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Competition between algae and fungi in a lake: a mathematical model

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Abstract

In this paper a mathematical model for handling water pollution is introduced. We assume that algae and fungi are in competition for resources that come from wastewater. Both algae and fungi need dissolved oxygen (DO) for their biologial process of growth. But there is a difference, indeed algae produce it too and in a higher quantity than the one they use. It is shown that if the coexistence equilibrium exists, it is stable without additional conditions. If the competition rate between algae and fungi is not high for a chosen set of parameters the stability of the coexistence equilibrium is reached even without an external constant input of DO in the system.

Key words: mathematical model, algae, fungi, competition, wastewater
MSC 2000: AMS codes (optional)

1 Introduction

Algae are important in a lake, they can improve the quality of the aquatic ecosystem. Under right conditions such as adequate nutrients (mostly phosphorus, but nitrogen is important too) they grow. The nutrients that are present in the wastewater can derive from agricultural and/or industrial discharges. Fungi can be used for biodegradation of organic pollutant in a waterbody, and they grow using the nutrients obtained from the biodegradation, [1]. Some mathematical models in the literature study the behavior of algae biomass in a waterbody in the presence of organic pollutants, [4, 5]. In [2, 3] the case of fungi has been addressed. In this paper we want to study what happens when both algae and fungi are present in the same waterbody, for example in a lake. Furthermore we suppose that they are in competition for the resources coming from the pollutants.

2 The mathematical model

In this paper we introduce a mathematical system that models the behavior of algae and fungi in a waterbody. The waterbody considered could be nutrient-rich waters, like municipal wastewater or some industrial effluents. Both algae and fungi can feed on these wastes and therefore purify the water, while also producing a biomass suitable for biofuels production. Thus algae and fungi are in competition for food, since both share the same resources. Further, fungi as well as algae need DO to thrive but we assume that the algae's production and input of DO into the system is much larger than their own use for their growth.

The model consists of three equations that describe the time evolution of the algae population, the fungi population and the DO respectively. The model, in which all the parameters are nonnegative, reads:

$$\begin{aligned} \frac{dA}{dt} &= r_A A - a_A A - b_A A^2 - cAF \\ \frac{dF}{dt} &= \frac{hOF}{k + k_O O} - a_F F - b_F F^2 - cAF \\ \frac{dO}{dt} &= q_O + gA - a_O O - f\frac{hOF}{k + k_O O}. \end{aligned} \tag{1}$$

In the first equation algae grow at a constant rate r_A and are washed out at a constant rate a_A. We assume that algae are in competition among themselves at a constant rate b_A and also experience interspecific competition with fungi at rate c.

In the second equation the fungi's growth depends on the presence of DO. They are washed out at rate a_F. The intraspecific competition occurs at rate b_F while c denotes the rate of the interspecific competition with the algae population.

The third equation shows the evolution in time of DO. We assume that it is supplied from external sources at rate q_O, but a part of it comes from the algae own production at rate g. We take further into account its washing out, at rate a_O and its depletion due to its assumption by fungi at rate $f \geqslant 1$.

3 The qualitative analysis of the model

First of all to find the equilibrium points of the model, we need to solve the system obtained by setting the right hand side of (1) to zero,

$$\begin{cases} A(r_A - a_A - b_A A - cF) = 0 \\ F\left(\dfrac{hO}{k + k_O O} - a_F - b_F F - cA\right) = 0 \\ q_O + gA - a_O O - f\dfrac{hOF}{k + k_O O} = 0. \end{cases} \tag{2}$$

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Further, for the stability analysis, we need to calculate the Jacobian matrix of the system (1)

$$J = \begin{bmatrix} r_A - a_A - 2b_A A - cF & -cA & 0 \\ -cF & -a_F - 2b_F F - cA + \dfrac{hO}{k + k_O O} & \dfrac{hkF}{(k + k_O O)^2} \\ g & -\dfrac{fhO}{k + k_O O} & -a_O - \dfrac{fhkF}{(k + k_O O)^2} \end{bmatrix}. \tag{3}$$

Solving (2) we obtain the analytic expression of three equilibrium points. In addition, we prove that two other equilibria exist. We also show that all these points are conditionally locally asymptotically stable, while the coexistence equilibrium is stable if it is feasible.

Proposition 1. The trivial equilibrium point, $E_0 = (0, 0, 0)$, exists if

$$q_O = 0. \tag{4}$$

Furthermore, it is stable if the following condition holds:

$$r_A < a_A. \tag{5}$$

Proof. For $A = F = O = 0$ in the system (2) we get that E_0 exists if $q_O = 0$. The characteristic polynomial associated to the matrix (3) evaluated at E_0 is

$$\det(J - \mu) = (r_A - a_A - \mu)(-a_F - \mu)(-a_O - \mu) = 0.$$

To have the stability of E_0 all the eigenvalues shoud be negative thus the condition (5) must hold.

□

Proposition 2. The fungi-and-algae-free point $E_1 = \left(0, 0, q_O a_O^{-1}\right)$ exist always. It is stable if the following conditions hold:

$$r_A < a_A \quad \text{and} \quad \frac{hq_O}{ka_O + k_O q_O} < a_F. \tag{6}$$

Proof. In fact for $A = F = 0$ in (2) from the last equation we get $O = q_O a_O^{-1}$. While the characteristic polynomial associated to E_1 is

$$\det(J - \mu) = (r_A - a_A - \mu)(-a_O - \mu)\left(\frac{hq_O}{ka_O + k_O q_O} - a_F - \mu\right) = 0.$$

To have all the eigenvalues negative the conditions (6) must hold. □

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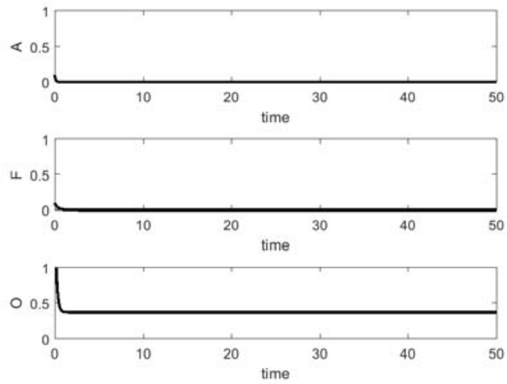

Figure 1: The equilibrium E_1 is stably achieved with the parameter values $r_A = 10.1273$, $a_A = 16.93072$, $b_A = 11.7382$, $c = 19.3012$, $h = 1.61592$, $k = 0.454245$, $k_O = 5.87845$, $a_F = 2.55798$, $b_F = 0.0344478$, $q_O = 2$, $g = 3.63317$, $a_O = 5.41771$, $f = 1$.

In Figure 1 one can see that for a chosen set of parameters the equilibrium E_1 is stably achieved.

Proposition 3. The fungi-free equilibrium $E_2 = \left(\frac{r_A - a_A}{b_A}, 0, \frac{q_O b_A + g(r_A - a_A)}{b_A a_O}\right)$ is feasible if

$$r_A > a_A \tag{7}$$

and it is stable if

$$\frac{hO_2}{k + k_O O_2} < a_F + cA_2 \tag{8}$$

hold.

Proof. If $F = 0$ in the system (2) we get

$$\begin{cases} r_A - a_A - b_A A = 0 \\ F = 0 \\ q_O + gA - a_O O = 0. \end{cases} \tag{9}$$

From the first equation of (9) it follows

$$A = \frac{r_A - a_A}{b_A}.$$

Thus for the nonnegativity of the algae population, (7) must hold. From the third equation instead we get the equilibrium value of the oxygen.

$$O = \frac{q_O b_A + g(r_A - a_A)}{b_A a_O}.$$

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The characteristic polynomial associated to E_1 is once again easily obtained,

$$\det(J-\mu) = (-r_A + a_A - \mu)(-a_O - \mu)\left(\frac{hO_2}{k+k_O O_2} - a_F - cA_2 - \mu\right) = 0,$$

as well as its eigenvalues

$$\begin{aligned} \mu_1 &= -r_a + a_A < 0 \\ \mu_2 &= -a_O < 0 \\ \mu_3 &= \frac{hO_2}{k+k_O O_2} - a_F - cA_2 \end{aligned}$$

Requiring $\mu_3 < 0$ we get (8). □

In Figure 2 we show that for a chosen set of parameters the stability of the fungi-free equilibrium, E_2, is attained.

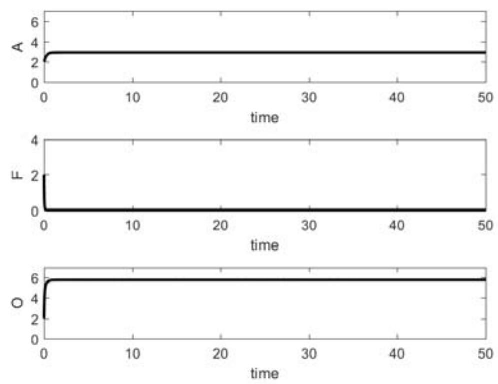

Figure 2: The equilibrium E_2 is stable for the parameter values $r_A = 8.90096$, $a_A = 4.14886$, $b_A = 1.62848$, $c = 0.0691916$, $h = 11.8402$, $k = 1.92602$, $k_O = 16.7554$, $a_F = 18.7713$, $b_F = 15.4976$, $q_O = 69.5303$, $g = 13.847$, $a_O = 19.037$, $f = 1$.

Proposition 4. The algae-free point is in fact a set of multiple equilibria, namely $(0, F_3, O_3)$, $(0, F_4, O_4)$ and $(0, F_5, O_5)$. Of them, only one is feasible, if

$$O > \frac{a_F k}{h - a_F k_O}, \tag{10}$$

and it is stable if

$$r_A < a_A + cF_3. \tag{11}$$

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Proof. **Part 1: existence**

For $A = 0$ the system (2) becomes

$$\begin{cases} A = 0 \\ \dfrac{hO}{k + k_O O} - a_F - b_F F = 0 \\ q_O - a_O O - f \dfrac{hOF}{k + k_O O} = 0. \end{cases} \quad (12)$$

Solving the second equation with respect to F we get

$$F = \frac{O(h - a_F k_O) - a_F k}{b_F (k + k_O O)}. \quad (13)$$

Condition (10) arises by requiring the positivity of the expression (13). Note that the opposite case obtained when $h - a_F k_O < 0$, cannot arise,

$$O < \frac{a_F k}{h - a_F k_O},$$

because from it, $O < 0$ follows, which is impossible.

Substituting the expression (13) for F into the third equation of the system (12) we obtain the following third degree equation in O

$$aO^3 + bO^2 + cO + d = 0, \quad (14)$$

with

$$\begin{aligned} a &= -a_O b_F k_O^2 < 0 \\ b &= q_O b_F k_O^2 - 2a_O b_F k k_O - f h^2 + f h a_F k_O \\ c &= 2q_O b_F k k_O - a_O b_F k^2 + f h a_F k \\ d &= q_O b_F k^2 > 0. \end{aligned}$$

Since $a < 0$ and $d > 0$ by Descartes' rule of signs the third degree polynomial (14) in O has at least one positive root. We are able to show that there is exactly one such root. In the next Table the only four possible cases are summarized.

Cases	a	b	c	d	number of real positive roots
1)	−	+	−	+	3 (impossible)
2)	−	+	+	+	1
3)	−	−	−	+	1
4)	−	−	+	+	1

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The first case is impossible, in fact assuming that $b > 0$ and $c < 0$ we find

$$a_O b_F k + \frac{fh^2}{k_O} < q_O b_F k_O - a_O b_F k + fha_F < -q_O b_F k_O.$$

But this is a contradiction, because the term in the middle should be less than a negative term (on the right) and greater than a positive one (on the left side).

Thus there is only one positive equilibrium

$$E_3 = \left(0, \frac{O_3(h - a_F k_O) - a_F k}{b_F(k + k_O O_3)}, O_3\right) \quad \text{if} \quad O_3 > \frac{a_F k}{h - a_F k_O},$$

with O_3 the real positive root of (14).

Part 2: stability

To study the stability of the equilibrium point we evaluate the Jacobian matrix (3) at E_3. The resulting characteristic polynomial is

$$\begin{aligned} \det(J - \mu) &= (r_A - a_A - cF_3 - \mu)\left\{\mu^2 + \left(a_F + 2b_F F_3 + a_O + \frac{fhkF_3}{(k + k_O O_3)^2} + \right.\right. \\ &\left.\left. - \frac{hO_3}{k + k_O O_3}\right)\mu + \left(a_O + \frac{fhkF_3}{(k + k_O O_3)^2}\right)(a_F + 2b_F F_3) - \frac{a_O h O_3}{k + k_O O_3}\right\} = 0. \end{aligned}$$

The eigenvalue $\mu_1 = r_A - a_A - cF_3$ is negative if (11) holds, while the roots of the quadratic polynomial in μ are negative with no further conditions. It turns out that both coefficients of the terms of the two lowest degrees in μ are positive. In fact, substituting F_3, (13), for the coefficient of μ we get

$$\begin{aligned} &\left(a_F + \frac{2hO_3}{k + k_O O_3} - \frac{2(O_3 a_F k_O + a_F k)}{k + k_O O_3} + a_O + \frac{fhkF_3}{(k + k_O O_3)^2} - \frac{hO_3}{k + k_O O_3}\right) \\ &= \left(a_F + \frac{hO_3}{k + k_O O_3} - 2a_F + a_O + \frac{fhkF_3}{(k + k_O O_3)^2}\right) \\ &= \frac{O_3(h - a_F k_O) - a_F k}{b_F(k + k_O O_3)} + a_O + \frac{fhkF_3}{(k + k_O O_3)^2} = F_3 + a_O + \frac{fhkF_3}{(k + k_O O_3)^2} > 0. \end{aligned}$$

Similarly, for the constant term, by dividing by a_O, denoting by H is a positive term, we have:

$$a_F + 2b_F F_3 - \frac{hO_3}{k + k_O O_3} + H > 0.$$

□

Figure 3 shows that for a chosen set of parameters the algae-free equilibrium, E_3, is stably achieved.

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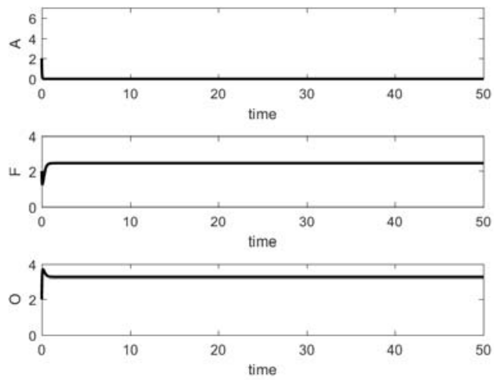

Figure 3: The equilibrium E_3 is stabile for the parameters $r_A = 4.85571$, $a_A = 13.2729$, $b_A = 11.6077$, $c = 12.7024$, $h = 11.4803$, $k = 2.3815$, $k_O = 1.11246$, $a_F = 1.40551$, $b_F = 1.95987$, $q_O = 65.3973$, $g = 16.7907$, $a_O = 15.374$, $f = 1$.

For the coexistence equilibrium point we have the following result

Proposition 5 There exists at least one feasible coexistence equilibrium $E_4 = (A^*, F^*, O^*)$ if the following three conditions hold:

$$r_A - a_A - b_A A^* > 0, \quad b_F b_A - c^2 > 0, \quad (a_F c + b_F)(k + k_O O^*)(r_A - a_A) > chO^* \tag{15}$$

and whenever it exists, it is stable.

Proof. To find the conditions for the existence of the coexistence equilibrium point from the first equation of the system (2) we get

$$F = \frac{r_A - a_A - b_A A}{c}$$

and substitute it into the remaining two equations. We solve these two equations with respect to A and we match the resulting expressions

$$A = \frac{(a_F c + b_F)(r_A - a_A)(k + k_O O) - chO}{(k + k_O O)(b_F b_A - c^2)} = \frac{fhO(r_A - a_A) + c(a_O O - q_O)(k + k_O O)}{cg(k + k_O O) + fhb_A O}.$$

Thus, we now have the following cubic polynomial in O:

$$a_1 O^3 + b_1 O^2 + c_1 O + d_1 = 0, \tag{16}$$

with

$$\begin{aligned}
a_1 &= -a_O k_O^2 (b_F b_A - c^2) < 0 \quad \text{for} \quad b_F b_A - c^2 > 0 \\
b_1 &= k_O(fhc + b_F k_O g)(r_A - a_A) + k_O(k_O q_O - 2a_O k)(b_F b_A - c^2) + \\
&\quad +(a_F k_O - h)(fhb_A + ck_O g) \\
c_1 &= k(fhc + 2b_F k_O g)(r_A - a_A) + k(2k_O q_O - a_O k)(b_F b_A - c^2) + \\
&\quad +chg(2a_F k_O - h) + a_F khb_A \\
d_1 &= a_F ck^2 g + k^2 q_O(b_F b_A - c^2) + b_F k^2 g(r_a - a_A) > 0.
\end{aligned}$$

Since for $b_F b_A - c^2 > 0$, $a_1 < 0$ and $d_1 > 0$ the polynomial (16) has at least one positive root O^* by the Descartes' rule of signs. For the feasibility of the equilibrium we need to have $F^* > 0$ and $A^* > 0$, providing thus the first and the third conditions in (8).

To study the stability, the characteristic polynomial of (3) evaluated at E_4 gives the cubic equation

$$\det(J - \mu) \;=\; \mu^3 + R\mu^2 + S\mu + P = 0$$

with

$$\begin{aligned}
R &= \left(b_A A^* + a_O + b_F F^* + \frac{fhkF^*}{(k + k_O O^*)^2} \right) > 0 \\
S &= \left(A^* F^* (b_A b_F - c^2) + b_F a_O F^* + \frac{fhkF^*(b_A A^* + b_F F^*)}{(k + k_O O^*)^2} + \frac{h^2 kF^* O^*}{(k + k_O O^*)^3} \right) > 0 \\
P &= \left((b_F b_A - c^2) \left(a_O A^* F^* + \frac{fhkF^{*2} A^*}{(k + k_O O^*)^2} \right) + \frac{b_A A^* h^2 kF^* O^*}{(k + k_O O^*)^3} + \frac{cghkA^* F^*}{(k + k_O O^*)^2} \right) > 0.
\end{aligned}$$

For $b_F b_A - c^2 > 0$, which holds by feasibility, the three eigenvalues are negative. Thus E_4 is stable whenever it is feasible. □

In Figure 4 we show that the coexistence equilibrium is stable for a selected set of parameter values.

In the next Table we summarize the feasibility conditions for the five equilibrium points of model (1).

In Figure 5 for a chosen set of parameters and the same initial conditions changing the value of q_O, the constant input of DO in the system, we obtain the stability of the coexistence equilibrium, E_4 on the left and of E_2 on the right. Thus, starting from the coexistence equilibrium, by decreasing the rate q_O at which oxygen is supplied into the system, we can obtain the fungi-free equilibrium.

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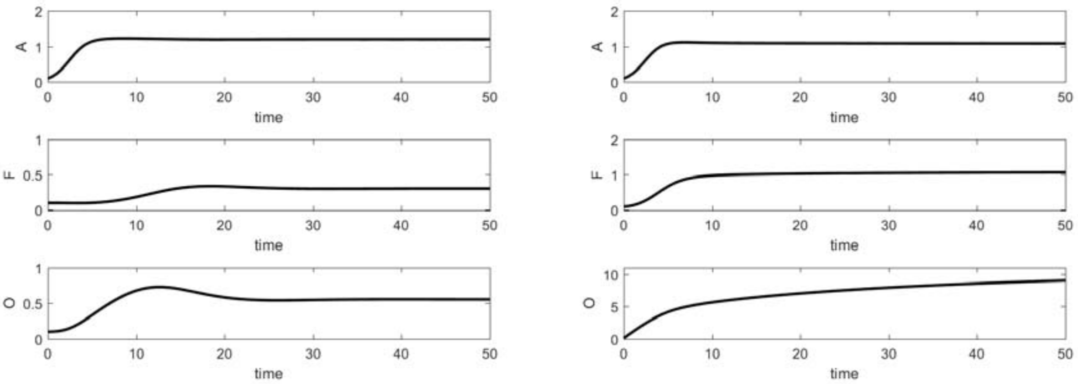

Figure 4: Two possible configurations for the equilibrium E_4, $q_O = 0$ and $q_O = 1$. It is stable in the following cases. Left: $q_O = 0$, $(1.20, 0.30, 0.55)$ is achieved for the parameters $r_A = 1$, $a_A = 0.001$, $b_A = 0.8$, $c = 0.1224$, $h = 1.1$, $k = 1$, $k_O = 1$, $a_F = 0.001$, $b_F = 0.8$, $q_O = 0$, $g = .1$, $a_O = 0.001$, $f = 1$. Right: $q_O = 1$, $(1.08, 1.08, 10.15)$ is obtained for the parameters $r_A = 1$, $a_A = 0.001$, $b_A = 0.8$, $c = 0.1224$, $h = 1.1$, $k = 1$, $k_O = 1$, $a_F = 0.001$, $b_F = 0.8$, $q_O = 1$, $g = .1$, $a_O = 0.001$, $f = 1$.

Eq.	Feasibility conditions	Stability conditions
E_0	$q_0 = 0$	$r_A < a_A$
E_1	none	$r_A < a_A$ and $\dfrac{hq_O}{ka_O + k_O q_O} < a_F$
E_2	$r_A > a_A$	$\dfrac{hO_2}{ka_O + k_O O_2} < a_F + cA_2$
E_3	$O_3 > \dfrac{a_F k}{h - a_F k_O}$	$r_A < a_A + cF_3$
E_4	$r_A - a_A - b_A A^* > 0, \quad b_F b_A - c^2 > 0$ $(a_F c + b_F)(k + k_O O^*)(r_A - a_A) > chO^*$	none

4 Conclusions and future work

A three dimensional, nonlinear mathematical model has been introduced and analysed. In addition to the trivial equilibrium, four additional equilibrium points have been found. Their stability was been completely analysed. For a chosen set of parameters with the same initial conditions we get the stability of the coexistence equilibrium, E_4, both in the absence, $q_O = 0$, and with full, $q_O = 1$, external oxygen supply, Figure 4. Thus the constant input of DO is not necessarily needed if the parameters are chosen appropriately, to have a viable

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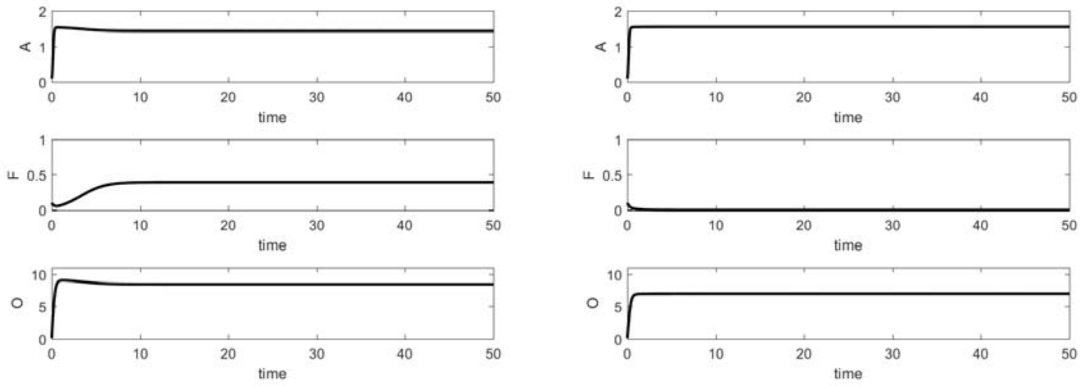

Figure 5: Left: the equilibrium E_4 is stable for $q_O = 30$; $r_A = 19.6445$, $a_A = 1.73234$, $b_A = 11.5828$, $c = 3.34324$, $h = 16.676$, $k = 12.4305$, $k_O = 0.517493$, $a_F = 3.04963$, $b_F = 1.3597$, $q_O = 30$, $g = 11.5323$, $f = 0.835718$, $a_O = 5.42261$ Right: the equilibrium E_2 at the stability $q_O = 20$ for $r_A = 19.6445$, $a_A = 1.73234$, $b_A = 11.5828$, $c = 3.34324$, $h = 16.676$, $k = 12.4305$, $k_O = 0.517493$, $a_F = 3.04963$, $b_F = 1.3597$, $q_O = 20$, $g = 11.5323$, $f = 0.835718$, $a_O = 5.42261$.

system. In fact algae contribution of DO to the system is enough for the fungi utilization. The simulations of Figure 5 instead show that the DO concentration should not drop below a critical threshold, because in such situation the fungi may disappear. Such a loss would be detrimental for the ecosystem.

One of the hypothesis of the model is the competition for food between algae and fungi, but in an indirect way the results indicate that algae help the fungi growth by producing DO.

In our future research we will compare the model introduced here (1) with another one in which the nutrient equation is also considered, as follows:

$$\begin{aligned}
\frac{dA}{dt} &= \frac{h_A N A}{k_A + k_N N} - e_A A - m_A A^2 \\
\frac{dF}{dt} &= k(N,O)F - e_F F - m_F F^2 \\
\frac{dN}{dt} &= q_N - r\frac{h_A N A}{k_A + k_N N} - sk(N,O)F - e_N N \\
\frac{dO}{dt} &= q_O + g_A A - e_O O - c_O k(N,O)F
\end{aligned} \tag{17}$$

with

$$k(N,O) = \frac{k_1 N O}{k_2 + k_3 N + k_4 O + k_3 k_4 N O}.$$

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