Differential equations and integrable models: the $SU(3)$ case

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Abstract

We exhibit a relationship between the massless $a_2^{(2)}$ integrable quantum field theory and a certain third-order ordinary differential equation, thereby extending a recent result connecting the massless sine-Gordon model to the Schrödinger equation. This forms part of a more general correspondence involving A_2-related Bethe ansatz systems and third-order differential equations. A non-linear integral equation for the generalised spectral problem is derived, and some numerical checks are performed. Duality properties are discussed, and a simple variant of the non-linear equation is suggested as a candidate to describe the finite volume ground state energies of minimal conformal field theories perturbed by the operators ϕ_{12}, ϕ_{21} and ϕ_{15}. This is checked against previous results obtained using the thermodynamic Bethe ansatz. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

A curious connection between certain integrable quantum field theories and the theory of the Schrödinger equation has been the subject of some recent work [1–4]. In this paper we extend these results by establishing a link between functional relations for A_2-related Bethe ansatz systems (see, for example, Refs. [5,6]) and third-order differential equations. Most of our analysis concerns a certain specialisation of the model, a particularly symmetric case that can also be related to the dilute A-model of [7].

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In the cases studied in [1–4], the most general differential equation was a radial Schrödinger problem with 'angular momentum' l and homogeneous potential x^{2M}, initially defined on the positive real axis $x \in (0, \infty)$:

$$\left(-\frac{d^2}{dx^2} + x^{2M} + \frac{l(l+1)}{x^2} \right) \psi(x, E) = E\psi(x, E). \tag{1.1}$$

The relevant integrable quantum field theories were the massless twisted sine-Gordon models or, equivalently, the twisted XXZ/6-vertex models in their thermodynamic limits, and their reductions. It is worth noting that these models are all related to the Lie algebra A_1. Spectral functions associated with (1.1) satisfy functional relations [8–11], and these were mapped into functional equations appearing in the context of integrable quantum field theory in [1–4]. We will follow a similar strategy here, taking a simple third-order ordinary differential equation as our starting-point and showing that the Stokes multipliers and certain spectral functions for this equation together satisfy relations which are essentially the analogues, for the Bethe ansatz systems treated in [6,7], of the T–Q systems which arise in the context of the integrable quantum field theories related to A_1 [12,13]. This is the subject of Section 2, while in Section 3 we borrow some other ideas from integrable quantum field theory in order to derive a non-linear integral equation for the spectral functions, an equation which is put to the test in a simple example in Section 4. Duality properties are discussed in Section 5, allowing us to find the equivalent of the angular-momentum term in (1.1) for the third-order equation. Connections with various perturbed conformal field theories are discussed and tested in Section 6. Finally, Section 7 discusses the most general A_2-related BA equations that arise in this context, and Section 8 contains our conclusions.

2. The differential equation

We begin with the following third-order ordinary differential equation:

$$y'''(x, E) + P(x, E)\, y(x, E) = 0, \tag{2.1}$$

and initially restrict ourselves to purely homogeneous 'potentials' x^{3M}, giving $P(x, E)$ the form

$$P(x, E) = x^{3M} - E. \tag{2.2}$$

These are the simplest higher-order generalisations of the $l = 0$ cases of (1.1), and so we expect that some of the properties of that equation, used in the analysis of [4], will be preserved. In particular, motivated by the results of [8,9] for second-order equations, we suppose that (2.1) has a solution $y = y(x, E)$ such that:

(i) y is an entire function of (x, E) though, due to the branch point in the potential at $x = 0$, x must in general be considered to live on a suitable cover of the punctured complex plane;

(ii) y, $y' = dy/dx$ and $y'' = d^2 y/dx^2$ admit, for $M > 1/2$, the asymptotic representations

$$y \sim x^{-M} e^{-\frac{1}{M+1} x^{M+1}}, \qquad y' \sim -e^{-\frac{1}{M+1} x^{M+1}}, \qquad y'' \sim x^M e^{-\frac{1}{M+1} x^{M+1}},$$

(2.3)

as x tends to infinity in the sector

$$|\arg x| < \frac{4\pi}{3M+3}.$$

(2.4)

Furthermore, these asymptotics, or even just the asymptotic of $y(x,E)$ with x remaining on the positive real axis, characterise y uniquely.

For $M \leqslant 1/2$, the story is complicated by the appearance of extra terms in the asymptotic (2.3). The behaviour of the solution which decays as $x \to +\infty$ can be more generally found from the formula

$$y(x,E) \sim P(x,E)^{-1/3} \exp\left(-\int_{x_0}^x P(t,E)^{1/3} dt\right),$$

(2.5)

with the constant x_0 being related to the normalisation of the solution. (This is the analogue of an approximate WKB solution of a Schrödinger equation.) Since to take $M \leqslant 1/2$ would bring other technical problems into the treatment to be given below, from now on, unless otherwise stated, we shall restrict ourselves to $M > 1/2$. This range is the analogue of the 'semiclassical domain' of [13] (see Refs. [1,2,4] for a discussion in the context of differential equations).

Given $y(x,E)$, bases of solutions to the third-order equation can be constructed just as in the second-order case. For general values of k, define

$$y_k(x,E) = \omega^k y(\omega^{-k} x, \omega^{-3Mk} E),$$

(2.6)

with

$$\omega = \exp\left(\frac{2\pi i}{3M+3}\right).$$

(2.7)

Then y_k solves

$$y_k'''(x,E) + e^{-2k\pi i} P(x,E) y_k(x,E) = 0,$$

(2.8)

and so when k is an integer it provides a (possibly new) solution to the original problem (2.1). However, since we will shortly need to consider fractional values, we will leave k arbitrary for now. It is convenient to define sectors $\mathscr{S}_k$ as

$$\mathscr{S}_k: \quad \left|\arg x - \frac{2k\pi}{3M+3}\right| < \frac{\pi}{3M+3}.$$

(2.9)

On the cover of the punctured complex plane on which x is defined, the sector $\mathscr{S}_k$ abuts the sectors $\mathscr{S}_{k-1}$ and $\mathscr{S}_{k+1}$, and the sector (2.4) is $\mathscr{S}_{-3/2} \cup \mathscr{S}_{-1/2} \cup \mathscr{S}_{1/2} \cup \mathscr{S}_{3/2}$. The pattern of dominance and subdominance of solutions is more involved than in the second-order case, since there are now three different behaviours for solutions at large $|x|$. In addition to a solution with leading behaviour $x^{-M} \exp(-x^{M+1}/(M+1))$ as $|x| \to +\infty$, there are also solutions which behave as $x^{-M} \exp(e^{\pm \pi i/3} x^{M+1}/(M+1))$.

(This is simply a consequence of the fact that the three third roots of -1 are -1, $e^{\pi i/3}$ and $e^{-\pi i/3}$.) Depending on the sector, either one or two of these solutions tend to zero at large $|x|$. We call 'subdominant' the solution which tends to zero *fastest* in a given sector; then, up to a scalar multiple, y_k is characterised as the unique solution to (2.8) subdominant inside $\mathcal{S}_k$.

The asymptotic (2.3) and the definition (2.6) together imply

$$y_k \sim \omega^{(M+1)k} x^{-M} e^{-\frac{1}{M+1}\omega^{-(M+1)k} x^{M+1}}, \qquad y_k' \sim -e^{-\frac{1}{M+1}\omega^{-(M+1)k} x^{M+1}},$$

$$y_k'' \sim \omega^{-(M+1)k} x^M e^{-\frac{1}{M+1}\omega^{-(M+1)k} x^{M+1}}, \tag{2.10}$$

for $|x| \to \infty$ with

$$x \in \mathcal{S}_{k-3/2} \cup \mathcal{S}_{k-1/2} \cup \mathcal{S}_{k+1/2} \cup \mathcal{S}_{k+3/2}. \tag{2.11}$$

Comparing y_k, y_{k+1} and y_{k+2} in the region $\mathcal{S}_{k+1/2} \cup \mathcal{S}_{k+3/2}$, where the asymptotics of all three are given by (2.10), establishes their linear independence. The set $\{y_k, y_{k+1}, y_{k+2}\}$ therefore forms a basis of solutions to (2.8) (and, for k integer, to (2.1)). Alternatively, we can examine

$$W_{k_1,k_2,k_3} = W[y_{k_1}, y_{k_2}, y_{k_3}], \tag{2.12}$$

where the generalised Wronskian $W[f,g,h]$ is defined to be

$$\mathrm{Det} \begin{bmatrix} f & f' & f'' \\ g & g' & g'' \\ h & h' & h'' \end{bmatrix}. \tag{2.13}$$

It is a standard result (see, for example, Ref. [14]) that, for f, g and h solving (2.1), $W[f,g,h]$ is independent of x, and that f, g and h are linearly independent if and only if $W[f,g,h]$ is non-zero. For $(k_1,k_2,k_3) = (-1,0,1)$, the asymptotic (2.10), used in $\mathcal{S}_{-1/2} \cup \mathcal{S}_{1/2}$, shows that

$$W_{-1,0,1} = 8i\sin\left(\frac{M}{3M+3}\pi\right)\sin\left(\frac{2M}{3M+3}\pi\right). \tag{2.14}$$

It is also the case that

$$W_{k_1+a,k_2+a,k_3+a}(E) = W_{k_1,k_2,k_3}(\omega^{-3Ma}E), \tag{2.15}$$

so $W_{k,k+1,k+2}$ is non-zero for all k, thus confirming the independence of $\{y_k, y_{k+1}, y_{k+2}\}$.

We now aim to generalise the analysis of [4] to this situation, guided in part by the treatment of A_2-related BA systems provided by [6]. Since y_1, y_2, y_3 form a basis, we can write

$$y_0 - S^{(1)}(E) y_1 + S^{(2)}(E) y_2 - y_3 = 0 \tag{2.16}$$

with

$$S^{(1)}(E) = \frac{W_{0,2,3}}{W_{1,2,3}}, \qquad S^{(2)}(E) = \frac{W_{1,0,3}}{W_{1,2,3}}. \tag{2.17}$$

The coefficient of y_3 in (2.16) is -1 by (2.15); $S^{(1)}$ and $S^{(2)}$ are Stokes multipliers for (2.1), and are analytic functions of E. Notice the formal similarity between this equation and Eq. (15) of Ref. [6].

Now suppose that k_1 and k_2 differ by an integer. Then y_{k_1} and y_{k_2} both solve (2.8) (with $e^{-2k\pi i} = e^{-2k_1\pi i} = e^{-2k_2\pi i}$), and it can be checked by direct substitution that the function

$$z_{k_1 k_2}(x,E) = y_{k_1} y'_{k_2} - y'_{k_1} y_{k_2} \tag{2.18}$$

solves

$$z'''_{k_1 k_2}(x,E) - e^{-2k\pi i} P(x,E) z_{k_1 k_2}(x,E) = 0. \tag{2.19}$$

This is just the equation adjoint to (2.8); the observation that the Wronskian of two solutions of a third-order ordinary differential equation satisfies the adjoint equation dates back at least to Birkhoff [15]. Observe also that if k_1 and k_2 are shifted by a half-integer, then a solution of the original equation (2.8) results:

$$z'''_{k_1 + \frac{1}{2}, k_2 + \frac{1}{2}}(x,E) + e^{-2k\pi i} P(x,E) z_{k_1 + \frac{1}{2}, k_2 + \frac{1}{2}}(x,E) = 0. \tag{2.20}$$

For $|k_1 - k_2| < 3$, the regions (2.11) for $k = k_1$ and $k = k_2$ have a non-empty overlap, and an asymptotic for $z_{k_1 k_2}$ is easily obtained from (2.10). In particular, for $k = 1,2,3$ we have

$$z_{-k/2, k/2}(x,E) \sim 2 i \sin(\pi k/3) \, x^{-M} e^{-2\cos(\pi k/3)\frac{1}{M+1} x^{M+1}}, \qquad x \to +\infty. \tag{2.21}$$

For $k = 1$, $z_{-1/2, 1/2}$ solves (2.1), and now from (2.21) we see that it shares (up to a proportionality factor) the asymptotic (2.3). By uniqueness, we deduce

$$z_{-1/2, 1/2}(x,E) = i\sqrt{3}\, y(x,E). \tag{2.22}$$

Unfortunately, this argument is not so effective for the other cases. At $k = 2$, the formula (2.21) shows only that $z_{-1,1}$ is not subdominant on the real axis, and this information is not enough to pin the function down. For $k = 3$, $\sin(\pi k/3) = 0$ and all that can be deduced is that the leading asymptotic of $z_{-3/2, 3/2}$ is subleading to the term over which we have control.

The next step is to manipulate (2.16) in order to eliminate either $S^{(1)}$ or $S^{(2)}$. We have

$$y'_1 y_0 - S^{(1)}(E) y'_1 y_1 + S^{(2)}(E) y'_1 y_2 - y'_1 y_3 = 0, \tag{2.23}$$

$$y_1 y'_0 - S^{(1)}(E) y_1 y'_1 + S^{(2)}(E) y_1 y'_2 - y_1 y'_3 = 0, \tag{2.24}$$

and, subtracting,

$$S^{(2)}(E) z_{12} = z_{01} + z_{13}. \tag{2.25}$$

For the reasons just explained, functions $z_{k_1 k_2}$ with $|k_1 - k_2| = 1$ are the most easily handled, so we use the identity $y_2 z_{13} = y_1 z_{23} + z_{12} y_3$ to rewrite (2.25) as

$$S^{(2)}(E) y_2 z_{12} = y_2 z_{01} + z_{12} y_3 + z_{23} y_1. \tag{2.26}$$

Likewise,

$$S^{(1)}(E) y_1 z_{12} = z_{12} y_0 + y_1 z_{23} + z_{01} y_2 . \tag{2.27}$$

Now the result (2.22) can be combined with shifts in E to $\omega^{15M/4}E$ and $\omega^{21M/4}E$ respectively to rewrite both (2.26) and (2.27) as

$$T(E) y_{-1/4} y_{1/4} = y_{-1/4} y_{5/4} + y_{-3/4} y_{3/4} + y_{-5/4} y_{1/4} , \tag{2.28}$$

where

$$T(E) = S^{(1)}(\omega^{15M/4}E) = S^{(2)}(\omega^{21M/4}E). \tag{2.29}$$

As a byproduct, this has established that the two Stokes multipliers $S^{(1)}$ and $S^{(2)}$ are related by an analytic continuation in E.

Finally, taking (2.28) at $x = 0$ yields a functional relation involving E alone. To absorb various phases, it is convenient to set

$$Q^+(E) = E^{-1/3M} y(0,E), \qquad Q_k^+(E) = Q^+(\omega^{-3Mk}E). \tag{2.30}$$

Then the relation is

$$TQ^+_{-1/4}Q^+_{1/4} = Q^+_{-1/4}Q^+_{5/4} + Q^+_{-3/4}Q^+_{3/4} + Q^+_{-5/4}Q^+_{1/4}. \tag{2.31}$$

This is very similar to the equations related to the dilute A model studied in [7,16]. An equation involving $y''(0,E)$ can also be derived. First, differentiate (2.25) twice with respect to x:

$$S^{(2)}(E) z''_{12} = z''_{01} + z''_{13} . \tag{2.32}$$

Using the fact that y_1, y_2 and y_3 all solve (2.1), we have $y''_2 z''_{13} = y''_1 z''_{23} + z''_{12} y''_3$, and so the previous steps can be repeated to find

$$T(E) y''_{-1/4} y''_{1/4} = y''_{-1/4} y''_{5/4} + y''_{-3/4} y''_{3/4} + y''_{-5/4} y''_{1/4} . \tag{2.33}$$

Again set $x = 0$, and define

$$Q^-(E) = \tfrac{1}{2}E^{1/3M} y''(0,E), \qquad Q_k^-(E) = Q^-(\omega^{-3Mk}E) \tag{2.34}$$

(the factor $\frac{1}{2}$ is included for later convenience). Then

$$TQ^-_{-1/4}Q^-_{1/4} = Q^-_{-1/4}Q^-_{5/4} + Q^-_{-3/4}Q^-_{3/4} + Q^-_{-5/4}Q^-_{1/4}. \tag{2.35}$$

There is no simple relation involving $y'(0,E)$ alone, but from (2.18) and (2.22) one can deduce

$$i\sqrt{3}\, y' = y_{-1/2} y''_{1/2} - y''_{-1/2} y_{1/2} , \tag{2.36}$$

which allows $y'(0,E)$ to be recovered once $y(0,E)$ and $y''(0,E)$ are known.

3. The non-linear integral equation

The functions $Q^\pm(E)$ are not single-valued, and to derive an integral equation it is more convenient to work with the functions $y(0,E)$ and $y''(0,E)$ directly. Set

$$D^+(E) = y(0,E), \qquad D^-(E) = \tfrac{1}{2}y''(0,E) \tag{3.1}$$

(so that $D^{\pm}(E) = E^{\pm \frac{1}{3M}} Q^{\pm}(E)$ and $Q_k^{\pm}(E) = \omega^{\pm k} E^{\mp \frac{1}{3M}} D^{\pm}(\omega^{-3Mk} E)$). These are entire functions of E and can be interpreted as spectral determinants for the third-order equation (2.1), since their zeroes coincide with the values of E for which the solution y, decaying at $x \to +\infty$ for all values of E, in addition either vanishes at $x = 0$ (for the zeroes of D^{+}), or has a vanishing second derivative at $x = 0$ (for the zeroes of D^{-}). (See, for example, Ref. [4] for a more detailed discussion of this point in the context of second-order equations.) For $M > 1/2$, the functions $D^{\pm}(E)$ have large-$|E|$ asymptotics

$$\ln D^{\pm}(E) \sim a_0(-E)^{\mu}, \qquad |E| \to \infty, \qquad |\arg(-E)| < \pi, \tag{3.2}$$

where $\mu = (M+1)/3M$, $a_0 = \kappa(3M,3)$, and

$$\kappa(a,b) = \int_0^{\infty} dx \left((x^a + 1)^{1/b} - x^{a/b} \right) = \frac{\Gamma\left(1 + \frac{1}{a}\right) \Gamma\left(1 + \frac{1}{b}\right)}{\Gamma\left(1 + \frac{1}{a} + \frac{1}{b}\right)} \frac{\sin \frac{\pi}{b}}{\sin\left(\frac{\pi}{b} + \frac{\pi}{a}\right)}. \tag{3.3}$$

The growth of $\ln D^{\pm}(E)$ is no larger on the positive real E-axis than elsewhere, so the orders of D^{+} and D^{-} as functions of E are both equal to μ, and are less than 1 for $M > 1/2$. Invoking the Hadamard factorisation theorem, we can write

$$D^{\pm}(E) = D^{\pm}(0) \prod_{k=1}^{\infty} \left(1 - \frac{E}{E_k^{\pm}}\right). \tag{3.4}$$

The precise values of the constants $D^{\pm}(0)$ are irrelevant for the treatment below, but some knowledge of the positions of the zeroes $\{E_k^{\pm}\}$ will be crucial. We conjecture that, for all $M > 0$, all of the zeroes of $D^{\pm}(E)$ lie on the positive real E-axis. Some numerical evidence in favour of this claim will be presented below.

The generalised T–Q relations (2.31), (2.35) taken at either $E \in \{\omega^{3M/4} E_n^{\pm}\}$ or $E \in \{\omega^{-3M/4} E_n^{\pm}\}$ imply

$$\frac{D^{\pm}\left(\omega^{-3M} E_n^{\pm}\right)}{D^{\pm}\left(\omega^{-3M/2} E_n^{\pm}\right)} = -\omega^{\mp 1} \frac{D^{\pm}\left(\omega^{3M} E_n^{\pm}\right)}{D^{\pm}\left(\omega^{3M/2} E_n^{\pm}\right)}, \tag{3.5}$$

an equation that can be written in a Bethe-ansatz form as

$$\prod_{k=1}^{\infty} \frac{E_k^{\pm} - \omega^{-3M} E_n^{\pm}}{E_k^{\pm} - \omega^{3M} E_n^{\pm}} = -\omega^{\mp 1} \prod_{k=1}^{\infty} \frac{E_k^{\pm} - \omega^{-3M/2} E_n^{\pm}}{E_k^{\pm} - \omega^{3M/2} E_n^{\pm}}. \tag{3.6}$$

This equation is at least not inconsistent with the conjectured reality of the E_n's, since both sides then reduce to pure phases. There are certainly other, complex, solutions to (3.6), so the reality property should be seen as a way of selecting the particular solution relevant to our differential equation, analogous to the selection of the ground state in an integrable model.

A non-linear integral equation, similar to those described in [13,17–19], can now be obtained for the quantity

$$d^{\pm}(E) = \omega^{\pm 1} \frac{D^{\pm}(\omega^{-3M} E)}{D^{\pm}(\omega^{3M} E)} \frac{D^{\pm}(\omega^{3M/2} E)}{D^{\pm}(\omega^{-3M/2} E)}. \tag{3.7}$$

We shall follow a path that completely parallels the treatment given in [13]. By (3.5), $d^{\pm}(E) = -1$ at the points $\{E_k^{\pm}\}$. (The value -1 might also occur at other points; we supplement our previous conjecture with the assumption that none of these points lie on the positive real axis.) The product representation (3.4) implies

$$\ln d^{\pm}(E) = \pm i\pi \frac{2}{3M+3} + \sum_{n=1}^{\infty} F(E/E_n), \tag{3.8}$$

where

$$F(E) = \ln \frac{(1 - E\omega^{-3M})}{(1 - E\omega^{3M})} \frac{(1 - E\omega^{3M/2})}{(1 - E\omega^{-3M/2})}. \tag{3.9}$$

The sum over the E_n in (3.8) can be written as a contour integral

$$\ln d^{\pm}(E) = \pm i\pi \frac{2}{3M+3} + \int_C \frac{dE'}{2i\pi} F(E/E') \partial_{E'} \ln(1 + d^{\pm}(E')) \tag{3.10}$$

with the contour C running from $+\infty$ to 0 above the real axis, winding around 0 and returning to $+\infty$ below the real axis. (It is at this point that the conjectures about the locations of the E_n's and of the other zeroes of $d^{\pm}(E) + 1$ are used.) If the new variable $\theta = \mu \ln E$ is introduced, the function F becomes

$$F(e^{3M\theta/(M+1)})$$

$$= \ln \left(\omega^{-3M/2} \frac{\sinh\left(\frac{3}{2}\frac{1}{1+\xi}\theta + i\pi\frac{\xi}{1+\xi}\right)}{\sinh\left(\frac{3}{2}\frac{1}{1+\xi}\theta - i\pi\frac{\xi}{1+\xi}\right)} \frac{\sinh\left(\frac{3}{2}\frac{1}{1+\xi}\theta + i\frac{\pi}{2}\frac{1}{1+\xi}\right)}{\sinh\left(\frac{3}{2}\frac{1}{1+\xi}\theta - i\frac{\pi}{2}\frac{1}{1+\xi}\right)} \right), \tag{3.11}$$

with $\xi = 1/M$. Now define

$$f^{\pm}(\theta) = \ln d^{\pm}(e^{3M\theta/(M+1)}), \tag{3.12}$$

use the property $d^{\pm}(E)^* = d^{\pm}(E^*)^{-1}$ and integrate by parts to recast (3.10) as

$$f^{\pm}(\theta) - \int_{-\infty}^{\infty} d\theta' R(\theta - \theta') f^{\pm}(\theta' - i0)$$

$$= \pm i\pi \frac{2}{3M+3} - 2i \int_{-\infty}^{\infty} d\theta' R(\theta - \theta') \operatorname{Im} \ln(1 + \exp(f^{\pm}(\theta' - i0))) \tag{3.13}$$

with

$$R(\theta) = \frac{i}{2\pi} \partial_{\theta} F(e^{3M\theta/(M+1)}). \tag{3.14}$$

The term $(1 - R) * f^{\pm}(\theta)$ on the l.h.s. of (3.13) is easily inverted using Fourier transforms. Using

$$i\partial_\theta \ln \frac{\sinh(h\theta + i\pi\tau)}{\sinh(h\theta - i\pi\tau)} = \frac{2h\sin(2\tau\pi)}{\cosh(2h\theta) - \cos(2\tau\pi)}, \tag{3.15}$$

$$\int \frac{d\theta}{2\pi} e^{-ik\theta} \frac{2h\sin(2\tau\pi)}{\cosh(2h\theta) - \cos(2\tau\pi)} = \frac{\sinh\left((1 - 2\tau)\dfrac{\pi k}{2h}\right)}{\sinh\left(\dfrac{\pi}{2}\dfrac{k}{h}\right)}, \tag{3.16}$$

we have

$$\tilde{R}(k) = \frac{\sinh\left(\dfrac{\pi}{3}(1 - \xi)k\right)}{\sinh\left(\dfrac{\pi}{3}(1 + \xi)k\right)} + \frac{\sinh\left(\dfrac{\pi}{3}\xi k\right)}{\sinh\left(\dfrac{\pi}{3}(1 + \xi)k\right)}$$

$$= \frac{2\sinh\left(\dfrac{\pi k}{6}\right)\cosh\left(\dfrac{\pi}{6}(1 - 2\xi)k\right)}{\sinh\left(\dfrac{\pi}{3}(1 + \xi)k\right)},$$

$$1 - \tilde{R}(k) = \frac{\sinh\left(\dfrac{\pi}{3}\xi k\right)\cosh\left(\dfrac{\pi}{2}k\right)}{\sinh\left(\dfrac{\pi}{3}(1 + \xi)k\right)\cosh\left(\dfrac{\pi}{6}k\right)}. \tag{3.17}$$

Transforming back to θ space and rewriting the imaginary part in terms of values above and below the real axis, the functions $f^{\pm}(\theta)$ solve

$$f^{\pm}(\theta) = \pm i\pi\alpha - ib_0 e^\theta + \int_{\mathscr{C}_1} \varphi(\theta - \theta')\ln\left(1 + e^{f^{\pm}(\theta')}\right) d\theta'$$

$$- \int_{\mathscr{C}_2} \varphi(\theta - \theta')\ln\left(1 + e^{-f^{\pm}(\theta')}\right) d\theta', \tag{3.18}$$

where $\alpha = 2/3$, the contours $\mathscr{C}_1$ and $\mathscr{C}_2$ run from $-\infty$ to $+\infty$, just below and just above the real θ-axis,

$$\varphi(\theta) = -\int_{-\infty}^{\infty} \frac{e^{ik\theta}\sinh\left(k\dfrac{\pi}{3}\right)\cosh\left(\dfrac{\pi}{6}k(1 - 2\xi)\right)}{\cosh\left(\dfrac{\pi}{2}k\right)\sinh\left(k\dfrac{\pi}{3}\xi\right)} \frac{dk}{2\pi}, \qquad \xi = \frac{1}{M}, \tag{3.19}$$

and the constant $b_0 = 2\sin(\pi\frac{M+1}{3M})a_0$ has been fixed using the asymptotic behaviour (3.2). (The corresponding zero-mode can be traced to the zero in $1 - \tilde{R}$ at $k = i$.) The

parameter α in (3.18) is analogous to the chemical potential (or twist) term in the equations of [13,17–19].

A first consistency check is immediate: in the large θ limit of (3.18) the driving term $b_0 e^\theta$ dominates, and so in this limit the functions $\exp(f^\pm(\theta))$ are -1 at the points $\theta = \theta_n{}^\pm$, or

$$E = E_n^\pm = e^{\theta_n / \mu} = \left((2n - 1 \pm \tfrac{2}{3}) \pi / b_0 \right)^{1/\mu} \quad (n = 1, 2, \dots). \tag{3.20}$$

The same limit can be treated directly using a WKB-like approach to the differential equation (2.1). Start from (2.5) with $x > x_0$ and fix x_0 to be at the inversion point $P(x_0, E) = 0$ ($x_0 = E^{1/3M}$). Now, using analytic continuation (see, for example, Section 47 in Ref. [20]), the dominant part in the region $x < x_0$ is

$$y(x, E) \sim |P(x,E)|^{-1/3} \exp\left(\frac{1}{2} \int_x^{x_0} |P(x,E)|^{1/3} dx \right)$$

$$\times \cos\left(\frac{\sqrt{3}}{2} \int_x^{x_0} |P(x,E)|^{1/3} dx - \frac{\pi}{3} \right). \tag{3.21}$$

Thus to have $y(0, E) = 0$, requires

$$\sqrt{3} \int_0^{x_0} \left(x^{3M} - E_n^+ \right)^{1/3} = b_0 \left(E_n^+ \right)^\mu = \left(2n - \tfrac{1}{3} \right) \pi, \quad n = 1, 2, \dots, \tag{3.22}$$

where the formula

$$\int_0^1 (1 - x^a)^{1/b} = \frac{\sin\left(\dfrac{\pi}{b} + \dfrac{\pi}{a} \right)}{\sin \dfrac{\pi}{b}} \kappa(a, b) \tag{3.23}$$

was used. The prediction (3.22) agrees perfectly with (3.20). In Fig. 1 the positions of the lowest zeroes of $D^+(E)$ are plotted in the range $0.1 < 3M < 7$, and compared with the WKB-like prediction. Evidence for the reality of the E_n at $3M = 1$ will be given in Section 4; in the meantime, we note that the levels continue smoothly away from that point, and the eigenvalues appear to remain real in the range studied. (The figure can be compared with Figs. 1 and 2 of Ref. [4], which illustrate cases where the spectrum does *not* remain real in the full range displayed.) The kernel $\varphi(\theta)$ given in (3.19) coincides with $i/2\pi$ times the logarithmic derivative of the scalar factor in the Izergin–Korepin S-matrix for the $a_2^{(2)}$ model [21] (cf. Eq. (3.21) of Ref. [22], though note that the normalisation of ξ used by Smirnov in [22] differs from ours: $\xi^{[\text{Smirnov}]} = \frac{2\pi}{3} \xi^{[\text{this paper}]}$.) This is an element of the advertised link between the differential equation (2.1) and the $a_2^{(2)}$ model, the parameters being related as $M = 1/\xi$ (with ξ related to the $a_2^{(2)}$ coupling γ as $\xi = \gamma/(2\pi - \gamma)$ [22]). When $3M$ is an integer the potential is analytic, and the associated scattering theory is diagonal; the same phenomenon was observed in the Schrödinger/sine-Gordon case in [1]. The similarity between the relations (2.31), (2.35) and (3.6) and those arising in the dilute A model [7,16] has already been
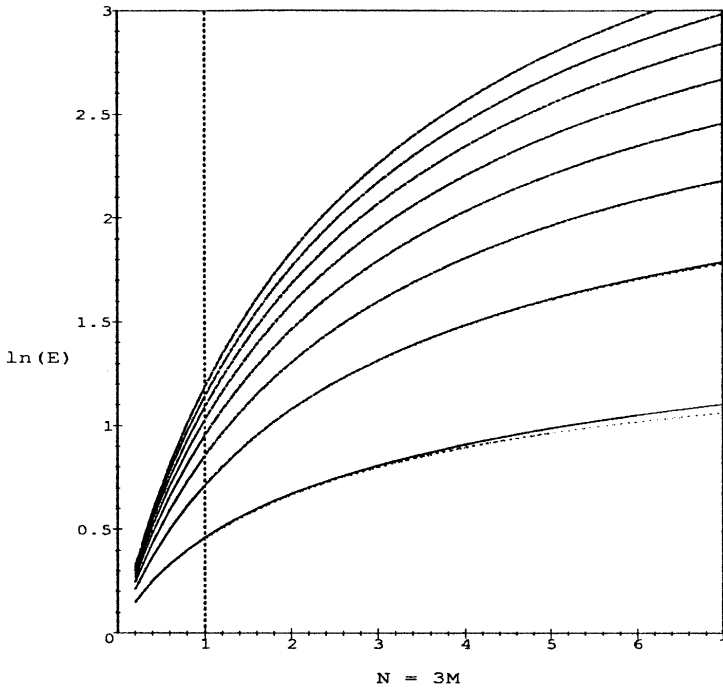
Fig. 1. The positions of the first eight zeroes of $D^+(E)$, plotted on a log scale. Dotted lines show the WKB-like predictions, and solid lines the results from the non-linear integral equation.

mentioned. Since the $a_2^{(2)}$ model is conjectured to be the continuum limit of the dilute A model (see, for example, Ref. [23]), the fact that elements of it emerge here is not a
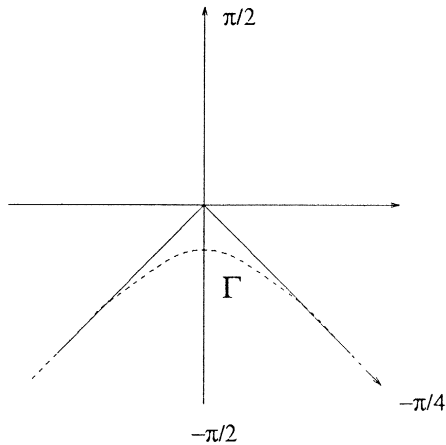

Fig. 2. The integration path.

complete surprise. Nevertheless, it is an encouraging signal that we are on the right track. We will return to this point in Section 6.

4. The linear potential

A simple but non-trivial example occurs when $M = 1/3$, and is the analogue of the 'Airy case' of the second-order problem, discussed in [1,24]. This lies outside the $M > 1/2$ zone treated so far, so we have to assume that the results obtained above continue to hold as the region of their initial derivation is left. The basic differential equation is

$$y'''(x,E) + xy(x,E) = Ey(x,E). \tag{4.1}$$

Setting $y(x,E) = \mathscr{A}(x - E)$, this becomes

$$\mathscr{A}'''(x) + x\mathscr{A}(x) = 0. \tag{4.2}$$

This equation is solvable via a complex-Fourier transform:

$$\mathscr{A}(x) = \sqrt{\frac{3}{2\pi}} \int_{\Gamma} e^{-ipx + \frac{1}{4}p^4} dp, \tag{4.3}$$

where the integration path Γ is represented in Fig. 2. (A curious feature of this case is that the function $T(E)$ is a constant, equal to 1.) Even though the problem is not self-adjoint, numerical evidence suggests that all the zeroes of $\mathscr{A}(x)$, $\mathscr{A}'(x)$ and $\mathscr{A}''(x)$ lie on the negative real axis (see Fig. 3), and so the zeroes of $y(0,E) \equiv \mathscr{A}(-E)$, and of $y'(0,E)$ and $y''(0,E)$, are positive and real. In the first columns of Tables 1 and 2, the positions of the first ten zeroes of $\mathscr{A}(-x)$ and $\mathscr{A}''(-x)$ are displayed.

The approximate positions of the zeroes of $\mathscr{A}(x)$ can be found from the WKB formula of the last section. As a check, we rederive them here via a saddle-point
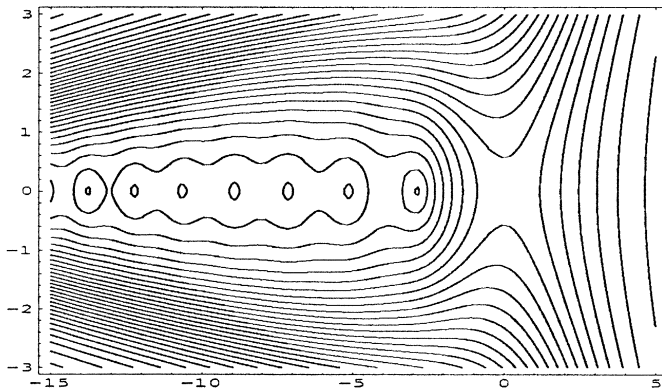

Fig. 3. A search for the zeroes of $\mathscr{A}(x)$ in the complex x-plane. The function plotted is $|A(x)|/(1 + |A(x)|)$, $A(x) = e^x \mathscr{A}(x)$.

Table 1
Zeroes of $\mathscr{A}(-x)$

k	E_k (Exact)	E_k (WKB)	E_k (NLIE)
1	2.8868281617697677	2.84467	2.886828161769766
2	5.1522519299627660	5.13866	5.152251929962763
3	7.1303732976716514	7.12265	7.130373297671650
4	8.9403621072563961	8.93513	8.940362107256395
5	10.635608688272157	10.6317	10.63560868827213
6	12.245164125544329	12.2421	12.24516412554437
7	13.787063381394688	13.7846	13.78706338139461
8	15.273489957153985	15.2714	15.27348995715393
9	16.713173447810789	16.7114	16.71317344781078

treatment of (4.3). The exponent of the integrand for $x = -|x| < 0$ has stationary points at

$$p_0 = i|x|^{1/3}, \qquad p_\pm = ie^{\pm i2\pi/3}|x|^{1/3}. \tag{4.4}$$

Deforming the contour Γ so that it touches the points $p_\pm$, we get

$$\mathscr{A}(-|x|) \sim \mathscr{N}\left(e^{\frac{3}{4}|x|^{4/3}e^{-i\pi/3}+i\frac{\pi}{3}} + e^{\frac{3}{4}|x|^{4/3}e^{i\pi/3}-i\frac{\pi}{3}}\right), \tag{4.5}$$

where the phases $\pm\pi/3$ are the contributions from the choice of the steepest descent directions, transforming the quadratic terms in the expansion near the saddle points into a pure Gaussian integral

$$\mathscr{N} = \sqrt{\frac{3}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{3}{2}|x|^{2/3}t^2}\,dt = |x|^{-1/3}. \tag{4.6}$$

Thus for large negative x we have

$$\mathscr{A}(-|x|) \sim 2|x|^{-1/3}e^{\frac{3}{8}|x|^{4/3}}\cos\left(\frac{3}{8}\sqrt{3}|x|^{4/3} - \frac{\pi}{3}\right). \tag{4.7}$$

For $x > 0$, the dominant saddle point is instead at $p_0 = -i|x|^{1/3}$, and

$$\mathscr{A}(x) \sim |x|^{-1/3}e^{-\frac{3}{4}|x|^{4/3}}. \tag{4.8}$$

(This agrees with the general asymptotic (2.5), since $y(x,E) = \mathscr{A}(x-E)$.) The dominant behaviours of $\mathscr{A}'(x)$ and $\mathscr{A}''(x)$ for x real and $|x|$ large are

$$\mathscr{A}'(-|x|) \sim -2e^{\frac{3}{8}|x|^{4/3}}\cos\left(\frac{3}{8}\sqrt{3}|x|^{4/3}\right), \qquad \mathscr{A}'(|x|) \sim -e^{-\frac{3}{4}|x|^{4/3}}, \tag{4.9}$$

$$\mathscr{A}''(-|x|) \sim 2|x|^{\frac{1}{3}}e^{\frac{3}{8}|x|^{4/3}}\cos\left(\frac{3}{8}\sqrt{3}|x|^{4/3} + \frac{\pi}{3}\right), \qquad \mathscr{A}''(|x|) \sim |x|^{\frac{1}{3}}e^{-\frac{3}{4}|x|^{4/3}}. \tag{4.10}$$

Table 2
Zeroes of $\mathscr{A}''(-x)$.

k	E_k (Exact)	E_k (WKB)	E_k (NLIE)
1	0.8909448213691012	0.85075	0.890944821369104
2	3.6439018898747455	3.66124	3.643901889874749
3	5.8171363674931786	5.82455	5.817136367493172
4	7.7377094704565507	7.74230	7.737709470456556
5	9.5084853382296383	9.51174	9.508485338229636
6	11.174539990553549	11.1770	11.17453999055357
7	12.761111499098718	12.7631	12.76111149909874
8	14.284208133811659	14.2859	14.28420813381167
9	15.754819677279076	15.7562	15.75481967727909

Continuing to differentiate, the general result for the approximate positions of the zeroes of the mth derivative $\mathscr{A}^{(m)}(x)$ is

$$\mathscr{A}^{(m)}(x) = 0: \quad x = -\left[\frac{4}{3\sqrt{3}}\left(2n - \frac{2m+1}{3}\right)\pi\right]^{3/4} \quad (n = 1, 2, \dots). \qquad (4.11)$$

At $m = 0$ (4.11) reduces to the $M = 1/3$ WKB prediction (3.22). In Tables 1 and 2 the results from formula (4.11) are compared with the 'exact' result from a numerical treatment of (4.3), and also against the results of the numerical solution of the non-linear integral equation (3.18). Clearly the agreement is very good.

5. Duality and a more general chemical potential

In this section we shall investigate the effect of a duality transformation on (2.1), analogous to that studied in [2] for the Schrödinger equation (1.1). In the Schrödinger case duality maps wavefunctions for confining potentials $(M > 0)$ to wavefunctions for singular potentials $(-1 < M < 0)$, in such a way that the theories with M and $\tilde{M} = -M/(M+1)$ are dual, their respective spectral problems being essentially equivalent. It also changes the coefficient of the 'angular momentum' term $l(l+1)/x^2$ in (1.1) in a non-trivial way; the same phenomenon here will allow us to guess the corresponding term for the third-order problem (2.1).

To implement the duality transformation, we begin with a Langer [25] type variable transformation

$$y(x) = e^z u(z), \qquad z = \ln x, \qquad (5.1)$$

after which (2.1) becomes

$$u'''(z) - u'(z) + (e^{(3M+3)z} - Ee^{3z})u(z) = 0. \qquad (5.2)$$

The duality $M \to \tilde{M}$ is now effected by interchanging the rôles of the two exponentials. Substituting

$$z \to \frac{z}{M+1} + \ln\left(\frac{M+1}{E^{1/3}}\right)$$

yields

$$u'''(z) - \frac{1}{(M+1)^2} u'(z) + (-e^{3z/(M+1)} - \tilde{E}e^{3z}) u(z) = 0, \tag{5.3}$$

where $\tilde{E} = -(M+1)^{3M}/E^{M+1}$.

Now transforming back results in the equation

$$\tilde{y}''' + \frac{M(M+2)}{(M+1)^2} \left(\frac{1}{x^2} \frac{d}{dx} - \frac{1}{x^3} \right) \tilde{y} + (-x^{-3M/(M+1)} - \tilde{E}) \tilde{y} = 0. \tag{5.4}$$

As promised, the confining 'potential' x^{3M} has been exchanged for a singular potential $-x^{-3M/(M+1)}$, and a new term, proportional to $(x^{-2}d/dx - x^{-3})y$, has been generated. This motivates us to enlarge the set of differential equations under consideration to

$$y''' - G \left(\frac{1}{x^2} \frac{d}{dx} - \frac{1}{x^3} \right) y + (x^{3M} - E) y = 0 \tag{5.5}$$

with G a new parameter, analogous to $l(l+1)$ for the Schrödinger equation. Duality maps (5.5) to

$$\tilde{y}''' - \tilde{G} \left(\frac{1}{x^2} \frac{d}{dx} - \frac{1}{x^3} \right) \tilde{y} + (-x^{3\tilde{M}} - \tilde{E}) \tilde{y} = 0, \tag{5.6}$$

where

$$\tilde{y}(x, \tilde{E}, \tilde{G}) = (M+1)^{-1} E^{1/3} x^{M/(M+1)} y((M+1) E^{-1/3} x^{1/(M+1)}, E, G) \tag{5.7}$$

and

$$\tilde{M} = -\frac{M}{M+1}, \qquad \tilde{E} = -\frac{(M+1)^{3M}}{E^{M+1}}, \qquad \tilde{G} = \frac{G - M(M+2)}{(M+1)^2}. \tag{5.8}$$

Duality therefore maps the 3-parameter family $(\{M, E, G\})$ of differential equations (5.5) onto itself. The analysis of Section 2 can now be repeated for these generalised problems. It is convenient to write $G = g(g+2)$ and to work mostly with g instead of G. We first enlarge the scope of (2.6) by setting

$$y_k(x, E, g) = \omega^k y(\omega^{-k} x, \omega^{-3Mk} E, g); \tag{5.9}$$

then y_k solves

$$y_k''' - G \left(\frac{1}{x^2} \frac{d}{dx} - \frac{1}{x^3} \right) y_k + e^{-2k\pi i} P(x, E) y_k = 0. \tag{5.10}$$

Next we define $z_{k_1 k_2}$ as in (2.18). If $e^{-2k_1\pi i} = e^{-2k_2\pi i} \equiv e^{-2k\pi i}$, then

$$z_{k_1 k_2}''' - G \left(\frac{1}{x^2} \frac{d}{dx} - \frac{1}{x^3} \right) z_{k_1 k_2} - e^{-2k\pi i} P(x, E) z_{k_1 k_2} = 0, \tag{5.11}$$

which is the equation adjoint to (5.10)[1]. We can recover the original problem by shifting k by a half-integer, and arguing just as before we find that

$$z_{-1/2, 1/2}(x, E, g) = i\sqrt{3}\, y(x, E, g) \tag{5.12}$$

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[1] Notice that the operator $x^{-2}d/dx - x^{-3}$ is by itself anti-self-adjoint; this gives some insight as to why it is a sensible generalisation of the x^{-2} term in the Schrödinger equation.

and then

$$T(E,g) \, y_{-1/4} \, y_{1/4} = y_{-1/4} \, y_{5/4} + y_{-3/4} \, y_{3/4} + y_{-5/4} \, y_{1/4}, \qquad (5.13)$$

where $T(E,g) = S^{(1)}(\omega^{15M/4}E,g) = S^{(2)}(\omega^{21M/4}E,g)$, and $S^{(1)}$ and $S^{(2)}$ are defined as in (2.16). A non-zero value of $G = g(g+2)$ causes (5.10) to be singular at the origin, so simply considering (5.13) at $x=0$ is not an option. Instead, we expand $y(x,E,g)$ as

$$y(x,E,g) = D^+(E,g) \, \chi_+ + D^0(E,g) \, \chi_0 + D^-(E,g) \, \chi_-, \qquad (5.14)$$

where $\{\chi_+,\chi_0,\chi_-\}$ forms a basis of solutions defined via behaviour near the origin:

$$\chi_i(x,E,g) \sim x^{\lambda_i} + O(x^{\lambda_i+3}), \quad i = +,0,-, \qquad (5.15)$$

with the λ_i's the roots of the indicial equation $(\lambda - 1)(\lambda(\lambda - 2) - g(g+2)) = 0$:

$$\lambda_+ = -g, \qquad \lambda_0 = 1, \qquad \lambda_- = g+2. \qquad (5.16)$$

Explicitly, the functions D^+, D^0 and D^- are

$$D^+ = \frac{W[y,\chi_0,\chi_-]}{W[\chi_+,\chi_0,\chi_-]}, \qquad D^0 = \frac{W[y,\chi_-,\chi_+]}{W[\chi_+,\chi_0,\chi_-]}, \qquad D^- = \frac{W[y,\chi_+,\chi_0]}{W[\chi_+,\chi_0,\chi_-]}, \qquad (5.17)$$

with $W[\chi_+,\chi_0,\chi_-] = 2(g+1)^3$. At $g = 0$ they reduce to $y(0,E)$, $y'(0,E)$ and $y''(0,E)/2$ respectively, in agreement with the notation of earlier sections.

Defining

$$Q^\pm(E,g) = E^{\mp(g+1)/3M} D^\pm(E,g), \qquad Q_k^\pm = Q^\pm(\omega^{-3Mk}E,g), \qquad (5.18)$$

the generalised T–Q relations (2.31), (2.35) have exactly the same form as before, and (3.6) becomes

$$\prod_{k=1}^{\infty} \frac{E_k^\pm - \omega^{-3M}E_n^\pm}{E_k^\pm - \omega^{3M}E_n^\pm} = -\omega^{\mp(g+1)} \prod_{k=1}^{\infty} \frac{E_k^\pm - \omega^{-3M/2}E_n^\pm}{E_k^\pm - \omega^{3M/2}E_n^\pm}. \qquad (5.19)$$

The arguments of Section 3 can now be repeated essentially verbatim, to discover that, so long as G is such that the conjectures of Section 3 about the zeroes of T and $D^\pm$ remain true, the quantities D^+ and D^- for the more general differential equation (5.5) are again described by the non-linear integral equation (3.18), but with chemical potential term now taking the value $\alpha = \frac{2}{3}(g+1)$.

At $M = 1$, $\omega^{\pm 3M} = -1$, the l.h.s. of (5.19) is 1 and the A_2-related BA equation 'collapses' onto one more closely linked with A_1. This leads to a rather surprising equivalence between spectral problems for a Schrödinger equation with potential $x^6 + l(l+1)/x^2$ and the third-order problem (5.5) at $M = 1$. For these special points, the quantities in this paper are related to those of [4] as

$$D^\pm(E,g)|_{M=1} \propto D^\pm(c^{-3/2}E,l)|_{M=3}^{\text{Ref.}[4]},$$

$$T(E,g)|_{M=1} = T_1(c^{-3/2}E,l)|_{M=3}^{\text{Ref.}[4]}, \qquad (5.20)$$

where $l + \frac{1}{2} = \frac{2}{3}(g+1)$, $c = \frac{27}{4} \frac{\Gamma(7/6)}{\sqrt{\pi}} \frac{x\Gamma(5/3)}{\Gamma(1/3)}$, and T_1 is one of the 'fused' T-operators discussed in Section 4 of Ref. [4].

6. A link with perturbed conformal field theory

We now return to the relation with the $a_2^{(2)}$ model, briefly mentioned at the end of Section 3. The analogy with results of [17–19] for the A_1-related models suggests that the quantity

$$c_{\text{eff}} = \frac{6ib_0}{\pi^2} \left(\int_{\mathscr{C}_1} e^\theta \ln(1 + e^{f^\pm(\theta)}) \, d\theta - \int_{\mathscr{C}_2} e^\theta \ln(1 + e^{-f^\pm(\theta)}) \, d\theta \right) \qquad (6.1)$$

should be interpreted as an effective central charge of an underlying conformal field theory. For general $\alpha = \frac{2}{3}(g+1)$, this would predict

$$c_{\text{eff}} = 1 - \frac{3}{M+1} \alpha^2 . \qquad (6.2)$$

Going further, it is natural to interpret (3.18) and (6.1) as the ultraviolet limit of the following 'massive' system:

$$f(\theta) = i\pi\alpha - ir\sinh\theta + \int_{\mathscr{C}_1} \varphi(\theta - \theta') \ln(1 + e^{f(\theta')}) \, d\theta'$$

$$- \int_{\mathscr{C}_2} \varphi(\theta - \theta') \ln(1 + e^{-f(\theta')}) \, d\theta' ,$$

$$c_{\text{eff}}(r) = \frac{3ir}{\pi^2} \left(\int_{\mathscr{C}_1} \sinh\theta \, \ln(1 + e^{f(\theta)}) \, d\theta - \int_{\mathscr{C}_2} \sinh\theta \, \ln(1 + e^{-f(\theta)}) \, d\theta \right). \quad (6.3)$$

This should encode finite-size effects in the massive $a_2^{(2)}$ theory, with $r = M_s R$, M_s the mass of the fundamental soliton and R the circumference of the (infinite) cylinder on which the theory is living. (Notice that there is no need to distinguish f^+ from f^- any more, since the mapping $\theta \to -\theta$ now has the effect of negating α.) There is now a natural scale, which can be related to an operator ϕ perturbing the ultraviolet conformal field theory. Standard considerations [26], based on the $\theta \to \theta + i2\pi(M+1)/3M$ periodicity of $f(\theta)$, suggest that so long as α is not an integer $c_{\text{eff}}(r)$ will have an expansion in powers of $r^{6M/(M+1)}$ (together with an irregular 'anti-bulk' term, irrelevant to the current discussion). This implies for ϕ either the conformal dimensions

$$\bar{h}_\phi = h_\phi = 1 - \frac{3M}{2M+2} \qquad (6.4)$$

and an expansion of $c_{\text{eff}}(r)$ in which only even powers of the coupling λ to the operator ϕ appear, or, alternatively, the conformal dimensions

$$\bar{h}_\phi = h_\phi = 1 - \frac{3M}{M+1} \qquad (6.5)$$

and an expansion which sees both even and odd powers of λ.

When α is an integer, the standard considerations of [26] may have to be modified. Absorbing the term $i\pi\alpha$ into a shift in $f(\theta)$, (6.3) becomes exactly odd under a negation of θ. This forces the shifted $f(\theta)$ to be zero at $\theta = 0$, even in the far ultraviolet, and so long as the would-be plateau value is non-zero, it splits the plateau region into two pieces, each of half the previous length. As a result, the regular expansion of $c_{\text{eff}}(r)$ is in powers of $r^{3M/(M+1)}$, and not $r^{6M/(M+1)}$. Formula (6.4) now describes the situation when both even and odd powers of λ appear in the expansion of c_{eff}, while for even powers only, the correct formula is $\bar{h}_\phi = h_\phi = 1 - 3M/(4M+4)$. Note though that this plateau-splitting effect does not occur at $\alpha = 0$, since for this case the plateau value of f is anyway zero, and imposing $f(0) = 0$ has no effect.

As explained in [22] (see also Refs. [28–30]), the $a_2^{(2)}$ model, when appropriately quantum-reduced, should correspond to the minimal models $\mathscr{M}_{p,q}$ (with p and q coprime integers and $p < q$) perturbed by either ϕ_{12}, ϕ_{21} or [31,32] ϕ_{15}. With ϕ_{12} the perturbing operator, the relation with the parameter ξ appearing in the kernel (3.19) is [22]

$$\frac{p}{q} = \frac{2\xi}{(1+\xi)}. \tag{6.6}$$

Since $M = 1/\xi$, the ultraviolet effective central charge for a given value of α, as predicted by (6.2), is

$$c_{\text{eff}} = 1 - \frac{3p}{2q}\alpha^2. \tag{6.7}$$

To recover ϕ_{21} perturbations one simply has to swap p and q in (6.6) and (6.7) [22], while to find ϕ_{15}, p/q should be replaced by $4p/q$ [31].

For the sine-Gordon model, naturally associated with the ϕ_{13} perturbing operator [33,34], it has been observed both analytically and numerically [13,17–19,35,36] that reduction is implemented at the level of finite-size effects and the non-linear integral equation via a particular choice of the chemical potential. The similarity between our equations and those in [17–19] suggests that the same should be true here. To decide which value of α will tune (6.3) onto the ground state of the relevant perturbed minimal model, we demand that the ultraviolet effective central charges match up; the predicted dimensions of the perturbing operators, and a comparison of results at non-zero values of r with those obtained via the thermodynamic Bethe ansatz method, will then provide some non-trivial tests of the proposal.

The effective central charge of the ground state of the theory $\mathscr{M}_{pq}$ is $c_{\text{eff}} = 1 - 6/pq$. Thus to have any chance of matching the vacua of the ϕ_{12}-perturbed models, we must set $\alpha = 2/p$. The required value of h_ϕ, namely $h_{12} = (3p/4q) - \frac{1}{2}$, is then matched by (6.4). The value just chosen for α being a non-zero integer if and only if $p = 2$, (6.4) will be the correct formula to use provided the regular parts of the ground state energies of the models $\mathscr{M}_{pq}$ perturbed by ϕ_{12} expand in even powers of λ for $p \geqslant 3$, and in even and odd powers for $p = 2$. This 'prediction' holds for all of the examples that we checked. We then compared numerical results for $(M, \alpha) = (4,1)$, $(\frac{5}{3}, \frac{2}{3})$, $(\frac{3}{2}, \frac{1}{2})$, $(\frac{4}{3}, \frac{1}{3})$ and $(1,0)$ against the tables of [27,37] for the $A_2^{(2)}$ (Yang–Lee), E_8, E_7, E_6 and D_4-related TBA equations, respectively, finding excellent agreement. Swapping p and q in (6.7),

Table 3
NLIE results versus TBA data from Refs. [27,37–39]

Model	(M,α)	r	TBA	NLIE
$A_2^{(2)} + \phi_{12}$	$(4,1)$	0.001	0.399999735051974	0.399999735051971
		0.002	0.399998953903823	0.399998953903824
$E_8 + \phi_{12}$	$(\frac{5}{3},\frac{2}{3})$	0.025	0.499926331494289	0.499926331494288
		0.05	0.499705463734389	0.499705463734387
$E_7 + \phi_{12}$	$(\frac{3}{2},\frac{1}{2})$	0.02	0.699928050129612	0.699928050129611
		0.04	0.699712371203531	0.699712371203531
$E_6 + \phi_{12}$	$(\frac{4}{3},\frac{1}{3})$	0.025	0.857016839032789	0.857016839032790
		0.05	0.856639509661813	0.856639509661819
$D_4 + \phi_{12}$	$(1,0)$	0.01	0.999972507850553	0.999972507850552
		0.02	0.999890328583463	0.999890328583464
$A_1 + \phi_{21}$	$(\frac{1}{2},\frac{1}{2})$	0.02	0.499697279140833	0.499697279140832
		0.04	0.498957654198721	0.498957654198722
$A_2 + \phi_{21}$	$(\frac{2}{3},\frac{1}{3})$	0.001	0.799999470103948	0.799999470103940
		0.002	0.799997907807646	0.799997907807649
$M_{35} + \phi_{21}$	$(\frac{1}{5},\frac{2}{5})$	0.1	0.596517064916761	0.596517064916762
		0.15	0.592881408017592	0.592881408017593
$M_{37} + \phi_{15}$	$(\frac{1}{6},\frac{1}{3})$	0.1	0.709591770021299	0.709591770021299
		0.15	0.705031895238354	0.705031895238357

the choice $\alpha = 2/q$ should capture the ϕ_{21} cases. The conformal weight of $h_{21} = (3q/4p) - \frac{1}{2}$ is matched by (6.4), provided the swap of p and q in (6.6) is remembered. This time α is never an integer, and the use of (6.4) is justified by the regular parts of the ground state energies of the ϕ_{21} perturbations always being in even powers of the coupling λ. For $(M,\alpha) = (\frac{1}{2},\frac{1}{2})$, $(\frac{2}{3},\frac{1}{3})$ and $(\frac{1}{5},\frac{2}{5})$ the results from the A_1 and A_2-related TBA equations [27] and the $\mathcal{M}_{35}$ model [38,39] were reproduced within our numerical accuracy[2]. Finally, replacing p/q by $4p/q$ in (6.2) ($q > 2p$), at $\alpha = 1/p$ the models $\mathcal{M}_{pq}$ perturbed by ϕ_{15} are recovered. This time it is (6.5) which predicts the correct value for h_ϕ, as expected given that ϕ_{15} perturbations expand in both even and odd powers of λ. TBA equations for a number of ϕ_{15}-perturbed models have been proposed in [31,38,40], but so far we have only compared (3.18) with the TBA for the ϕ_{15} perturbation of the $\mathcal{M}_{37}$ model given in [38]. A selection of our numerical results for all of the cases just mentioned is presented in Table 3, together with thermodynamic Bethe ansatz data taken from Refs. [27,37–39]. To facilitate the comparison, we took $r = M_1 R$ throughout, with M_1 the mass of the fundamental particle in the reduced scattering theory. For $A_2^{(2)}$ and E_8 this is the first breather in the unreduced theory, and M_1 is equal to $2\cos(5\pi/12)M_s$ and $2\cos(3\pi/10)M_s$ respectively. In all of the other models in the table, M_1 is equal to M_s. The results strongly support the claim that the system (6.3) encodes the ground state energies of ϕ_{12}, ϕ_{21} and ϕ_{15} perturbations of minimal models.

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[2] Beware of a misprint in Eq. (7) of Ref. [38]: the (minus) sign before $\sum_{j=1}^{2}$ should be reversed.

In previously-studied examples, equations similar in form to those for the ground state have been found to describe excited states (see, for example, Refs. [13,35,41–43]). We expect that the same will be possible here, but we will leave investigation of this point for future work. Finally, we remark that it would be interesting to derive a non-linear integral equation for the $a_2^{(2)}$ model directly from finite lattice BA equations. We understand that progress is currently being made in this direction [44].

7. General A_2-related BA equations

In this section, we discuss the effect of adding a term proportional to x^{-3} to the differential equation (5.5). The equation becomes

$$y''' - G\left(\frac{1}{x^2}\frac{d}{dx} - \frac{1}{x^3}\right)y + P(x,E,L)\,y = 0, \tag{7.1}$$

where

$$P(x,E,L) = x^{3M} - E + \frac{L}{x^3}. \tag{7.2}$$

Duality acts on M and G as before, and transforms L as

$$L \to \tilde{L} = \frac{L}{(M+1)^3}. \tag{7.3}$$

The relation (5.7), apart from the appearance of L and $\tilde{L}$ as arguments of y and $\tilde{y}$ respectively, is unchanged. The earlier treatment can be generalised by defining

$$y_k \equiv y_k(x,E,g,L) = \omega^k y(\omega^{-k}x, \omega^{-3Mk}E, g, \omega^{-3(M+1)k}L), \tag{7.4}$$

so that

$$y_k''' - G\left(\frac{1}{x^2}\frac{d}{dx} - \frac{1}{x^3}\right)y_k + e^{-2k\pi i}P(x,E,L)\,y_k = 0. \tag{7.5}$$

If we also define $z_{k_1k_2}$ as in (2.18), then, for k_1 and k_2 differing by an integer,

$$z_{k_1k_2}''' - G\left(\frac{1}{x^2}\frac{d}{dx} - \frac{1}{x^3}\right)z_{k_1k_2} - e^{-2k\pi i}P(x,E,L)\,z_{k_1k_2} = 0, \tag{7.6}$$

(with $e^{-2k_1\pi i} = e^{-2k_2\pi i} \equiv e^{-2k\pi i}$) which is the adjoint to (7.5). Again we can recover the original problem by shifting k by a half-integer. As before, we find that $z_{-1/2,1/2}(x,E,g,L) = i\sqrt{3}\,y(x,E,g,L)$ and (cf. (2.28))

$$T(E,g,L)\,y_{-1/4}\,y_{1/4} = y_{-1/4}\,y_{5/4} + y_{-3/4}\,y_{3/4} + y_{-5/4}\,y_{1/4}. \tag{7.7}$$

The presence of a non-vanishing L has however introduced an extra complication. To see this explicitly, shift k by $\pm 1/4$ to get

$$T^{\pm} y_0\, y_{\pm 1/2} = y_0\, y_{\pm 3/2} + y_1\, y_{-1/2} + y_{-1}\, y_{1/2}, \tag{7.8}$$

with $T^{\pm} = T(E\omega^{\mp 3M/4}, g, \mp iL)$. If this equation is rewritten in terms of the function $y(x, E, g, L)$, both signs of L appear: for k integer or half-integer we have

$$y_k = \omega^k y\left(x\omega^{-k}, E\omega^{-3Mk}, g, (-1)^{2k} L\right). \tag{7.9}$$

Notice that the same does not happen for the argument g (or G), which is why this problem did not arise before. From an analytic point of view, $y(x, E, g, L)$ and $y(x, E, g, -L)$ are just two distinct points of the same function, but in the derivation of the non-linear integral equation it is only the analyticity in E that is used. To proceed, it is best to consider L to be held fixed once and for all, and to treat the pair of functions $v_k = \omega^k y(\omega^{-k}x, \omega^{-3Mk}E, g, L)$ and $\bar{v}_k = \omega^k y(\omega^{-k}x, \omega^{-3Mk}E, g, -L)$ independently. Then (7.8) becomes

$$T^{\pm} v_0 \bar{v}_{\pm 1/2} = v_0 \bar{v}_{\pm 3/2} + v_1 \bar{v}_{-1/2} + v_{-1}\bar{v}_{1/2}. \tag{7.10}$$

This equation is very reminiscent of those given in [6] for the A_2-lattice model. There remains an x-dependence in (7.10) which can be eliminated, once again, by expanding

$$v = D^+(E, g, L)\, \chi_+ + D^0(E, g, L)\, \chi_0 + D^-(E, g, L)\, \chi_-, \tag{7.11}$$

$$\bar{v} = \bar{D}^+(E, g, L)\, \bar{\chi}_+ + \bar{D}^0(E, g, L)\, \bar{\chi}_0 + \bar{D}^-(E, g, L)\, \bar{\chi}_-, \tag{7.12}$$

where $\{\chi_+, \chi_0, \chi_-\}$ and $\{\bar{\chi}_+, \bar{\chi}_0, \bar{\chi}_-\}$ are alternative bases defined via the behaviour near the origin

$$\chi_i(x, E, g, L) \sim x^{\lambda_i} + O(x^{\lambda_i + 3}), \quad i = +, 0, -, \tag{7.13}$$

$$\bar{\chi}_i(x, E, g, L) \sim x^{\bar{\lambda}_i} + O(x^{\bar{\lambda}_i + 3}), \quad i = +, 0, -, \tag{7.14}$$

and the λ_i's and $\bar{\lambda}_i$'s are respectively solutions of the indicial equations

$$(\lambda - 1)(\lambda(\lambda - 2) - g(g + 2)) + L = 0,$$
$$(\bar{\lambda} - 1)(\bar{\lambda}(\bar{\lambda} - 2) - g(g + 2)) - L = 0. \tag{7.15}$$

If the labelling is chosen consistently with that of section Section 5, so that the λ's and the $\bar{\lambda}$'s reduce to the quantities in (5.16) when $L = 0$, then

$$T^{\pm} Q_0^{\pm} \bar{Q}_{\pm 1/2}^{\pm} = Q_0^{\pm} \bar{Q}_{\pm 3/2}^{\pm} + Q_1^{\pm} \bar{Q}_{-1/2}^{\pm} + Q_{-1}^{\pm} \bar{Q}_{1/2}^{\pm}, \tag{7.16}$$

with

$$Q^{\pm}(E, g, L) = E^{(\lambda_{\pm} - 1)/3M} D^{\pm}(E, g, L), \quad Q_k^{\pm} = Q^{\pm}(\omega^{-3Mk}E, g, L), \tag{7.17}$$

$$\bar{Q}^{\pm}(E, g, L) = E^{(\bar{\lambda}_{\pm} - 1)/3M} \bar{D}^{\pm}(E, g, L), \quad \bar{Q}_k^{\pm} = \bar{Q}^{\pm}(\omega^{-3Mk}E, g, L). \tag{7.18}$$

This leads to two coupled sets of BA equations

$$\prod_{k=1}^{\infty} \frac{E_k^{\pm} - \omega^{-3M} E_n^{\pm}}{E_k^{\pm} - \omega^{3M} E_n^{\pm}} = -\omega^{2\lambda_{\pm} - \bar{\lambda}_{\pm} - 1} \prod_{k=1}^{\infty} \frac{\bar{E}_k^{\pm} - \omega^{-3M/2} E_n^{\pm}}{\bar{E}_k^{\pm} - \omega^{3M/2} E_n^{\pm}} \,, \tag{7.19}$$

$$\prod_{k=1}^{\infty} \frac{\bar{E}_k^{\pm} - \omega^{-3M} \bar{E}_n^{\pm}}{\bar{E}_k^{\pm} - \omega^{3M} \bar{E}_n^{\pm}} = -\omega^{2\bar{\lambda}_{\pm} - \lambda_{\pm} - 1} \prod_{k=1}^{\infty} \frac{E_k^{\pm} - \omega^{-3M/2} \bar{E}_n^{\pm}}{E_k^{\pm} - \omega^{3M/2} \bar{E}_n^{\pm}} \,. \tag{7.20}$$

Generalising the analysis of Section 3, it should be possible to derive a non-linear integral equation relevant to this more general case. We expect that this equation will coincide with the $a_2^{(1)}$-related case of the equations found in [45,46], in its massless limit, but we will leave a detailed investigation for future work.

8. Conclusions

We have continued to study the relationship between integrable quantum field theories and ordinary differential equations, and in the process have obtained a novel non-linear integral equation which is able to describe the ϕ_{12}, ϕ_{21} and ϕ_{15} perturbations of minimal models within a unified framework. We have also found a natural generalisation of the duality symmetry enjoyed by the Schrödinger/massless sine Gordon system [2]. A major theme has been that the A_2 structures hidden inside certain third-order ordinary differential equations, and also inside certain integrable quantum field theories and BA systems, are very closely related. It seems clear that the correct way to generalise to yet further models is to look to differential equations of even higher order. While this might appear to be a task of ever-increasing complexity, there are some reasons to suppose that a more unified picture will ultimately emerge. ADE structures have been observed in many different, but related, settings in the context of integrable models (see, for example, Refs. [5,26,45–51] and references therein). One might hope that the process of generalisation will reveal similar phenomena on the differential equations side of the correspondence, but more case-by-case analysis will certainly be required before this can be confirmed.

Notes added

(i)   The 'massless' non-linear integral equation derived in Section 3 has appeared previously, in connection with the Izergin–Korepin model, in [52].

(ii)   A conjecture due to Kausch et al. [40,53] states that the ϕ_{12} perturbation of $\mathcal{M}_{p,q}$ and the ϕ_{15} perturbation of $\mathcal{M}_{p',q'}$ have identical ground-state scaling functions if (and only if) $p' = p/2$, $q' = 2q$. (This implies $p = 2 \bmod 4$, since (p,q) and (p',q') must both be coprime; such pairs are called 'type II' in [40].) It is easily checked that this equality follows from the recipe for finding ground-state scaling functions given in Section 6: the values of M and α that should be used in the two cases are identical, and so both are described by the

same non-linear integral equation. We take this as additional support both for our conjectures and for that of Ref. [40].

(iii)   In a recent paper [54], Suzuki has independently remarked the relevance of higher-order ordinary differential equations to integrable models associated with the algebra A_n, though with a slightly different emphasis from that adopted above.

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