# STICK-BREAKING REPRESENTATIONS OF 1/2-STABLE POISSON-KINGMAN MODELS


Stefano Favaro[1], Maria Lomeli[2], Bernardo Nipoti[1] and Yee Whye Teh[4]

[1] Università di Torino and Collegio Carlo Alberto
Corso Unione Sovietica, 218/bis, 10134 Torino, Italy
(e-mail: `stefano.favaro@unito.it`)
(e-mail: `bernardo.nipoti@carloalberto.org`)

[2] Gatsby Unit, University College London
17 Queen Square, London WC1N3AR, United Kingdom
(e-mail: `mlomeli@gatsby.ucl.ac.uk`)

[3] University of Oxford
1 South Parks Road, Oxford OX13TG, United Kingdom
(e-mail: `y.w.teh@stats.ox.ac.uk`)



ABSTRACT. We propose a novel constructive definition for a large class of random probability measures termed 1/2-stable Poisson-Kingman models. Our result extends a characterization recently proposed in the literature and limited to a special case of this class.


## 1 INTRODUCTION

Random probability measures (RPMs) play a fundamental role in Bayesian nonparametrics as their distributions act as nonparametric priors. The most notable example is the Dirichlet process in Ferguson (1973). Many nonparametric priors used in practice admit different, though equivalent in distribution, representations. Some of these are convenient to investigate theoretical properties of the prior, others are more useful for modelling and computation. In terms of the latter, the so-called stick-breaking representations certainly stand out. Indeed, they allow to define efficient simulation algorithms. Furthermore, they represent a useful tool for defining priors capable of incorporating certain forms of dependence for the observables and hence, go beyond the exchangeability setting. The stick-breaking representation of the Dirichlet process was proposed in Sethuraman (1994). In order to describe it, let $\mathcal{P}_0$ be a nonatomic probability measure on a complete and separable metric space $\mathbb{X}$ equipped with the Borel $\sigma$-field $\mathscr{X}$, and let $(V_i)_{i\geq 1}$ be independent random variables identically distributed as a Beta distribution with parameter $(1,\theta)$. Based on such $V_i$'s define the random probabilities $(P_i)_{i\geq 1}$ as $P_1 = V_1$ and

$$P_i = V_i \prod_{j=1}^{i-1}(1-V_j),$$

for any $i > 1$, and let $(Z_i)_{i\geq 1}$ be random variables, independent of $(P_i)_{i\geq 1}$, and independent and identically distributed as $\mathcal{P}_0$. Then, a Dirichlet process with parameter $(\theta, \mathcal{P}_0)$ coincides in distribution with the RPM

$$\tilde{Q}_\theta(\cdot) = \sum_{i\geq 1} P_i \delta_{Z_i}(\cdot),$$

where $\delta_a$ denotes the point mass at $a$. The name of this construction refers to the fact that, in order to define the $P_i$'s, one can think of breaking a stick of length one in two parts of lengths $V_1$ and $(1-V_1)$, respectively. The first part is $P_1$ whereas the second part is further split into two parts of lengths $V_2(1-V_1)$, that will coincides with $P_2$, and $(1-V_2)(1-V_1)$ that, in turn, will be split to generate $P_3$, and so on. Another noteworthy nonparametric prior admitting a stick-breaking representation is the two parameter Poisson-Dirichlet process introduced in Perman et al. (1992).

The stick-breaking representations of the Dirichlet process and the two parameter Poisson-Dirichlet process naturally lead to define a general class of discrete RPMs, the so-called stick-breaking priors, obtained by allowing independent stick-breaking random variables $V_i$'s with an arbitrary probability distribution supported on the set $(0,1)$. Such an issue has been addressed in Ishwaran and James (2001). Nonetheless, as discussed in detail in Favaro *et al.* (2012), most of the nonparametric priors proposed in the literature are not part of the class of stick-breaking priors introduced in Ishwaran and James (2001). Indeed, they do not admit a stick-breaking representation in terms of a collection of independent $V_i$'s. As an example, in Favaro *et al.* (2012) a stick-breaking representation is provided for the normalized inverse Gaussian process introduced by Lijoi *et al.* (2005). To the best of our knowledge, the normalized inverse Gaussian process is the first example of a discrete RPM admitting a stick-breaking representation in terms of a collection of dependent $V_i$'s which are no more distributed as Beta distributions.

In this paper we derive the stick-breaking representation for a large class of RPMs introduced by Pitman (2003) and termed $1/2$-stable Poisson-Kingman models. Such a representation leads to a novel constructive definition for $1/2$-stable Poisson-Kingman models. In particular, due to the flexible definition of this class of nonparametric prior models, the proposed stick-breaking representation allows to recover as special cases the stick-breaking representations of several RPMs well-known in the literature such as, the two parameter Poisson-Dirichlet process with discount parameter equal to $1/2$ and the normalized inverse Gaussian process.

## 2 $1/2$-STABLE POISSON-KINGMAN MODELS

We start by recalling the concept of completely random measure (CRM) due to Kingman (1967). A CRM $\tilde{\mu}$ is a random element with values on the space of boundedly finite measures on $\mathbb{X}$ and such that for any $A_1,\dots,A_n$ in $\mathscr{X}$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, the random variables $\tilde{\mu}(A_1),\dots,\tilde{\mu}(A_n)$ are mutually independent. A CRM is uniquely characterized by a measure $\nu$ on $\mathbb{R}^+ \times \mathbb{X}$ that is referred to as the Lévy intensity of $\tilde{\mu}$. Kingman (1967) showed that a CRM $\tilde{\mu}$ is almost surely discrete and, accordingly, it can be represented by random masses $(\tilde{J}_i)_{i\geq 1}$ at random locations $(Y_i)_{i\geq 1}$, i.e.

$$\tilde{\mu}(\cdot) = \sum_{i\geq 1} \tilde{J}_i \delta_{Y_i}(\cdot).$$

CRMs provides a fundamental tool to define almost surely discrete RPMs through a normalization approach. See, e.g., Pitman (2003), James et al. (2009) and references therein for a detailed account.

The class of 1/2-stable Poisson-Kingman models is defined hierarchically in terms of an underlying normalized 1/2-stable CRM $\tilde{\mu}_{1/2}$ which is suitably mixed with respect to the normalizing total mass. Specifically, let $\tilde{\mu}_{1/2}$ be a 1/2-stable CRM, that is a CRM characterized by the Lévy intensity

$$\nu(\mathrm{d}s, \mathrm{d}y) = \rho_{1/2}(s)\mathrm{d}s\mathcal{P}_0(\mathrm{d}y) = \frac{s^{-3/2}}{2\sqrt{\pi}}\mathrm{d}s\mathcal{P}_0(\mathrm{d}y),$$

and let $T_{1/2} = \sum_{i\geq 1}\tilde{J}_i$ be the total mass of $\tilde{\mu}_{1/2}$. Hence $T_{1/2}$ is a positive 1/2-stable random variable with density function

$$f_{1/2}(x) = \frac{1}{2\sqrt{\pi}}x^{-\frac{3}{2}}\mathrm{e}^{-\frac{1}{4x}}.$$

Intuitively one can define an almost surely discrete RPM $\tilde{P}_{1/2}$ by normalizing $\tilde{\mu}_{1/2}$ with respect to $T_{1/2}$. Formally, one obtains

$$\tilde{P}_{1/2}(\cdot) = \frac{\tilde{\mu}_{1/2}(\cdot)}{T_{1/2}} = \sum_{i\geq 1}\tilde{P}_i\delta_{Y_i}(\cdot),$$

with $\tilde{P}_i = \tilde{J}_i/T_{1/2}$ and where $(Y_i)_{i\geq 1}$ are random variables, independent of $(\tilde{P}_i)_{i\geq 1}$, and independent and identically distributed as $\mathcal{P}_0$. The RPM $\tilde{P}_{1/2}$ is termed normalized 1/2-stable process with base distribution $\mathcal{P}_0$. See, e.g., Pitman (2003) and references therein for details on the RPM $\tilde{P}_{1/2}$.

A 1/2-stable Poisson-Kingman model is defined as a generalization of $\tilde{P}_{1/2}$ obtained by suitably deforming (tilting) the distribution of $T_{1/2}$. Let $(P_{(i)})_{i\geq 1}$ be the decreasing rearrangement of $(\tilde{P}_i)_{i\geq 1}$ and let $T_{1/2,h}$ be a nonnegative random variable with density function of the form $hf_{1/2}$ for a nonnegative measurable function $h$. Denoting by $\mathrm{PK}(\rho_{1/2}\,|\,t)$ the conditional distribution of $(P_{(i)})_{i\geq 1}$ given $T_{1/2,h}$, let

$$\mathrm{PK}\left(\rho_{1/2}, hf_{1/2}\right) = \int_0^{+\infty}\mathrm{PK}\left(\rho_{1/2}\,|\,t\right)h(t)f_{1/2}(t)dt$$

be a mixture distribution which we refer to as the 1/2-stable Poisson-Kingman distribution with parameter $h$ as in Pitman (2003). Then, a 1/2-stable Poisson-Kingman model with parameter $h$ and base distribution $\mathcal{P}_0$ is defined as an almost surely discrete RPM $\tilde{P}_{1/2,h}$ of the form

$$\tilde{P}_{1/2,h}(\cdot) = \sum_{i\geq 1}P_{(i)}\delta_{Y_i}(\cdot),$$

where $(P_{(i)})_{i\geq 1}$ are random probabilities distributed as a 1/2-stable Poisson-Kingman distribution with parameter $h$, and $(Y_i)_{i\geq 1}$ are random variables, independent of $(P_{(i)})_{i\geq 1}$, and independent and identically distributed as $\mathcal{P}_0$. Suitable choices of $h$ allows us to recover as special cases of $\tilde{P}_{1/2,h}$ some well-known RPMs such as, for instance, the normalized 1/2-stable process, the two parameter Poisson-Dirichlet process with a discount parameter equal to $1/2$ and the normalized inverse Gaussian process.

## 3 Main result

The next result can be read from Theorem 2.1 in Perman et al. (1992) and it provides a stick-breaking representation for the class of 1/2-stable Poisson-Kingman models. See also Pitman (2003) for details.

**Lemma 1** *Let $\tilde{P}_{1/2,h}$ be a 1/2-stable Poisson-Kingman model and let $(P_{(i)})_{i\geq 1}$ be the characterizing decreasing ordered random probabilities. Then,*

$$\tilde{P}_{1/2,h}(\cdot) = \sum_{i\geq 1} P_i \delta_{Z_i}(\cdot),$$

*where $(P_i)_{i\geq 1}$ is the size-biased random permutation of $(P_{(i)})_{i\geq 1}$ and $(Z_i)_{i\geq 1}$ are independent random variables identically distributed as $\mathcal{P}_0$. Moreover, one has*

$$P_i = V_i \prod_{j=1}^{i-1}(1-V_j),$$

*where $(V_i)_{i\geq 1}$ are random variables such that the conditional distribution of $V_i$ given $T_{1/2,h}$ and $(V_1,\ldots,V_{i-1})$ has a density function on $(0,1)$ of the form*

$$g_{V_i|V_1,\ldots,V_{i-1},T_{1/2,h}}(v_i \,|\, v_1,\ldots,v_{i-1},t) = \frac{1}{2\sqrt{\pi}} \frac{\exp\left\{-\frac{v_i}{4tw_i(1-v_i)}\right\}}{(1-v_i)^{\frac{3}{2}}(tv_iw_i)^{\frac{1}{2}}}, \tag{1}$$

*with respect to the Lebesgue measure, for any index $i \geq 1$ and where we defined $w_i = \prod_{j=1}^{i-1}(1-v_j)$ with $w_1 = 1$. Finally, $(P_i)_{i\geq 1}$ is independent of $(Z_i)_{i\geq 1}$.*

Lemma 1 provides the distribution of the stick-breaking random variables $V_i$'s given the total mass $T_{1/2,h}$. This is coherent with the hierarchical definition of Poisson-Kingman distributions. In particular, after specifying the parameter $h$, the conditional distribution of $V_i$ given $(V_1,\ldots,V_{i-1})$ can be derived from (1) by integrating out the random variable $T_{1/2,h}$ with density function $hf_{1/2}$. Lemma 1 leaves open the problem of finding a straightforward description for the distribution of the stick-breaking random variables $V_i$'s. Here we present a characterization of density function (1) in terms of a suitable transformation (normalization) involving a Gamma random variable and an inverse Gamma random variable. Specifically, let $(V_{1/2,i})_{i\geq 1}$ be a sequence of random variables defined as

$$(V_{1/2,i} \,|\, V_{1/2,1},\ldots,V_{1/2,i-1},T_{1/2,h}) \stackrel{\mathrm{d}}{=} \frac{X_i^{\frac{1}{2}}}{X_i^{\frac{1}{2}}+Y_i^{\frac{1}{2}}}, \tag{2}$$

where, for any index $i \geq 1$, the random variables $X_i$ and $Y_i$ are assumed to be independent and distributed as

$$X_i \sim \mathcal{G}\left(\frac{3}{4},1\right) \tag{3}$$

and

$$Y_i \sim \mathcal{IG}\left(\frac{1}{4}, \frac{\frac{1}{4^3\left(T_{1/2,h}\right)^2}}{\prod_{j=1}^{i-1}(1-V_{1/2,j})^2}\right), \tag{4}$$

with the proviso that the empty product is defined to be unity and with $\mathcal{G}$ and $\mathcal{IG}$ denoting the Gamma distribution and the inverse Gamma distribution, respectively. The next theorem introduces a novel constructive definition for the class of 1/2-stable Poisson-Kingman models. In particular, it provides a generalization of Proposition 1 in Favaro *et al.* (2012) to the entire class of 1/2-stable Poisson-Kingman models.

**THEOREM 1** *Let* $(V_{1/2,i})_{i\geq 1}$ *be the sequence of random variables introduced in* (2). *Then, one has*

$$\tilde{P}_{1/2,h}(\cdot) = \sum_{i\geq 1} V_{1/2,i} \prod_{j=1}^{i-1}(1-V_{1/2,j})\delta_{Z_i}(\cdot),$$

*where* $(Z_i)_{i\geq 1}$ *are independent random variables identically distributed as* $\mathcal{P}_0$ *and independent of* $(V_{1/2,i})_{i\geq 1}$.

**Proof.** We start by focusing on the distribution of the random variable $V_{1/2,1}\,|\,T_{1/2,h}$. In this case (1) simplifies to

$$g_{V_{\frac{1}{2},1}|T_{\frac{1}{2},h}}(v_1\,|\,t) = \frac{1}{2\sqrt{\pi}} \frac{\exp\left\{-\frac{v_1}{4t(1-v_1)}\right\}}{(1-v_1)^{\frac{3}{2}}(tv_1)^{\frac{1}{2}}}. \tag{5}$$

We need to proove that the density function (5) coincides with the density function of the random variable

$$S_1 \stackrel{\mathrm{d}}{=} \frac{X_1^{\frac{1}{2}}}{X_1^{\frac{1}{2}} + Y_1^{\frac{1}{2}}}, \tag{6}$$

where $X_1$ and $Y_1$ are the random variables in (3) and (4), respectively. Recall that $Y_1$ is independent $X_1$. Then, by making the transformation (6), the density function of $S_1$ coincides with

$$\begin{aligned} g_{S_1}(s_1) &= 4\frac{s_1^{\frac{1}{2}}(1-s_1)^{-\frac{3}{2}}\left(4^3t^2\right)^{-\frac{1}{4}}}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)} \int_0^{+\infty} \exp\left\{-\left((zs_1)^2 + \frac{\frac{1}{4^3t^2}}{(z(1-s_1))^2}\right)\right\}\mathrm{d}z \\ &= 4\frac{s_1^{\frac{1}{2}}(1-s_1)^{-\frac{3}{2}}\left(4^3t^2\right)^{-\frac{1}{4}}}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)} \frac{\sqrt{\pi}}{2s_1}\exp\left\{-\frac{s_1}{4t(1-s_1)}\right\} \\ &= \frac{1}{2\sqrt{\pi}} \frac{\exp\left\{-\frac{s_1}{4t(1-s_1)}\right\}}{(1-s_1)^{\frac{3}{2}}(ts_1)^{\frac{1}{2}}}. \end{aligned} \tag{7}$$

The resulting expression for $g_{S_1}$ coincides with the density function of the random variable $V_{1/2,1}\,|\,T_{1/2,h}$ in (5). According to (1), for any index $i > 1$ the density function of the random variable

$$V_{1/2,i}\,|\,V_{1/2,1},\ldots,V_{1/2,i-1},T_{1/2,h}$$

coincides with the density function (7) in which $t$ is replaced by the term $t\prod_{j=1}^{i-1}(1-v_j)$. Therefore, such a density function will coincide with the density function of the random variable

$$W_i \overset{\mathrm{d}}{=} \frac{X_i^{\frac{1}{2}}}{X_i^{\frac{1}{2}}+Y_i^{\frac{1}{2}}}, \tag{8}$$

for any index $i > 1$, where the random variables $X_i$ and $Y_i$ in (8) have the same distributions of $X_1$ and $Y_1$ in (6), respectively, with $t$ replaced by the term $t\prod_{j=1}^{i-1}(1-v_j)$. The proof is completed. □

## 4 Conclusions

Our result is interesting from both theoretical, modelling and computational points of view. Firstly, it completes the study of the well-known class of 1/2-stable Poisson-Kingman models, by providing a constructive definition in terms of a latent random variable and a collection of dependent stick-breaking weights defined by means of Gamma and inverse Gamma random variables. Secondly, it suggests a simple way to define new Bayesian nonparametric models based on the class of 1/2-stable Poisson-Kingman models by modifying well-established models based on the stick-breaking definition of the Dirichlet process. Finally, our representation allows to extend to the class of 1/2-stable Poisson-Kingman models various recently proposed simulation algorithms such as, for instance, the slice sampling and the retrospective sampling. Indeed, both of these simulation algorithms assume to have access to a stick breaking representation of the underlying RPM.

## References


Favaro, S., Lijoi, A. and Prünster, I. (2012): On the stick-breaking representation of normalized inverse Gaussian priors. *Biometrika*, **99**, 663–674.

Ferguson, T. S. (1973). A Bayesian analysis of some nonparametric problems. *Ann. Statist.*, **1**, 209–230.

Ishwaran, H. and James, L.F. (2001): Gibbs sampling methods for stick-breaking priors. *J. Amer. Stat. Ass.*, **96** 161–173.

James, L.F., Lijoi, A. and Prünster, I. (2009): Posterior analysis for normalized random measures with independent increments *Scand. J. Statist.*, **36** 76–97.

Kingman, J.F.C (1967): Completely random measures. *Pacific J. Math.*, **21**, 59–78.

Lijoi, A., Mena, R.H. and Pruňster, I. (2005): Hierarchical mixture modelling with normalized inverse Gaussian priors. *J. Amer. Stat. Assoc.*, **100**, 1278–1291.

Perman, M., Pitman, J. and Yor, M. (1992): Size-biased sampling of Poisson point processes and excursions. *Probab. Theory Related Fields*, **92**, 21–39.

Pitman, J. (2003). Poisson-Kingman partitions. *Science and Statistics: A Festschrift for Terry Speed* (D.R. Goldstein, Ed.) *Lecture Notes Monograph Series* **40**, 1–34. IMS, Beachwood, OH.

Sethuraman, J. (1994): A constructive definition of Dirichlet priors. *Statist. Sinica*, **4**, 639–650.