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# Dynamic oligopolies with contingent workforce and investment costs

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**Abstract**

Cournot oligopolies are examined with two kinds of output adjustment costs, which model the use of contingent work force and additional investments. The best responses of the firms are first determined and the partial adjustment toward best responses is assumed in formulating a dynamic model. The steady states are first characterized and the dynamic behavior of the output trajectories is demonstrated by computer simulation. With small number of firms and low speeds of adjustments the trajectories converge to a steady state. This convergence is lost with increasing number of firms and/or larger speeds of adjustment giving the possibility of cycles and even chaotic behavior.

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## 1. Introduction

One of the most frequently discussed models of mathematical economy is the classical Cournot oligopoly [2]. It represents an industry with a few firms producing the same goods or offering the same services to a homogeneous market. This situation is usually modeled as a noncooperative game, and the solution of the game is the Nash equilibrium. Initially the existence and uniqueness of the equilibrium was the main research issue, and computational methods were developed to find the equilibria. Okuguchi [12] offered a comprehensive summary of the major results up to the mid 70s, and their multiproduct extensions with some practical applications were discussed in [13]. In dynamic extensions linear models were first introduced and examined, the dynamic properties of which are relatively simple and local asymptotical stability implies global asymptotical stability. The local stability of nonlinear systems can be investigated by linearization, however the study of global dynamics required the development of new methodology. In the case of discrete time scales the critical curve method has been shown to be a powerful tool. [1] has an introduction to this methodology as well as its applications to a large variety of nonlinear dynamic oligopolies. The classical oligopoly models as most models introduced in mathematical economics, are based on certain assumptions in order to have a mathematically tractable problem. Several of the traditional assumptions have been already criticized in the literature. Some of the special models discussed in [1] take some of these critical comments and suggestions into account. Based on the critical comments nonlinearities were introduced into the models. For example, [8] modeled joint ventures, partial equity interests and two types of indirect shareholdings between the firms as partially cooperative Cournot oligopolies. In [9] and [10] antitrust tresholds were introduced into oligopolies with linear and isoelastic price functions based on the Herfindal-Hirschmann index as an indicator of violation of antitrust regulations. The firms were assumed to partially cooperate until this index reached a certain threshold showing violation, then stop cooperating resulting in flip-flop dynamics. Analytic conditions of local stability were derived and the global asymptotic behavior of the steady states were examined by computer simulation.

In this paper we will introduce production adjustment costs into the oligopoly models. Only very few works have been done in this direction in the past. Linear inverse demand function and quadratic production and adjustment functions were assumed by [6] and the authors showed that the adjustment costs had no effect on the equilibrium, it had only a stabilizing effect on the market. In the dynamic game of [7] the additional adjustment cost had only a limited effect on the output levels of the firms. In the model of [14] quadratic adjustment cost function was assumed with zero value at zero so the profit of the firms remained continuous and differentiable. He considered only continuous time scales. In [4] it was assumed that the output adjustment cost depended on the current output levels and their derivatives in a duopoly. Discrete time scales with continuous adjustment cost functions were assumed in [17], and both continuous and discrete time models were examined in [13]. A linear-quadratic differential game model was introduced in [15] with a special focus on the equilibria and feedback strategies. The continuity assumption of the adjustment cost function was dropped in [20] where the best responses of the firms were determined and the equilibria set was described. The dynamic extension of this model was analyzed in [11], and the complex dynamic behavior of the firms was illustrated by computer simulations. All previous models assumed that the firms face additional cost at each time period when they increase their output levels. This assumption is realistic especially in the case when contingent workforce is employed by the firms (see for example, [19]). Increasing the output may also require additional investment including the purchase of new machineries, equipments and even constructing new buildings. In such cases additional cost arises when the output level exceeds the capacity limit that was already build up by the firm. That is, this type of cost arises when the current output of the firm exceeds the maximum output occurred in the previous time periods. In this paper we will introduce both types of production adjustment costs into the classical oligopoly model. For the sake of mathematical simplicity single-product models without product differentiation will be considered, more complex oligopolies can be examined in a similar way. The combination of the two types of adjustment cost is the main contribution of this paper.

This paper develops as follows. In Section 2 the mathematical model is introduced and the best response functions of the firms are determined. Section 3 constructs the dynamical model with partial adjustment toward best responses, and the steady states of these dynamic systems are found. Simulation studies are reported in Section 4 showing the large variety and the complexity of the long-term behavior of the state trajectories. Conclusions and further research directions are drawn in the final Section 5.

## 2. The Mathematical Model and Best Responses

Consider a single-product oligopoly without product differentiation. Let $n$ denote the number of firms, $x_k(t)$ the output of firm $k$ at time period $t$, $L_k$ the absolute maximal production level of firm $k$, which might be infinity and can never be exceeded, and $S(t) = \sum_{k=1}^{n} x_k(t)$ the total output of the industry. The price function is assumed to be linear: $p(S) = A - BS$ with $A, B > 0$, and the cost function of firm $k$ is given as $C_k(x_k) = c_k + d_k x_k$ with $d_k > 0$ and $c_k \geq 0$. Two kinds of adjustment costs are assumed. In the first case adjustment cost is assumed if the firm increases its production level from the previous time period:

$$\overline{C}_k\left(x_k, x_k\left(t-1\right)\right) = \begin{cases} 0 & \text{if } x_k \le x_k\left(t-1\right) \\ \gamma_k\left(x_k - x_k\left(t-1\right)\right) & \text{otherwise,} \end{cases} \tag{1}$$

where $\gamma_k > 0$ is a given constant. For example, having additional workforce can be modeled by this equation. The second type of adjustment cost arises when the output of a firm exceeds the capacity limit being built up by the firm during the previous time periods:

$$\overline{\overline{C}}_k\left(x_k, X_k\left(t-1\right)\right) = \begin{cases} 0 & \text{if} \quad x_k \le X_k\left(t-1\right) \\ \alpha_k\left(x_k - X_k\left(t-1\right)\right) & \quad \text{otherwise,} \end{cases} \tag{2}$$

where $\alpha_k > 0$ is a positive constant and

$$X_k\left(t-1\right) = \max_{0 \le \tau \le t-1} \left\{x_k\left(\tau\right)\right\} \tag{3}$$

is the maximum built-up capacity of firm $k$ up to time period $t-1$. For example, the purchase of additional machinery has this kind of cost function. Notice that equation (2) can be intepreted as investment in capacity, the depreciation of it could be also included in the formula. This additional feature would not change the model significantly, so for the sake of mathematical simplicity we did not take it into account. In this paper we consider adjustment costs only when the firm increases its output level. In principle a firm could also face an adjustment cost even though it reduces the production level with respect to the previous period. This is the case for example, when workforce is laid off and the firm is obliged to give severance pay.

The profit of firm $k$ at time period $t$ can be obtained as follows:

$$\begin{aligned} \Pi_k\left(x_k, x_k\left(t-1\right), X_k\left(t-1\right)\right) = \;& x_k\left(A - Bx_k - Bs_k\right) - \left(c_k + d_k x_k\right) \\ & - \overline{C}_k\left(x_k, x_k\left(t-1\right)\right) - \overline{\overline{C}}_k\left(x_k, X_k\left(t-1\right)\right), \end{aligned} \tag{4}$$

where $s_k = \sum_{l \neq k} x_l$ is the output of the rest of the industry. With fixed values of $x\left(t-1\right)$ and $X\left(t-1\right)$ this profit function is concave in $x_k$.

In order to determine the best response function of firm $k$ we have to consider several cases. Their condition numbers are indicated in Figure 1, where the different possible shapes of this profit function are shown.

If $\partial \Pi_k / \partial x_k \le 0$ at $x_k = 0$, then the best response is $x_k = 0$. This is the case when

$$A - Bs_k - d_k \le 0,$$

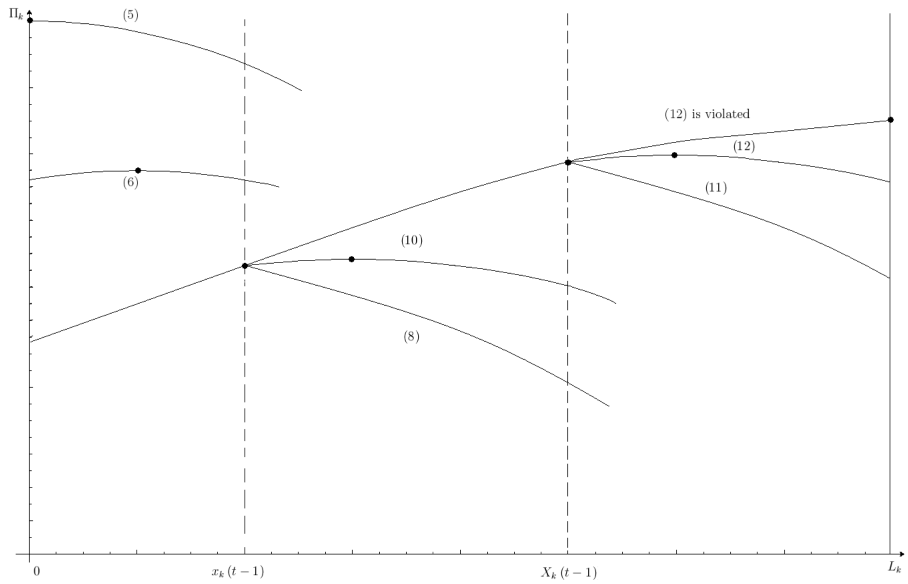


Figure 1: The possible shapes of the profit functions

that is, when

$$s_k \geq \frac{A - d_k}{B}. \tag{5}$$

Otherwise we have two possibilities. The first occurs when $\partial\Pi_k/\partial x_k \leq 0$ at $x_k = x_k(t-1)$ from the left hand side. This is the case, when

$$A - 2Bx_k(t-1) - Bs_k - d_k \leq 0,$$

or

$$s_k \geq \frac{A - d_k}{B} - 2x_k(t-1). \tag{6}$$

In this case the best response is the stationary point between 0 and $x_k(t-1)$:

$$x_k = -\frac{s_k}{2} + \frac{A - d_k}{2B}. \tag{7}$$

In the second case the left hand side derivative of $\partial\Pi_k/\partial x_k$ at $x_k = x_k(t-1)$ is positive. For the right hand side derivative we have two possibilities. It is nonpositive if

$$A - 2Bx_k(t-1) - Bs_k - d_k - \gamma_k \leq 0,$$

that is, when

$$s_k \geq \frac{A - d_k - \gamma_k}{B} - 2x_k\left(t-1\right), \tag{8}$$

and the best response is $x_k = x_k\left(t-1\right)$. Otherwise (8) is violated and we have again two cases. If the left hand side derivative $\partial\Pi_k/\partial x_k$ is nonpositive at $X_k\left(t-1\right)$, then the best response is the stationary point between $x_k\left(t-1\right)$ and $X_k\left(t-1\right)$:

$$x_k = -\frac{s_k}{2} + \frac{A - d_k - \gamma_k}{2B}. \tag{9}$$

It is clearly the case, when

$$A - 2BX_k\left(t-1\right) - Bs_k - d_k - \gamma_k \leq 0,$$

which can be rewritten as

$$s_k \geq \frac{A - d_k - \gamma_k}{B} - 2X_k\left(t-1\right). \tag{10}$$

Assume next that (10) is violated. Then we have again two possibilities concerning the right hand side derivative $\partial\Pi_k/\partial x_k$ at $X_k\left(t-1\right)$. Assume first that it is nonpositive, that is,

$$A - 2BX_k\left(t-1\right) - Bs_k - d_k - \gamma_k - \alpha_k \leq 0,$$

or

$$s_k \geq \frac{A - d_k - \gamma_k - \alpha_k}{B} - 2X_k\left(t-1\right). \tag{11}$$

In this case the best response is $x_k = X_k\left(t-1\right)$. Otherwise we have to consider the value of the left hand side derivative $\partial\Pi_k/\partial x_k$ at the final point $L_k$. It is nonpositive if

$$A - 2BL_k - Bs_k - d_k - \gamma_k - \alpha_k \leq 0,$$

or

$$s_k \geq \frac{A - d_k - \gamma_k - \alpha_k}{B} - 2L_k. \tag{12}$$

This relation always holds if $L_k = \infty$. Under this condition the best response is the stationary point between $X_k\left(t-1\right)$ and $L_k$:

$$x_k = -\frac{s_k}{2} + \frac{A - d_k - \gamma_k - \alpha_k}{2B}. \tag{13}$$

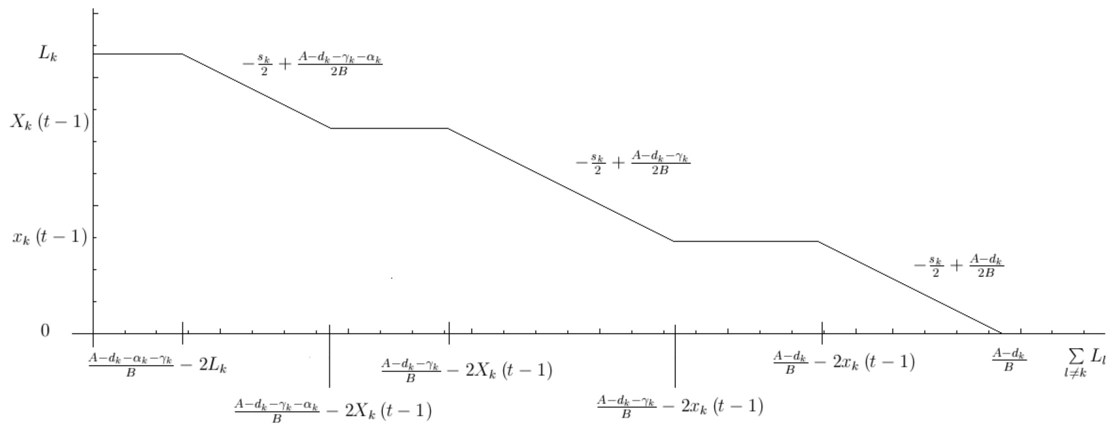


Figure 2: Best response function of firm $k$

Otherwise (12) is violated and the best response is $x_k = L_k$.

We can summarize the best response of firm $k$ as follows

$$R_k(s_k, x_k(t-1), X_k(t-1)) =$$

$$= \begin{cases} L_k & \text{if } 0 \leq s_k < \frac{A-d_k-\gamma_k-\alpha_k}{B} - 2L_k \\ -\frac{s_k}{2} + \frac{A-d_k-\gamma_k-\alpha_k}{2B} & \text{if } \frac{A-d_k-\gamma_k-\alpha_k}{B} - 2L_k \leq s_k < \frac{A-d_k-\gamma_k-\alpha_k}{B} - 2X_k(t-1) \\ X_k(t-1) & \text{if } \frac{A-d_k-\gamma_k-\alpha_k}{B} - 2X_k(t-1) \leq s_k < \frac{A-d_k-\gamma_k}{B} - 2X_k(t-1) \\ -\frac{s_k}{2} + \frac{A-d_k-\gamma_k}{2B} & \text{if } \frac{A-d_k-\gamma_k}{B} - 2X_k(t-1) \leq s_k < \frac{A-d_k-\gamma_k}{B} - 2x_k(t-1) \\ x_k(t-1) & \text{if } \frac{A-d_k-\gamma_k}{B} - 2x_k(t-1) \leq s_k < \frac{A-d_k}{B} - 2x_k(t-1) \\ -\frac{s_k}{2} + \frac{A-d_k}{2B} & \text{if } \frac{A-d_k}{B} - 2x_k(t-1) \leq s_k < \frac{A-d_k}{B} \\ 0 & \text{if } \frac{A-d_k}{B} \leq s_k \leq \sum_{l \neq k} L_l. \end{cases} \tag{14}$$

The graph of this function is shown in Figure 2 with the assumption that $0 < x_k(t-1) < X_k(t-1) < L_k$, otherwise one or more segments of the function are merged. It is interesting to note that the best response functions (14) are piecewise linear, and nonautonomous, since they change during the game as a result of the different output levels selected during the history of the game.

## 3. Dynamic Model and Steady States

Assume that the firms adjust their outputs toward their best responses. Let $K_k$ denote the speed of adjustment of firm $k$, then the output of this firm at time period $t$ becomes

$$x_k(t) = x_k(t-1) + K_k\left(R_k\left(s_k(t-1), x_k(t-1), X_k(t-1)\right) - x_k(t-1)\right) \tag{15}$$

and the new value of $X_k$ is the following:

$$X_k(t) = \max\left\{x_k(t), X_k(t-1)\right\}. \tag{16}$$

Notice that in (15) the firm assumes that the output of the rest of the industry is $s_k(t-1)$ which is usually called the naïve or static expectation.

The steady state of this dynamic system is a $2n$-dimensional vector $\left(\overline{x}_1, \ldots, \overline{x}_n, \overline{X}_1, \ldots, \overline{X}_n\right)$ such that for all $k$,

$$\begin{cases} \overline{x}_k \leq \overline{X}_k \\ \overline{x}_k = R_k\left(\sum_{l \neq k} \overline{x}_l, \overline{x}_k, \overline{X}_k\right) \end{cases} \tag{17}$$

The second condition is satisfied if and only if the left hand side derivative $\partial\Pi_k/\partial x_k$ is nonnegative and the right hand side derivative $\partial\Pi_k/\partial x_k$ is non-positive at $x_k = x_k(t-1) = \overline{x}_k$ and $X_k(t-1) = \overline{X}_k$. This is the case when

$$A - 2B\overline{x}_k - B\sum_{l \neq k} \overline{x}_l - d_k \geq 0 \geq A - 2B\overline{x}_k - B\sum_{l \neq k} \overline{x}_l - d_k - \begin{cases} \gamma_k & \text{if } \overline{x}_k < \overline{X}_k \\ \gamma_k + \alpha_k & \text{if } \overline{x}_k = \overline{X}_k. \end{cases} \tag{18}$$

for all $k$, where the left-hand side is the left-hand side derivative $\partial\Pi_k/\partial x_k$ and the right-hand side is the right-hand side derivative $\partial\Pi_k/\partial x_k$ . In a slight increase of $\overline{x}_k$ the adjustment cost (1) always occurs, however the cost (2) occurs only if $\overline{x}_k = \overline{X}_k$. If we are interested in the equilibria in terms of the output quantities, then it is sufficient to consider the case when $\overline{x}_k = \overline{X}_k$ for all $k$. So the set of all equilibrium output vectors $(\overline{x}_1, \ldots, \overline{x}_n)$ forms a nonempty convex polyhedron with usually infinitely many points.

The equilibrium of the corresponding Cournot equilibrium without output adjustment costs is clearly a steady state of system (15)-(16) with $\overline{X}_k = \overline{x}_k$

for all $k$. Usually there are infinitely many solutions of relation (18). In the case of duopolies, relation (18) simplifies as

$$A-2B\overline{x}_k-B\overline{x}_l-d_k \geq 0 \geq A-2B\overline{x}_k-B\overline{x}_l-d_k-\begin{cases} \gamma_k & \text{if } \overline{x}_k<\overline{X}_k \\ \gamma_k+\alpha_k & \text{if } \overline{x}_k=\overline{X}_k \end{cases} \tag{19}$$

for $k,l=1,2$ and $l \neq k$. In the case of $\overline{x}_k=0$ the left hand side is omitted and if $\overline{x}_k=L_k$, then the right hand side has to be omitted from relations (18) and (19). In the case of the classical Cournot model $\alpha_k=\gamma_k=0$, so an output vector $(\overline{x}_1,\dots,\overline{x}_n)$ is an equilibrium if and only if for all $k$,

$$\begin{aligned} &A-B\sum_{l\neq k}\overline{x}_l-d_k \leq 0 && \text{if} \quad \overline{x}_k=0; \\ &A-2BL_k-B\sum_{l\neq k}\overline{x}_l-d_k \geq 0 && \text{if} \quad \overline{x}_k=L_k; \end{aligned}$$

and

$$A-2B\overline{x}_k-B\sum_{l\neq k}\overline{x}_l-d_k=0 \quad \text{if} \quad 0<\overline{x}_k<L_k.$$

Notice that they are the well-known equilibrium conditions in the classical Cournot model (see for example, [1]).

The dynamic behavior of system (15)-(16) cannot be examined by the usual methodology of stability analysis, since instead of isolated equilibria we face a nonempty convex equilibria set with usually infinitely many points. Instead of convergence to a single point we have to deal with convergence to a certain set, which problem can be treated mathematically by using the theory of differential manifolds. In this paper therefore no analytical study is presented, the dynamic behavior of the system will be illustrated by computer simulation.

## 4. Dynamic Analysis

In the computer simulation study we selected $A=20$ and $B=2$ for the price function, and considered the semisymmetric case where $n-1$ firms had identical cost functions with $c_k=0$ and $d_k=1$ and for firm $n$, $c_n=0$ and $d_n=2$. This type of the semisymmetric case is very common in the literature (see for example [1]), however more general cases can be studied in the same way. For the sake of comparing our results with those known from the literature we selected this special case. We also considered both the homogeneous case, in which all firms were identical, and the nonsymmetric case, where half of the firms had identical cost functions with $c_k = 0$ and $d_k = 1$ and for the remaining firms, $c_l = 0$ and $d_l = 2$. The results were quite similar to the semisymmetric case in all but one case, and therefore they are not reported. The only case which gave different results will be discussed. Identical adjustment cost functions were selected: $\alpha_k \equiv \alpha$ and $\gamma_k \equiv \gamma$ for all $k$. The common speed of adjustment $K$ was varied between 0 and 1, and it was selected as the bifurcation parameter. We also varied the number $n$ of firms. For each case, for the sake of comparison, we computed three model variants. Each of them models a different assumption on the production adjustment costs. The first case with $\alpha = 1.0$ and $\gamma = 0.0$ occurs when increasing production requires only additional investment; the second one, $\alpha = \gamma = 0.5$ corresponds to the case in which both additional investment and contingent workforce is necessary, and the final case, with $\alpha = 0.0$ and $\gamma = 1.0$ requires only contingent workforce to increase production level.

For $n = 2$ and $n = 3$ the trajectories always converge to a steady state. The limit depends on the initial outputs levels of the firms. In these cases no bifurcation occurs. As we will see later, the increasing number of firms will produce instability, and the associated bifurcation is called the "border collision bifurcation" (see for example [3]), since the response functions are piece-wise linear and nonautonomous.

If $n = 4$, then for small values of $K$ the trajectory always converges to a steady state. By increasing the common value of the speed of adjustment this convergence is lost and two-period cycles emerge. The amplitude of the cycle is small initially, and as $K \simeq 0.8372$ the amplitude suddenly widens and further increases with larger values of $K$. This is illustrated in Figure 3 by the bifurcation diagram, in which the trajectory values are shown for $t > 2500$ with zero initial outputs and varying values of the bifurcation parameter $K$. The vertical axis represents the common output of the identical $n - 1$ firms. Parts a), b) and c) correspond to the three scenarios mentioned above. Notice that in the three scenarios the value of $\overline{K}$ where convergence is lost is slightly different: a) $\overline{K} = 0.8352$ with only investment costs, b) $\overline{K} = 0.8191$ with investment and contingent workforce costs and c) $\overline{K} = 0.7998$ with only contingent workforce costs.

The dynamic behavior remains similar for $n < 9$, as both convergence and 2-period cycles can be observed. The bifurcation diagram is also similar to the one depicted in Figure 3.

For $n = 9$ and $n = 10$ the bifurcation diagrams are quite similar. In Figure 4 the case of $n = 10$ is depicted. Positive contingent workforce costs generate

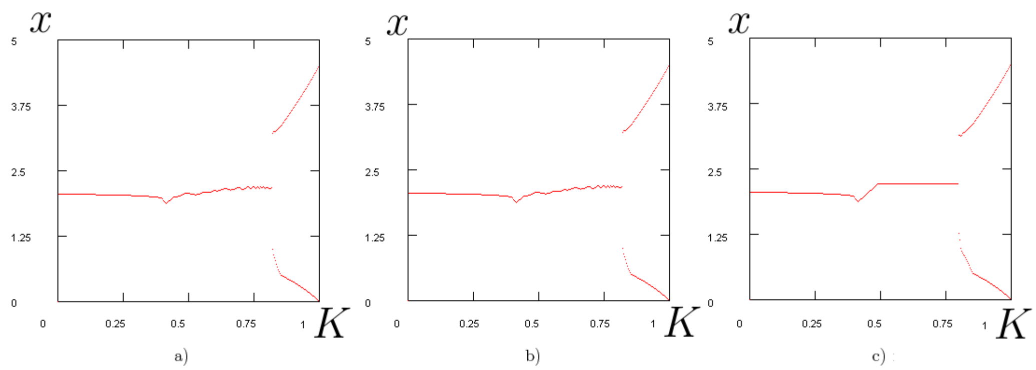


Figure 3: The case of $n = 4$ firms. a) $\alpha = 0.0$, $\gamma = 1.0$ b) $\alpha = 0.5$, $\gamma = 0.5$ c) $\alpha = 1.0$, $\gamma = 0.0$

two nonconnected intervals of the adjustment speed with convergence and also with two nonconnected intervals with 2-period cycles. With increasing values of $K$ the 2- period cycles become wider.

When the number of firms is $n = 11$, the dynamics becomes more complex as illustrated in Figure 5. If $K$ is small then there is convergence with limit point depending on the initial outputs of the firms. By increasing the value of $K$, 2-period cycles appear. If $K$ increases further, then 4-period cycles are observed. Finally, when $K$ increases even further then 2-period cycles can be found again. Notice that in cases a) and b) where the contingent workforce costs are nonzero there is a small interval around $K = 0.7714$ and $K = 0.7821$ respectively with convergence. After this small interval the 2- period cycles reappear. While no chaotic behavior is shown so far, this depends on the choice of parameters we have considered. In fact still considering $n = 11$ firms, but with group sizes 8 and 3 it is possible to observe chaotic bands as shown in Figure 6. It can be also observed that increasing investment costs seems to reduce chaotic behavior.

When the number of firms increases up to 12, the dynamics becomes much more complex as shown in Figure 7 by the appearance of a chaotic region for all the three cases.

If the number of firms increases even further, then the dynamics becomes more complex. Figures 8 and 9 illustrate the semisymmetric and nonsymmetric cases for $n = 14$ firms. The bifurcation diagrams show a larger variety of complex dynamic behaviors with, in some cases, even a second chaotic

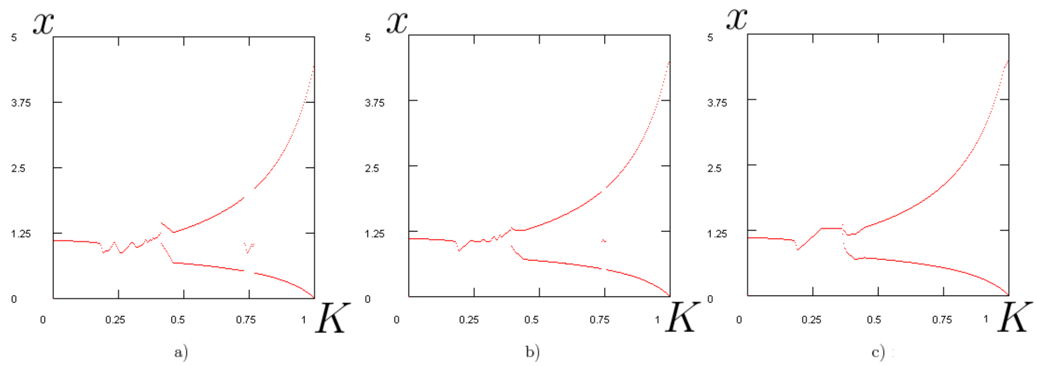


Figure 4: The case of $n = 10$ firms. a) $\alpha = 0.0$, $\gamma = 1.0$ b) $\alpha = 0.5$, $\gamma = 0.5$ c) $\alpha = 1.0$, $\gamma = 0.0$

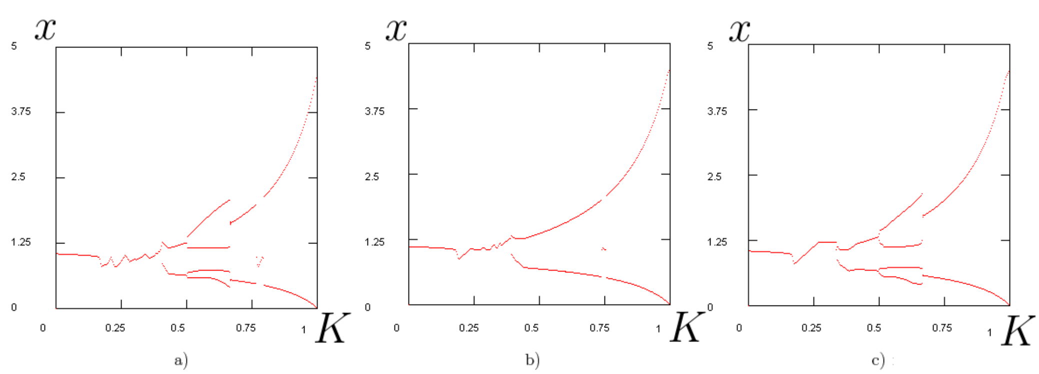


Figure 5: The case of $n = 11$ firms. a) $\alpha = 0.0$, $\gamma = 1.0$ b) $\alpha = 0.5$, $\gamma = 0.5$ c) $\alpha = 1.0$, $\gamma = 0.0$

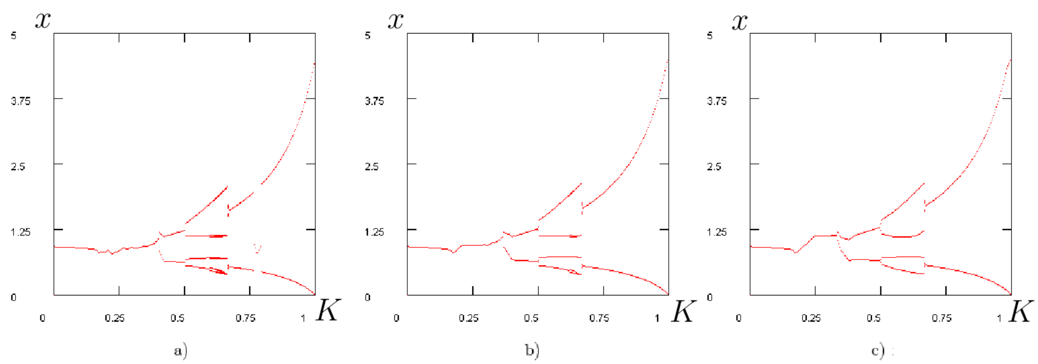


Figure 6: The case of $n = 11$ firms partitioned into two groups of 8 and 3 members. a) $\alpha = 0.0$, $\gamma = 1.0$ b) $\alpha = 0.5$, $\gamma = 0.5$ c) $\alpha = 1.0$, $\gamma = 0.0$

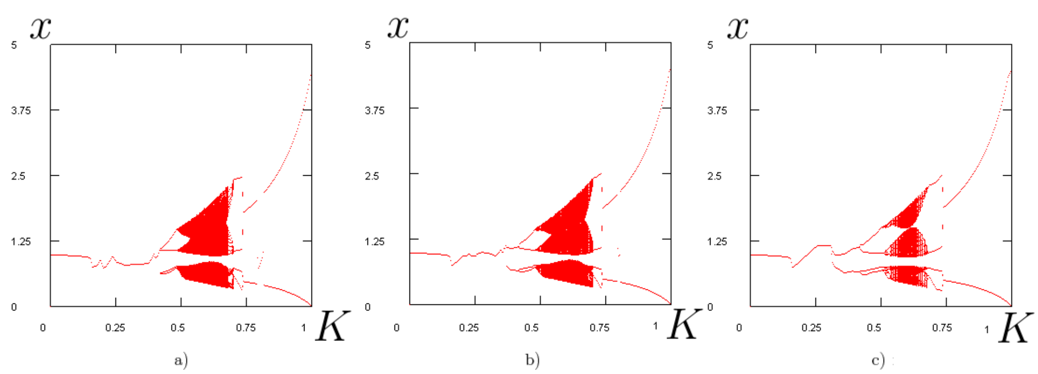


Figure 7: The case of $n = 12$. a) $\alpha = 0.0$, $\gamma = 1.0$ b) $\alpha = 0.5$, $\gamma = 0.5$ c) $\alpha = 1.0$, $\gamma = 0.0$

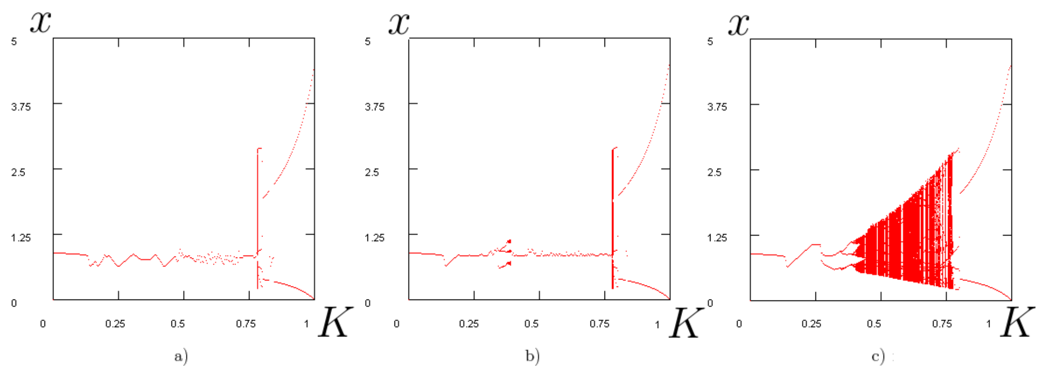


Figure 8: The case of $n = 14$ firms. a) $\alpha = 0.0$, $\gamma = 1.0$ b) $\alpha = 0.5$, $\gamma = 0.5$ c) $\alpha = 1.0$, $\gamma = 0.0$

region. Furthermore, the bifurcation diagram of the nonsymmetric case is different from the semisymmetric one and we can observe that the presence of the adjustment type (1) seems to have a stronger destabilizing effect than type (2). One reason could be that type (1) adjustments are likely to affect the dynamics more often than the others which are more structural. Finally, it must be observed that considering null initial outputs makes the dynamics more complex. In fact, considering initial outputs larger than zero would mean that part of the adjustment cost would have already made resulting in a less complex dynamics as illustrated in Figure 10.

## 5. Conclusions

The classical Cournut oligopoly model was extended to include production adjustment costs. As in modeling the use of contingent work force the increase from the previous output level is first considered and then the additional cost of increase from the past largest output level was included into the model, which is the case of additional investments. Each firm has a unique best response depending on the output of the rest of the industry, its output at the previous time period as well as its maximal past output level.

Partial adjustment toward best responses was next assumed in formulating the corresponding dynamic extension which has at least one steady state. Their number is usually infinity, and the set of the steady states was characterized by a system of linear inequalities. Computer simulation confirmed that this model can lead to convergence to steady states, cycles and even

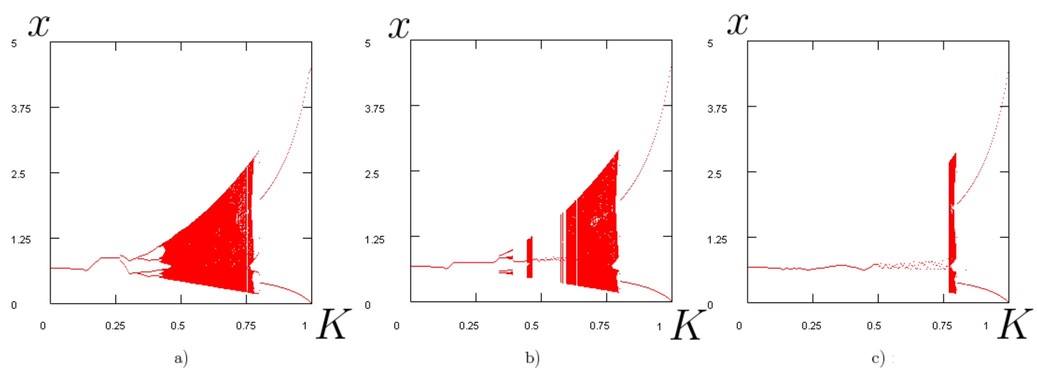


Figure 9: The case of $n = 14$ firms, nonsymmetric case where 7 firm have cost functions with $c_k = 0$ and $d_k = 1$ and the remaining firms $c_k = 0$ and $d_k = 2$. a) $\alpha = 0.0$, $\gamma = 1.0$ b) $\alpha = 0.5$, $\gamma = 0.5$ c) $\alpha = 1.0$, $\gamma = 0.0$

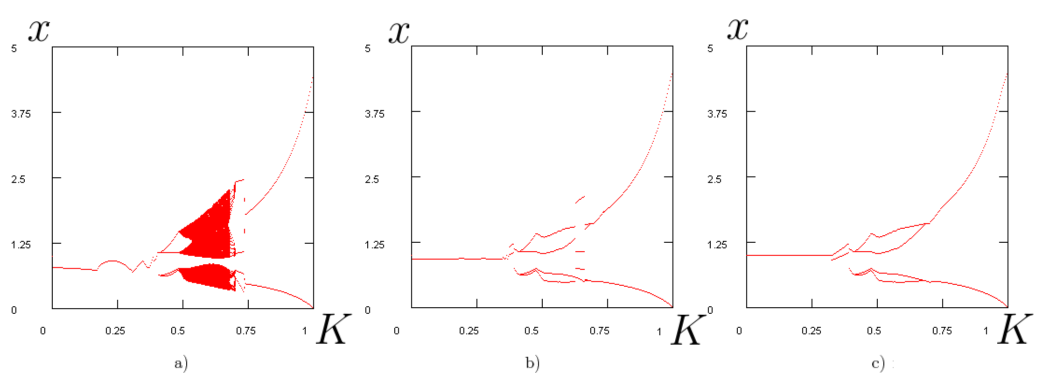


Figure 10: The case of $n = 12$, with initial outputs $x_k(0) = 1, \forall k$. a) $\alpha = 0.0$, $\gamma = 1.0$ b) $\alpha = 0.5$, $\gamma = 0.5$ c) $\alpha = 1.0$, $\gamma = 0.0$

chaotic behavior if the number of firms and/or the speeds of adjustments are sufficiently large. Similarly to the classical Cournot model an increase in the number of firms and/or in the speed of adjustments makes the asymptotic behavior of the system more complex ([1], [5], [12], [13], [16] [18]). The profit functions assumed in this paper, similarly to most studies of the literature, were continuous. However the increase of production capacity might require fixed cost such as building construction, land purchase, etc. This additional assumption would result in discontinuous profit functions which would make the steady states and dynamic analysis more complicated. In our future research we will consider such discontinuous models, nonlinear price and cost functions. Also it will be interesting to consider partial cooperation between the firms and introduce antitrust thresholds as in [8, 9, 10].

## Acknowledgments

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Referee's report on the paper "Dynamic oligopolies with contingent workforce and investment costs" By Ugo Merlone and Ferenc Szidarovszky
Submitted to Mathematics and Computers in Simulation

The paper addresses the issue of adjustments costs within the classic Cournot oligopoly framework without product differentiation. These adjustments costs are related to the increments of production over the last period of time or over the entire duration of the game. In the first part of the paper, the authors derive the best reply functions of the players. This derivation is very clearly explained and Figure 1 is helpful in working out the various cases. The second part of the work presents numerical results related to the best reply dynamics of the oligopoly in a semi-symmetric setting. The paper as well as several results are interesting. In the following, I try to provide some comments to improve the overall presentation of the paper.

Main remarks

In my opinion there is a big gap in the way the paper is presented, namely between the first part (Sections 2 and 3) and the last part (Section 4). The first part presents a nice derivation of the model and the best replies, which are then employed to obtain the dynamical system in terms of best reply dynamics with naïve expectations. On the other hand, the second part appears just as a collection of bifurcation diagrams with varying speed of adjustment. This second part outlines through numerical examples the increasing instability in the model as the agents' speed of adjustment as well as the number of competitors is increased. Although the latter is a classic topic within the framework of Cournot oligopoly (Theocaris, Hahn, McManus, Seade,...), the authors do not quote any of the works related to this issue, even though this literature is strictly related to their paper.

*In the revised version we quote these important results which are related to our findings.*

For instance, for N=2,3 (page 9) it seems that the classic results are retrieved also with adjustments costs. However, there is no way to understand how robust this is, based on the numerical experiments of the paper. By the first part of the paper, it is clear that the best reply functions are nonautonomous, in the sense that they change during the game as a result of the different quantities played during the history of the game. Moreover, the best reply functions (and hence the map defined in (15)) are clearly piecewise linear and the possible bifurcations of (15) should be Border Collision Bifurcations. However, the authors do not recall this fact (the expressions "piecewise-linear" and "border collision bifurcation" do not appear in the text, as well as quotes to the related literature, see for instance Di Bernardo et al. 2008) and do not discuss the difficulties arising from the analysis of the map (15). I would encourage the authors to provide some analytical results on the stability of the Nash equilibrium or, at least, to discuss the problems related to this analysis, before providing the numerical examples.

*In the revised version we mention the border-collision bifurcation providing a reference to as suggested. Furthermore, at the end of Section 3 we discuss the difficulties related to the theoretical analysis and explain why we provide numerical examples.*

Minor remarks

Page 1, line 5 of the Introduction: I would avoid to say "in the first stage", as this expression seems to be related to a multi-stage game;

*The expression has been modified.*

Page 2: a review of the literature on adjustment costs is provided. However, it would be useful to discuss better how the quoted works relate to the present paper;

*In the revised version we provide a better discussion which links how the critiques to the traditional models have been approached.*

Page 3: the maximum capacity level Lk can never be exceeded. However, in the long-run, any level can, in principle, be exceeded, as firms can invest in increasing the production capacity. Do the authors have in mind a specific example of oligopoly where the proposed cost functions are a reasonable assumption?

*In the revised version, after equation (3) we discuss the assumptions underlying our formalization.*

Page 4: formula (4) represents the profit that firm k expects for time period t. However, it should be said that firm k has naïve expectations when assessing the future output of the rest of the industry;

*In the revised version we mention this point later on, when we discuss the dynamical model, after equation (16).*

Page 4: equation (2) is interpreted as an investment in capacity; however, here the capital does not depreciate; would it be possible/feasible/interesting to embed in Xk(t-1) [formula (3)] a depreciation term?

*Clearly it would be interesting and will make the analysis more difficult. Nevertheless we keep in mind this suggestion for further research as we say in the conclusions DA FARE*

Page 6, first line: “we have have two possibilities”

*We corrected the typo, thank you.*

Page 7: better explain equation (18)
What happens when the assumption of a semi-symmetric model is relaxed? What if the players are equally split in two groups of equal size? What about the case of completely homogeneous players (i.e. an unidimensional map if also the initial outputs coincide)?

*In preparing the revised version we repeated our analyis considering also homogenous players and two groups of equal size. As the results do not change, we do not report the bifurcation diagrams. Nevertheless, we discuss this point and provide the bifurcation diagram for the case in which the nonsymmetric case is different from the semi-symmetric.*

In principle a firm could face an adjustment cost even though it reduces the production with respect to the previous period (e.g. when workforce is laid off a firm could be obliged to give severance pay). The authors could briefly mention this point.

*Right before Equation (4) of the revised version we mention this point.*

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***Response to Reviewers**

Referee Report on
Dynamic Oligopolies with Contingent Workforce and Investment Costs
by Merlone and Szidarovsky

This paper considers inertial best response dynamics in Cournot oligopolies with adjustment costs based on excess output relative to last output (work- force adjustment) and relative to the maximal output in the past (capacity adjustment). The resulting dynamical system is piecewise smooth and has typically a continuum of fixed points. By means of numerical simulation the dynamic properties of the system are explored and it is shown that for an increasing number of firms cycles and chaotic dynamics occur for certain ranges of the adjustment speed parameter of firms.

I appreciate the idea of the authors to introduce different types of ad justment costs into the heavily studied myopic best reply Cournot models. The analysis is competently done and results seem to be correct (obviously the numerical findings could not be checked in detail). However, the substance of the contribution does not become very clear in the current version.
From a technical perspective the study is rather standard, so in my opinion the new economic insights have to become more clear. I think this aspect should be improved to make the paper publishable and give detailed comments below.

1.) The basic observation that an increase in the number of firms leads to instability of the steady state(s) and, under consideration of (non-negativity) contraints on prices and outputs, cycles and maybe more complex patters might emerge under myopic best response dynamics is not really new (e.g. Theocharis (1960)). So, it has to be made clear what kind of qualitatively new phenomenon arises due to the introduction of these two types of adjustment costs. Also, the different implications of the two types of adjustment costs (i.e. the differences between cases a) and c)) should be discussed in more detail. Is there any economic intuition to explain the different patterns we observe in a) vs. c) in figs 7. - 9.?

*In the revised version we provide some economic intuition to explain the different effects on the dynamics between the two kinds of adjustment costs.*

2.) The assumption that firms only consider current profit when making their output decision is quite standard in this literature, but it seems to me that in this particular setting the assumption is more heroic than usual. It is assumed that production capacities stay intact without any depreciation forever, which implies that firms in their planing should distribute their investment costs over the anticipated number of periods in which they will produce such output or higher. In that sense it seems that in the current formulation firms systematically underinvest in capacities. Some discussion of this point and some defense of this assumption should be provided.

*In the revised version we discuss why we do not take into account depreciation.*

3.) From a more mathematical perspective I am wondering whether no analytical conditions can be derived to characterize the threshold for the number of firms (as a function of parameters) where a steady state losses stability.

*In the revised version we explain why it is so difficult to obtain analytical conditions.*

4.) The exposition could be more clear at several points. Some Examples:

In (15) on p7. the term sk is used on the right hand side. I guess this should be sk (t − 1) with sk (t − 1) = l=k xl (t − 1) but this should be explicitly stated.

*In the revised version we corrected the typo and address this point.*

On p8 it is explained that in the simulations n−1 firms are assumed to be identical and one firm is different. Why is this setup used? Would the dynamics look qualitatively different if a completely symmetric (or a much more heterogeneous) setup would be considered?

*In preparing the revised version we repeated our analyis considering also homogenous players and two groups of equal size. As the results do not change, we do not report the bifurcation diagrams. Nevertheless, we discuss this point and provide the bifurcation diagram for the case in which the nonsymmetric case is different from the semi-symmetric.*

The assumption that initial output is zero might not be innocent. Due to the well known overshooting this should imply large outputs in the first period and this might take care of all capital adjustment costs in the following periods. Would steady states look different for larger initial output?

*The analysis has been repeated considering initial output larger than zero. In the revised version we show how the initial condition plays such a role. Nevertheless, as the paper investigates the role of adjustment costs, considering large outputs in the first period which would take care of all capital adjustment costs at the beginning making the dynamics less interesting from the economic point of view.*

On p9 it is explained that in Figure 3 the smallest and largest trajectory value for t > 2500 is shown. This seems to suggest that for each value of K we see exactly two points (which might coincide) in the graph. Whereas this is true for figures 3 and 4, in figs. 5 - 9 we see multiple points for a given K-value. Hence, these figures must have been generated differently from the explanation on p9. Please explain what they exactly show.

*In the revised version we provide a better explanation. The previous version was unclear and we are grateful to the reviewer for pointing out this.*

References:
Theocharis, R. (1960): "On the Stability of the Cournot Solution on the Oligopoly Problem," The Review of Economic Studies, 27, 133-134.