Systematic Approach to N -Person Social Dilemma Games: Classification and Analysis

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(Article begins on next page)

Systematic approach to N-person social dilemma games

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Abstract

This paper presents a new systematic review of N-person social dilemma games using a new approach based on dynamic properties of the corresponding system. Traditionally N-person social dilemma games are classified by relative orders of magnitude of payoff parameters. Without border-line cases 24 are identified. The new approach introduced in this paper categorizes the social dilemma games in cases with different number and asymptotic properties of the equilibria. In these cases the solution structure or the trajectory of the percentage of cooperators is readily apparent. These cases also provide the modeler with additional information concerning the choice of parameters. The example of a simple cartel illustrates this methodology.

Keywords: Social dilemmas; Agent-based simulation; N-person games; Pavlovian agents; equilibrium.

1. Introduction

Agent-based simulation has been used extensively to study N-person social dilemma games. Their applications in the literature include the study of artificial societies [2], the examination of collective communication within a socio-geographic community [7] and the investigation of mass transit usage in a large city [12]. There is a significant amount of research in understanding steady state solutions, agent behavioral traits and percentage of cooperator trajectories for the same model used in this paper [14, 15, 10, 11, 13, 9, 5] N-person social dilemma games are an extension of the traditional two-player social dilemma games. The general payoff matrix for two-player social dilemma games is shown in Table 1. The parameters T, R, P and S are associated with decisions made by each player where the rows show the decisions of player 1 and the columns to player 2. They are typically referred to as Reward (R), Punishment (P), Sucker's Bet (S) and Temptation (T) as defined in the prisoner's dilemma game. Prisoner's dilemma game occurs when the parameters of the payoff functions satisfy the relation $T > R > P > S$ [6]. Since the relative order of magnitude of these parameters have $4! = 24$ possibilities there are usually 24 different games identified in the literature [8].

1 \ 2	Cooperate	Defect
Cooperate	R,R	S,T
Defect	T,S	P,P

Table 1: Payoff matrix for two-player game.

An N-person game involves more than two individual agents where the collective behavior in society influences the payoff of each agent and the actions of the individual agents determine the collective behavior of the society. Each agent may choose to cooperate for the collective best interest or defect to pursue its own self-interest. There are several variables that can be used when modeling N-person games including the number of agents in the society, payoff functions, agent decision making rules or personality, environmental influence or neighborhood, and simultaneous or sequential decision making. In this paper we will look at a specific N-person model, identify the difficulties involved for the modeler in selecting values for payoff function parameters, and present an alternative method to classify games which significantly aids the modeler in the selection of these parameters. Agents are given a reward or punishment for their action. The amount of reward or punishment an agent receives for a decision in each iteration is called the payoff function. An example of linear payoff functions is shown in Figure 1. In these payoff functions x is the percentage of cooperators, $C(x)$ is the payoff for those agents that are cooperating and $D(x)$ is the payoff for those agents that are defecting. These linear payoff functions are consistent with the prisoner's dilemma game since $T > R > P > S$.

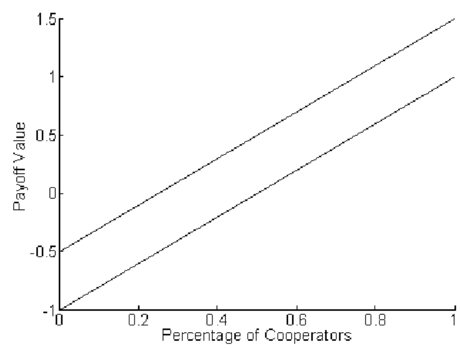

Figure 1: Linear payoff functions for Cooperators ($C(x)$) and Defectors ($D(x)$).

We will deal with the Pavlovian agent type which uses reinforcement learning [1, 4, 3] in its decision making process. The Pavlovian agent has a certain probability of cooperating in each time period, which changes for the next time period or iteration by a proportion of the reward or punishment received. The probability that an agent i will be cooperating at time period t can be computed as

$$p_i(t) = \begin{cases} p_i(t-1) + \alpha C(x(t-1)) & \text{if the agent cooperated at time period } t-1 \\ p_i(t-1) - \beta D(x(t-1)) & \text{if the agent defected at time period } t-1 \end{cases} \quad (1)$$

where $x(t-1)$ is the percentage of cooperating agents in the population at time period $t-1$, α is the proportion or learning factor for cooperators and β is the learning factor for defectors. Since $p_i(t)$ is a probability it must be between zero and one. So whenever $p_i(t)$ becomes larger than one it is adjusted to be one. Likewise, whenever $p_i(t)$ becomes negative it is adjusted to be zero.

When a modeler is looking for appropriate parameter values for a specific application, the values of the model parameters are typically based on their practical meaning. In the two-player prisoner's dilemma game jail time is typically used to assign these parameters. For an N-person prisoner's dilemma example, consider a cartel. The players or agents decide to cooperate by setting an optimum production and price as if it were a monopoly.

In the case of cooperation, each agent agrees to charge the monopoly price and limit production to set forth quantities. The temptation for each agent is to defect by lowering the price and producing more. The defecting agent has a higher profit or payoff since his market share increases significantly while the remaining cooperating agents lose market share while holding the same monopoly price. If all agents defect, then all are worse off since they would all be charging the same lower price with the same market share as if they all would have cooperated. In this example the assessed values of P, R, S and T may be based on profit estimates for cooperating and defecting agents. The modeler needs to be careful in determining the parameter values since the solution structure can change significantly based on slight and seemingly irrelevant differences. For example, Figure 2 shows the solution structures for an N-person prisoner's dilemma game. The only difference between the figures is that the payoff functions are shifted up and down while keeping the relative distance between them constant. This situation creates a difficult problem to develop and validate a model since any one of these solution structures could be the simulation result if this is not considered. So the same game with slight changes in model parameters might have very different properties.

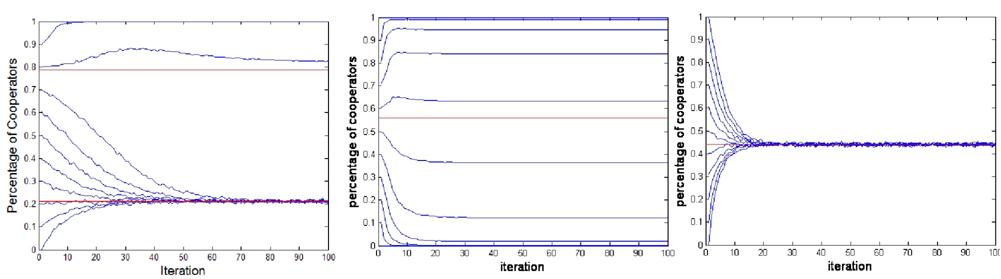

Figure 2: Solution structures for N-person prisoner's dilemma games.

Therefore there is a need to characterize the N-person binary games through their properties including number of steady states as well as the asymptotical properties of the state trajectories. The main objective of this paper is to introduce such a classification. In addition to theoretical analysis some important cases will be illustrated by simulation studies.

This paper develops as follows. After the mathematical methodology is briefly summarized, we will introduce a complete description of the possible dynamical properties of the system and 13 cases will be identified. Four important cases will be then analyzed in detail. An example of a simple cartel will illustrate the methodology. The last section is devoted to conclusions and further research directions.

2. Mathematical methodology

The new approach is to plot the state transition formula of the percentage of cooperators against the 45 degree line. With linear payoff functions this transition formula is represented by a quadratic curve. The derivation of this quadratic curve is presented in [5] and is given as

$$x^{\text{new}} = Ax^2 + Bx + C \tag{2}$$

where

$$A = \alpha R - \beta P - W$$
$$B = W + 2\beta P + 1$$

and

$$C = -\beta P,$$

with

$$W = \alpha S - \beta T.$$

The steady state equilibrium solution and the trajectory of percentage of cooperators is readily apparent when this quadratic curve is plotted with the 45 degree line. For example, Figure 3 shows the case diagram and simulation results for an example with parameters $\alpha = 0.05$, $\beta = 0.05$, $P = -4$, $R = 3$, $S = -8$ and $T = 3$. In this case the blue line is the quadratic curve, the green line is the 45 degree line and the red line shows the trajectory for the percentage of cooperators. The steady state equilibria are the stars. Steady states occur when the two curves intersect. The trajectory of percentage of cooperators is determined by starting at any initial state and plotting the successive iterations by moving horizontally from the quadratic curve to the 45 degree line and then vertically back to the quadratic curve. In this case the trajectory converges to the attractor solution when the initial percentage of cooperators is below the repeller solution and repels to 100% cooperation if the initial percentage of cooperators is above the repeller solution. That is, $x(t) \to x_A^*$ when $x(0) \leq x_R^*$, and $x(t) \to 1$ when $x(0) > x_R^*$. The simulation results on the right hand side verify this solution. This solution structure in this case is Attractor/Repeller, which is defined as a case when both attractor and repeller steady states occur between zero and one.

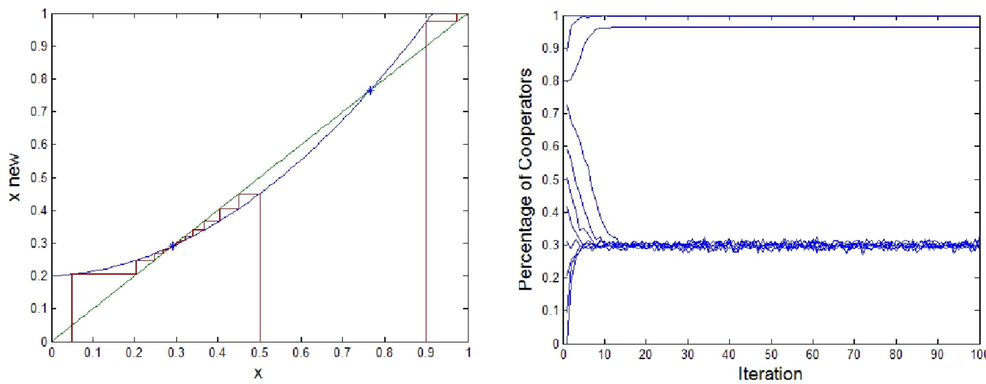

Figure 3: Case diagram and simulation results example.

The first question to answer is can A, B and C be any values given the social dilemma parameters α, β, P, R, S and T. In order to answer this question we need to evaluate how all the parameters relate to each other. In equation (2) the parameters α, β, P, R, S, T and W from the social dilemma game can be related to A, B and C by equation

$$\begin{pmatrix} \alpha & -\beta & -1 \\ 0 & 2\beta & 1 \\ 0 & -\beta & 0 \end{pmatrix} \begin{pmatrix} R \\ P \\ W \end{pmatrix} = \begin{pmatrix} A \\ B-1 \\ C \end{pmatrix}.$$

Since the learning factors α and β are nonzero, the matrix in this linear set of equations is nonsingular. This means that all real values of A, B and C can be attained from a social dilemma game given proper selections of α, β, P, R, S, T and W. That is, the right hand side of equation (2) can be any parabola, linear function (with $A = 0$, $B \neq 0$) or constant (with $A = B = 0$). The asymptotical behavior of the system depends on the relative location of the graph of this function and the 45 degree line. We also note that if the right hand side of the equation is larger than one, then x^{new} is adjusted to 1, and if it is negative, then x^{new} is adjusted to zero value.

Before reviewing the different cases we mention some simple facts. They will be used in determining the criteria or constraints for model parameters.

Let

$$\varphi(x) = Ax^2 + Bx + C, \tag{3}$$

then

$$\varphi'(x) = 2Ax + B$$

implying that

$$\varphi(0) = C = -\beta P, \quad \varphi(1) = A + B + C = \alpha R + 1$$

and

$$\varphi'(0) = B = W + 2\beta P + 1, \quad \varphi'(1) = 2A + B = -W + 2\alpha R + 1.$$

The steady state of the system solves the fixed point problem $x = Ax^2 + Bx + C$, which is equivalent to equation $Ax^2 + (B-1)x + C = 0$. Let now

$$\psi(x) = Ax^2 + (B-1)x + C, \tag{4}$$

then

$$\psi'(x) = 2Ax + B - 1,$$

and so

$$\psi'(0) = B - 1 = W + 2\beta P, \quad \psi'(1) = 2A + B - 1 = -W + 2\alpha R.$$

The number of real solutions is determined by the discriminant

$$\begin{aligned} D &= (B-1)^2 - 4AC \\ &= (\alpha S - \beta T)^2 + 4\alpha\beta PR \\ &= W^2 + 4\alpha\beta PR. \end{aligned} \tag{5}$$

Notice that the vertex of $\varphi(x)$ is $-B/(2A)$ and the vertex of $\psi(x)$ is $(1-B)/(2A)$.

These simple facts can be used in finding the conditions of the different cases.

3. Overview of cases

In this section we will identify and systematically review all the possible cases that may occur. Without loss of generality we assume $A > 0$ because if payoff parameters and learning curves are such that A is negative then we can simply interchange the definition of cooperation and defection in the game and in the redefined reverse game A' will become positive. For example, Table 2 shows the payoff matrix for a two-player game with the definition of cooperation and defection switched from Table 1. Now the common payoff if both agents cooperate is P (instead of R), the common payoff if both agents defect is R (instead of P), the payoff for cooperating if other agent defects is T (instead of S), and the payoff for defecting if the other agent cooperates is S (instead of T). So to find the reverse game we simply have to switch $R \leftrightarrow P$ and $S \leftrightarrow T$. For the N-player extension we also switch $\alpha \leftrightarrow \beta$.

1 \ 2	Defect	Cooperate
Defect	R,R	S,T
Cooperate	T,S	P,P

Table 2: Payoff matrix with definition of cooperation and defection switched.

In equation (2) we have

$$A = \alpha\,(R - S) + \beta\,(T - P)\,,$$

and after switching $R \leftrightarrow P$, $S \leftrightarrow T$ and $\alpha \leftrightarrow \beta$, the value of A' becomes

$$A' = \beta\,(P - T) + \alpha\,(S - R) = -A.$$

So, if A is negative, then we can switch the definition of cooperation and defection and in this equivalent game A' will become positive. Since we can assume A is positive without the loss of generality, the quadratic curve for any case will be convex. We will now show all possible ways that a convex quadratic curve can relate to the 45 degree line in terms of intersections and relative values. There are 13 cases. These cases include those when the quadratic curve is linear or even constant as well with $A = 0$ or $A = B = 0$. Section 3.1 shows the cases where there is no steady state solution. Section 3.2 presents cases with one steady state solution. Section 3.3 shows cases of two steady state solutions.

3.1. Cases without steady state solution

Figure 4 shows the cases without steady state solutions. There are two such cases. The first occurs when the entire quadratic curve is above the 45 degree line (Case 1) and the second occurs when the entire quadratic curve is below the 45 degree line (Case 2).

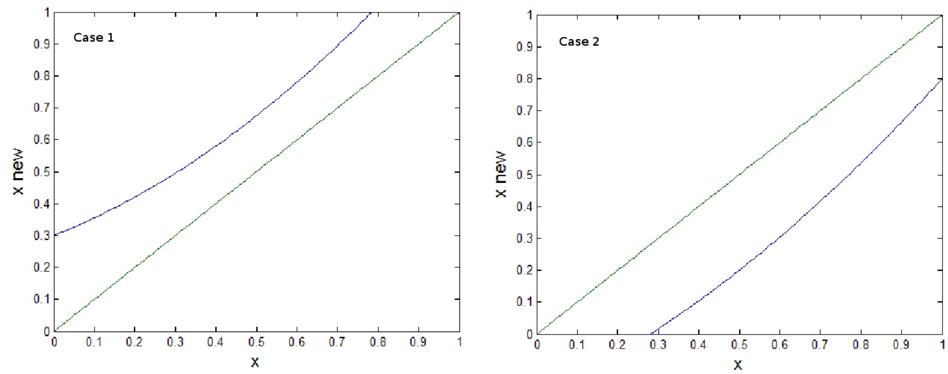

Figure 4: Cases without steady state solution.

3.2. Cases with a unique steady state solution

Figure 5 shows seven cases with unique steady state solutions. Three of these cases occur when the entire quadratic curve is above the 45 degree line except for one point. This single point occurs at 0 for Case 7, between 0 and 1 for Case 3, and at 1 for Case 4. Two cases occur when the entire quadratic curve is below the 45 degree line except for one point. This single point is at 0 for Case 6 and at 1 for Case 8. Finally, two more cases are possible when the magnitude of the quadratic curve changes relative to the 45 degree line and a single intersection occurs when x is between 0 and 1. In Case 5 the quadratic curve is above the 45 degree line before the intersection point and is below the 45 degree line after the intersection point. In Case 9 the quadratic curve is below the 45 degree line before the intersection point and is above the 45 degree line after the intersection point.

3.3. Cases with two steady state solutions

Figure 6 shows the four cases with two steady state solutions. Case 10 occurs when the quadratic curve is above the 45 degree line for low x values, intersects and moves below the 45 degree line for mid x values, and then intersects again and moves above the 45 degree line for high x values. Case 11 occurs when the quadratic curve is above the 45 degree line for low x values, then intersects and moves below the 45 degree line until it intersects again at $x = 1$. Case 12 occurs when the first intersection occurs at $x = 0$ and the quadratic curve is below the 45 degree line for low x values, then intersects and moves above the 45 degree for high x values. Finally, Case 13 occurs when the quadratic curve is entirely below the 45 degree line except at the two intersection points $x = 0$ and $x = 1$.

4. Analysis of selected cases

In reviewing the thirteen cases there are six different solution structures identified. These are Total Cooperation, Total Defection, Single Attractor, Single Repeller, Attractor/Repeller and Oscillation. Table 3 summarizes which cases are applicable for each solution structure. For example, Case 1, Case 4 and Case 7 are Total Cooperations. Total Cooperation is the case when the trajectory for all initial percentages of cooperators converges to all agents cooperating. The table also shows the conditions which should be satisfied by model parameters in each case.

All cases can be analyzed in the same manner, in this section we choose four of them which do not result in unique analytic behavior. A complete summary of the analysis of all 13 cases is given in Table 4.

4.1. Case 3

This case occurs when the quadratic curve lies entirely above the 45 degree line except for one point between 0 and 1 as shown in Figure 5. The 45 degree line is the tangent line of the parabola at this point. It is clear that $\varphi(0) > 0$, $\varphi(1) > 1$, $\varphi'(0) < 1$, $\varphi'(1) > 1$ and the discriminant of $\psi(x)$ is zero in this case. This requires $P < 0$, $R > 0$, $W + 2\beta P < 0$, $W - 2\alpha R < 0$ and $W^2 + 4\alpha\beta PR = 0$. Notice that this case is impossible for $A = 0$. We will now show that $W + 2\beta P = 0$ and $W - 2\alpha R < 0$ cannot occur simultaneously. Relation $W^2 + 4\alpha\beta PR = 0$ can be rewritten as $W^2 - (2\alpha R)(-2\beta P) = 0$. Since $W = -2\beta P, W^2 - (2\alpha R)(-2\beta P) = W^2 - (2\alpha R)W = W(W - 2\alpha R) = 0$. We now see the contradiction as neither W nor $W - 2\alpha R$ can equal zero. We can also show that $W + 2\beta P < 0$ and $W - 2\alpha R = 0$ are impossible together using the same logic. In addition $W + 2\beta P = 0$ and $W - 2\alpha R = 0$ together are also invalid. If we simply add the two equations we find $2W - 2\alpha R + 2\beta P = -W + \alpha R - \beta P = A = 0$. If $A = 0$ then we have a linear curve without intersection between zero and one since $\varphi(0) > 0$ and $\varphi(1) > 1$. There are two subcases of the trajectory, one if $\varphi(0) \leq x^*$ and the other if $\varphi(0) > x^*$. These subcases will be evaluated separately and called Case 3a and Case 3b respectively. In both of them there is a single intersection point that is an attractor from below and a repeller from above. This is a result of having two identical solutions. One solution is an attractor and the other is a repeller. The percentage of cooperators in Case 3a is converging to the solution when the initial percentage of cooperators is less than the solution and repelling to 100% cooperation when the initial percentage of cooperators is larger than the solution. That is, in the limit $x(t) \to x^*$ when $x(0) \leq x^*$, and $x(t) \to 1$ when $x(0) > x^*$. The left hand side of Figure 7a shows the case diagram with parameters $\alpha = 0.05$,$\beta = 0.05$, $P = -1$, $R = 4$, $S = -3$ and $T = 1$ with the initial percentages of cooperators 0.1 and 0.5. The right hand side shows simulation results with the same parameter values and initial cooperating probabilities 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0. The case diagram and simulation results both show that the percentage of cooperators is attracting from below and repelling from above. We will refer to this type of solution structure as Attractor/Repeller since there is both an attractor and repeller solution between 0 and 1.

The trajectory for Case 3b is similar to that of Case 3a except if $\varphi(0) > x^*$ with $x(0) \leq x^*$ when there is a large rise in cooperation from low initial percentages of cooperators. Large enough, in fact, that the percentage of cooperation goes above x^* and the trajectory repels from the solution thereafter. This creates two different trajectories when $x(0) \leq x^*$ in addition to the trajectory when $x(0) > x^*$. Following is a summary of the limits of trajectories as they occur from low to high percentages of cooperation:

- $x(t) \to 1$ when $x(0) \leq x^*$, $\varphi(x(0)) > x^*$
- $x(t) \to x^*$ when $x(0) \leq x^*$, $\varphi(x(0)) \leq x^*$
- $x(t) \to 1$ when $x(0) > x^*$.

Figure 7b shows the case diagram and simulation results with parameters $\alpha = 0.2$, $\beta = 0.2$, $P = -4$, $R = 1$, $S = -3.5$ and $T = 0.5$ with the initial percentages of cooperators 0.05 and 0.4. It is clear that the percentage of cooperators is attracting from below when the first iteration stays below the solution and jumps over to the repeller portion of the curve if the first iteration goes above the solution. The simulation results show the same reaction for low percentages of cooperators where the percentage jumps significantly above the repeller solution and then eventually repels to 100% cooperation. This solution structure is also Attractor/Repeller.

4.2. Case 5

This case occurs when the quadratic curve is above the 45 degree line before the intersection point and is below the 45 degree line after the intersection point as shown in Figure 5. It is clear that $\varphi(0) > 0$ and $\varphi(1) < 1$. This requires $P < 0$ and $R < 0$. There are four subcases of the trajectory. These occur when $x_v \leq 0$, $0 < x_v \leq x^*$, $x^* < x_v \leq 1$, and $x_v \geq 1$ where x_v is the vertex of the quadratic curve. These subcases will be evaluated separately and be called Case 5a, Case 5b, Case 5c, and Case 5d respectively. Case 5a occurs when $x_v \leq 0$. This requires $W + 2\beta P + 1 \geq 0$. The trajectory in Case 5a converges to an attractor solution with all initial percentages of cooperators. That is, $x(t) \to x^*$ as $t \to \infty$. The left hand side of Figure 8 shows the case diagram with parameters $\alpha = 0.05$, $\beta = 0.05$, $P = -4$, $R = -2$, $S = -3$ and $T = 1$ with the initial percentages of cooperators 0.1 and 0.9. The right hand side shows simulation results with the same parameter values and initial cooperating probabilities 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1.0. The case diagram and simulation results both show all trajectories converging to the attractor solution. We refer to this solution structure as Single Attractor since all trajectories converge to an attractor solution.

Case 5b occurs when $0 < x_v \leq x^*$. This requires $W + 2\beta P + 1 < 0$ and $W^2 + 4\alpha\beta PR \leq 1$. The trajectory for Case 5b is similar to that of Case 5a except if $\varphi(x(0)) > x^*$ and $x(0) \leq x^*$ with an overshoot of the attractor solution before converging. Figure 8b shows the case diagram and simulation results with parameters $\alpha = 0.2$, $\beta = 0.2$, $P = -7.5$, $R = -0.2$, $S = 0$ and $T = 2.5$ with the initial percentages of cooperators 0.3 and 0.65. This solution structure is also Single Attractor.

Case 5c occurs when $x^* < x_v \leq 1$. This requires $W^2 + 4\alpha\beta PR > 1$ and $W - 2\alpha R \leq 1$. The trajectory for Case 5c is similar to 5b except oscillation occurs. Figure 8c shows this case with parameters $\alpha = 0.2$, $\beta = 0.2$, $P = -4$, $R = -0.5$, $S = 0$ and $T = 10$ with the initial percentage of cooperators 0.1. This solution structure is called Oscillation.

Case 5d occurs when $x_v > 1$. This requires $W - 2\alpha R > 1$. The trajectory for Case 5d is similar to 5c except that the oscillation is more extreme and does not converge to the attractor solution. Figure 8d shows this case with parameters $\alpha = 0.2$, $\beta = 0.2$, $P = -6$, $R = -4.5$, $S = -4$ and $T = -1$ with the initial percentage of cooperators 0.4. This solution structure is also Oscillation. In this case the trajectory reaches a completely cyclic behavior or converges to all agents cooperating or defecting.

4.3. Case 10

This case occurs when the quadratic curve is above the 45 degree line for low x values, intersects and moves below the 45 degree line for mid x values, and then intersects again and moves above the 45 degree line for high x values as shown in Figure 6. It is clear that $\varphi(0) > 0$, $\varphi(1) > 1$, $\varphi'(0) < 1$, $\varphi'(1) > 1$ and the discriminant of the solution to $\psi(x)$ is positive in this case. This requires $P < 0$, $R > 0$, $W + 2\beta P < 0$, $W - 2\alpha R < 0$ and $W^2 + 4\alpha\beta PR > 0$. Similarly to Case 3 we can prove that $W + 2\beta P = 0$ and $W - 2\alpha R < 0$ cannot occur simultaneously as well as $W + 2\beta P < 0$ and $W - 2\alpha R = 0$ are invalid together. In addition, $W + 2\beta P = 0$ and $W - 2\alpha R = 0$ are also impossible together. There are four subcases of the trajectory. These subcases will be evaluated separately and be called Case 10a, Case 10b, Case 10c and Case 10d. In all cases there is an attractor solution x^*_A and a repeller solution x^*_R. The first subcase 10a occurs when $\varphi(x(0)) \leq x^*_A$ and $x_v \leq x^*_A$. In this case $W^2 + 4\alpha\beta PR \leq 1$. The trajectory converges to the attractor solution when the initial percentage of cooperators is below the repeller solution and to repel to 100% cooperation if the initial percentage of cooperators is above the repeller solution. That is, in the limit $x(t) \to x^*$ when $x(0) \leq x^*_R$, and $x(t) \to 1$ when $x(0) > x^*_R$. The left hand side of Figure 9a shows the case diagram with parameters $\alpha = 0.05$, $\beta = 0.05$, $P = -4$, $R = 3$, $S = -8$ and $T = 3$ with the initial percentages of cooperators 0.05, 0.5 and 0.9. It is clear from the case diagram that the percentage of cooperators is attracting to x^*_A from below the repelling solution x^*_R, and repelling to 100% cooperation from above. The right hand side shows simulation results with the same parameter values and initial cooperating probabilities 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0. These simulation results show the same trajectories as well. This solution structure is Attractor/Repeller.

The second subcase 10b occurs when $x^*_A < \varphi(x(0)) \leq x^*_R$ and $x_v \leq x^*_A$. This case is similar to case 10a except for low initial percentages of cooperators the trajectory jumps above x^*_A and then converges down to the attractor solution. The final result is the same as in Case 10a. Figure 9b shows this case with parameters $\alpha = 0.05$, $\beta = 0.05$, $P = -16$, $R = 3$, $S = -20$ and $T = 3$ with the initial percentages of cooperators 0.1 and 0.9. This solution structure is Attractor/Repeller.

The third subcase 10c occurs when $\varphi(x(0)) > x^*_R$ and $x_v \leq x^*_A$. This case is similar to case 10b except for low initial percentages of cooperators the trajectory jumps above x^*_R to the repeller portion of the solution structure where agents repel upward and cooperate 100% of the time. This creates two different trajectories when $x(0) \leq x_R^*$ in addition to the case when $x(0) > x_R^*$. Following is a summary of the limits of these trajectories as they occur from low to high percentages of cooperation:

- $x(t) \to 1$ when $x(0) \leq x_R^*$, $\varphi(x(0)) > x_R^*$
- $x(t) \to x_A^*$ when $x(0) \leq x_R^*$, $\varphi(x(0)) \leq x_R^*$
- $x(t) \to 1$ when $x(0) > x_R^*$.

Figure 9c shows the case diagram and simulation result with parameters $\alpha = 0.05$, $\beta = 0.05$, $P = -21$, $R = 6$, $S = -25$ and $T = 0$ with the initial percentages of cooperators 0.1 and 0.2. It is clear that for low percentages of cooperators the trajectory is jumping above x_R^* and converging to 100% cooperation. This solution structure is Attractor/Repeller.

Case 10d occurs when the vertex of the quadratic curve is larger than the attractor solution, $x_v > x_A^*$. This requires $W^2 + 4\alpha\beta PR > 1$. Cases 10a, 10b and 10c presented above all assumed the vertex is smaller than the attractor solution. In this case, similarly to Case 5c and Case 5d, oscillations occur. Figure 9d shows this case with parameters $\alpha = 0.05$, $\beta = 0.05$, $P = -26$, $R = 6$, $S = -75$ and $T = 0$ with the initial percentage of cooperators 0.13. It is clear that the percentage of cooperators is oscillating. It should be noted that the system may slowly converge to the attractor solution while oscillating. This solution structure is Oscillation.

4.4. Case 11

This case occurs when the quadratic curve is above the 45 degree line for low x values, then intersects and moves below the 45 degree line until it intersects again at $x = 1$ as shown in Figure 6. It is clear that $\varphi(0) > 0$, $\varphi(1) = 1$, $\varphi'(0) < 1$ and $\varphi'(1) > 1$. This requires $P < 0$, $R = 0$ and $W < 0$. There are three subcases of the trajectory. These occur when $x_v \leq 0$, $0 < x_v \leq x^*$ and $x^* < x_v \leq 1$. These subcases will be evaluated separately and called Case 11a, Case 11b and Case 11c respectively. Case 11a occurs when $x_v \leq 0$. This requires $W + 2\beta P + 1 \geq 0$. In this case the trajectory is an attractor solution for all initial percentages of cooperators. The left hand side of Figure 10a shows the case diagram with parameters $\alpha = 0.05$, $\beta = 0.05$, $P = -4$, $R = 0$, $S = -10$ and $T = -3$ with the initial percentages of cooperators 0.1 and 0.9. The right hand side shows simulation results with the same parameter values and initial cooperating probabilities

0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0. The case diagram and simulation results both show trajectories converging to the attractor solution. This solution structure is Single Attractor.

Case 11b occurs when $0 < x_v \leq x^*$. This requires $W + 2\beta P + 1 < 0$ and $W^2 \leq 1$. The trajectory for Case 11b is similar to the previous case except if $\varphi(x(0)) > x^*$ then for low initial percentages of cooperation the trajectory will overshoot the attractor solution before converging, and the limit steady state is the same as in Case 11a. Figure 10b shows this case with parameters $\alpha = 0.2$, $\beta = 0.2$, $P = -7.5$, $R = 0$, $S = 0$ and $T = 2.5$ with the initial percentages of cooperators 0.1, 0.3 and 0.65. If the initial percentage of cooperators is selected such that at the next iteration it becomes unity, then it remains unity for all future times. This is seen for initial percentage of cooperators 0.1 in Figure 10b. The simulation results also show the trajectory overshooting the attractor solution before converging. The exact 100% cooperation rate is not attained in the simulation for initial percentage of cooperators 0.1 because Pavlovian agents do have a stochastic nature preventing them from getting 100% cooperation rate when their probabilities are less than one. This solution structure is Single Attractor. The single point at $x = 1$ is a repeller.

Case 11c occurs when $x^* < x_v \leq 1$. This requires $W^2 > 1$ and $W \leq 1$, so just $W < -1$. The trajectory for this case is similar to that of Case 11b except oscillation occurs. Figure 10c shows this case with parameters $\alpha = 0.2$, $\beta = 0.2$, $P = -4$, $R = 0$, $S = 0$ and $T = 10$ with the initial percentage of cooperators 0.1, where the solution oscillates. It even shows an overshoot before settling into an oscillation state. It should be noted that the system may slowly converge to the attractor solution while oscillating. This solution structure is Oscillation.

We have shown four cases in detail in the prevous discussions. The other 9 cases can be similarly viewed and related to the classical 24 game types based on the relative order of magnitudes of P, R, S, T. The results are summarized in Table 4.

5. An application

As an example we will develop a model for a simple cartel. Suppose there is an industry where agents can choose to either cooperate with other agents to collude to set prices as a monopoly or defect by selling product at free market prices. In this industry about half of agents cooperate by colluding to set monopolistic prices, but the percentage fluctuates dramatically between 25%–75% depending on various changing environmental conditions such as market conditions, costs and agent financial stability. We note here that OPEC is a cartel and that the cooperating nations produce 40%–50% of the world's oil. Suppose also that in the steady state a cooperating agent can gain $3M by changing its decision to defection. The defecting agent would increase its market share by lowering its price and selling more product. Of course it would be bad for the cartel as a whole since if the entire set of the cooperators changed their decision then they all would end up lowering their price without gaining any competitive advantage or market share. This social dilemma game is the prisoner's dilemma with $T > R > P > S$. If the modeler assigns parameters satisfying only condition $T > R > P > S$ and performs agent-based simulations, then the results may not look anything like the industry. The results could have all agents defecting or have the percentage of cooperators repelling from a solution instead of attracting to 50% cooperation as would be required for model validation. Attempting to develop a model that fluctuates or oscillates between $25\% - 75\%$ cooperation rates would be very difficult without additional information. Now we will use the concepts introduced in this paper to develop the model. First we see that we are looking at Case 5c. In this case the system oscillates around an attractor solution. So in addition to the condition $T > R > P > S$ we have the following information:

1. $P < 0$
2. $R < 0$
3. $W^2 + 4\alpha\beta PR > 1$
4. $W - 2\alpha R \leq 1$
5. $\varphi(x_v) = 0.25$
6. $\varphi(\varphi(x_v)) = 0.75$.

Also, in order to give some meaning to the payoff functions we set the defecting payoff function to be 3 units higher than the cooperating payoff function. That is,

$$P - S = 3 \text{ and } T - R = 3. \tag{6}$$

This is an N-person prisoner's dilemma game with the defecting curve being three units above the cooperating curve for all percentages of cooperation. We arbitrarily assign equal learning factor of 0.1. Conditions 1 and 2 show that both P and R are negative. If we look at Case 5c we can see that the graph could meet the assumptions if the vertex of the quadratic curve is approximately at $x = 0.80$ with a vertex value of 0.25. Solving the vertex equation $-B/(2A) \approx 0.80$ with the above assumptions leads to equation

$$0.6W - 0.16R - 0.04P \approx 1. \tag{7}$$

We also know that $\varphi(vertex \approx 0.80) \approx 0.64A + 0.08B + C \approx 0.25$, which implies

$$0.16W + 0.064R - 0.004P \approx -0.55. \tag{8}$$

A set of parameters that meets equations (6), (7) and (8) is $P = -11$, $R = -7$, $S = -14$ and $T = -4$. Constraints 3 and 4 above are also met with these parameters. Figure 11 shows the solution structure and simulation result with these parameter values. Indeed this does meet the required model. Note that the parameters were found without any agent-based simulation and we really only needed to look at the solution structure for verification. In fact, to develop a model one could simply draw in a quadratic curve required to produce the desired solution structure, determine quadratic curve constants A, B, C through linear regression or other means, and then determine parameters P, R, W by solving the three equations for A, B, C for the three unknowns P, R, W. There is still some flexibility with this method since S and T can be set to meet other application requirements as long as $W = \alpha S - \beta T$ is met.

Now the modeler has a model that is validated, so it may be of interest to perform experiments to see the effects of changing parameters. Let us say we want to identify the impact of increasing the temptation to defect by 50%. This could be done by raising P and T by 1.5 units which increases the separation of the payoff functions from 3 to 4.5. Figure 12 shows the case diagram and simulation result for these new parameter values $P = -9.5$, $R = -7$, $S = -14$ and $T = -2.5$. It is seen from both the solution structure and the simulation results that when the temptation to defect is increased then the percentage of cooperators on average goes down from 50% to 40% and the oscillation is not as prevalent. Both of these changes make sense from a practical standpoint. More people are going to defect if the temptation is greater and a wider gap in payoff values would lead to more certainty about a decision resulting in less oscillation. Understanding these market reactions would be good for the cartel participates and government agencies since there might be policies and decisions that could be made to influence the temptation level and thus produce a more stable environment.

6. Conclusions

Developing and validating a model for a particular N-person social dilemma game based on only traditional classification information is difficult. The traditional game classification provides only an ordering of the parameters P, R, S and T. For example, it was shown by agent-based simulation that at least three significantly different types of trajectories or solution structures can be attained for a prisoner's dilemma game by simply shifting the payoff functions up and down while keeping the separation between them identical. This paper developed a technique where the trajectory can be evaluated by graphing a quadratic curve derived from the discrete dynamic system against the 45 degree line with a set of parameters P, R, S and T. From this graph the solution structure is readily apparent. It was presented that there are thirteen different cases. These cases could be systematically analyzed where the solution structure and additional constraints for P, R, S and T were identified. Four cases were presented in detail in this paper. Viewing possible applications from this new perspective will greatly assist the modeler in determining proper values for P, R, S and T when a given solution structure is desired. This is typically the case when developing and validating a model. In this situation the modeler can simply find the case that applies to the desired solution structure and implement the identified additional constraints. An example on how this new perspective can be applied was presented for a cartel which is an N-person prisoner's dilemma game.

In our future research we will examine the cases of nonlinear payoff functions as well as other types of behavioral patterns of the agents.

Acknowledgements

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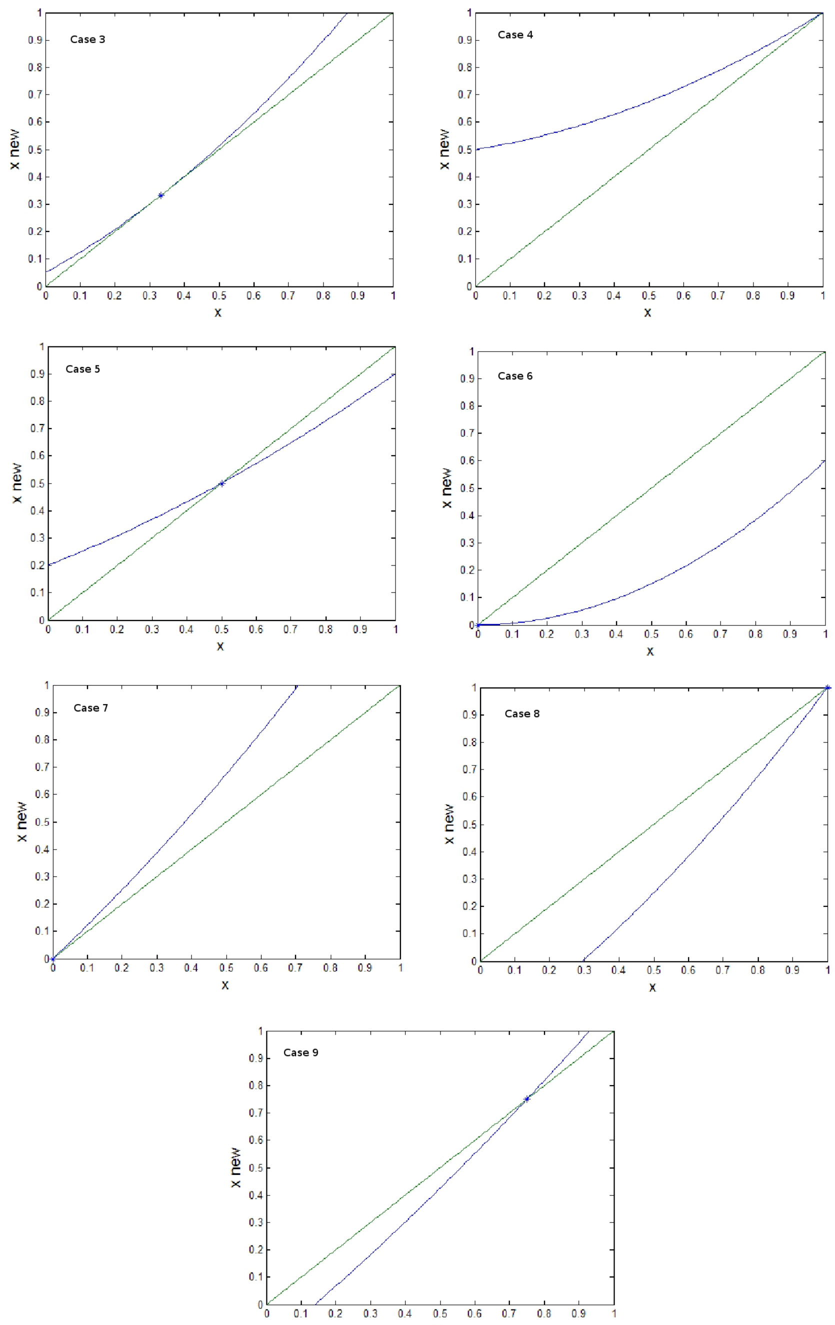

Figure 5: Cases with unique steady state solution.

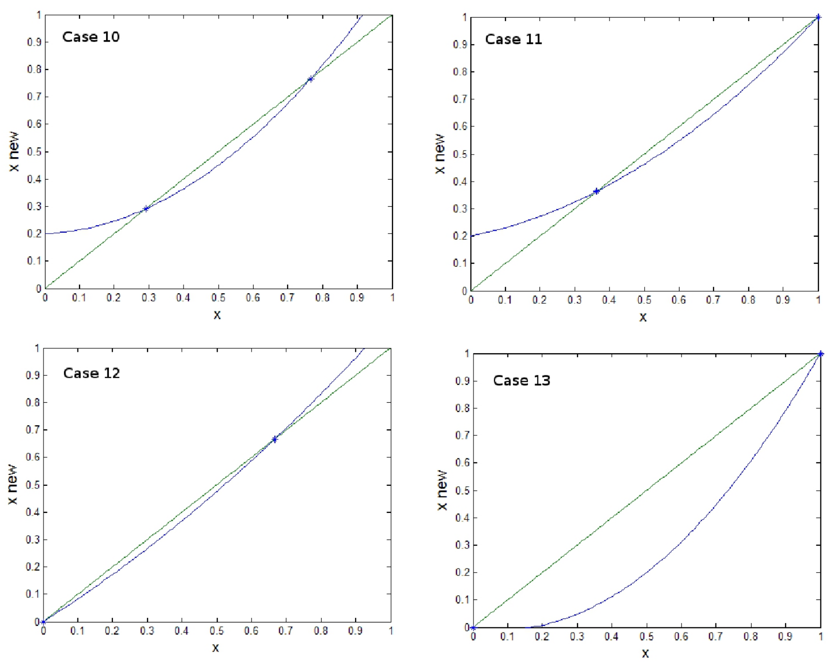

Figure 6: Cases with two steady state solutions.

Social Game Solution Structure	Case	Case Constraints
Total Cooperation	1	$P<0,\ R>0,\ W^2+4\alpha\beta PR<0$ $P<0,\ R>0,\ W^2+4\alpha\beta PR<0,\ W+2\beta P>0,\ W-2\alpha R\leq 0$ $P<0,\ R>0,\ W^2+4\alpha\beta PR<0,\ W+2\beta P\leq 0,\ W-2\alpha R>0$
	4	$P<0,\ R=0,\ W\geq 0$
	7	$P=0,\ R>0,\ W\geq 0$
Total Defection	2	$P=0,\ R=0$
	6	$P=0,\ R<0,\ W<0$
	8	$P>0,\ R=0,\ W+\beta P<0$
	13	$P=0,\ R=0,\ W<0$
Single Attractor	5a	$P<0,\ R<0,\ W+2\beta P+1\geq 0$
	5b	$P<0,\ R<0,\ W+2\beta P<0,\ W^2+4\alpha\beta PR\leq 1$
	11a	$P<0,\ R=0,\ W<0,\ W+2\beta P+1\geq 0$
	11b	$P<0,\ R=0,\ W<0,\ W+2\beta P+1<0,\ W^2\leq 1$
Single Repeller	9	$P>0,\ R>0,\ W-2\alpha R<0$
	12	$P=0,\ R>0,\ W<0$
Attractor/ Repeller	3a	$P<0,\ R>0,\ W+2\beta P<0,\ W-2\alpha R<0,\ W^2+4\alpha\beta PR=0,\ \varphi(0)\leq x^*$
	3b	$P<0,\ R>0,\ W+2\beta P<0,\ W-2\alpha R<0,\ W^2+4\alpha\beta PR=0,\ \varphi(0)>x^*$
	10a	$P<0,\ R>0,\ W+2\beta P<0,\ W-2\alpha R<0,\ 0<W^2+4\alpha\beta PR\leq 1,\ \varphi(x(0))\leq x_A^*$
	10b	$P<0,\ R>0,\ W+2\beta P<0,\ W-2\alpha R<0,\ 0<W^2+4\alpha\beta PR\leq 1,\ x_A^*<\varphi(x(0))\leq x_B^*$
	10c	$P<0,\ R>0,\ W+2\beta P<0,\ W-2\alpha R<0,\ 0<W^2+4\alpha\beta PR\leq 1,\ \varphi(x(0))>x_B^*$
Oscillation	5c	$P<0,\ R<0,\ W^2+4\alpha\beta PR>1,\ W-2\alpha R\leq 1$
	5d	$P<0,\ R<0,\ W-2\alpha R>1$
	10d	$P<0,\ R>0,\ W+2\beta P<0,\ W-2\alpha R<0,\ W^2+4\alpha\beta PR>1$
	11c	$P<0,\ R=0,\ W<-1$

Table 3: Summary of cases sorted by solution structure.

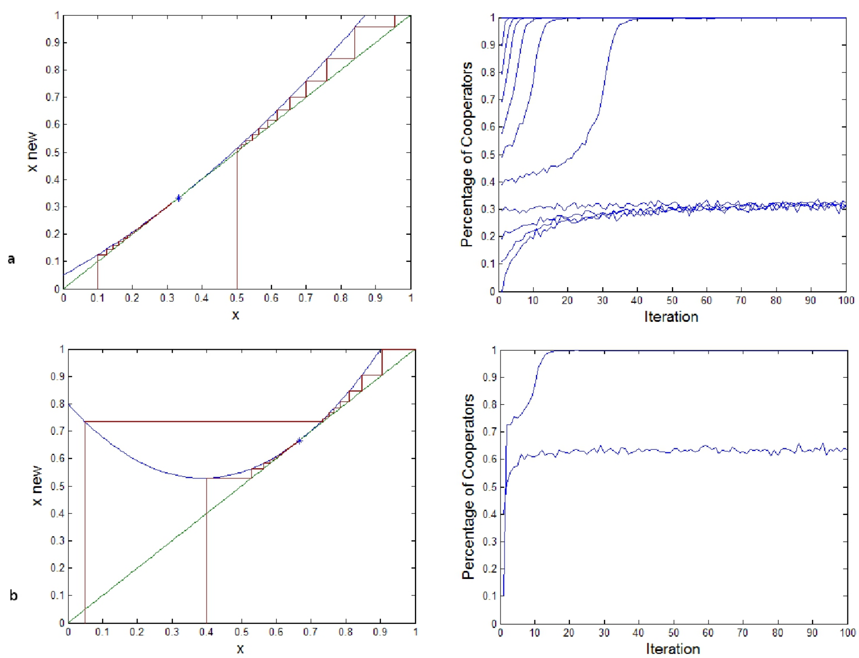

Figure 7: Case diagram and simulation results for Case 3.

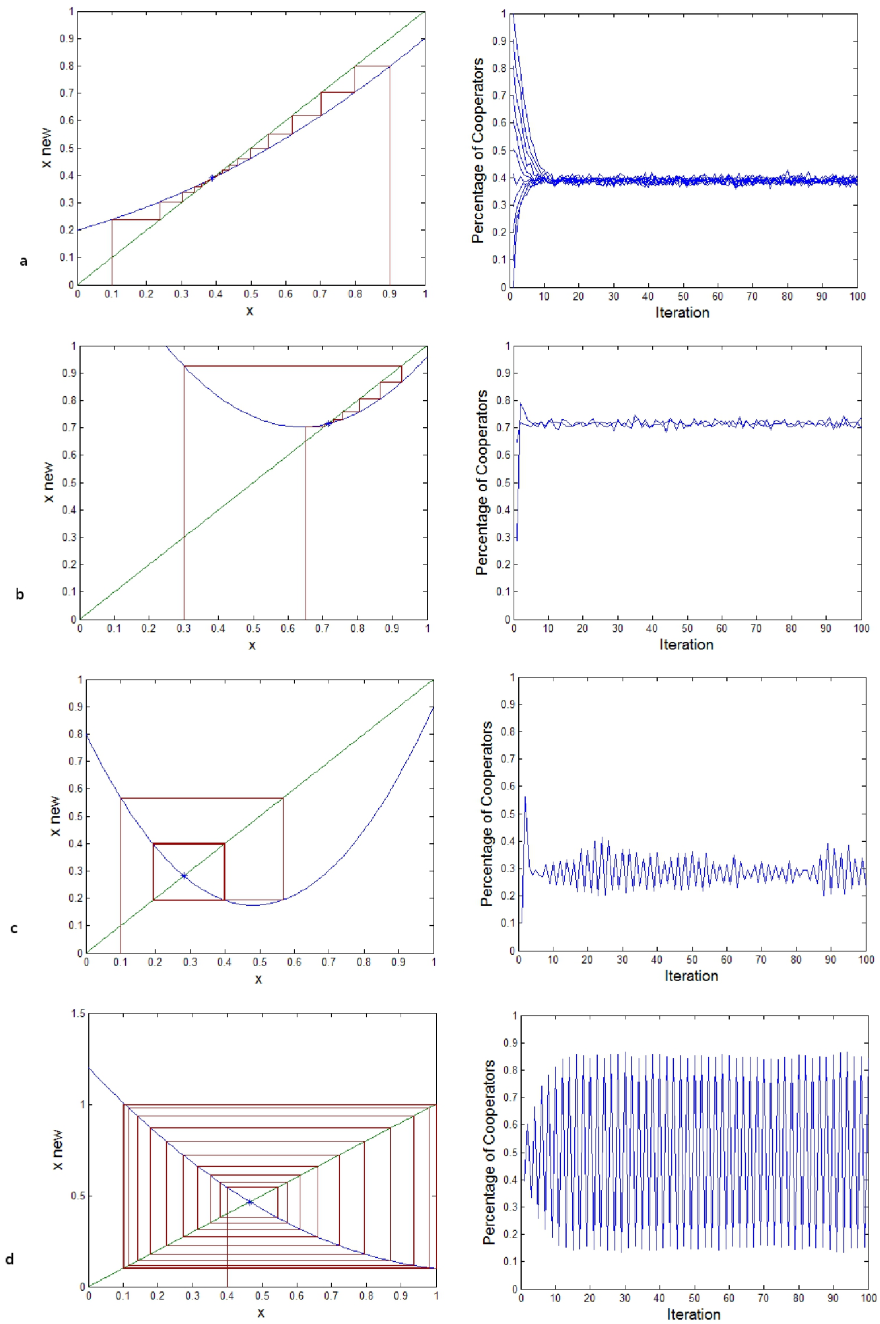

Figure 8: Case diagram and simulation results for Case 5.

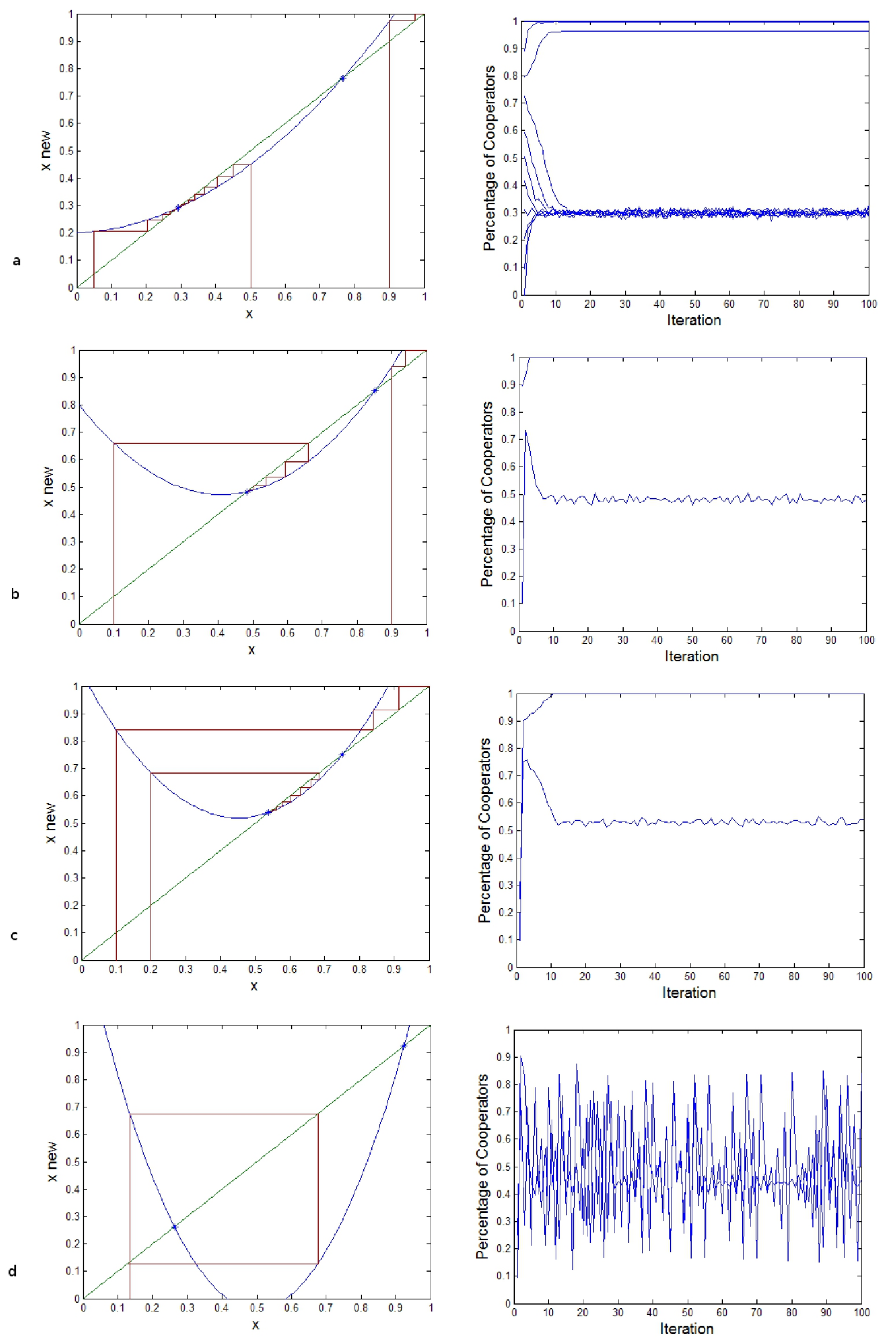

Figure 9: Case diagram and simulation results for Case 10.

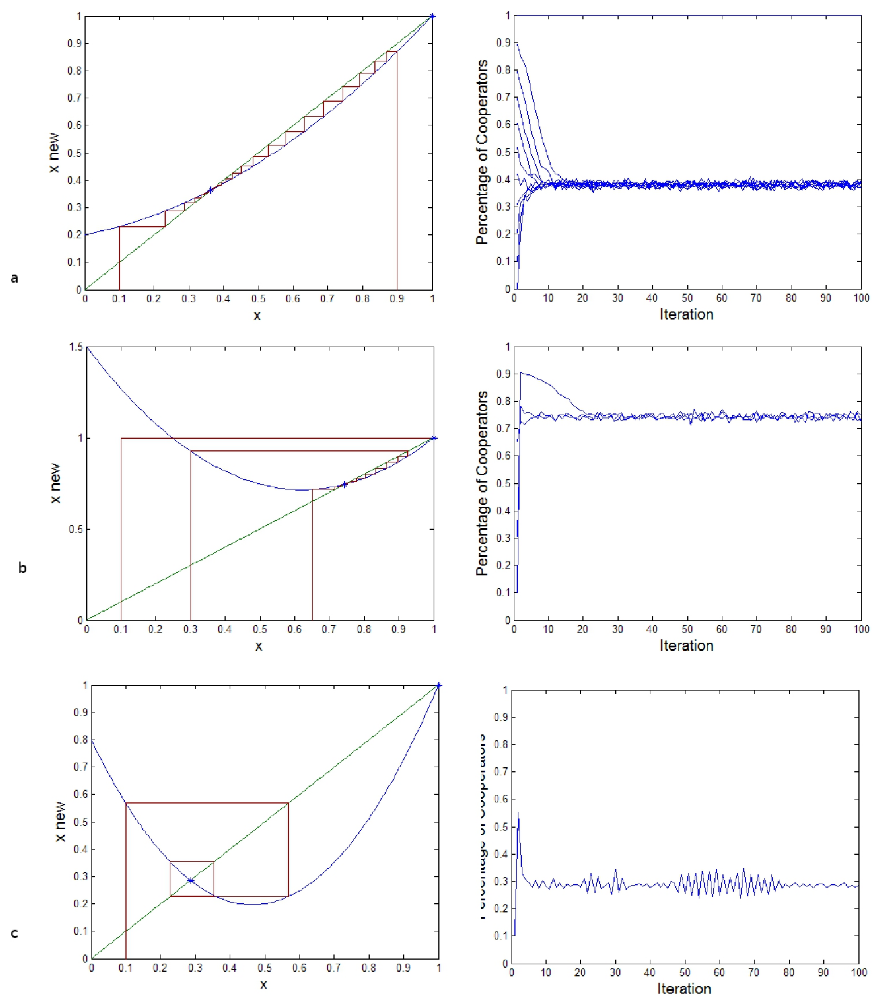

Figure 10: Case diagram and simulation results for Case 11.

Game Number	Inequality	Social Dilemma Game	Total Cooperation	Total Defection	Single Attractor	Single Repeller	Attractor/Repeller	Oscillation
1	$T > R > P > S$	Prisoner's Dilemma	X		X	X	X	X
2	$T > R > S > P$	Chicken	X		X	X	X	X
3	$T > S > R > P$	Leader	X		X	X	X	X
4	$S > T > R > P$	Reverse Battle of the Sexes	X		X	X	X	X
5	$R > T > P > S$	Stag Hunt	X		X	X	X	X
6	$R > T > S > P$	Harmony	X		X	X	X	X
7	$R > S > T > P$	Harmony	X		X	X	X	X
8	$S > R > T > P$	Unnamed	X		X	X	X	X
9	$R > P > T > S$	Coordination	X		X	X	X	X
10	$R > P > S > T$	Coordination	X		X	X	X	X
11	$R > S > P > T$	Unnamed	X		X	X	X	X
12	$S > R > P > T$	Reverse Deadlock	X		X	X	X	X
13	$S > P > R > T$	Reverse Prisoner's Dilemma		X	X	X		X
14	$S > P > T > R$	Reverse Chicken		X	X	X		X
15	$S > T > P > R$	Reverse Leader		X	X	X		X
16	$T > S > P > R$	Battle of the Sexes		X	X	X		X
17	$P > S > R > T$	Reverse Stag Hunt		X	X	X		X
18	$P > S > T > R$	Reverse Harmony		X	X	X		X
19	$P > T > S > R$	Reverse Harmony		X	X	X		X
20	$T > P > S > R$	Unnamed		X	X	X		X
21	$P > R > S > T$	Reverse Coordination		X	X	X		X
22	$P > R > T > S$	Reverse Coordination		X	X	X		X
23	$P > T > R > S$	Unnamed		X	X	X		X
24	$T > P > R > S$	Deadlock		X	X	X		X

Table 4: Summary of the feasible solution structures for each game.

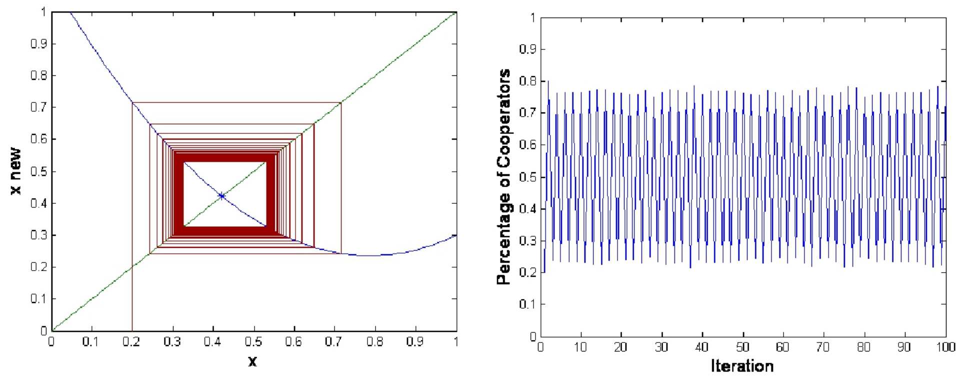

Figure 11: Solution structure and simulation results for the cartel example.

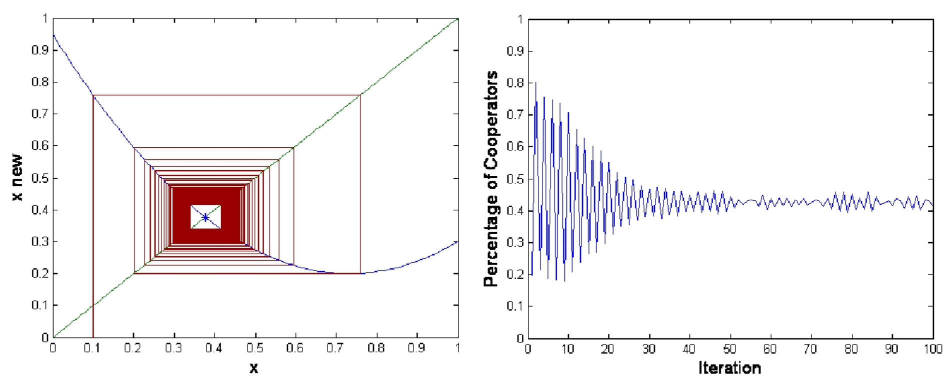

Figure 12: Solution structure and simulation results for example with higher temptation to defect.