$
\begin{aligned}
H_0(\mathfrak{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T}^o)) = &\left(\bigoplus_{\gamma\in\mathcal{T}^o_0}\bigoplus_{\tau\in E(\gamma)} \mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\tau)\right)\\
&\Bigg/\left(\sum_{(\tau,\tau')\in P(\gamma),\gamma\in\mathcal{T}^o_0} K^{\boldsymbol{T}}_{\tau,\tau',\gamma} + \sum_{\tau\in E_h(\gamma),\tau'\in E_v(\gamma),\gamma\in\mathcal{T}^o_0} K^{\boldsymbol{T}}_{\tau,\tau',\gamma} + \sum_{\tau=(\gamma,\gamma')\in\mathcal{T}^o_1} K^{\boldsymbol{T}}_{\tau,\gamma,\gamma'}\right),
\end{aligned}
$$

where

$$
K^{\boldsymbol{T}}_{\tau,\tau',\gamma} := \left\{ q \in \bigoplus_{\tilde{\gamma}\in\mathcal{T}^o_0}\bigoplus_{\tilde{\tau}\in E(\tilde{\gamma})} \mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\tilde{\tau}) : q_{\gamma,\tau} = -q_{\gamma,\tau'} \text{ and } q_{\tilde{\gamma},\tilde{\tau}} = 0 \text{ for each } (\tilde{\gamma},\tilde{\tau}) \notin \{(\gamma,\tau),(\gamma,\tau')\} \right\},
$$

$$
\begin{aligned}
K^{\boldsymbol{T}}_{\tau,\gamma,\gamma'} := \Bigg\{ q \in \bigoplus_{\tilde{\gamma}\in\mathcal{T}^o_0}\bigoplus_{\tilde{\tau}\in E(\tilde{\gamma})} \mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\tilde{\tau}) : q_{\gamma,\tau} = -q_{\gamma',\tau} \text{ if } \gamma,\gamma' \in \mathcal{T}^o_0&\\
\text{and } q_{\tilde{\tau},\tilde{\gamma}} = 0 \text{ for each } (\tilde{\gamma},\tilde{\tau}) \notin \{(\gamma,\tau),(\gamma',\tau)\}&\Bigg\},
\end{aligned}
$$

and where $q_{\gamma,\tau}$ stands for the component of q associated with the vertex γ and the edge τ.

We can further simplify the expression of $H_0(\mathfrak{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T}^o))$ by using the concept of maximal interior segments. The next proposition generalizes [29, Proposition 2.5] and can be proved with the same line of arguments by taking into account the analogy between Proposition 3.7 and [29, Proposition 2.4].

Proposition 3.8. *For the complex $\mathfrak{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T}^o)$ in (2.5), we have*

$$
H_0(\mathfrak{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T}^o)) = \left(\bigoplus_{\rho\in\mathrm{MIS}(\mathcal{T})} \mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\rho)\right)\Bigg/\left(\sum_{\gamma\in\mathcal{T}^o_0} K^{\boldsymbol{T}}_{\gamma}\right),
$$

where $\mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\rho) := \mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\tau)$ for any $\tau \subseteq \rho$,

$$
K^{\boldsymbol{T}}_{\gamma} := \left\{ q \in \bigoplus_{\rho\in\mathrm{MIS}(\mathcal{T})} \mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\rho) : \exists \tilde{q} \in \widetilde{K}^{\boldsymbol{T}}_{\gamma} \text{ such that } q_\rho = \tilde{q}_\rho \text{ for each } \rho \right\},
$$

$$
\widetilde{K}^{\boldsymbol{T}}_{\gamma} := \left\{ \tilde{q} \in \bigoplus_{\rho\in\mathrm{MS}(\mathcal{T})} \mathbb{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\rho) : \tilde{q}_{\rho_h(\gamma)} = -\tilde{q}_{\rho_v(\gamma)} \text{ and } \tilde{q}_\rho = 0 \text{ for each } \rho \notin \{\rho_h(\gamma),\rho_v(\gamma)\} \right\},
$$

and where q_ρ stands for the component of q associated with the maximal segment ρ.

Example 3.2. *Consider the T-mesh $\mathcal{T}$ in Figure 3, and the constant smoothness $r(\tau) = 1$ for every $\tau \in \mathcal{T}^o_1$. Moreover, let $\boldsymbol{T} = (\mathbb{P}_2, \mathbb{P}_2)$, so $p_h = p_v = 2$. Then, the space $K^{\boldsymbol{T}}_{\gamma}$ for $\gamma_8, \gamma_3, \gamma_2$, is given, respectively, by*

- $K^{\boldsymbol{T}}_{\gamma_8} = \{\, (0,0,0) \,\}$;
- $K^{\boldsymbol{T}}_{\gamma_3} = \{\, (a(x-x_2)^2(y-y_3)^2, 0, 0) : a \in \mathbb{R} \,\}$;

- $K_{\gamma_2}^{\boldsymbol{T}} = \{\, (a(x-x_2)^2(y-y_2)^2, 0, -a(x-x_2)^2(y-y_2)^2) : a \in \mathbb{R} \,\}$.

We are now going to bound $\dim\big(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big)$ for general T-meshes (see Theorem 3.1). Let ι be a given ordering of $\textsc{mis}(\mathcal{T})$. For any $\rho \in \textsc{mis}(\mathcal{T})$, we denote by $\Gamma_\iota(\rho)$ the set of vertices of ρ which do not belong to $\rho' \in \textsc{mis}(\mathcal{T})$ with $\iota(\rho') > \iota(\rho)$. The cardinality of such a set is denoted by $\lambda_\iota(\rho)$.

Definition 3.2 (Weight of MIS)**.** *Given an ordering ι of $\textsc{mis}(\mathcal{T})$, the weight of a maximal interior segment $\rho \in \textsc{mis}(\mathcal{T})$ is defined as*

$$\omega_\iota(\rho) := \begin{cases} \sum_{\gamma\in\Gamma_\iota(\rho)}(p_h - r_h(\gamma)), & \text{if } \rho \in \textsc{mis}_h(\mathcal{T}) \\ \sum_{\gamma\in\Gamma_\iota(\rho)}(p_v - r_v(\gamma)), & \text{if } \rho \in \textsc{mis}_v(\mathcal{T}) \end{cases}.$$

Example 3.3. *Consider the T-mesh $\mathcal{T}$ in Figure 3 and Example 2.1. The ordering ι of $\textsc{mis}(\mathcal{T})$ is given by $\iota(\rho_j) = j$, $j = 1, 2, 3$. In this case, we have*

$$\Gamma_\iota(\rho_1) = \{\gamma_1, \gamma_3\}, \quad \Gamma_\iota(\rho_2) = \{\gamma_4, \gamma_6\}, \quad \Gamma_\iota(\rho_3) = \{\gamma_2, \gamma_5\},$$

so $\lambda_\iota(\rho_1) = \lambda_\iota(\rho_2) = \lambda_\iota(\rho_3) = 2$. Moreover, let $p_h = p_v = 2$ and let $r_h(\gamma) = r_v(\gamma) = 1$ for every $\gamma \in \mathcal{T}_0^o$. Then, the corresponding weights of the maximal interior segments are

$$\omega_\iota(\rho_1) = \omega_\iota(\rho_2) = \omega_\iota(\rho_3) = 2(2-1) = 2.$$

Indeed, $p_\ell - r_\ell(\gamma) = 2 - 1 = 1$ for every $\rho \in \textsc{mis}_\ell(\mathcal{T})$, $\ell = h, v$.

Definition 3.3 ($\boldsymbol{r}$-sum property on $\mathcal{T}$)**.** *Given a smoothness distribution $\boldsymbol{r}$ on $\mathcal{T}$, we say that $\boldsymbol{T} := (\mathbb{T}_{p_h}^h, \mathbb{T}_{p_v}^v)$ has the $\boldsymbol{r}$-sum property on $\mathcal{T}$, if each of its components $\mathbb{T}_{p_\ell}^\ell([a_\ell, b_\ell])$ with $\ell = h, v$ has the $\boldsymbol{d}$-sum property (see Definition 3.1) for any subvector $\boldsymbol{d}$ of the vector $\boldsymbol{r}_\ell := (r_\ell(\gamma))_{\gamma\in\mathcal{T}_0^o}$.*

In the next theorem we give bounds for the dimension of $H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))$. Its proof makes use of the characterization of the space provided in Proposition 3.8, and is a slightly reformulated version of the proof in [29, Theorem 3.7].

Theorem 3.1. *Let ι be a given ordering of $\textsc{mis}(\mathcal{T})$, and assume that $\boldsymbol{T}$ has the $\boldsymbol{r}$-sum property on $\mathcal{T}$. It holds*

$$\begin{aligned} 0 \le \dim\big(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big) \le & \sum_{\rho\in\textsc{mis}_h(\mathcal{T})} (p_h + 1 - \omega_\iota(\rho))_+ \, (p_v - r(\rho)) \\ & + \sum_{\rho\in\textsc{mis}_v(\mathcal{T})} (p_h - r(\rho)) \, (p_v + 1 - \omega_\iota(\rho))_+, \end{aligned}$$

where $(x)_+ := \max(x, 0)$.

Proof. The lower bound is trivial, so we just focus on the upper bound. Let $\rho_1, \dots, \rho_N$ be the maximal interior segments of $\mathcal{T}$ where the indices are ordered according to the given ordering ι.

Any element $q \in \bigoplus_{\rho\in\textsc{mis}(\mathcal{T})} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\rho) =: R^{\boldsymbol{T}}$ can be seen as a vector $(q_{\rho_1}, \dots, q_{\rho_N})$ where $q_{\rho_i} \in \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\rho_i)$. Note that the space $K_\gamma^{\boldsymbol{T}}$ defined in Proposition 3.8 is a subspace of $R^{\boldsymbol{T}}$ consisting of the elements q whose components satisfy the following conditions:

- if $\gamma = (\bar{x}, \bar{y})$ is the intersection of the maximal interior segments ρ_i and ρ_j, then

$$q_\rho = 0 \quad \text{for each } \rho \notin \{\rho_i, \rho_j\}, \qquad \text{and} \qquad q_{\rho_i} = -q_{\rho_j},$$
$$D_x^k D_y^l q_{\rho_i}(\bar{x}, \bar{y}) = D_x^k D_y^l q_{\rho_j}(\bar{x}, \bar{y}) = 0, \qquad 0 \le k \le r_h(\gamma) \text{ or } 0 \le l \le r_v(\gamma);$$

- if $\gamma = (\bar{x}, \bar{y})$ is the intersection of the maximal interior segment ρ_i with a maximal segment intersecting $\partial\Omega$, then

$$q_\rho = 0 \quad \text{for each } \rho \ne \rho_i,$$
$$D_x^k D_y^l q_{\rho_i}(\bar{x}, \bar{y}) = 0, \qquad 0 \le k \le r_h(\gamma) \text{ or } 0 \le l \le r_v(\gamma).$$

For any $q \in R^{\boldsymbol{T}}$, we define $F(q)$ as the element in $R^{\boldsymbol{T}}$ having all zero components except $q_{\rho_{i_{\max}}}$, which is the component of q with the maximal index such that $q_{\rho_i} \neq 0$. Let $F(K^{\boldsymbol{T}})$ be the space spanned by $\{F(q) : q \in K^{\boldsymbol{T}}\}$ where $K^{\boldsymbol{T}} := \sum_{\gamma \in \mathcal{T}_0^o} K_\gamma^{\boldsymbol{T}}$. It is clear that $\dim\bigl(F(K^{\boldsymbol{T}})\bigr) \leq \dim\bigl(K^{\boldsymbol{T}}\bigr)$, so by Proposition 3.8 we have

$$\dim\bigl(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\bigr) = \dim\bigl(R^{\boldsymbol{T}}/K^{\boldsymbol{T}}\bigr) \leq \dim\bigl(R^{\boldsymbol{T}}/F(K^{\boldsymbol{T}})\bigr). \tag{3.11}$$

Moreover, for each $\rho_i \in \text{MIS}(\mathcal{T})$, the space $F(K^{\boldsymbol{T}})$ contains the vectors

$$\begin{array}{ccccccccc} (\,0, & \ldots, & 0, & \ldots, & 0, & q_{\rho_i}, & 0, & \ldots, & 0\,), \\ & & \uparrow & & & \uparrow & & & \\ & & j & & & i & & & \end{array} \tag{3.12}$$

where q_{ρ_i} satisfies

$$D_x^k D_y^l q_{\rho_i}(\bar{x}, \bar{y}) = 0, \qquad 0 \leq k \leq r_h(\gamma) \text{ or } 0 \leq l \leq r_v(\gamma), \tag{3.13}$$

for some $\gamma = (\bar{x}, \bar{y}) \in \Gamma(\rho_i)$. Note that γ is either the intersection of the maximal interior segments ρ_i and ρ_j with $i > j$ or the intersection of the maximal interior segment ρ_i with a maximal segment intersecting $\partial\Omega$. Let Q_i be the space spanned by the vectors of the form (3.12)–(3.13). More precisely, if $\rho_i \in \text{MIS}_h(\mathcal{T})$ belongs to the line $y = \bar{y}$ then

$$Q_i = \Biggl\{ q \in \bigoplus_{\rho \in \text{MIS}(\mathcal{T})} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\rho_i) : q_\rho = 0 \text{ if } \rho \neq \rho_i \text{ and } q_{\rho_i}(x, y) \in \widehat{Q}_i \Biggr\},$$

where

$$\widehat{Q}_i := \Biggl(\sum_{\gamma = (\bar{x}, \bar{y}) \in \Gamma(\rho_i)} \mathbb{I}_p^{T_h, r_h(\gamma)}(\bar{x}) \Biggr) \otimes \mathbb{I}_p^{T_v, r_v(\gamma)}(\bar{y}).$$

A similar characterization holds if $\rho_i \in \text{MIS}_v(\mathcal{T})$. As a consequence, by Definition 3.1, the dimension of Q_i is given by

$$\dim(Q_i) = \dim(\widehat{Q}_i) = \begin{cases} \min(p_h + 1, \omega_\iota(\rho_i))(p_v - r(\rho_i)), & \text{if } \rho_i \in \text{MIS}_h(\mathcal{T}) \\ \min(p_v + 1, \omega_\iota(\rho_i))(p_h - r(\rho_i)), & \text{if } \rho_i \in \text{MIS}_v(\mathcal{T}) \end{cases},$$

and from (3.2) we get

$$\dim\bigl(\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\rho_i)/\widehat{Q}_i\bigr) = \begin{cases} (p_h + 1 - \omega_\iota(\rho))_+(p_v - r(\rho_i)), & \text{if } \rho_i \in \text{MIS}_h(\mathcal{T}) \\ (p_h - r(\rho_i))(p_v + 1 - \omega_\iota(\rho))_+, & \text{if } \rho_i \in \text{MIS}_v(\mathcal{T}) \end{cases}. \tag{3.14}$$

Since $\sum_{i=1}^N Q_i \subseteq F(K^{\boldsymbol{T}})$ and $\dim\bigl(\sum_{i=1}^N Q_i\bigr) = \sum_{i=1}^N \dim(Q_i)$, we obtain

$$\dim\bigl(R^{\boldsymbol{T}}/F(K^{\boldsymbol{T}})\bigr) \leq \dim\Biggl(\Biggl(\bigoplus_{\rho \in \text{MIS}(\mathcal{T})} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\rho)\Biggr) \Big/ \Biggl(\sum_{i=1}^N Q_i\Biggr)\Biggr) = \sum_{i=1}^N \dim\bigl(\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\rho_i)/\widehat{Q}_i\bigr). \tag{3.15}$$

By combining (3.11) and (3.15), we arrive at the upper bound for $\dim\bigl(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\bigr)$ by taking into account (3.14). $\square$

4. Dimension formulas for Tchebycheffian spline spaces over T-meshes

In this section we address the dimension problem in case of Tchebycheffian spline spaces over T-meshes. First, we state a general dimension formula involving a homology term, and we give a lower and upper bound. Then, we consider certain conditions on the T-mesh and/or the Tchebycheffian spline space so that the homology term vanishes.

4.1. A dimension formula with a homology term

Using the results in the previous sections, we are able to state a general dimension formula for any Tchebycheffian spline space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})$ defined over a T-mesh $\mathcal{T}$.

Theorem 4.1. *We have*

$$\begin{aligned}\dim\bigl(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})\bigr) = &\sum_{\sigma\in\mathcal{T}_2}(p_h+1)(p_v+1) - \sum_{\tau\in\mathcal{T}_1^{o,h}}(p_h+1)(r(\tau)+1) - \sum_{\tau\in\mathcal{T}_1^{o,v}}(r(\tau)+1)(p_v+1)\\ &+\sum_{\gamma\in\mathcal{T}_0^o}(r_h(\gamma)+1)(r_v(\gamma)+1) + \dim\bigl(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\bigr). \end{aligned} \tag{4.1}$$

Proof. By combining Propositions 3.1 and 3.4 with the equality (2.7), we arrive at the formula (4.1). □

Since $\dim\bigl(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\bigr) \geq 0$, we immediately get the same lower bound for the dimension of $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})$ as in the algebraic polynomial case, see [29, Section 3]:

$$\begin{aligned}\dim\bigl(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})\bigr) \geq &\sum_{\sigma\in\mathcal{T}_2}(p_h+1)(p_v+1) - \sum_{\tau\in\mathcal{T}_1^{o,h}}(p_h+1)(r(\tau)+1) - \sum_{\tau\in\mathcal{T}_1^{o,v}}(r(\tau)+1)(p_v+1)\\ &+\sum_{\gamma\in\mathcal{T}_0^o}(r_h(\gamma)+1)(r_v(\gamma)+1). \end{aligned} \tag{4.2}$$

Moreover, given an ordering ι of $\textsc{mis}(\mathcal{T})$, from Theorem 3.1 we also obtain an upper bound for the dimension of $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})$, under the assumption that $\boldsymbol{T}$ has the $\boldsymbol{r}$-sum property on $\mathcal{T}$:

$$\begin{aligned}\dim\bigl(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})\bigr) \leq &\sum_{\sigma\in\mathcal{T}_2}(p_h+1)(p_v+1) - \sum_{\tau\in\mathcal{T}_1^{o,h}}(p_h+1)(r(\tau)+1) - \sum_{\tau\in\mathcal{T}_1^{o,v}}(r(\tau)+1)(p_v+1)\\ &+\sum_{\gamma\in\mathcal{T}_0^o}(r_h(\gamma)+1)(r_v(\gamma)+1) + \sum_{\rho\in\textsc{mis}_h(\mathcal{T})}(p_h+1-\omega_\iota(\rho))_+\,(p_v-r(\rho))\\ &+\sum_{\rho\in\textsc{mis}_v(\mathcal{T})}(p_h-r(\rho))\,(p_v+1-\omega_\iota(\rho))_+. \end{aligned} \tag{4.3}$$

We say that a smoothness distribution $\boldsymbol{r}$ on $\mathcal{T}$ is *constant* if there exist $\boldsymbol{\mu} := (\mu_h, \mu_v)$ such that

$$r(\tau_v) = \mu_h, \quad \forall \tau_v \in \mathcal{T}_1^{o,v}, \qquad r(\tau_h) = \mu_v, \quad \forall \tau_h \in \mathcal{T}_1^{o,h}. \tag{4.4}$$

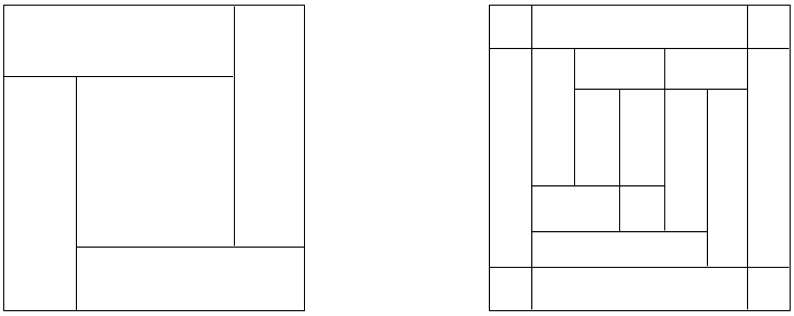
Example 4.1. *Given a T-mesh $\mathcal{T}$ with a constant smoothness distribution $\boldsymbol{r}$ as in* (4.4)*, the dimension formula* (4.1) *simplifies to*

$$\begin{aligned}\dim\bigl(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})\bigr) = &(p_h+1)(p_v+1)f_2 - (p_h+1)(\mu_v+1)f_1^h - (\mu_h+1)(p_v+1)f_1^v\\ &+(\mu_h+1)(\mu_v+1)f_0 + \dim\bigl(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\bigr). \end{aligned} \tag{4.5}$$

The formula (4.5) *corresponds to the formula found in [29] for the algebraic polynomial spline space defined on the same T-mesh. In this case, the homology term can be bounded as*

$$\begin{aligned}0 \leq \dim\bigl(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\bigr) \leq &\sum_{\rho\in\textsc{mis}_h(\mathcal{T})}(p_h+1-(p_h-\mu_h)\lambda_\iota(\rho))_+\,(p_v-\mu_v)\\ &+\sum_{\rho\in\textsc{mis}_v(\mathcal{T})}(p_h-\mu_h)\,(p_v+1-(p_v-\mu_v)\lambda_\iota(\rho))_+, \end{aligned}$$

under the assumption that $\boldsymbol{T}$ has the $\boldsymbol{r}$-sum property on $\mathcal{T}$.



(a)

(b)

Figure 4: (a) A T-mesh with no cycles of MIS but with cycles. (b) A T-mesh with no cycles but with cycles of MIS.

4.2. Dimension formulas without homology term

Under certain conditions on the T-mesh and/or the Tchebycheffian spline space, the homology term in the dimension formula (4.1) is zero, so that the dimension of $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})$ agrees with the lower bound in (4.2). In the following we discuss some noteworthy cases and examples. The first is a direct consequence of Proposition 3.8.

Corollary 4.1. *If the T-mesh $\mathcal{T}$ has no maximal interior segments, then* $\dim\big(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big) = 0$.

Note that a T-mesh with no maximal interior segments is a so-called *quasi-cross-cut partition* of the domain Ω (see, e.g., [7]). Indeed, in such a T-mesh each edge $\tau \in \mathcal{T}_1^o$ extends to the boundary $\partial\Omega$. An edge that extends to the boundary is referred to as a *pseudo-boundary edge* in [30].

Example 4.2. *Let $\mathcal{T}$ be a tensor-product mesh defined on the domain $[a_h, b_h] \times [a_v, b_v]$ by the partitions $a_h = x_0 < \cdots < x_{k+1} = b_h$ and $a_v = y_0 < \cdots < y_{l+1} = b_v$. One can easily check that there are no maximal interior segments. As a consequence, the dimension of $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})$ agrees with the lower bound in* (4.2)*. For instance, taking a constant smoothness distribution $\boldsymbol{r}$ as in* (4.4)*, the dimension formula* (4.5) *simplifies to*

$$\dim\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})\big) = \big((p_h+1)(k+1) - k(\mu_h+1)\big)\big((p_v+1)(l+1) - l(\mu_v+1)\big).$$

In the algebraic polynomial context, it is known (see [29, 32]) that if the degree is large enough with respect to the smoothness, then the dimension of $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{r}}(\mathcal{T})$ agrees with the lower bound in (4.2). This extends to the setting of Tchebycheffian splines, as stated in Corollary 4.2 and Corollary 4.3. To this end, we recall, respectively from [29] and [32], the concepts of *hierarchical T-mesh* and *cycle*. Note that the former is also referred to as *LR-mesh* in [5, 12]. In addition, we define the concept of *cycle of MIS*.

Definition 4.1 (Hierarchical T-mesh)**.** *A hierarchical T-mesh is either an axis-aligned rectangular domain Ω or a T-mesh obtained by splitting a cell of another hierarchical T-mesh along a vertical or horizontal line.*

Definition 4.2 (Cycle)**.** *A segment ϱ of a T-mesh is called a composite edge if all the vertices lying in its interior are T-vertices and if it cannot be extended to a longer segment with the same property. A sequence $\varrho_1, \dots, \varrho_n$ of composite edges in a T-mesh is said to form a cycle if each ϱ_i has one of its endpoints in the interior of ϱ_{i+1} (we assume $\varrho_{n+1} := \varrho_1$).*

Definition 4.3 (Cycle of MIS)**.** *A sequence $\rho_1, \dots, \rho_n$ of maximal interior segments in a T-mesh is said to form a cycle of MIS if each ρ_i has one of its endpoints in the interior of ρ_{i+1} (we assume $\rho_{n+1} := \rho_1$).*

The above three definitions lead to three different families of T-meshes: the family of hierarchical T-meshes, the family of T-meshes without cycles, and the family of T-meshes without cycles of MIS.

Remark 4.1. *The families of T-meshes without cycles and of T-meshes without cycles of MIS do not coincide and none of them includes the other. In fact, there are T-meshes having no cycles of MIS but having cycles, and vice versa (see Figure 4). On the other hand, the family of hierarchical T-meshes is strictly included in both the family of T-meshes without cycles and the family of T-meshes without cycles of MIS. The T-mesh in Figure 4(a) has no cycles of MIS and is not hierarchical; the T-mesh in Figure 4(b) has no cycles and is not hierarchical as well.*

We observe that a T-mesh $\mathcal{T}$ without cycles of MIS allows us to define an ordering ι of $\textsc{mis}(\mathcal{T})$ such that $\rho, \rho' \in \textsc{mis}(\mathcal{T})$ with ρ having one endpoint in the interior of ρ' implies that $\iota(\rho') < \iota(\rho)$. For instance, the ordering of $\textsc{mis}(\mathcal{T})$ in Example 3.3 satisfies this condition. The following algorithm generates such an ordering.

Algorithm 4.1. *Given a T-mesh $\mathcal{T}$ without cycles of MIS, the ordering ι of $\textsc{mis}(\mathcal{T})$ is constructed as follows. Initialize $\mathcal{A} = \textsc{mis}(\mathcal{T})$ and $\mathcal{B} = \emptyset$. Then, repeat as long as $\mathcal{A}$ is not empty:*

1. *select any $\rho \in \mathcal{A}$ such that for each endpoint of ρ belonging to $\rho' \in \textsc{mis}(\mathcal{T})$ we require $\rho' \in \mathcal{B}$;*
2. *set $\iota(\rho) = n + 1$ with n the cardinality of the set $\mathcal{B}$;*
3. *remove ρ from the set $\mathcal{A}$ and insert it in the set $\mathcal{B}$.*

It is clear that the algorithm starts the ordering by selecting a maximal interior segment whose both endpoints do not belong to any other maximal interior segment.

Lemma 4.1. *Algorithm 4.1 terminates in a finite number of steps, and it generates an ordering ι of $\textsc{mis}(\mathcal{T})$ such that $\rho, \rho' \in \textsc{mis}(\mathcal{T})$ with ρ having one endpoint in the interior of ρ' implies that $\iota(\rho') < \iota(\rho)$.*

Proof. We show by contradiction that the algorithm terminates in a finite number of steps. Suppose the set $\mathcal{A}$ is not empty and $\mathcal{A}$ does not contain any element that satisfies the condition in Step 1. This would imply that each $\rho \in \mathcal{A}$ has at least one endpoint belonging to another $\rho' \in \mathcal{A}$. Since the number of elements in $\mathcal{A}$ is finite, this means that there must be a cycle of MIS, which contradicts with the assumption on the T-mesh $\mathcal{T}$. Finally, it is easy to see from Step 1 that the algorithm generates an ordering satisfying the stated condition. ☐

Note that the above algorithm terminates in a finite number of steps if and only if there are no cycles of MIS in the T-mesh. Indeed, from Lemma 4.1 we know that the algorithm terminates in a finite number of steps when there are no cycles of MIS in the T-mesh. On the other hand, if there is a cycle of MIS, say $\boldsymbol{\rho} := \{\rho_1, \ldots, \rho_n\}$, then the set $\mathcal{A}$ can never become empty (and so the algorithm does not terminate) because, by Definition 4.3, none of the elements of $\boldsymbol{\rho}$ satisfies the condition in Step 1. As a consequence, the algorithm can also be used to check whether or not there are cycles of MIS in a T-mesh.

In the next corollary we address certain Tchebycheffian spline spaces defined over T-meshes without cycles of MIS. It is a generalization of [29, Proposition 4.3] in two directions, namely to the Tchebycheffian spline setting and to a family of T-meshes which includes hierarchical T-meshes (see Remark 4.1).

Corollary 4.2. *Let $\mathcal{T}$ be a T-mesh without cycles of MIS, and let $\boldsymbol{r}$ be a smoothness distribution on $\mathcal{T}$. Then, assuming that $p_h \geq 2r(\tau) + 1$ for all $\tau \in \mathcal{T}_1^{o,v}$ and $p_v \geq 2r(\tau) + 1$ for all $\tau \in \mathcal{T}_1^{o,h}$, we have $\dim\big(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big) = 0$.*

Proof. Let ι be an ordering of $\textsc{mis}(\mathcal{T})$ generated by Algorithm 4.1. Therefore, both endpoints of any $\rho \in \textsc{mis}(\mathcal{T})$ belong to $\Gamma_\iota(\rho)$, which must contain at least 2 elements. As a consequence, for any $\rho \in \textsc{mis}_h(\mathcal{T})$ we have

$$2\omega_\iota(\rho) = \sum_{\gamma \in \Gamma_\iota(\rho)} 2(p_h - r_h(\gamma)) = \sum_{\gamma \in \Gamma_\iota(\rho)} p_h + p_h - 2r_h(\gamma) \geq \sum_{\gamma \in \Gamma_\iota(\rho)} p_h + 1 \geq 2p_h + 2,$$

which means that $\omega_\iota(\rho) \geq p_h + 1$. Similarly, we deduce that $\omega_\iota(\rho) \geq p_v + 1$ for any $\rho \in \textsc{mis}_v(\mathcal{T})$. From Proposition 3.2 we see that $\boldsymbol{T}$ has the $\boldsymbol{r}$-sum property on $\mathcal{T}$, and so Theorem 3.1 completes the proof. ☐

Example 4.3. *Consider the T-mesh $\mathcal{T}$ in Figure 3, and the constant smoothness distribution $\boldsymbol{r}$ on $\mathcal{T}$ as in* (4.4) *with $\boldsymbol{\mu} = (1,1)$. The space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})$ with $\boldsymbol{p} = (3,3)$ has dimension* 32*. This immediately follows from the mesh numbers in Example 2.1, the dimension formula in Example 4.1 and Corollary 4.2.*

An analogous result for T-meshes without cycles was implicitly proved in [32] for the algebraic polynomial setting. The result can be generalized to our non-polynomial setting by following the same line of arguments as in [6] for spaces of generalized splines over T-meshes, which are Tchebycheffian splines belonging piecewisely to tensor-products of the spaces considered in Example 2.5. Although the original formulation was given for a constant smoothness distribution on $\mathcal{T}$, its proof can be easily extended to a general smoothness distribution on $\mathcal{T}$.

Corollary 4.3. *Let $\mathcal{T}$ be a T-mesh without cycles, and let $\boldsymbol{r}$ be a smoothness distribution on $\mathcal{T}$. Then, assuming that $p_h \geq 2r(\tau)+1$ for all $\tau \in \mathcal{T}_1^{o,v}$ and $p_v \geq 2r(\tau)+1$ for all $\tau \in \mathcal{T}_1^{o,h}$, we have* $\dim\big(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big) = 0$.

Finally, we observe that the properties of the subdivision algorithm presented in [29, Algorithm 4.4] hold in our Tchebycheffian spline context too. Let us start by recalling the algorithm.

Algorithm 4.2. *Given a T-mesh $\mathcal{T}$ and two positive integers k_h, k_v, the (k_h,k_v)-weighted subdivision rule is defined as follows. For any cell $\sigma \in \mathcal{T}_2$ marked to be subdivided:*

1. *split σ by adding the new edge τ;*
2. *if inserting τ does not extend an existing edge, then extend τ so that the maximal segment containing τ, say $\rho(\tau)$, either intersects $\partial\Omega$ or satisfies $\omega_\iota(\rho(\tau)) \geq k_h$ with $\rho(\tau) \in \mathrm{MIS}_h(\mathcal{T})$ or $\omega_\iota(\rho(\tau)) \geq k_v$ with $\rho(\tau) \in \mathrm{MIS}_v(\mathcal{T})$.*

It is clear that applying the (k_h,k_v)-weighted subdivision rule to T-meshes satisfying

$$\begin{aligned} \omega_\iota(\rho) &\geq k_h, \qquad \forall \rho \in \mathrm{MIS}_h(\mathcal{T}),\\ \omega_\iota(\rho) &\geq k_v, \qquad \forall \rho \in \mathrm{MIS}_v(\mathcal{T}), \end{aligned} \tag{4.6}$$

always gives T-meshes satisfying (4.6). In particular, if we choose $k_h \geq p_h + 1$ and $k_v \geq p_v + 1$, then Theorem 3.1 implies that $\dim\big(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big) = 0$, under the assumption that $\boldsymbol{T}$ has the $\boldsymbol{r}$-sum property on $\mathcal{T}$. Therefore, the dimension of the corresponding space depends only on the number of cells, interior edges and interior vertices. Note that the study of this algorithm allows us to obtain the dimension formula for spaces of LR Tchebycheffian splines, i.e., the extension of LR-splines (see [12] and also [5]) to the setting of Tchebycheffian splines.

Example 4.2 and Corollaries 4.2–4.3 show that for a large class of Tchebycheffian spline spaces we have

$$\dim\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})\big) = \dim\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{r}}(\mathcal{T})\big). \tag{4.7}$$

Even though there are cases where this equality does not hold (see Section 5), we make the following conjecture inspired by the concept of generic embeddings used in [2] in the context of dimensions of polynomial spline spaces on triangulations.

Conjecture 4.1. *The equality in* (4.7) *holds generically for all Tchebycheffian spline spaces on T-meshes. This means that if for a given space $\mathbb{P}_{\boldsymbol{p}}^{\boldsymbol{T}}$ and a T-mesh $\mathcal{T}$ with a smoothness distribution $\boldsymbol{r}$ the equality in* (4.7) *does not hold, then there exists an arbitrarily small perturbation of the vertices of $\mathcal{T}$ making the equality true.*

5. Instability in the dimension of Tchebycheffian spline spaces over T-meshes

In this section we show that the dimension of the Tchebycheffian spline space $\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})$ can depend not only on the topological information of $\mathcal{T}$ but also on the geometry of the T-mesh. This particular behavior is usually referred to as *instability in the dimension* of the considered space. Examples of instability in the dimension are well known for polynomial spline spaces either defined over triangulations [11] or over T-meshes [1, 20].

We focus on the T-mesh in Figure 1, which was already considered in [20] in the case of polynomial splines. For this T-mesh we have

$$f_2 = 24, \quad f_1^h = f_1^v = 22, \quad f_0 = 21.$$

Moreover, we take $\boldsymbol{p} = (2,2)$ and a constant smoothness distribution $\boldsymbol{r}$ as in (4.4) with $\boldsymbol{\mu} = (1,1)$. From (4.5) we obtain

$$\dim\big(\mathbb{S}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T})\big) = 36 + \dim\big(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big). \tag{5.1}$$

So, we still have to determine $\dim\big(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big)$. For this, we can use the characterization of $H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))$ provided in Proposition 3.8.

Let $\{\psi_{\bar{x},i}^h\}_{i=0}^2$ and $\{\psi_{\bar{y},i}^v\}_{i=0}^2$ be Taylor-like bases of the spaces $\mathbb{T}_2^h$ and $\mathbb{T}_2^v$, respectively. From (2.2) and (2.3) we know that every element in $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\rho_v)$ can be written as

$$q_{\rho_v}(x,y) = \psi_{\bar{x},2}^h(x) \sum_{j=0}^{2} a_{2,j}^{\rho_v} \psi_{\bar{y},j}^v(y),$$

for some $\bar{y}$, if ρ_v is a vertical maximal interior segment and $x = \bar{x}$ is the abscissa of any point of ρ_v. Similarly, every element in $\mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\rho_h)$ can be written as

$$q_{\rho_h}(x,y) = \psi_{\bar{y},2}^v(y) \sum_{i=0}^{2} a_{i,2}^{\rho_h} \psi_{\bar{x},i}^h(x),$$

for some $\bar{x}$, if ρ_h is a horizontal maximal interior segment and $y = \bar{y}$ is the abscissa of any point of ρ_h. Therefore, requiring

$$q_{\rho_v}(x,y) = -q_{\rho_h}(x,y)$$

implies that $\sum_{j=0}^2 a_{2,j}^{\rho_v} \psi_{\bar{y},j}^v(y)$ is a multiple of $\psi_{\bar{y},2}^v(y)$ and $\sum_{i=0}^2 a_{i,2}^{\rho_h} \psi_{\bar{x},i}^h(x)$ is a multiple of $\psi_{\bar{x},2}^h(x)$. In other words, there exists a coefficient $a^{\rho_h,\rho_v} \in \mathbb{R}$ such that

$$q_{\rho_v}(x,y) = -q_{\rho_h}(x,y) = a^{\rho_h,\rho_v} \psi_{\bar{x},2}^h(x) \psi_{\bar{y},2}^v(y).$$

The T-mesh in Figure 1 has two horizontal maximal interior segments belonging to the straight lines $y = t_3$ and $y = t_4$, respectively, and two maximal vertical interior segments belonging to the straight lines $x = s_3$, $x = s_4$, respectively. The maximal interior segments are ordered as follows

$$y = t_3, \; x = s_3, \; y = t_4, \; x = s_4.$$

We denote by $\gamma_{i,j}$ the vertex obtained by intersecting the lines $x = s_i$ and $y = t_j$. The vertices belonging to the maximal interior segments are ordered as follows

$$\gamma_{4,3}, \; \gamma_{1,3}, \; \gamma_{2,3}, \; \gamma_{3,3}, \; \gamma_{3,1}, \; \gamma_{3,2}, \; \gamma_{3,4}, \; \gamma_{2,4}, \; \gamma_{5,4}, \; \gamma_{4,4}, \; \gamma_{4,2}, \; \gamma_{4,5}.$$

From Proposition 3.8 it follows that

$$\dim\big(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big) = \dim\Bigg(\bigoplus_{\rho \in \mathrm{MIS}(\mathcal{T})} \mathbb{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\rho)\Bigg) - \dim\Bigg(\sum_{\gamma \in \mathcal{T}_0^o} K_\gamma^{\boldsymbol{T}}\Bigg) = 4 \cdot 3 - \dim\Bigg(\sum_{\gamma \in \mathcal{T}_0^o} K_\gamma^{\boldsymbol{T}}\Bigg). \tag{5.2}$$

The elements of $\sum_{\gamma\in\mathcal{T}_0^\circ} K_\gamma^{\boldsymbol{T}}$ are linear combinations of the rows of the following matrix

$$\begin{pmatrix}
\psi^h_{s_4,2}(x)\psi^v_{t_3,2}(y) & 0 & 0 & -\psi^h_{s_4,2}(x)\psi^v_{t_3,2}(y)\\
\psi^h_{s_1,2}(x)\psi^v_{t_3,2}(y) & 0 & 0 & 0\\
\psi^h_{s_2,2}(x)\psi^v_{t_3,2}(y) & 0 & 0 & 0\\
\psi^h_{s_3,2}(x)\psi^v_{t_3,2}(y) & -\psi^h_{s_3,2}(x)\psi^v_{t_3,2}(y) & 0 & 0\\
0 & -\psi^h_{s_3,2}(x)\psi^v_{t_1,2}(y) & 0 & 0\\
0 & -\psi^h_{s_3,2}(x)\psi^v_{t_2,2}(y) & 0 & 0\\
0 & -\psi^h_{s_3,2}(x)\psi^v_{t_4,2}(y) & \psi^h_{s_3,2}(x)\psi^v_{t_4,2}(y) & 0\\
0 & 0 & \psi^h_{s_2,2}(x)\psi^v_{t_4,2}(y) & 0\\
0 & 0 & \psi^h_{s_5,2}(x)\psi^v_{t_4,2}(y) & 0\\
0 & 0 & \psi^h_{s_4,2}(x)\psi^v_{t_4,2}(y) & -\psi^h_{s_4,2}(x)\psi^v_{t_4,2}(y)\\
0 & 0 & 0 & -\psi^h_{s_4,2}(x)\psi^v_{t_2,2}(y)\\
0 & 0 & 0 & -\psi^h_{s_4,2}(x)\psi^v_{t_5,2}(y)
\end{pmatrix}.$$

Hence, the dimension of $\sum_{\gamma\in\mathcal{T}_0^\circ} K_\gamma^{\boldsymbol{T}}$ is given by the dimension of the space spanned by the rows of the following matrix

$$M := \begin{pmatrix}
\psi^h_{s_4,2}(x) & 0 & 0 & \psi^v_{t_3,2}(y)\\
\psi^h_{s_1,2}(x) & 0 & 0 & 0\\
\psi^h_{s_2,2}(x) & 0 & 0 & 0\\
\psi^h_{s_3,2}(x) & \psi^v_{t_3,2}(y) & 0 & 0\\
0 & \psi^v_{t_1,2}(y) & 0 & 0\\
0 & \psi^v_{t_2,2}(y) & 0 & 0\\
0 & \psi^v_{t_4,2}(y) & \psi^h_{s_3,2}(x) & 0\\
0 & 0 & \psi^h_{s_2,2}(x) & 0\\
0 & 0 & \psi^h_{s_5,2}(x) & 0\\
0 & 0 & \psi^h_{s_4,2}(x) & \psi^v_{t_4,2}(y)\\
0 & 0 & 0 & \psi^v_{t_2,2}(y)\\
0 & 0 & 0 & \psi^v_{t_5,2}(y)
\end{pmatrix}. \tag{5.3}$$

In the polynomial case, i.e. $\mathbb{T}_2^h = \mathbb{T}_2^v = \mathbb{P}_2$, we have

$$\psi^h_{\bar{x},i}(x) = \frac{(x-\bar{x})^i}{i!}, \qquad \psi^v_{\bar{y},j}(x) = \frac{(y-\bar{y})^j}{j!}, \qquad i,j = 0,1,2,$$

and the dimension of $\sum_{\gamma\in\mathcal{T}_0^\circ} K_\gamma^{\boldsymbol{T}}$ with $\boldsymbol{T} = (\mathbb{P}_2,\mathbb{P}_2)$ is given by the dimension of the space spanned by the rows of the following matrix

$$M_{\mathrm{poly}}(s_1,\ldots,s_5;t_1,\ldots,t_5) := \begin{pmatrix}
1 & s_4 & {s_4}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_3 & {t_3}^2\\
1 & s_1 & {s_1}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & s_2 & {s_2}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & s_3 & {s_3}^2 & 1 & t_3 & {t_3}^2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & t_1 & {t_1}^2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & t_2 & {t_2}^2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & t_4 & {t_4}^2 & 1 & s_3 & {s_3}^2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & s_2 & {s_2}^2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & s_5 & {s_5}^2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & s_4 & {s_4}^2 & 1 & t_4 & {t_4}^2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_2 & {t_2}^2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_5 & {t_5}^2
\end{pmatrix}.$$

It is clear that $\mathrm{rank}(M_{\mathrm{poly}}(s_1,\ldots,s_5;t_1,\ldots,t_5)) \geq 11$. The matrix $M_{\mathrm{poly}}(s_1,\ldots,s_5;t_1,\ldots,t_5)$ has been analyzed in [20] where it has been proved that $\det(M_{\mathrm{poly}}(s_1,\ldots,s_5;t_1,\ldots,t_5)) = 0$ if and only if

$$\frac{(s_3-s_1)(s_5-s_4)}{(t_3-t_1)(t_5-t_4)} = \frac{(s_4-s_1)(s_5-s_3)}{(t_4-t_1)(t_5-t_3)}; \tag{5.4}$$

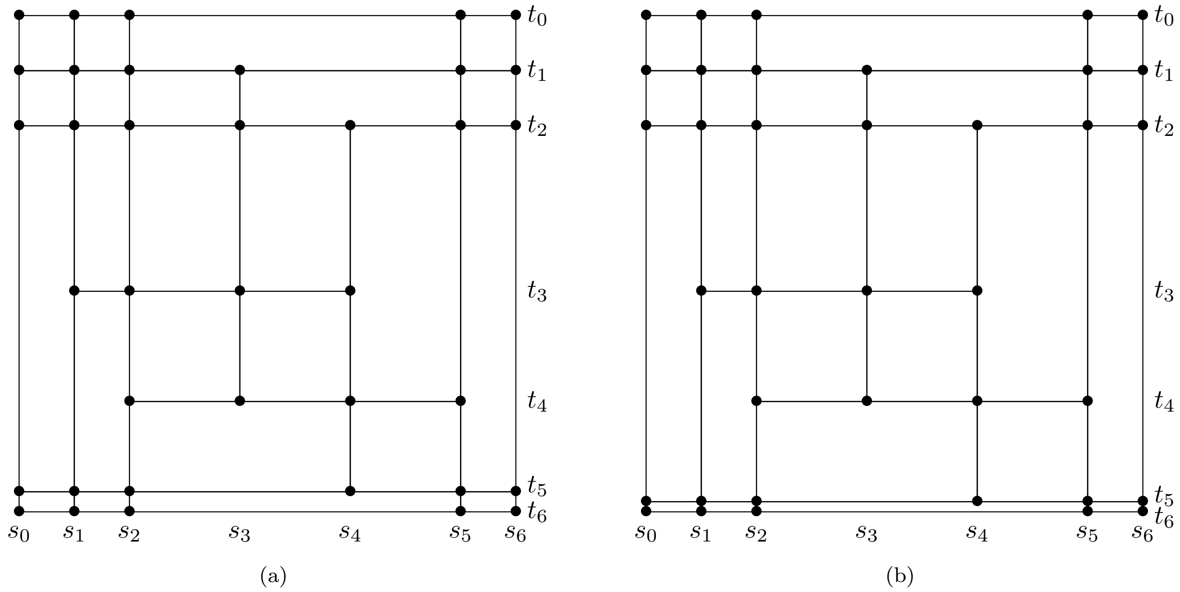


Figure 5: (a) A T-mesh such that (5.4) holds but (5.6) does not. (b) A T-mesh such that (5.6) holds but (5.4) does not. In both cases $\alpha = 1$.

see Figure 1 and Figure 5(a) for two examples. Therefore, from the formula (5.2) with $\boldsymbol{T} = (\mathbb{P}_2, \mathbb{P}_2)$ we get

$$\dim\bigl(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\bigr) = \begin{cases} 1, & \text{if (5.4) holds,} \\ 0, & \text{otherwise.} \end{cases}$$

In other words, the dimension of the quadratic C^1 polynomial spline space over the T-mesh in Figure 1 depends on the geometry of $\mathcal{T}$ according to the validity of (5.4). This result has also been obtained in [20] by means of a different approach, the so-called *smoothing cofactor method*.

Let us now consider the following space of exponential functions,

$$\mathbb{G}_{2,\alpha}^{\exp} := \langle 1, \mathrm{e}^{\alpha x}, \mathrm{e}^{-\alpha x} \rangle, \qquad 0 < \alpha \in \mathbb{R}.$$

It is easy to check that this space is the kernel of the differential operator $D_x(D_x^2 - \alpha^2 I)$, so it is an extended Tchebycheff space on $\mathbb{R}$ (see Example 2.4 or Example 2.5). A Taylor-like basis for $\mathbb{G}_{2,\alpha}^{\exp}$ is given by

$$\begin{aligned} \psi_{0,\bar{x}}(x) &= 1, \\ \psi_{1,\bar{x}}(x) &= \frac{1}{2\alpha}\mathrm{e}^{\alpha(x-\bar{x})} - \frac{1}{2\alpha}\mathrm{e}^{-\alpha(x-\bar{x})}, \\ \psi_{2,\bar{x}}(x) &= -\frac{1}{\alpha^2} + \frac{1}{2\alpha^2}\mathrm{e}^{\alpha(x-\bar{x})} + \frac{1}{2\alpha^2}\mathrm{e}^{-\alpha(x-\bar{x})}. \end{aligned}$$

Therefore, we may conclude from (5.3) that the dimension of $\sum_{\gamma\in\mathcal{T}_0^o} K_\gamma^{\boldsymbol{T}}$ with $\boldsymbol{T} = (\mathbb{G}_{2,\alpha}^{\exp}, \mathbb{G}_{2,\alpha}^{\exp})$ is given by the dimension of the space spanned by the rows of the following matrix

$$
M_{\exp}(s_1,\dots,s_5;t_1,\dots,t_5) := \begin{pmatrix}
1 & \mathrm{e}^{\alpha s_4} & \mathrm{e}^{-\alpha s_4} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathrm{e}^{\alpha t_3} & \mathrm{e}^{-\alpha t_3} \\
1 & \mathrm{e}^{\alpha s_1} & \mathrm{e}^{-\alpha s_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \mathrm{e}^{\alpha s_2} & \mathrm{e}^{-\alpha s_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \mathrm{e}^{\alpha s_3} & \mathrm{e}^{-\alpha s_3} & 1 & \mathrm{e}^{\alpha t_3} & \mathrm{e}^{-\alpha t_3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \mathrm{e}^{\alpha t_1} & \mathrm{e}^{-\alpha t_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \mathrm{e}^{\alpha t_2} & \mathrm{e}^{-\alpha t_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \mathrm{e}^{\alpha t_4} & \mathrm{e}^{-\alpha t_4} & 1 & \mathrm{e}^{\alpha s_3} & \mathrm{e}^{-\alpha s_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathrm{e}^{\alpha s_2} & \mathrm{e}^{-\alpha s_2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathrm{e}^{\alpha s_5} & \mathrm{e}^{-\alpha s_5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathrm{e}^{\alpha s_4} & \mathrm{e}^{-\alpha s_4} & 1 & \mathrm{e}^{\alpha t_4} & \mathrm{e}^{-\alpha t_4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathrm{e}^{\alpha t_2} & \mathrm{e}^{-\alpha t_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathrm{e}^{\alpha t_5} & \mathrm{e}^{-\alpha t_5}
\end{pmatrix}.
$$

Again it is easy to see that $\operatorname{rank}(M_{\exp}(s_1,\dots,s_5;t_1,\dots,t_5)) \geq 11$. Moreover, when considering the substitution

$$
x_i := \mathrm{e}^{\alpha s_i}, \quad y_i := \mathrm{e}^{\alpha t_i}, \quad i = 1,\dots,5,
$$

it can be verified that

$$
\left(x_2 y_2 \prod_{i=1}^{5} x_i y_i \right) \det(M_{\exp}(s_1,\dots,s_5;t_1,\dots,t_5)) = \det(M_{\mathrm{poly}}(x_1,\dots,x_5;y_1,\dots,y_5)).
$$

Taking into account (5.4), this means that in the exponential case the condition for rank deficiency is

$$
\frac{(\mathrm{e}^{\alpha s_3} - \mathrm{e}^{\alpha s_1})(\mathrm{e}^{\alpha s_5} - \mathrm{e}^{\alpha s_4})}{(\mathrm{e}^{\alpha t_3} - \mathrm{e}^{\alpha t_1})(\mathrm{e}^{\alpha t_5} - \mathrm{e}^{\alpha t_4})} = \frac{(\mathrm{e}^{\alpha s_4} - \mathrm{e}^{\alpha s_1})(\mathrm{e}^{\alpha s_5} - \mathrm{e}^{\alpha s_3})}{(\mathrm{e}^{\alpha t_4} - \mathrm{e}^{\alpha t_1})(\mathrm{e}^{\alpha t_5} - \mathrm{e}^{\alpha t_3})}; \tag{5.5}
$$

see Figure 1 and Figure 5(b) for two examples. Therefore, from the formula (5.2) with $\boldsymbol{T} = (\mathbb{G}^{\exp}_{2,\alpha}, \mathbb{G}^{\exp}_{2,\alpha})$ we get

$$
\dim\bigl(H_0(\mathfrak{I}^{\boldsymbol{T},\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T}^o))\bigr) = \begin{cases} 1, & \text{if (5.5) holds,} \\ 0, & \text{otherwise.} \end{cases}
$$

Note that the space $\mathbb{G}^{\exp}_{2,\alpha}$ is equal to $\langle 1, \sinh(\alpha x), \cosh(\alpha x)\rangle$ and the condition in (5.5) is equivalent to

$$
\frac{\sinh\bigl(\frac{\alpha(s_3-s_1)}{2}\bigr)\sinh\bigl(\frac{\alpha(s_5-s_4)}{2}\bigr)}{\sinh\bigl(\frac{\alpha(t_3-t_1)}{2}\bigr)\sinh\bigl(\frac{\alpha(t_5-t_4)}{2}\bigr)} = \frac{\sinh\bigl(\frac{\alpha(s_4-s_1)}{2}\bigr)\sinh\bigl(\frac{\alpha(s_5-s_3)}{2}\bigr)}{\sinh\bigl(\frac{\alpha(t_4-t_1)}{2}\bigr)\sinh\bigl(\frac{\alpha(t_5-t_3)}{2}\bigr)}. \tag{5.6}
$$

Indeed, the above relation can be obtained by multiplying both sides of (5.5) by

$$
\frac{\mathrm{e}^{-\alpha(s_1+s_3+s_4+s_5)/2}}{\mathrm{e}^{-\alpha(t_1+t_3+t_4+t_5)/2}},
$$

and by using the identity $\sinh y = (\mathrm{e}^{y} - \mathrm{e}^{-y})/2$.

Finally, let us consider the following space of trigonometric functions,

$$
\mathbb{G}^{\mathrm{trig}}_{2,\alpha} := \langle 1, \sin(\alpha x), \cos(\alpha x)\rangle, \qquad 0 < \alpha(b-a) < \pi,
$$

which is an extended Tchebycheff space on the interval $[a,b]$. By following the same arguments as before, we can show that there is an instability in the dimension of the space $\sum_{\gamma\in\mathcal{T}^o_0} K^{\boldsymbol{T}}_\gamma$ with $\boldsymbol{T} = (\mathbb{G}^{\mathrm{trig}}_{2,\alpha}, \mathbb{G}^{\mathrm{trig}}_{2,\alpha})$ characterized by the condition

$$
\frac{\sin\bigl(\frac{\alpha(s_3-s_1)}{2}\bigr)\sin\bigl(\frac{\alpha(s_5-s_4)}{2}\bigr)}{\sin\bigl(\frac{\alpha(t_3-t_1)}{2}\bigr)\sin\bigl(\frac{\alpha(t_5-t_4)}{2}\bigr)} = \frac{\sin\bigl(\frac{\alpha(s_4-s_1)}{2}\bigr)\sin\bigl(\frac{\alpha(s_5-s_3)}{2}\bigr)}{\sin\bigl(\frac{\alpha(t_4-t_1)}{2}\bigr)\sin\bigl(\frac{\alpha(t_5-t_3)}{2}\bigr)}. \tag{5.7}
$$

As a consequence,

$$\dim\big(H_0(\mathfrak{I}_{\boldsymbol{p}}^{\boldsymbol{T},\boldsymbol{r}}(\mathcal{T}^o))\big) = \begin{cases} 1, & \text{if (5.7) holds,} \\ 0, & \text{otherwise.} \end{cases}$$

We remark that both rank deficiency conditions in (5.6)–(5.7) approach (5.4) as the parameter α tends to 0. On the one hand, we see that there exist special geometric configurations of $\mathcal{T}$ such that the 0-homology term in (5.1) exceeds 0 for the polynomial, exponential and trigonometric case (see, e.g., Figure 1). On the other hand, there also exist geometric configurations of $\mathcal{T}$ such that the 0-homology term in (5.1) exceeds 0 for the polynomial case but not for the exponential/trigonometric case and vice versa (see, e.g., Figure 5).

6. Conclusions

In this paper we have defined Tchebycheffian spline spaces over planar T-meshes and we have addressed the problem of determining their dimension. We have provided combinatorial lower and upper bounds for the dimension, and we have shown that these bounds coincide if the dimensions of the underlying extended Tchebycheff section spaces are large enough with respect to the smoothness, under some mild conditions on the T-mesh. Moreover, we have illustrated that the dimension of Tchebycheffian spline spaces over T-meshes can depend on the exact geometry of the given T-mesh.

Some of the results have been proved under a technical assumption on the underlying extended Tchebycheff spaces, see Definition 3.1 and Definition 3.3. However, as conjectured in Section 3.1, we think that any extended (complete) Tchebycheff space possesses this property, just like the algebraic polynomial space does. This means that the assumption should not restrict our results in practice.

As expected, it turns out that Tchebycheffian spline spaces and polynomial spline spaces over planar T-meshes behave completely similarly from the dimension point of view. Even stronger, we conjecture in Section 4.2 that the dimensions of these two spline spaces agree generically.

The main ingredient of the paper is the extension of the homological characterization proposed in [29] to the Tchebycheffian spline context. This extension is non-trivial because the ring structure of algebraic polynomials cannot be used anymore in this general setting. Nevertheless, basically all the results obtained for polynomial splines have been rephrased, and sometimes improved, for Tchebycheffian splines. This strengthens the structural similarity between polynomial and Tchebycheffian spline spaces, which originates in the same upper bounds on the number of (real) roots of any non-trivial element of the space.

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