

# ICTMT 11

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## Conference Proceedings

edited by

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## PREFACE

The Eleventh International Conference on Technology in Mathematics Teaching took place from Tuesday July 9th to Friday July 12th, for the first time in Italy, at the University of Bari, in collaboration with the University of Torino.

As in the spirit of the ICTMT Conferences, it aimed to bring together lecturers, teachers, curriculum designers, mathematics researchers, learning technologists and educational software designers, who shared an interest in improving the quality of teaching and learning by an effective use of technology.

Digital technologies are now becoming ubiquitous, so nobody can ignore them. It is extremely urgent to analyse their influence on the education systems and to explore and pave the way for other possible research.

The Conference aimed therefore to be a forum in which researchers and practitioners could exchange and discuss better practices, theoretical know-how, innovation and perspectives on educational technologies and their impact on maths teaching and learning.

Five plenary speakers were involved to broaden the focus of the conference.

Michal Yerushalmy, arguing that digital books offer new kinds of flexibility, participation, and personalization, speculated about challenges for a new textbook culture.

Regina Bruder, describing and analysing some results of the project CALIMERO, pointed out that, apart from the availability of technology and suitable teaching and learning materials, the training of didactical-methodical competencies of the teachers in this field is crucial for the efficient technology-based teaching.

The third keynote has been devoted to the experience of a secondary school teacher and a primary school teacher. The former, Dave Murrels, explained how he had become increasingly aware of the interdependence of some key factors upon the successful use of ICT in his classroom and, looking forward, why his long term aim is for students to develop their mathematical thinking skills through, and eventually, independently of the technology. The latter, Ketty Savioli, reflected on the potential of motion detectors when used to introduce the concept of function with children, suggesting that if we want to be able to exploit the potential success of the integration of technology in school mathematics teaching, the search for why(s) is a (not only didactical) enterprise that is still our responsibility.

Gilles Aldon, finally, underlined that the observation of a changing world where “digital natives” have a different perception of things is often made but the question of how to change teaching and learning in order to adapt schools and society is an open question. He suggested a methodology, the incident analysis, to analyze the modifications of class dynamics and to better understand the students and teacher's joint action.

Last but not least, more than sixty delegates (coming from 25 different Countries) exposed and discussed their papers (in forms of oral presentations, workshops or posters) on the following themes:

- Curriculum, *The impact of technology on the mathematics curriculum*
- Assessment, *New possibilities for assessment in mathematics*
- Students, *Technology to motivate and support students to learn mathematics*

- Teachers, *Technology for mathematics teachers' professional development*
- Innovation, *New development in technology for learning and teaching mathematics*
- Applications, *Technology as a bridge between mathematics and other subjects*
- Software, *Design, evaluate and choose software to learn and teach mathematics*

The high level of papers submitted has been guaranteed by the International Scientific Committee. Two peer reviewers have examined and accepted all different proposals.

This book is gratefully dedicated to all the Scientific Committee Members and Conference delegates. Special thanks have to be given to the Organizing Committee and all the people who have contributed to the success of the Conference.

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# **Plenary Keynotes**



## HOP-O'-MY-THUMB AND MATHS EDUCATION

Gilles Aldon

*École Normale Supérieure de Lyon, France*

*In the new world of information and communication new students come into classrooms and universities with a different relation with knowledge. Teachers' roles become more brokers' roles leading to use and to give sense to knowledge much more than transmitting it. Different didactical tools take into account the new paradigm of teaching; particularly, the documental genesis which is interested in the process of acquisition and transformation of resources into the teachers' documental system of resources. On an other hand, the notion of didactic incidents should give tools to analyze the modifications of class dynamics and to better understand the students and teacher's joint action.*

### INTRODUCTION

Improving the quality of teaching and learning by effective use of technology is a common goal that brings together teachers, researchers, students and more widely citizens. The last ICTMT conferences have shown very interesting and convergent studies about the potentialities of software and their added values for the teaching and learning of mathematics: algebra, calculus, modeling,... In the same time, the new role and the new necessary skills of teachers have been highlighted as well as the opportunities to catch mathematical objects through different and intertwined representations. The multirepresentation and the multimodality seen as the two faces, technological and cognitive, of the same coin, appear to be facilitated by the use of technology (Arzarello, Robutti, 2010).

But the roads leading to this goal are often quite different. In the fairy tale, Hop-o'-My-Thumb's parents walked in the forest and abandoned their seven children in a place where nobody could return, but Hop-o'-My-Thumb had always a mean to save himself and his brothers. The world changes, Hop-o'-My-Thumb, *Petite Poucette*, (Serres, 2012) is abandoned in the knowledge world with her tools.

Walking a short way with Hop-o'-My-Thumb in the mathematics knowledge world, we can try to understand who is now coming at school or at university because before teaching anything to whosoever, at least we must know who he/she is: "Today's students have not just changed *incrementally* from those of the past, [...] a really big *discontinuity* has taken place" (Prensky, 2001). But what exactly change? Information, resources, documents are available, knowledge is everywhere, within reach of a click, answers to school questions are, most of the time, available on the web and reachable through research engines. Students in their everyday life dispose of these tools and use them widely and freely; most of the time, for their everyday life, they don't need to know because they know where the knowledge is available; or more precisely, they believe that they don't need to know because they believe that the available knowledge is understandable.

Following Michel Serres, three main revolutions occurred in the history of humanity; these three revolutions concerns the dialectic between support and content of informations that men have to exchange. At the beginning was an oral world, and the first revolution appeared when writing appeared; fundamentally, civilizations are the daughters of this first revolution through the development of sciences, organization of the world, appearance of structured religions,... The second revolution is the dissemination of writing, say the apparition of printing. From that revolution appears the democratization of knowledge and fundamentally a new commercial organization, the Reformation when everybody can read directly in the Bible disrupts the religious order; it is also the beginning and the support of experimental sciences. Adding digital to the words

“resource” and “document” leads to a new revolution whose consequences will be surely as important as those that have been reached by the two first revolutions. Particularly, the question of education is crucial in this time of fundamental society changes. For example, the most important schools and universities have started programs to invest MOOCs. But what kind of teaching, instruction, assessment does these MOOCs take? Is it the opportunity to flip classrooms? As Harry E. Pence (2013) asked: if MOOCs are the answer, *what is the question?*

In this conference I would like to dwell on the paradigm shift in education that occurs from information and communication technology and to have a glance to such didactical tools and their application in ordinary classrooms. The first question that I would like to address is the question of the available tools given by education research to understand, to accompany and to improve mathematics education in this new world. The second question that seems to me important to deal with is the question of the renewing of teaching and learning mathematics with technology and the relationships between technology and an experimental part of mathematics.

### **DOCUMENTATIONAL GENESIS AND DIDACTIC INCIDENTS**

The notion of resources and documents have to be clarified especially as the adjective “digital” is added. The word resource comes from the Latin *resurgere* which means to rise again and has been used by d'Alembert as a possibility given by the language to express subtly a difficult notion. Resource is defined in the on-line Oxford dictionary as *a stock or supply of money, materials, staff, and other assets that can be drawn on by a person or organization in order to function effectively*. The meaning is wide and resources concern all that can be useful for somebody in order to do something. Resources can be either a discussion with a colleague, a reading, a software, or anything that flow into the actors' intentions of doing something. We will use this word in this wide sense. It is important to note that a resource is not directly connected with a particular intention. For example, if I read the proof of a theorem, it will enrich my set of resources but I perhaps never use it for teaching or I can build on the idea of this proof a problem for my students.

At the opposite, a document is attached to a particular intention and has a *value of evidence* as well as a *value of information*: an identity document is the administrative proof of your existence and gives information on your birth place, date, and so on, civil law is a reference document for lawyers but also the document where everybody can find information about the law of his/her country, etc. The noun document comes from the Latin verb *docere* which means *to teach*. A document can then be seen as a resource that has been transformed by an actor including in itself his/her particular intentions.

Changes in documentary activity can be seen as a process of modernization (Pedauque, 2006). Keeping the traditional functions, the modernization has enriched them by the modification of the ratio content/container but documents possess within themselves the dimensions of creativity, organization of ideas, mnemonics and communication. The two cognitive functions (mnemonics and organization of ideas) are directly linked with the *value of information*, that is to say, the ideas, the important elements have to be hoarded and organized in such a way that it will be easy to find information. These two dimensions of documentary are widely intertwined and the result is part of the organization of the actor's set of resources. The creativity which is attached to a resource, is also directly linked, in our concern, to the multiple representation of mathematical objects. The different representations and the links between them express, through semiotic registers, the creativity of the resource. Finally, the transmission function of a documentary production takes into account both the support and the subject and the new possibilities bring this function in front of the stage, directly linked with the mediation of the production:

- in an individual level, the documentation remains private and is built and used by a certain person,
- in a collective level, the documentation is shared within a delimited community, say a group of persons working together, a class (students and teachers), a community of practice, etc.
- in a public level, the documentation evades the control of authors.

Concerning education, teachers' resources come from different *sources* and include their conceptions, knowledge and beliefs about mathematics as well as their conceptions about teaching and learning mathematics. It's an evidence to say that a same resource will not be used in the same way by two persons having different conceptions of teaching and learning; which means that there is a transformation of the resource between its creation or discovering and its effective use; this transformation is called by Gueudet and Trouche (2012) the *documentational genesis* in reference to the *instrumental genesis* (Rabardel, 1995).

In the *documentational approach*, resources become documents after a long process where the resource modifies the actor's behavior (*instrumentation*) and the actor modifies and shapes the resource (*instrumentalization*). The result of the process is called a document and can be described by the formula:

Document = Resources + Scheme of utilization (id. p. 205).

In this definition of a document, the two previous main values remain present, the value of proof being linked with the implementation of the resource in a particular context; the transformation of the document into a new resource that augments the actor's set of resources is related to the *information value* of the document.

As well as teachers build documents for their teaching, students in the digital world have also the opportunity to transform the available resources into their own documents; they amass their own set of resources which they can keep private or share in a collective level of mediation. As well as teachers, they include their conceptions, beliefs, knowledge. Therefore the *documentational genesis* of teacher and students are developed simultaneously, with sometimes some intersections or common views

Different examples should be taken showing how the *documentational genesis* can catch the process of transformation of a resource into a document build in a particular goal. Particularly, I would like to take attention to the fact that the same resources leads for different actors to different documents depending on the goal and intentions to teach or to learn or to succeed or to pass... The notion of *didactic incidents* (Aldon, 2013) should then give tools to analyze the modifications of class dynamics and to better understand the students and teacher's joint action and so, to enhance teachers' professional development.

Considering a classroom as a dynamic system where interactions provoke knowledge constructions through mathematical situations, it is rightful to be interested in the particular moments where something happens which disturb and modify this dynamic. We called these particular moments *didactic incidents* because they are fundamentally attached to the didactical situation and they affect the didactical system, that is to say the relationships between knowledge, student and teacher. There were different types of didactical incidents, each of which causing perturbations in the classroom's dynamic; observing the perturbations don't give clues about the reason of these perturbations whereas looking at incidents and at their causes can bring useful information for teachers in a particular situation. Briefly, it is possible to distinguish five types of incidents:

1. an outside incident corresponds to an event not directly linked to the situation but often important in the classroom. This type of incident can strengthen a previously caused perturbation,

2. a syntactic incident is linked to the conversion between semiotic registers of representations; in a technological environment, these incidents mainly come from the feedback of the machine,
3. a friction incident corresponds to the confrontation of two situations in the interactions between students and teachers,
4. a contract incident occurs when an event breaks or modifies significantly the didactical contract; this modification is strongly correlated with the appearance of didactical bifurcations where students invest a situation different from the teacher's intended situation,
5. a mathematical incident when a mathematical question is asked without answers.

Without entering too deeply into the concept of didactical incidents, that could be read in (Aldon, 2013), I would like to focus on didactic incidents directly linked with the documentational approach and taking into account both the students and teacher point of view. The different properties owned by digital resources, organization of ideas, mnemonics, creativity and communication are developed by actors through documentational genesis to become for a particular aim a digital document which is working for particular actors in a particular situation.

## **TECHNOLOGY AND THE EXPERIMENTAL PART OF MATHEMATICS**

Scientific phenomena can be expressed as experimental insofar they are qualified as object. Mathematics is not an exception and the relationship between reality and mathematical objects must be specified. In a Platonic perspective, mathematics exist in itself and mathematicians discover and explore like an explorer reveals a new continent. In contrast, the Kantian position offers mathematics as a human construction, which must determine and define its objects *a priori*.

Taking this point of view, an *experiment* in mathematics, appears to be a work on naturalized representations of mathematical objects which are defined in a system of signs. The word *naturalized* meaning that internal changes within a register of semiotic representation of an object or conversions from a register to an other (Duval 1993) are mastered. The experiment's purpose is then to define or to explore the properties of a mathematical object in connection with a theory. Thus, mathematics concepts, although produced in mind, have sense only in their relations with empirical phenomena. These philosophical considerations about the nature of mathematical objects are very important particularly when using technology. Experiments with technology are build not on mathematical objects, which have a synthetic nature, but on some of their representations. *Multirepresentations* in technological environments from numerical, graphical, algebraic,... registers can support students' processes of mathematics learning and their consequent *multimodal* productions, that is to say the different modalities students use in an experimental approach of mathematics (Arzarello & Robutti, 2010). The dialectic resource-document has then to be linked with mathematical objects and their representations: as resources become a document when adding a scheme of utilization, the mathematical object is constructed progressively through experiments on its representations.

## **APPLICATION IN THE CLASSROOM**

### **Documentational genesis and incidents**

In this part I would like to show how this theoretical frameworks can be useful for the analysis of a class situation, both for teachers and researchers: for teachers, because the analysis allows to understand the reasons of students' observed attitudes and the gap between the actual learning and the teaching's intentions; for researchers, because the framework allows a deep understanding of the interactions in a didactic situation, from its design to its implementation and opens new fields of investigation.



The context of the observations that are related above is an experiment in French classrooms in ordinary classrooms where students were equipped the TI-Nspire CAS calculator and software which include multiple representations properties, and allow students to organize files into an arborescence of directories. In this research, I wanted to follow during the year the documentational genesis of teachers and students and it is for this reason that I asked students to send me regularly the content of their calculators and that I meet regularly the teachers to discuss their teaching choices. The class observations were videotaped and transcribed in order to facilitate an analysis of incidents occurring during the course. I complete the information with students' interviews. The above examples were extracted of observations and completed with interviews.

The first context is a particular class of a highschool involved in a global project of using technology in scientific classes. However and even if the teacher, Marie, was experienced and volunteer in this project, she has very low competencies in integrating technology in her classes.

The second class is in the particular context of the EdUatics project<sup>1</sup>: in this project, partners worked in a pair, a University with a school working with a similar pair in an other country. The pair ENS de Lyon-lycée Parc Chabrière was coupled with the pair University of Torino-liceo scientifico Copernico. In the perspective of cross experimentation, situations were designed by the two teams and adapted to the local context. The French teacher, called Jean in the text that follows, started from the original proposal of Italian colleagues to build his own didactic situation, taking into account the French curriculum and his sixteen-seventeen years old students in a scientific class. Before and after the lesson Jean took part in an interview and the lessons were videotaped. The analysis has been constructed from these interviews and on the transcripts of the lessons.

As previously said, I just look at didactic incidents directly correlated to documentational genesis, and particularly incidents created by the gaps between the private, collective and public use of calculators which had been highlighted by the glance at the content of calculators and activity of students achieving a task. It is quite clear that the calculator belongs to the set of resources of the teacher and in this sense that when preparing the lesson, it becomes for the teacher a useful document for the design of the situation; at the same time, it belongs to the students' set of resources and takes place in the didactic situation. And more precisely, the calculator can become a document useful for students in the research time of the given problem if and only if it is a teacher's document and has been included in the situation's design. In other words, in the perspective of integration of the calculator in the mathematics lesson, it remains compulsory to negotiate the didactic contract including the different properties of calculators, as a tool becoming an instrument in specific situations but also as a resource becoming a document available in the set of resources of students and teacher.

In the following example the global consequence of an incident is illustrated; E5 and E6 are two Jean's students who do not want to use the TI-Nspire and prefer their old calculator, in fact a TI-82 (E5) and a Casio Graph 35 (E6). In the interview, they declare:

E6: Well, for the functions, with my old calculator, I type the function, Graph and I have the curve, whereas, with this one, I don't know, you must define it...

E5: There are many steps...

E6: Yes, there is a lot of things to do, just for one result, whereas with my calculator, you type your calculation, you have your result, that's all!

E5: It's faster...

I: And do you remember the moment you said: I don't want this calculator!

E5: Very quickly, yes, we must use menu, then this place, then click everywhere, we had a long course to do a calculation that can be done very quickly with our calculator.

E6: Yes, it was a lesson at the beginning of the year, about functions, we spent two hours with the calculator, it really bugged me, it put me off this calculator.

It is interesting to set this dialogue against the following observation in an observed lesson at the beginning of the year; the teacher is speaking to the whole class whilst students work with their calculator:

*Jean: Then you open the catalog and type the first letter of the command, well for the moment, R and you just have to go down, OK, you see Randint, it's here. Well. (he is doing on the computer whilst speaking)*

*Jean: Well. I have simulated the throw of a dice. The question now is: how are you going to simulate the throw of two dice and how will you obtain the value of the difference of the greater minus the smaller?*

*E1: We have to type a blank.*

*E2: Do you think that?*

*E1: It's six.*

*E2: Yes, randint one six minus randint one six?*

*E1: And, how do you type the absolute value?*

*E1: It doesn't work.*

*E2: (watching to the screen of E1's calculator) Missing?*

*E1: and now it gives six, Ahhh!*

*E2: Ahhhh!*

*E1: It doesn't work!*

*E2: Too many arguments!*

*E1: I can't do that!*

The gap between the talk of the teacher and the students' difficulties is clear. The syntactic incident is caused by incomprehension of the machine's feedback. At first, instead typing `randint(1,6)`, E1 typed `randint 1 6`; the feedback of the machine was `Missing`), but the bracket was not read by the students. E1 tried to type brackets but finally obtained fresh feedback which he cannot interpret. This kind of incident may lead to a rejection of the technology as E5 and E6 said.

Clearly the syntactic incidents are inherent in the use of technology in the class. Taking into account the perturbations, consequences of incidents are essential to limit their effects on the long range:

- from the point of view of professional development by increasing the response repertoire [Clark-Wilson, 2010, p.185]
- from the point of view of students by increasing the registers of representation of studied mathematical objects.

In other circumstances, incidents come from a different documental genesis and the position of the technology in the different level of mediation. Marie, in a class observation disconnected the experiment on the calculator and the mathematical knowledge which provokes a didactic incidents:

E1: Do we save our work?

Marie: You save if you want, but tomorrow we do the theoretical part.

She doesn't see the calculator as part of the student's set of resources and doesn't promote documental genesis. As a consequence, students investigate the memorization property of the technology privately which, later, provokes a deep perturbation in the integration of the calculator in the maths course, as expressed by Marie in an interview:

Sometimes it's difficult because they (the students) do not know their lessons. They do not learn because they believe that they have everything inside their calculators.

The same resource became a different document for students and teacher. A great lack of understanding prevents the use of the calculator in a collective level of mediation.

In the contrary the knowledge construction is reinforced when the collective level of mediation is taken into account:

Sometimes, I don't remember how to do, it happened once, I didn't remember the names, you know, it was, a x plus b plus c, or something like that, I was not able to go on; then I typed on my calculator. I didn't explain the result, but I had one and I was able to answer the next questions. Otherwise, I would have had nothing right in the exercise.

The examples discussed in the previous paragraph show that the documentary activity of students in relation to the properties of memorization, organization of ideas, creativity and communication contribute to the dynamic of knowledge construction. The instrumental approach focuses on the transformation of an artifact into an instrument, whereas the documentational approach allows to consider the calculator through specific documentary properties. Observations show that the role given to the calculator in the private domain by students and teachers is not necessarily shared. Therefore, documentaries genesis become distinct, unrelated and sometimes divergent which causes didactic incidents generating disturbances.

An other aspect that is important to notice is the school characteristic of the calculator which builds a barrier between the usual digital tools that students use in their everyday life in a collective level of mediation and the school tools that have to be used in the specific school community or in a private level of mediation. Different interviews show that students don't want to mixed the two worlds.

### **Experiments in mathematics**

The first example leans on a class (Jean) observation and more particularly on a group of four students searching the problem: “what are the natural numbers which are sum of at least two consecutive positive integers?” Students experiments with integers which are naturalized objects. They found that 2, 4, 8 are not reached and try to generalize:

E2 (using her calculator): Two to the power six is 64, two to the power eight is 256, OK but after?

E1: 2048, euh... 4096

E2: How do you compute so quickly without calculators?

E3: just times 2

E1: Ah yes, not stupid,... go on!

In this small extract the status of objects is different for E1 and her mate. For E1, the nth power of two is defined as the multiplication of n times 2 whereas E2 and E3 refer to the recursive definition of the power of a number. Students challenge their knowledge through the experiment with numbers through two different definitions taking place in two different representations of the same object.

The second example is taken also from the Jeann's classroom. The problem that students are confronted with comes from the Edumatics project and is called the “walking problem”:

Pjotr moves at constant speed along a square ABCD with center O (intersection of the two diagonals) and given side L, starting from vertex point A. Pjotr wants to describe how his distance from the center O of the square changes while he is moving along the square. How can you help him/her?

After having experimented on a dynamic geometry software, a student suggests that the arch of curve when Pjotr moves along a side of the square is a parabola. The teacher returns the question to the class and ask to work by group for five minutes. The above dialog takes place after these five minutes:

E: I find the result.

T: You find what?

E: That's minus the absolute value of sine of x plus the square of the length...

T: Yes, and how do you find this result?

E: Ben, with my calculator...

T: Yes, but how?

E: modulo and sine that's similar and there is a cusp then I choose the absolute value and I adjust...

T: and it fits?

E: Yes perfectly!

The questioning on the result is here fundamental for the understanding of the mathematical fact. Indeed the proposed solution is not acceptable but the refutation of the solution depends on a back and forth to the theory. The graphical possibilities of the calculator are not sufficient to discredit the solution as the end of the dialog shows us. What is here important is not the experiment in itself, but the reflexion about the experiment that achieve the complete experimental approach of mathematics.

## CONCLUSION

The observation of a changing world where “digital natives” have a different perception of things is often made but the question of how to change teaching and learning in order to adapt schools and society is an open question. However, looking forward to the documentational genesis of actors and crossing with incidents analysis brings a powerful tool to understand why Hop-o'-My-Thumb find or not his path back home. Two opposite phenomena were observed:

- incident causing a disturbance that comes in contradiction with the teacher's intentions,
- incident triggering a disturbance favoring a devolution of the situation and finally, the learning process.

Finally, didactic incidents may be considered in a broader perspective and the contribution of documentational genesis led to consider computing environments as part of the documentary system of teachers and students.

In a theoretical point of view, this research allows to build a methodology allowing to cross the didactical analysis with the documentary properties; the incident analysis can thus be seen as a methodology for analyzing a regular classroom situation, both for the researcher who can build a grid highlighting individual trajectories of students in the class but also for teachers who will have at their disposal tools for the understanding of phenomena observed in the classroom, especially when the situation seems out of control.

## NOTES

1. 0324-UK-2009-COMENIUS-CMP; European development for the use of mathematics technology in classrooms, Lifelong Learning Program, EU. <http://www.edumatics.eu>

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# CAS USE IN SECONDARY SCHOOL MATHEMATICS - TEACHING STYLE AND MATHEMATICAL ACHIEVEMENT

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*Report on the basic ideas and elements of a teaching concept for the CAS-use starting in grade 7 in the project CALiMERO (2005-2010) and results of an evaluation study on this project. On the basis of aggregated data from lesson journals correlations between elements of the teaching concept achieved in the different learning groups and performance evaluated in the tests are analyzed.*

## TARGETS OF THE PROJECT CALiMERO

CALiMERO is the name of a long term project on the use of computer algebra software (CAS) calculators in German secondary school mathematics. It was supported by the Ministry of Education of Lower Saxony and Texas Instruments. The project started in 2005 in 29 classes in the 7th grade and ended in 2010 when students finished secondary school.

The objectives of the project were

- to investigate the use of a handheld CAS system in German secondary maths education from grade 7 to grade 10,
- with the aim of implementing and evaluating a teaching concept that focuses on sustainable learning of mathematical competencies and basic knowledge,
- while the evaluation concentrates on the pupils' mathematical performance, teaching methods, and the pupils' and teachers' views on mathematics, teaching, and technology.

With the use of CAS in the classroom - the students worked with the CAS-Calculator TI-89 or TI-Voyage – it was assumed that it would ensure forms of inquiry-based learning (see e.g. Kahn & O'Rourke, 2005) and a multirepresentational approach to mathematical concepts, especially functions. Thus, the object of this investigation was to examine the effects of CAS on mathematical achievement in secondary mathematical education as part of a consistent teaching concept for sustainable learning.

## THE TEACHING CONCEPT OF THE PROJECT CALiMERO

The need to provide and systematically implement a suitable teaching concept for competence-oriented maths lessons with technology use is the consequence of research results on e-learning obtained in the past years: “the use of digital media does not automatically lead to better results than conventional educational concepts. On the contrary: their unreflected use is often leading to low acceptance by the learners, poor learning success and efficiency.“ (Kerres 2001, p.85). A suitable didactical-methodical framework had to be found which would take up both former teaching culture and the experience of maths teachers by making use of the potential of the applied technology. In the teaching methodology there is broad consensus that handheld calculators (GC, CAS) can contribute to the development of mathematical competencies as follows:

- **Reduction** of schematic processes (release of cognitive burden)
- Supporting the **discovery** of mathematical issues
- Supporting **individual** preferences and approach

- Promoting the **comprehension** of mathematical correlations
- Supporting the need for security by providing **control possibilities**.

In Germany maths lessons are concentrating on the working with tasks. In a comprehensive analysis of maths tasks as part of the CIMS project in Hamburg regarding the CAS-use in upper secondary school the following tasks formats could be identified with respect to their “calculator potential“:

### **How can a handheld calculator (HC) be used in a task?**

- 0 - HC not allowed or not reasonable
- 1 - HC assumes control function for simple calculations or justifications
- 2 - HC-use reduces formal calculation or construction effort; task is still solvable without HC
- 3 - Calculator supports experimental situations, checking of assumptions etc.
- 4 - the task is not (effectively) solvable without HC any more, due to the quantity of data or the complexity of modelling
- 5 - by using the HC new mathematical correlations are explored.

This typology of calculation potential can be adopted to analyze already existing tasks (e.g. in a schoolbook or in a test) or to specifically design new tasks. For the construction of tasks intended for a wide-ranging cognitive activation of the students a further typification was made.

Tasks are understood as a request to start learning activities, with the three components initial situation, possible transformations and final situation. These three components can either be known (✓) or unknown (-) in their details. This allows a classification of tasks in eight target types and a learning environment for sustained maths learning should include these tasks formats:

- Solved problems (as examples) (✓✓✓)
- Basic tasks (✓✓-)
- Modifications of basic tasks (as reverse tasks (- ✓✓), open respectively approach or/and result, admitting different approaches, evaluating and ensuring comprehension
- Extensions of basic tasks (including contents from former subjects, ensuring availability of basic knowledge, e.g. (✓- -), (- ✓-) and (✓✓✓)
- Complex tasks with standard or problem character (✓- -), (✓- ✓), (- - ✓) (optional choice)
- Problem situations ((-) - -) (open questions, applicable for internal differentiation, with optional contexts and difficulty).

Background for this sample of task-formats is a concept of exercise (with empirical evidence) based on the activity theory with three phases: First Exercises for a basic understanding, Varied (intelligent) Exercises and Complex Exercises and Applications (cf. Bruder, 2008).

The task classification is a very good method for the teacher to check the variety or top-heaviness of one’s own teaching with respect to the chosen tasks. Changing perspectives on mathematical learning contents in different contexts, as described in the eight task types allows and requires exactly those cognitive activities that are necessary for an understanding and cross linked-learning. On the basis of many years experience in teacher training and teacher further training we can say that an orientation based on these eight structure types leads to a greater methodical variety in the planning of teaching units, and offers students the possibility of understanding the learning content far beyond formal reproduction and application. These eight structure types may serve as a guideline

and construction heuristics for the design of units for self-directed learning, learning by stages, expert puzzles or for the provision of learning conditions to stimulate cognitive activation.

In the project CALiMERO learning materials were developed in regular meetings together with the participating teachers. These learning materials were intended for the multiple use of TC to support the learning process by taking into account the task variety described above. This development of tasks was structured by so-called typical teaching situations, i.e. phases with a dominating didactic function in the lessons (cf. Steinhöfel, Reichold & Frenzel, 1985) as follows:

- securing of the initial level for the progress of understanding and exercise,
- introduction of new learning contents,
- design of exercise processes and
- systematization of mathematical learning contents and their applications.

The concept developed on the basis of these teaching situations structures the teaching unit by offering suitable learning opportunities in specific moments for the elaboration, reflection, exercise and repetition of current and previous learning contents. Key issues of the concept and the technology-based teaching were to enhance the self-responsibility of students as well as to provide detailed and conceptual knowledge in the field of functions and algebra. Certain methodical elements were explicitly integrated into this concept, for example the learning journal, the check list for diagnostic purposes and regular mental maths to keep available basic knowledge and skills without technology support.

#### Learning journal

The learning journal is a learning tool to determine the current level of understanding of new learning contents (diagnosis) and, simultaneously, support the accessible anchoring of new learning contents (enhancement). Usually the learning journal is used in connection with a new teaching topic after the first lessons. Students are working on easy questions which are reflecting new contents learned in the past lessons. These questions are typically asking to describe the example of introduction, application context or typical mistakes, they also include a *basic and a reverse task*.

#### Check list

By means of a check list the students are assessing their own basic competencies with respect to the current teaching content. It is helpful to go through the check list at the end of a teaching unit and before a test. The list contains several statements, each beginning with „I can...“, which refer to elementary knowledge components and are generally substantiated by basic tasks.

#### Regular mental maths

This ritual learning instrument in the form of ten basic and reverse tasks is intended to keep alive mathematical knowledge from previous learning contents which were already understood and which are elementary for the further learning. Tasks for mental exercise should be done ‘in the head’ and aim at basic understanding. They only require 10 minutes and should be practised at least once every two weeks per teaching unit. Justification of this methodical element cf. Bruder & Pinkernell (2010).

One of the project's objectives was the implementation of specific methods to support basic skills. This seems vital for the acceptance of technology in school as we see a growing public concern about an alarming lack of basic mathematical skills with first year university students, which is often ascribed to the use of CAS or other handhelds in school. Project and control groups show a parallel increase in the test performance in grades 8, 9 and 10, thus suggesting that an extensive use



of CAS handhelds in class, accompanied by specific supporting measures, does not necessarily result in lesser basic mathematical skills (Pinkernell, Ingelmann & Bruder, 2009).

### **DEVELOPMENT OF THE STUDENT PERFORMANCE AFTER THE IMPLEMENTATION OF THE CONCEPT BY THE TEACHERS**

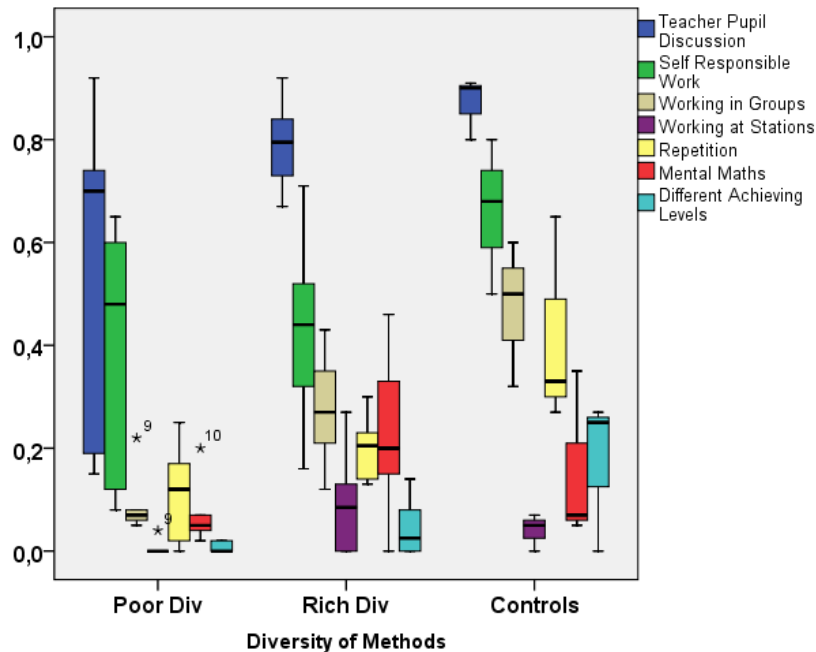
A math performance test, adapted every year, was used as evaluating instrument in the classes 7 to 10, in connection with a student interview (from class 8 with technical aid), and a yearly adapted basic knowledge test in the classes 8 to 10 (without technical aid). This range of test methods was designed for a comparative study of control groups and did not cover specific knowledge and skills especially fostered by the use of CAS in the present teaching concept. First of all it can be stated that there was a generally high and stable acceptance of HC use by teachers and students. The students who participated in the project considered the HC use as positive. They are of the opinion that the HC helps to learn mathematics. In this regard there were no significant differences with the control groups who had worked with a Grafic calculator (GC). After some time, the pleasure of working with the HC is generally decreasing, stabilizing on a neutral position. This effect is still stronger in the control group. It is reported that growing confidence in own capabilities and better results in the tests are leading to less difficulties and more pleasure to handle the HC. This is understandable because the three aspects support each other. While self-confidence and maths achievement are improving, the consideration of the HC as an important tool to learn mathematics is decreasing. In the light of the positive evaluation of the HC during the years of the project it can be resumed that this evaluation is rather made without reference to self-perception and level of performance. In a maths test for all classes the evaluation study was able to prove that the learning groups of the project CALiMERO had known a clear performance development over the school year, there was positive development also in basic knowledge without technical aid. Assumptions that deficits of maths basic knowledge, often complained by secondary education institutions, are the result of HC use were not confirmed in this project. The learning progress of the control group who had worked with GC surpassed the usual knowledge increase within one school year. The experimental design of the study could not show particular effects of the CAS compared with the GC in lower secondary school. There were no negative effects either.

Of particular interest is how the teachers implemented the developed teaching concept and the effects on the achievement of the students. As it was not possible to videograph the lessons over many years a learning journal was applied. The students registered methods recognized in the lessons and noted also the use of HC. From the point of view of the students the teaching concept is described as a variety of methods. However, immediate effects of the teaching concept or its theoretical roots, for example the task design or diagnostic aspects, cannot be recorded empirically, neither with lesson journals nor with videos. A suitable index had to be found to make learning groups comparable. According to Helmke (2009) there is no metric for the determination of an optimum of instructional variation. It is also known that variety of methods in itself does not guarantee effective lessons.

The measure for our project developed by Pinkernell is based on the Shannon index of biodiversity, which we modified for our purposes. Unlike the original biodiversity index the frequent occurrence of a method as well as its complete absence would lead to devaluation. Top assessment would be attributed to lessons where every specified method would be used in around 40% of the recorded units. Of course a differentiating evaluation of teaching lessons is not possible this way. The calculated measure is only a criterion to classify evaluated classes in “method-rich” and “method-poor” learning groups.

An index value was calculated for every project class with sufficient “log data”, which means that a teaching issue had been continuously documented. The calculation of the value included information of the lesson journals concerning the frequency of the following teaching methods: Teacher student discussion, Self-responsible work, Working in groups, Working at stations, Repetition of earlier learning contents, Mental maths (without calculator), Different tasks for different achieving levels (optional).

Afterwards the project classes were divided into two groups: classes showing „rich“ method diversity according to the index (N=10) and classes which must be characterized as rather ”method-poor“(N=5). A frequency profile created for both project groups and for the control groups (N=3) regarding the determined methods is revealed in the following picture (Fig.1):



**Fig. 1: Frequency profiles of methods used in two parameters (“poor“ and “rich“) in the project groups and control groups**

The index identifies those classes as method-rich where a teacher-student discussion takes place in almost every lesson. This seems to be obvious, however the profile of diversity-poor classes shows that this is not automatically the case. Moreover the students in method-rich classes had the opportunity to work independently in almost half of the lessons. Working in groups is practised clearly more often than in diversity-poor classes, mental maths as frequently as recommended in the teaching concept. Diversity-rich classes are working at stations in approximately every tenth lesson whereas diversity-poor classes did not practise this teaching method. The repetition of earlier learning contents was exercised in 20% of the lessons.

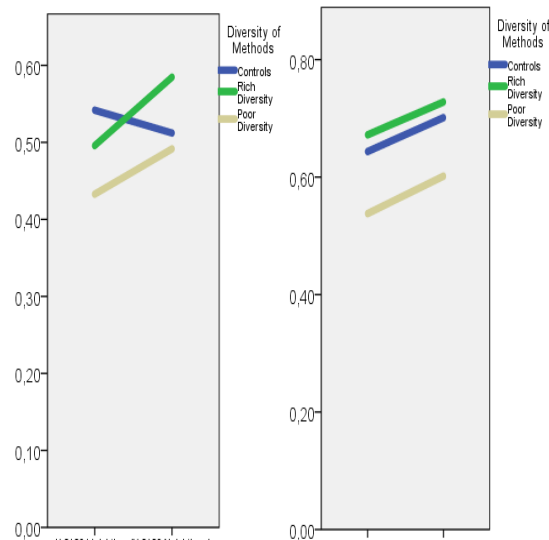
It is rather eye-catching that the method profile of diversity-rich project classes is closer to the profile of referenced control classes with above-average performance than to that of diversity-poor classes. In our understanding this shows that the index really helps to differentiate project classes depending on the diversity of teaching methods used.

As can be seen in figure 2, diversity-rich learning groups are reaching a higher level of performance than diversity-poor groups. A variance analysis (ANOVA) with repeated measures confirms the significance of small differences between the test performance of diversity-poor and diversity-rich project groups, as well with respect to group membership (ANOVA(group)  $F(1,147)=16.394$ ,

$p < 0.01$ ,  $\eta^2 = 0.1$ ) as regarding the interaction of repeated measures and group membership (ANOVA(time\*group)  $F(3,441)=4.826$ ,  $p < 0.01$ ,  $\eta^2 = 0.022$ ).

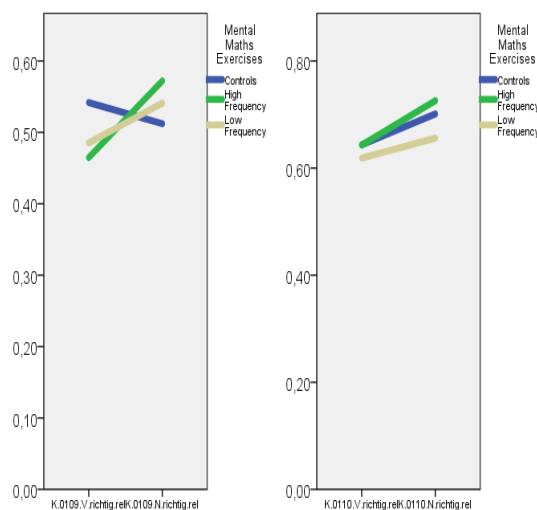
The same applies to the correlation between diversity of methods and achievement in general maths tests (ANOVA(group)  $F(1,145)=8.152$ ,  $p < 0.01$ ,  $\eta^2 = 0.053$  respectively ANOVA(time\*group)  $F(3,431)=3.771$ ,  $p=0.011$ ,  $\eta^2 = 0.025$ ).

Thus a correlation between teaching methodology and the achievement of the students can be assumed. It seems that the described diversity of methods as part of the developed teaching concept has positive, however small effects on the level of performance.



**Fig.2: Comparison between method-poor and method-rich project groups and control groups in class 9 (on the left) and class 10 (on the right) with respect to the performance achieved in the pre and post test “basic knowledge without calculator“**

A positive correlation could also be ascertained for the frequency of TC use and maths performance, however, again, with only small effects. Regular mental maths is one of the methodical elements of the teaching concept that provided the didactic framework for the elaboration of learning materials for the project groups.



**Fig. 13: Comparison between control group students and project students who exercised mental maths regularly or irregularly, in class 9 (on the left) and class 10 (on the right) with respect to the performance achieved in the pre and post test “basic knowledge without calculator“**

It was recommended to practise 10 minutes of mental maths every second week to keep alive mathematical basic knowledge which is not required or treated in the current lessons. By means of the lesson journals it was possible to see how often mental maths practice had been done during the period of lesson recording. The project groups were classified in terms of whether they met the recommended frequency of about 20% of the lessons (for three maths lessons per week in class 9) or not. This was the case for N=68 students, but not for N=81. Figure 3 shows that after starting with a similar level at the beginning of class 9 those students knew higher performance increase who had mental maths documented in their classes, in line with the recommended frequency. In the course of class 10 there is a growing achievement gap between both groups. Again, ANOVA is confirming significantly better results in the form of a small interaction effect between group membership and repeated measures (ANOVA(time\*group)  $F(3,441)=4.826$ ,  $p<0.01$ ,  $\eta^2 = 0.032$ ).

In contrast to the two previous investigations on teaching methods, the two groups of project students are starting from the same performance level in class 9. It can be supposed that in this case the effect of improved achievement in the basic knowledge test without calculator support will arise only after regular mental maths practice. This presumption is confirmed by the fact that such a growing performance gap in a maths test where technical aid is admitted cannot be observed for neither of the two project group types. Hence, the practice of regular mental maths seems to systematically support the availability of basic knowledge without auxiliary tools.

## SUMMARY

On the basis of quantitative data from lesson journals the project classes were classified with respect to teaching methods, frequency of use of the TC and frequency of use of mental maths. It was possible to show that a basically concept-conforming diversity of teaching methods, the almost permanent use of the TC for issues with technological potential and the frequent practice of mental maths came along with a slightly increased level of performance of the project students. As for mental maths the improved level of performance can be attributed to their frequent practice as recommended by the teaching concept.

Our results are particularly pointing out that apart from the availability of technology and suitable teaching and learning materials the training of didactical-methodical competencies of the teachers in this field is crucial for the efficient technology-based teaching.

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# EVALUATING THE IMPACT OF GRAPHING SOFTWARE ON LEARNING AND STUDENTS MATHEMATICAL THINKING

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*The main aim of this action research project was to evaluate the impact of investigative tasks involving the use of the dynamic graphing software Autograph v3.2 upon students' learning and mathematical thinking. A range of research tools and instruments were created and used to generate and analyse both quantitative and qualitative data. The results show a wide range of impact upon students' mathematical thinking and show which areas of mathematical thinking are most impacted upon. Viewing the results through the themes of other research into the use of graphing software and theories of algebraic thinking allows for critical reflection upon the authors current and future practice.*

## LITERATURE REVIEW

In studies focussing on the impact of graphing technology upon teaching and learning within the classroom, researchers have sought to highlight the key themes or patterns in the successful use of this technology across a range of different classrooms, teachers and classes. In their classroom-based study of the use of graphing calculators, Doerr and Zangor (2000), found “five patterns and modes of graphing calculator tool use” - as a computational tool, a transformational tool, a data collection and analysis tool, a visualising tool, and a checking tool. In a study “analysing the pedagogical ideas underpinning teachers accounts of the successful use of computer-based tools and resources to support the teaching and learning of mathematics”, Ruthven and Hennessy (2002) developed a practitioner model which identified ten “operational themes”. They refined this model in 2003, focussing on six broader themes which each “point[s] to important ways in which teachers considered that the use of ICT tools and resources could support teaching and learning” (2003, p20). These six themes are:

- Effecting working practices and improving production
- Supporting processes of checking, trialling and refinement
- Overcoming pupil difficulties and building assurance
- Focussing on over-arching issues and accentuating important features
- Enhancing the variety and appeal of classroom activity
- Fostering pupil independence and peer exchange

In 2009, Ruthven and Hennessy used these themes as a framework for their study of the contribution of graphing software to teaching and learning. They found evidence of teachers viewing graphing software as contributing positively to each of the six themes of supporting teaching and learning.

Another key theme of studies is the potential epistemic effects of the use of graphing software. Fey's (1989) analysis of then recent progress in the application of “electronic information technology to creation [sic] of new environments for intellectual work in mathematics” (p237). Whilst finding no “hard research data” to support this at the time, Fey reported on “popular and successful” graphing activities such as displaying series of graphs for related function rules to reveal the patterns associated with various rule types and the effects of parameters in each type. (p249). Ruthven and Hennessy (2009) state that mathematics educators have seen mathematical graphing software as

having “an epistemic, concept-building function, through supporting exploration of symbolic-graphic relationship”.

Another key theme found in research into the use of graphing technology in mathematics classrooms is the impact upon the roles of the teacher and of the students within classrooms employing this technology. Fey (1989) found a “notable change in the roles and interactions of teachers and students” (p250) in classrooms where computer-generated graphs were the focus of discussions. He argues that “The teacher role shifts from demonstration of "how to" produce a graph to explanations and questions of "what the graph is saying" about an algebraic expression or a situation it represents.”, and that “Student tasks shift from plotting of points and drawing curves to writing explanations of key graph points or global features” (p250). Goodwin and Sutherland (2004) highlight the interdependence of the software tools, the classroom environment, the teacher, the individual students and whole-class interactions. They highlight the vital role of the teacher in drawing out and building upon the individual learning of students and conclude that “The only way forward is for teachers to interact with students to develop an emergent and collective mathematical knowledge community” (p150)

These themes provide a framework within which the research question of “how does the use of investigative tasks using graphing software impact upon the learning of mathematics and upon students mathematical thinking?” can be addressed.

## **METHODOLOGY**

### **The Learning Goal**

Goldenberg (2000) argues that “only when one is clear about the learning goal of the problem can one make a clear decision about what technology to use”. The learning goal underpinning my decision to use Autograph was the development of students’ mathematical thinking, and more specifically, algebraic thinking. In order to evaluate the impact of the tasks using Autograph, I had to be very clear about what I was looking for. My research has been grounded in the theories of algebraic thinking proposed by Mason (1996) and Kieran (2004). Mason (1996) argues that “Detecting sameness and difference, making distinctions, repeating and ordering, classifying and labelling are expressions of the urge to prepare for the future, to minimise demands on attention. They are the basis for what I call algebraic thinking and the root of what we have come to call algebra” (p83) Kieran (2004) categorises school algebra according to activities that the students are engaged in. She describes meta-level mathematical activities as ones where “algebra is used as a tool but which are not exclusive to algebra”. These activities include “problems solving, modelling, noticing structure, studying change, generalising, justifying, proving, and predicting” (p142). The over-arching learning goal was for students to be able to make generalisations about the relationship between the symbolic form of linear equations and their graphical forms.

### **The Technology**

Graphing software was selected as an appropriate ICT resource for a number of key reasons underpinned by relevant research and theories. The theme of *Focussing on overarching issues and accentuating important features* from Ruthven and Hennessy's practitioner model (2002,2003) resonates most with the focus of my research. In the conclusion of their research, Ruthven and Hennessy (2009) highlight how graphing software can help achieve this “through helping to bring out the effects of altering particular coefficients or parameters in an equation on the properties of its graph, and through facilitating comparison of gradients and examination of limiting trends” (p293). Another reason for selecting graphing software was my consideration of Goldenberg's (2000) *Answer vs. Analysis* principal. My intention was for the software to provide the ‘answer’ by

producing the correct graph for given equations, allowing students attention to be drawn to the 'analysis' of the similarities and differences between graphs of different linear functions.

### The Tasks

The early phase of using mathematical software is called the instrumentation phase by Guin and Trouche (1999) who describe this process as “transforming any tool into a mathematical instrument” (p195). In this phase the focus is on becoming familiar with the functionality of the software. French (2004) argues that “students need to know three things in order to begin using a graph plotter productively:

- How to enter and plot a function
- How to modify the scales on the axes
- How to delete graphs that are incorrect or no longer needed

In addition to this, I would argue that students need to know how to enter and vary a constant in an equation (in the case of Autograph, how to use the 'constant controller'). In order to allow students time to become used to the software, I used a series of short tasks which involved step-by-step instructions. Through the use of the interactive whiteboard and discussions with pairs of students I was able to ensure that all members of the class had developed these four skills. My aim here was to ensure that students had undergone enough instrumentation to allow their time to be spent focussing on mathematics rather than the software.

The phase that follows instrumentation is instrumentalisation – it is in this phase that students can use the software as a tool to focus on the mathematics. In creating the main two sets of tasks, I again considered Goldenberg's (2000) principals of deciding how to use the software. In terms of *high order or low order thinking skills*, the tasks were designed to focus on high order thinking skills such as hypothesising, testing and generalising. According to Goldenberg (2000), creating appropriate tasks is crucial - “It is the problems that are posed, not the technology with which they are attacked, that make all the difference.” With the previously mentioned benefits of graph plotters in mind, two sets of tasks were created.

The first set of tasks focussed on supporting students in being able to articulate the general properties of linear graphs of the form  $y = mx + c$ . Six tasks were created, each at three different levels (see appendix 3). The six tasks (1,2,3,4,5, and 6) involved students investigating the impact of changing the parameters  $m$  and  $c$ , in the general equation, on the key features of their graphical form – the gradient, the  $y$ -intercept and the  $x$ -intercept. The three levels of tasks (a, b, and c) differed in terms of the level of structure and instruction. The lower (a) level tasks gave students explicit instructions of which graphs to plot and prompts to consider key features. The middle level tasks (b) still guided students, but were less structured. The higher level tasks (c) required students to demonstrate their understanding of the general form of linear graphs by plotting particular graphs which met particular criteria or could be manipulated to demonstrate this understanding. Students were put into pairs based on their self-assessment of their current understanding of linear graphs and were allowed to choose whether they began working on tasks a, b, or c. Students were given the option to move from one group of tasks to another if they found them too easy or too challenging.

The second set of tasks were used in the following lesson and focussed on supporting students in generalising the properties of linear graphs of the form  $ax+by=c$ . Based on feedback from students at the end of the first lesson, the second set of tasks were created at just two levels – one more structured (s) and one much more free and investigative in nature (f). These tasks were presented to students in one pack so that they could choose to work on some tasks at the structured level and others at the investigative level. Additional challenge cards were provided to those students who



were able to articulate the key features of the general form  $ax+by=c$ . These challenges were similar in nature to the (c) level tasks from the previous lesson in that they required students to create autograph files that demonstrated their understanding.

### **Tools and procedures**

This research is based upon the experiences of a class of 32, year 10 students whose end of year 9 attainment ranged from low level 7's to mid-level 8's. Every member of the class took part in the research by working on a range of tasks using Autograph to investigate the general properties of the graphs of linear functions given in the forms  $y = mx + c$ , and,  $ax + by = c$ . Autograph was selected as a specific tool as I have a solid understanding of the software and it is available to all students within the school. In total, two 100 minute lessons formed the basis of this research. The research took place in the regularly timetabled classrooms using laptop computers which have access to Autograph 3.2. Students worked on two sets of tasks in pairs. Students were already familiar with linear graphs of the form  $y = mx + c$  having studied them in both year 8 and year 9.

With the aim of capturing quantitative data from each student, I created a questionnaire (see appendix 5) which asked students to rate the impact of autograph on their learning and mathematical thinking in eight particular areas. A scale of 1 (low) to 10 (high) was used to rate the impact and allow for statistical analysis of the data generated. The questionnaires were completed by every student after each of the two main lessons outlined above. The eight areas on the questionnaire were based on my synthesis of Mason (1996) and Kieran's (2004) writing about thinking algebraically. The eight key aspects relating to algebraic thinking that I selected for the questionnaire were:

- Linking symbolic and graphical forms
- Manipulating the graphs and
- Getting a feel for the effects of changing part of an equation
- Hypothesising about the behaviour of linear graphs
- Testing whether a hypothesis is true
- Getting-a-sense-of the similarities and differences between different linear graphs
- Articulating patterns, similarities and differences observed
- Generalising about the effects of a graph of changing a constant

Based on the analysis of students mean scores outlined above, key students were selected for interview. Two students from the top quintile, two from the middle quintile and two from the bottom quintile were selected. The aim main objective of the interviews was to inform my future plans to use Autograph with this, and other, classes. In order to inform such decisions the interviews set out to try to identify similarities and differences in the experiences of these students and to gain an understanding of why they rated the impact of the software as they did. All students were asked the same questions, with the wording of one question slightly altered depending on which quintile they were in. The questions focussed on four main areas: perceived differences between the software based activities and the use of pen and paper only, reasoning for student's particular ratings, thoughts about how ratings might change, and what would cause ratings to change over time. Further unplanned questions and prompts were used depending upon the responses to each of the main questions. The interviews were recorded using a digital audio capture device so that responses could be analysed at a later stage via playback – both student-by-student and question-by-question.

## RESULTS

The whole class average (mean) and spread (standard deviation) for the two sets of tasks are shown in the table below.

|                          |      | make links between symbolic and graphical forms | manipulate | get a feel for the effects of changing part of the equation | come up with a hypothesis | test whether something you suspected to be true actually was? | get a sense of the similarities and differences | articulate or explain | make generalisations | Average of 8 categories |
|--------------------------|------|---|------------|---|---------------------------|---|---|-----------------------|----------------------|-------------------------|
| Task set 1:<br>$y=mx+c$  | Mean | 7.5   | 8.0        | 7.9   | 7.0                       | 8.0   | 7.4   | 7.0                   | 7.3                  | 7.5                     |
|                          | SD   | 1.2   | 1.1        | 1.3   | 1.4                       | 1.7   | 1.3   | 1.6                   | 1.8                  |                         |
| Task set 2:<br>$ax+by=c$ | Mean | 8.1   | 8.5        | 8.2   | 7.7                       | 8.2   | 7.9   | 7.5                   | 7.7                  | 8.0                     |
|                          | SD   | 1.3   | 1.1        | 1.1   | 1.2                       | 1.6   | 1.2   | 1.4                   | 1.8                  |                         |

**Table 1: average score for each category across the two tasks**

The average ratings based on students experiences of the first task ( $y=mx+c$ ) ranged from 7.0 (hypothesising and articulating) to 8.0 (manipulating and testing). The average ratings increased in all 8 categories based on students experiences of the second set of tasks ( $ax+by=c$ ) with the lowest average rating being 7.5 (articulating) and the highest being 8.5 (manipulating). Looking at the whole class average across all eight categories shows an increase from 7.5 for the first set of tasks to 8.0 for the second set of tasks. The standard deviation fell slightly in six of the eight categories from the first set of tasks to the second.

### Students perceived impact on mathematical thinking

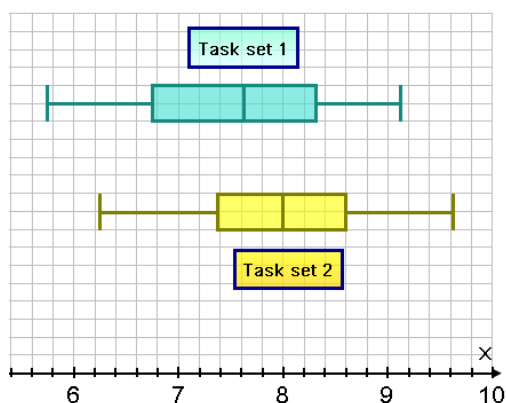
Using the whole class average for each of the eight categories allows them to be ranked in order of the students' perceived impact upon their learning. The table below shows the rankings for the two sets of activities.

| Rank | Task set 1: $y=mx+c$                                  | Task set 2: $ax+by=c$                                 |
|------|---|---|
| 1st  | Manipulate (8.0)                                      | Manipulate (8.5)                                      |
| 2nd  | Test (8.0)  | Test (8.2)  |
| 3rd  | get a feel (7.9)                                      | get a feel (8.2)                                      |
| 4th  | make links (7.5)                                      | make links (8.1)                                      |
| 5th  | get a sense of the similarities and differences (7.4) | get a sense of the similarities and differences (7.9) |
| 6th  | make generalisations (7.3)                            | make generalisations (7.7)                            |
| 7th  | articulate or explain (7.0)                           | Hypothesis (7.7)                                      |
| 8th  | Hypothesis (7.0)                                      | articulate or explain (7.5)                           |

**Table 2: Ranking of average score for each category across the two tasks**

There is a clear consistency across the two sets of tasks with the ranking of the first six categories being the same across the two tasks.

## Comparing the distribution of overall scores for task 1 and task 2



**Figure 1: box plots comparing students overall scores for the two tasks**

The most striking feature of these box plots in relation to this research is the similarity of the range between the two sets of tasks (3.4 for both tasks). This shows a continued significant difference in the experiences of the two students who rated the impact of the autograph tasks highest and lowest. The inter-quartile range for the first and second set of tasks is 1.5 and 1.2 respectively. This again highlights a significant difference in the experiences of different students. These differences formed the basis of the selection of students for interviews and impacted upon the questions which they were asked.

The data generated by the interviews with the six students is very rich and deep. Whilst the comments of every student interviewed have had an impact upon my plans for future teaching and learning I have tried here to pick out common themes exposed by all students and to try and identify differences between the experiences of the students in the different quintiles. When considering the differences between using software and pen and paper to look at general features of linear functions, five of the six students commented on the time benefits associated with using Autograph. Another key theme across the range of students was the ability to see patterns as a result of using the software. The two students from the middle quintile and one student from the lower quintile commented on the fact that there was no doubt that the graphs produced were the correct graphs for their associated equations. When considering whether their ratings would change as they used the software all six students responded affirmatively – five of them stating that they expected this to happen as they became more confident at using the software and one student (from the middle quintile) saying that her ratings would increase if the tasks themselves were to change. The two students from the bottom quintile offered very different reasons for their low ratings of the software's impact upon their learning. One of them spoke positively about their experiences of the software but highlighted difficulties in interpreting the phrasing of the questions. The other student highlighted the difference made by the change of partner between the two tasks and explained that they hadn't been able to see the links between the symbolic and graphical forms. They stated that the software couldn't explain the links and therefore hadn't helped them.

## DISCUSSION

The results outlined in the previous section can be reflected on by viewing them within the context of the themes of studies highlighted in the literature review and the theories of algebraic thinking described in the methodology section. The increase in the whole-class average ratings of all eight aspects relating to algebraic thinking shown by analysis of the questionnaire data highlights the need

to consider the processes of instrumentation and instrumentalisation when planning to use Autograph. The qualitative data generated by the interviews shows the potential for the software to increasingly impact upon students learning and highlights the needs of the students to increase their software based skills in order to allow this learning to take place. An example of this was highlighted by two of the students interviewed when they explained that they had been unable to make generalisations about the x-intercept of the graphs as they had not known how to make the software show them the intercept.

All of the students who were interviewed referred to the time saving aspect of the use of Autograph compared to pen and paper - highlighting the fact that the software quickly produced the correct graph for them with no need to work it out and that it was easier to see patterns as a result. These observations fit within Doerr and Zangor's (2000) pattern or mode of the use of graphical calculators as a *computational tool* and similarly within Ruthven and Hennessy's (2009) theme of *Effecting working processes and improving production*. In terms Doerr and Zangor's (2000) mode of a *visualising tool*, analysis of the questionnaire data shows relatively high ratings for the three associated aspects of algebraic thinking - linking symbolic and graphical forms, getting a feel for the effects of changing part of the equations, and getting a sense of the similarities and differences between linear graphs. These three aspects of algebraic thinking are also closely linked to Ruthven and Hennessy's (2009) theme of *Focussing on overarching issues and accentuating important features*. In terms of Doerr and Zangor's (2000) mode of a *checking tool*, and Ruthven and Hennessy's theme of *Supporting processes of checking, trialling and refinement*, analysis of the questionnaire data shows high ratings for testing whether a hypothesis is true, but relatively low ratings for hypothesising about the behaviour of linear graphs. One of the students also referred to these processes in the interviews saying that they could 'guess' what a graph would look like before using the software and thereby "test yourself".

Both Ruthven and Hennessy (2009) and Doerr and Zangor (2004) stress the interdependence of key factors such as the software tool, the classroom environment, the tasks, the teacher, the individual students and the interactions between them. A common theme in the responses of some students when asked what would need to change if they were to find the software more helpful was the need to interact outside of their working pair. One student commented that it would help them if they were able to compare their findings with those of other groups. Another student felt that it would help them, and other students, if she was able to try and "teach it to them". Another key feature is the nature of the tasks in terms of how structured or open they are. The interviews with students have highlighted the important role that the tasks themselves have upon students learning. This is stressed by Goldenberg (2000) when he argues that "It is the problems that are posed, not the technology with which they are attacked, that make all the difference." One of the students explained how working on the 'more instructional' tasks had helped them "see the links between the graphs" and that this had been a key reason for them rating the software's impact so highly. Another student commented that they found it very difficult to make any progress when they couldn't understand what a particular task was asking them to do. If I were to do this research again, I would like to be able to see if there was a link between the students rating and the tasks that they worked on.

## CONCLUDING FINDINGS

Through this research I have been able to evaluate the impact of investigative tasks involving the use Autograph 3.2 upon students learning and mathematical thinking. The high average scores of the class and the rise in scores from the first set of tasks to the second show a significant impact upon students learning and support a decision to continue the use of Autograph by students in this class. By viewing the results of this research through the filter of the themes of other research into

the use of graphing technology and theories of algebraic thinking, I have been able to critically reflect upon my current practices and to consider how they will need to change in the future if I am to successfully use Autograph to improve my students learning of mathematics. I have become increasingly aware of the interdependence of key factors upon the successful use of ICT in my classroom. The three main factors which I intend to address are in the short to medium term are; instrumentation – helping students to develop confidence and fluency in using the software, the nature of the tasks – providing a balance of challenge and support, and the interactions of students beyond their working pairs – allowing them greater opportunities to seek help from each other and to share their findings with the whole class. Looking forward, my long term aim is for students to develop their mathematical thinking skills through, and eventually, independently of the technology.

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## **AN EXPERIENCE AT PRIMARY SCHOOL: GRAPHING MOTION IN THE MATHEMATICS CLASSROOM**

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*In this paper I discuss a longitudinal study on graphing motion at primary school through the aid of motion detectors. I took part in the study as a teacher of a class of 16 children that was involved in the study from grade 2 to 5. Referring to some instances from written protocols produced by the children or from actions in the classroom, I reflect about the potential of motion detectors when used to introduce the concept of function with children, and whether we can say that the motion detectors “work” in the mathematics classroom and, especially, what they work or do not work for. I will also discuss the other side of the study, which is more concerned with my teaching practice and children’s engagement.*

### **INTRODUCTION AND THEORETICAL BACKGROUND**

The last ICMI study, devoted to the integration of ICT in mathematics education, pointed out that today the influence of digital technology affects most education systems, if not all of them (Hoyles & Lagrange, 2010). On the other hand, the US National Council of Teachers of Mathematics in its position statement claims that “Technology is an essential tool for learning mathematics in the 21<sup>st</sup> century, and all schools must ensure that all their students have access to technology” (NCTM, 2008, quoted in Drijvers, 2012). In 2008, UNESCO produced the ICT Competency Standards for Teachers, in which it is pointed out that “Today’s classroom teachers need to be prepared to provide technology-supported learning opportunities for their students. Being prepared to use technology and knowing how that technology can support student learning have become integral skills in every teacher’s professional repertoire. Teachers need to be prepared to empower students with the advantages technology can bring. Schools and classrooms, both real and virtual, must have teachers who are equipped with technology resources and skills and who can effectively teach the necessary subject matter content while incorporating technology concepts and skills.” (UNESCO 2008, p. 1). In the digital era, stakeholders and policy-makers pay more and more attention to the introduction of technology at school and to its use in teaching and learning mathematics. However, the issue of its integration in the didactical practice as an effective resource remains open. Teachers are often and feel poorly prepared to deal with the societal challenges that are offered to their educational role. So, one aspect of major concern for the educational world should be the relapse that the wide amount of studies about teaching and learning mathematics with technologies can have for teacher professional development (both for in service and pre-service teachers).

In a lecture at the International Congress on Mathematical Education, (Seoul, July 2012), Paul Drijvers has proposed a kind of meta-reflection on some research on the use of technological tools in mathematics teaching and learning. He took well-known studies in the field and their theoretical constructs and analysed them from a different perspective. What is interesting in Drijvers’ study is attention to a more general issue about the integration of technology at school, and a more relevant one for teachers too, that is, the issue about why we can say that digital technology in mathematics education works (or does not). In particular, Drijvers asks what exactly the potential of ICT for teaching and learning is, how to exploit the potential of ICT in mathematics education, whether digital technology really works and why, and which factors are decisive in making digital technology work or preventing it from working (Drijvers, 2012). The term “work” is meant from different points of view: improving students’ learning, inviting more efficient or more effective

learning, empowering teachers to better teach mathematics, helping the researcher to better understand the phenomenon, etc.

In this lecture, I will explore some of these issues through the particular case of graphing motion at primary school. Indeed, I have been involved for many years in research studies that regarded the use of motion detectors for a graphical approach to the concept of function at primary school (Ferrara & Savioli, 2009; 2011; Ferrara, accepted).

The choice of motion detectors is not accidental. It is instead relevant with respect to research that has investigated them as efficient tools to work on graphs and on the concept of function at different school levels (e.g. Nemirovsky *et al.*, 1998; Robutti, 2006; Ferrara, 2006). The analyses carried out about this kind of activities have shown, on the one hand, the value of experiences in which it is possible to bodily interact with tools and get feedback on mathematical objects (particularly, on the graphs' shape) as a consequence of these interactions. On the other hand, they give reason to the fundamental role of the sensuous and corporeal aspects of mathematical cognition, which have been studied in more recent research behind the influence of theories of embodiment and neuroscience over mathematics education (Radford *et al.*, 2009; Ferrara, accepted).

As a primary school teacher I became very interested in further developing these aspects with children, and I started to participate in the long-term research project about the use of motion detectors for an approach to function. The relevance of bodily activity in mathematical cognition shifted attention to the possibility of such an approach at early stages. Moreover, children are devoid of formal knowledge but, in the same time, they do not possess strong misconceptions. There is also a pedagogical aspect related to what I am saying here. That is, an approach of the kind I am referring to is not a traditional approach to function and to the idea of relation between variables. Even though relations are wide part of the curriculum (at least in Italy), in the primary grades they are too often exhausted with exercises on sets and correspondences, and little when nothing have to do with functional thinking, whose development the research project was tending to. This point of view is very important, due to the fact that functional thinking will be the basis of the later formation of Calculus concepts in the following school years.

### **MATH IN MOTION: PROJECT AND STUDY**

I have been involved as a primary teacher in a longitudinal research study that lasted 4 years. My class of 16 children was involved from grade 2 to grade 5 in this study (Ferrara & Savioli, 2009; 2011). The school is located in the suburbs of Chieri, a little town in the surroundings of Torino, in Northern Italy. I was in the classroom as a teacher-researcher, that is, as active observer, while a researcher was leading the activities.

The entire project is called "Math in Motion" from the kind of activities that were first designed and then implemented by the research and myself working together. In the activities, movements were performed using a motion detector and graphs were explored of these movements. Actually, two types of motion detectors were used in the course of the study. The first type is a one-dimension motion detector (CBR, that is, Calculator Based Ranger); the second type is instead specific software to gather data from the two-dimension movement of a coloured object (Motion Visualizer DV). The children only worked with the CBR in grade 2, while for the remaining of the study they were acting and interacting with the Motion Visualizer. The motion detectors were both used in the classroom in real time, giving an on-line feedback to children's movements in terms of displayed graphs. These graphs were position-time graphs related to: the distance of the moving subject versus time, in the case of the CBR; the position of the coloured moving object on the plane versus time, in the case of the Motion Visualizer. In this way, the shapes of the graphs could be associated to the qualities of a given motion, being it performed along a linear trajectory or along a planar one. The

opposite task was also considered in the classroom, by asking the children to produce movements for getting given graphs. So, the tools were involved in the activities both in an interpretative phase and in an expecting phase, that is, in the passage from model to motion as well as in the inverse passage, from motion to model.

Didactically speaking, the main purpose of the study was that of developing in the long period mathematical literacy regarding graph sense and relations between variables, and the concept of function as well. The starting point has always been the everyday phenomenon of motion, of which children have experience since their very first years. This choice is concerned with the awareness that the relationship between kinematics and calculus concepts (like function and derivative) is not easily addressed in mathematics at school, despite its being a crucial part of the mathematics curriculum in the secondary grades. The rationale of the study just comes from the expectation that, through the link between kinematics and calculus, we can create possibilities for the children to experiment a phenomenon from the mathematical side of the story, starting to create understanding of the concepts that inhabit such a story.

All the children (8 males and 8 females) participated in the activities working together, most of the time, as a group, but not avoiding occasions for individual or pair work. Moreover, no kind of work was abandoned: both oral interaction and written tasks were part of the study. A classroom discussion was always led by the research, as a final stage of a meeting or as a starting stage of the following meeting. In each grade, the experimental activities were carried out for about ten weekly meetings.

The study can be thought of as a bridge between curricular standards and problem solving, as well as between experimental methodology and technology. The presence of technology has been fundamental. First, it acquires a certain appeal for the children that showed to be tempted to discovery and searching for relations between the phenomenon and the graphical representations produced by the software. Secondly, it permits to approach these relations thanks to the possibility of watching in real time the effect of the movement actions on the shape of the graphs. Perceptions and sensations become part of a more conscious way of constructing meanings for the graphs and understanding the covariance of variables. Finally, the use of technology confers a specific nature to the mathematics proposed in the classroom. It marks that, instead of being related to algorithms, mathematics is naturally related to everyday phenomena. In fact, the experiences in which students can interact with a tool to create phenomena usually help them to understand the mathematics connected to those phenomena.

In what follows, based on episodes from the concrete work in the classroom, I will make some reflections about: the potential of motion detectors when used to introduce the children to the concept of function; whether we can say that the motion detectors “work”, and especially what they work or do not work for. I will also spend some words about what I have learnt from this experience and about the engagement of the children.

## **EXPERIENCES WITH THE CBR**

The experiences with the CBR have mainly regarded children’s walking or running in front of the motion detector. Various experiments were performed: movements in only one direction, backward and forward movements, as well as absence of movement. The graphs that we considered essentially were slanted and horizontal straight lines, and piecewise lines (like “mountains”, according to the children’s language). The properties of the movements were connected with the shape of the graphs. For example, how to change movement to change the number of the mountains or their heights was discussed, even involving issued about speed. Moreover, we dealt with the impossibility of having a vertical straight line. From the cognitive point of view, one main difficulty lies in understanding the



horizontal straight line as a model of absence of movement, since, in functional terms, it marks that distance is always the same while time goes on. As a consequence, the graph moves even though the moving subject does not move. Despite their inherent complexity, straight line pieces are well-known as *cognitive pivots*. Indeed, once the meaning of the shape has been understood in terms of motion, it is easier to understand the remaining connections between other pieces of the graph and portions of the movement (see e.g. Ferrara & Robutti, 2002). Thanks to these experiments, the children can come to have a covariational view of the graph (and of the concept of function): that is, position and time are associated in a dynamic way, depending on each other (Slavit, 1997).

In order to give an example, I take one individual written task that the children were asked to face after some experiences concerning the use of the CBR. The text of the task says:

*“After the experiments of these days, we have learnt new things. Tell, draw and explain everything... mathematical you have understood!! You can use the drawing below (or make others by yourself!). Explain always your reasoning.”*

The “drawing below”, to which the worksheet is referring, is given by a Cartesian plane that resembles the one shown on the screen of the graphic calculator linked to the sonar: two orthogonal axes, a vertical one for position ( $D$ , in meters) and a horizontal one for time ( $t$ , in seconds). The children can decide whether they want to use the axes or to support their reasoning with any other kind of representation.

Beniamino’s protocol is very interesting for his choice to draw a horizontal straight line (Fig. 1), which models one of the most difficult situations to be addressed, as discussed above. The written protocol well shows the covariational nature of Beniamino’s thought and his understanding of the relations between the variables. Figure 1 marks on the left the complete protocol with its translation, on the right the zoomed in sketch that Beniamino drew to refer to the way the calculator works.

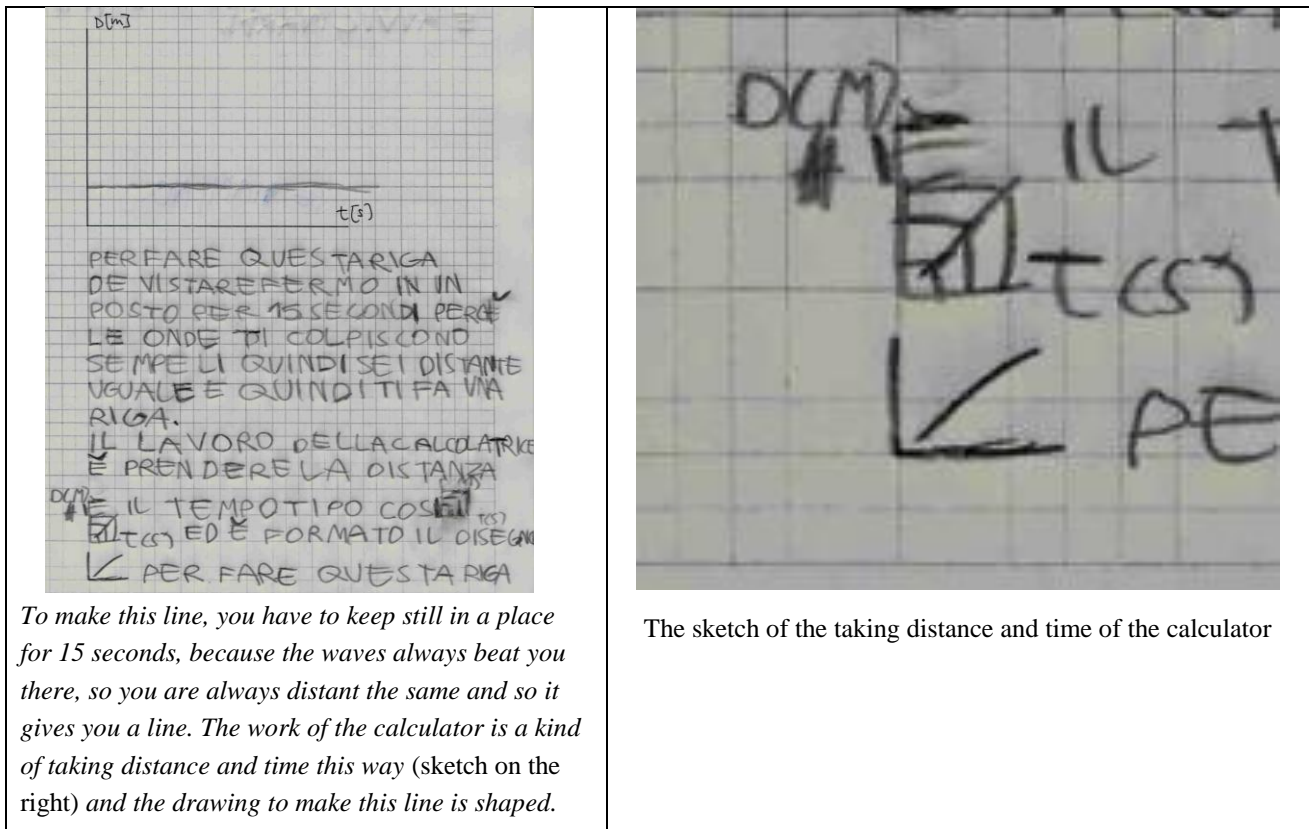
Beniamino has usually hard time with written arguments. But, in this case, his explanation is very rich and, surprisingly, complete. First, there is clear reference to the technological tools, by means of the “waves” that “beat you” and “the work of the calculator”.

The transition from motion to its graphical representation clearly occurs: the “you have to keep still in a place” is merely referred to absence of movement, and to what is perceptuo-motor in nature, while “you are always distant the same” embeds the mathematical meaning of a non changing distance, which becomes the shape of “a line”. The “keep still” in the phenomenon has its graphical counterpart in “the same” of the distance. The “always” beating “there” of the waves links the two sides. Also, the use of “always” is crucial to give that general character to the shape of a horizontal straight line: the independent variable (time) goes on, while the dependent variable (distance) always remains the same.

A reflection is needed at this point: Beniamino’s use of the word “work” is not at all intended in Drijvers’ meaning. Working has for Beniamino the sense of the way the calculator functions, that is, “a kind of taking distance and time this way”. Nonetheless, in his speaking about the “work” of the calculator, it is apparent the understanding of the covariance of the two variables, distance and time. What I want to argue is that for Beniamino, the CBR, together with the graphic calculator, *worked* (in the sense of Drijvers) for conceptualizing the idea of covariance as a basis for the concept of function. The diagram in Figure 1 expresses the covariance by means of the vertical and horizontal segment lines that relate every point in the plane to its (horizontal and vertical) components. Written expressions (like, for example, “you have to keep still in a place for 15 seconds”, “always there”, “so you are always distant the same”) also draw attention to the relations between the two variables. There is a sense here for which I can see that the potential of the sensor was exploited to overcome

the difficulty of understanding the meaning of a horizontal straight line (something –the graph– moves even though the moving subject keeps still).

This is only an instance of the many cases in which we can discuss about the potential of the CBR and the fact that it worked in my mathematics classroom.

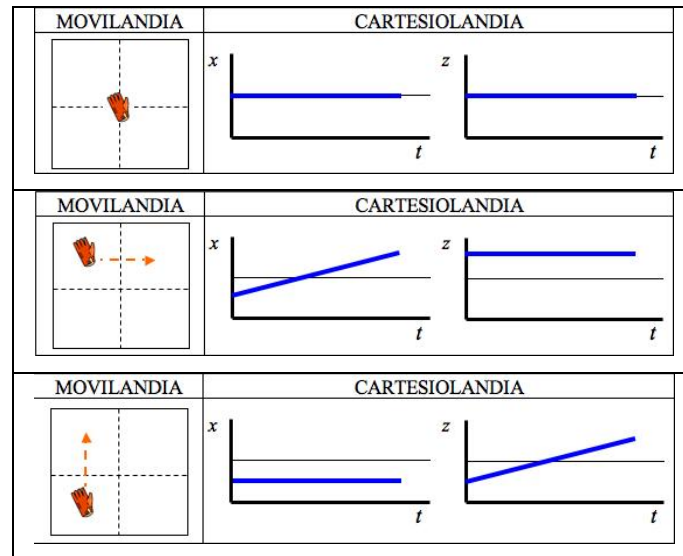


**Figure 1. Beniamino's protocol on the horizontal straight line**

## EXPERIENCES WITH THE MOTION VISUALIZER

The experiences with the Motion Visualizer were carried out from grade 3 to 5. The children always worked with two worlds: Movilandia (the land of Movement) and Cartesiolandia (the land of Descartes). The movements are performed in Movilandia using coloured objects, like an orange glove or an orange ball attached to a stick. Cartesiolandia displays the two graphs of position versus time related to the components of motion. The two worlds were easily created in the classroom. Movilandia was the space of a big paper on the wall; Cartesiolandia was given by the projection of the computer screen with the graphs resulting by the use of the software. Two blackboards were present in the classroom. In Cartesiolandia there also are a model of the room with the motion trajectory and a digital video of the movement performed with the trajectory again, superimposed on the video. Like in the case of the CBR, the origin of the graphs in Cartesiolandia can be set on line, so that the children can watch in real time how the Motion Visualizer works. The components of position, horizontal and vertical, are called  $x$  and  $z$  respectively. So, the two graphs displayed by the Motion Visualizer are the graphs of  $x(t)$  and  $z(t)$ . The children knew  $x$  and  $z$  as two secret agents (Mister  $x$  and Mister  $z$ ) that are hidden in the orange glove or in the orange ball and that speak a language through which they show qualities of the movement performed in Movilandia (the so-called "Cartesiolandese"). The challenge given to the class is that of understanding this strange language Cartesiolandese performing movements in front of a webcam through which the computer elaborates the motion data of coloured objects.

Situations of absence of motion, and of motion trajectories with geometrical shape (like horizontal, vertical and slanted segment lines, squares, regular polygons, the circumference) were faced by the children with the aim to arrive in grade 5 at sinusoidal functions related to a circular movement. The three instances of absence of motion, and vertical and horizontal trajectories are shown in Figure 2 with reference to the use of the orange glove. On the right, the corresponding graphs are given. When the glove is kept still in a certain position, two horizontal straight lines are displayed for the two graphs. When the glove is moved horizontally, the vertical position does not change while the horizontal position changes. When the glove is moved vertically, the horizontal position does not change and the vertical position changes (so horizontal and vertical positions exchange their role with respect to the previous case). A change in the direction of motion would entail a change in terms of the sign of the slanted straight lines' slope.

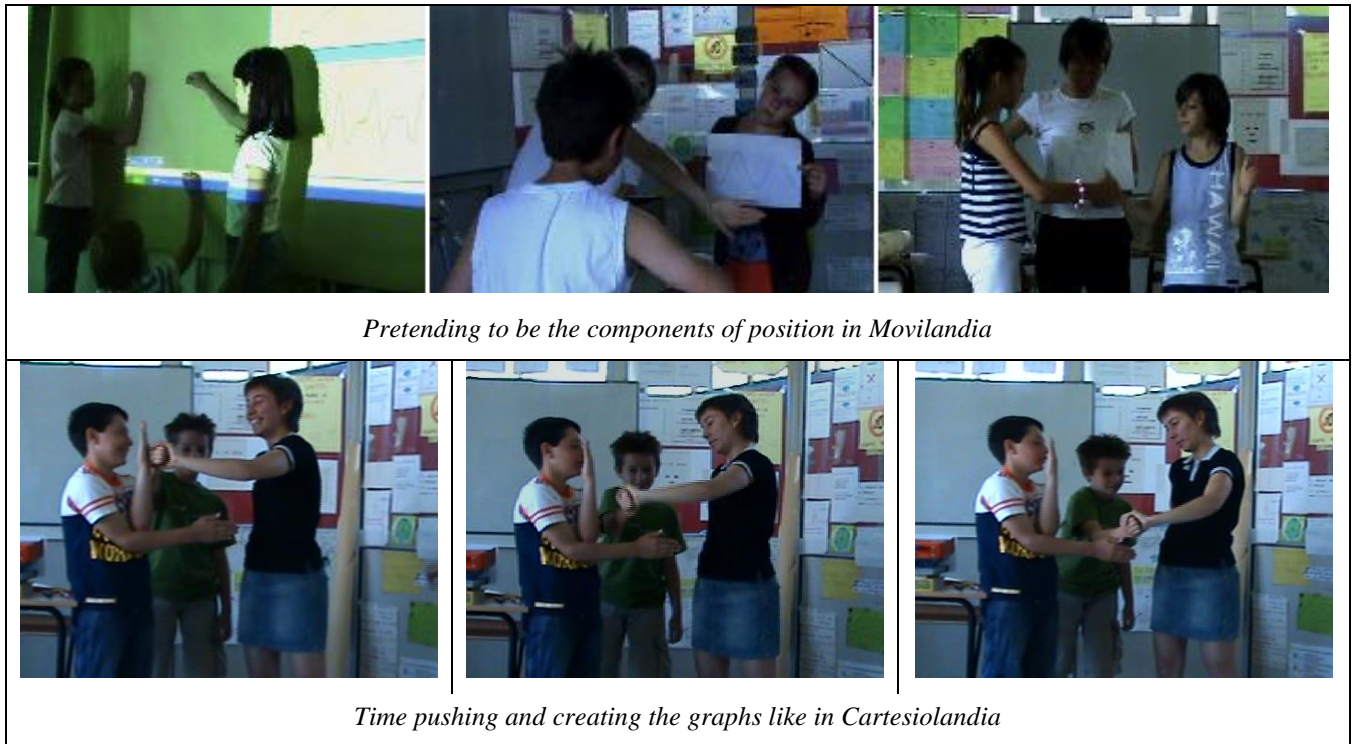


**Figure 2. The graphs associated to absence of motion, horizontal movement and vertical movement**

The main difficulty in using the Motion Visualizer for the children is concerned with the fact that the two components of position are represented on a vertical axis in Cartesiolandia. The vertical component ( $z$ ) has a natural correspondence with its 'vertical' representation on a graph, since its contribution to motion is already vertical. If we think of a centre in Movilandia we can also think that the vertical component contributes to motion in terms of being above/below the centre, while the horizontal component in terms of being on the right/on the left of the centre. The point is what happens when the components are transferred in Cartesiolandia and put on the vertical axis of the Cartesian plane. For the vertical position, being above/below (the centre) in Movilandia still means being above/below (the origin) in Cartesiolandia. Instead, for the horizontal position, being on the right/on the left (of the centre) in Movilandia corresponds to being above/below (the origin) in Cartesiolandia. This is the most complex thing for the children that have to understand the way the Motion Visualizer works to conceptualize the decomposition of motion in mathematical terms.

To overcome this complexity, at a certain point of the study, the researcher and I decided to adopt a bodily strategy to work with the children on interpreting the graphs and their shapes in relation to a certain motion trajectory. This strategy could work without the direct use of the Motion Visualizer, but it was entailed by the use of such kind of software and by its way of working. Pairs of children were asked to imagine to be the components of position and to move according to some motion trajectory produced by another child or by the researcher. While the motion trajectory is mimed, the two children who mime the components move their hands according to the trajectory. Then they are asked to remember their movements and to reproduce them without the presence of the trajectory anymore. In so doing, they are still imagining what happens in Movilandia. In order to shift to

Cartesiolandia, where position is displayed together with time, the idea we had was that of thinking of time as “pushing” (“pushing” was coming out from a classroom discussion in which the children used this way of looking at the ‘work’ of time in the graphs of position versus time). So, the next step was to ask the children to continue each to reproduce the movement of their component, transferring it along the vertical direction on which the components are displayed in Cartesiolandia. Adding time that pushes in the positive direction (through the help of a new hand) is the last step to reconstruct the shape of the graphs that the Motion Visualizer would have given in Cartesiolandia. Figure 3 shows instances of this process of constructing meanings for the graphs with the children pretending to be the components of position in Movilandia, or the idea of time pushing in the world of Cartesiolandia enacted by the researcher.



**Figure 3. Instances of the process of construction of meaning for the graphs in Cartesiolandia**

Consequently to the use of the Motion Visualizer, this new way of doing and knowing was born and shared in the classroom, like a sort of ‘didactics of gestures’, which then became integral part of the way the children dealt with similar tasks. In this perspective, I want to stress that understanding how the Motion Visualizer works (in its usual meaning) was the first step toward the success of these kind of activities. The Motion Visualizer *worked* (in the sense of Drijvers) for conceptualizing two things in the present case: on the one hand, the different contributions to motion of the components of position; on the other hand, their respective covariance with time. So, I claim that the potential of the Motion Visualizer was exploited in this case to overcome the complexity of distinguishing the different roles of the components of position in the movement, as well as that of representing them on the Cartesian plane in order to model a bi-dimensional motion.

## CONCLUSIONS

In the previous sections, I have discussed, according to my experience, why the motion detectors, which I have used for four years in my mathematics classroom, worked and also what they worked for. If, on the one hand, the CBR permitted to convey the understanding of the covariance between the variables of position and time, on the other, the Motion Visualizer permitted to use this covariance as the base for looking at the crucial role of time in the graphs of  $x(t)$  and  $z(t)$  when

moving on the  $xz$  plane. Practically, in this way, I was able to work with my primary children on the concept of function, on the sense of the graph, and even on the parametric functions related to a bi-dimensional motion. But there is also another side of the story, the one that is more concerned with the kind of reflections that this long experience entailed on my teaching practice, and on the children's engagement as well.

In the recent years, as a teacher, I have been both an observer and a mediator of important changes. Working with new generations of digital natives has surely modified the methods I adopt, but also the attention times, the kind of engagement, the use of tools and the finding of sources and resources. Now, the challenge is the conscious and suitable use of technology, as well as the keeping of control of mistakes and correct information. There is an even bigger challenge that I can feel as a teacher: that is, remaining anchored to people, sensibilities and intelligences. Indeed, technology *transforms*, shortens the way, engages us, but it will never *substitute us*. We can trust elaborations and algorithms but the search for *why(s)* is a (not only didactical) enterprise that is still our responsibility as people. This is the secret, I think, if we want to be able to exploit the potential success of the integration of technology in school mathematics teaching.

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## LEARNING MATHEMATICS WITH DIGITAL TEXTBOOKS: CHALLENGING AUTHORIZING AND AUTHORITY

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*A textbook is a special type of book that is part of institutionalized schooling, usually used in a particular way. Traditionally it has a specific structure and is designed to contain the message of the professional community about what students should learn. It also represents the ideas of the author about how the content should be taught and learned. Digital books offer new kinds of flexibility, participation, and personalization – properties that are in contrast to the traditionally authoritative structure of the textbook and the passiveness of the reader. To study the implications of the proposed affordances of digital textbooks, this paper explore three scenarios, analyzing the challenges involved in each and speculating about challenges for a new textbook culture.*

### TEXTBOOKS: THE OBJECT AND THE CULTURE

A textbook is a message from the professional community to students about what they should learn. It also represents the ideas of the author about how the content should be taught and learned. It plays a central role in school pedagogy and classroom norms, and its authoritative image has been the dominant aspect of the common classroom culture often identified as *textbook culture*. Traditional textbook culture assumes that teachers make sure their students learn the content in the book, usually in the specified order, because the book acts as a model for standards and the way standards are assessed. It is often argued that the external authority speaks *through* the textbook rather than the book speaks *to* its readers, whether the teacher or the students. Love and Pimm's (1996) "text on texts" is an important resource for understanding the features of textbooks that are usually assumed: textbooks are closed in the sense that both text and images have been created in the past; they include problems for exercising but not aim at questioning the content; they are linear and follow the "linear textual flow of reading" (p. 381); and usually they consist of cycles of expositions, examples, and exercises. Textbooks are supposed to provide guidance and present opportunities for students to learn, making the objectives and ideas of the curriculum more readily apparent. For teachers it also provides guidance in bringing their teaching in line with the expectations of the external authority which may be the school, the syllabus, or some central assessment. In this function, the textbook serves as syllabus and timekeeper, and its author is considered to be the authorized entity charged with delivering content and pedagogy. Note the direct etymological link between "author" and "authority" (as further described by Young, 2007), underscoring the authoritarian position of the textbook as written by a recognized expert author or group of authors. As an *author(iz)ed object*, the textbook is considered to be a solid personal resource for the learner. Learning with a textbook in this manner is often referred to as "learning by the book," that is, a passive type of learning. "Teaching by the book" refers to teaching that treats the textbook as an authority that should be fully accepted. Both are known components of *textbook culture*, which varies only slightly across different contexts worldwide.

Whether in the form of dedicated hardware, tablets, or some software format, digital books have challenged the object we used to refer to as a "book." Clearly, the 21st century reader will increasingly be reading materials in digital formats, and find them both useful and attractive. For a growing community of readers digital books have already changed the book culture. It did not take too long for the textbook publishing industry to follow these global changes and offer digital textbooks. At first they addressed the higher education audience, and recently have been targeting

schools and school teachers, both as authors and readers. Textbook publishers are addressing a wide range of expected changes in the affordances of the digital object, starting from material aspects of weight and cost, the quality and attractiveness of the material, the richness of the modes of presentation, and the opportunities for personalization. As the textbooks will be rapidly changing from print to digital formats, it is assumed that the ways in which the textbooks will be used will change as well.

A view that I would like to present in this paper is that the changes offered, especially those that concern teaching from open educational resources that advocate communally evolving writing, challenge the accepted functions of textbooks. Educational systems must understand better what they are up against if they are to realize the hope that digital textbooks will produce a change in the textbook culture.

## SCHOOL MATHEMATICS TEXTBOOKS

Herbel-Eisenmann's analysis of the sources of textbook authority (2009) distinguishes between the textbook being an objective representation of knowledge, so that the authority is an intrinsic property of the text, and the participatory relations between textbook and teaching. Teachers use various practices to confer authority onto the text and simultaneously onto themselves. The pioneering work of the mathematics teaching standards (NCTM, 1991) attempts to cause a shift in teacher's authority toward "guided inquiry" teaching. Although mathematics classrooms have increasingly adopted various formats of constructivism, reviews of post-reform math textbooks do not find that the practice of using textbooks has changed. Drawing a direct link between the author, the central authority that authorized the textbook, and the way teachers teach and students learn appears to be negotiable. Post-reform studies that examined the teachers' adoption of new curricula found that although teachers attempt to reform their pedagogy to emphasize guided inquiry, the student-teacher-book relations have not changed and the textbook remains a central formal authoritative resource for the teacher and the learner. Analyzing newly developed textbooks, Nathan et al. (2007) found that many new books have remained similar to the traditional ones. Reyes, Reyes, Tarr & Chavez (2006) studied for three years how newly developed textbooks supported by the NSF influenced math teaching and learning in a US middle school. The authors report that half the teachers declared that "My math book is my Bible", whereas the other half were influenced more by the state-determined curriculum and assessment materials. The teachers covered 60-70% of lessons of the specially designed-to-reform textbook, just as they did when using non-reform textbooks. McNaught (2009) presents similar findings in a study of the Core Plus curriculum project as an example of integrated content textbook, demonstrating that around 60% of the content of the textbook was taught but not necessarily from the textbook, and about one third of the teaching was based on other supplementary materials. Studying Swedish teachers guiding students who are solving tasks from the textbook, Johansson (2007) shows how the perception of the textbook as a resource of ultimate correctness has affected the teachers' decision not to question a textbook solution (2007, p. 49).

In sum, the brief attempt to characterize the current *textbook culture* has become more challenging than expected as it suggests the following: (a) the textbook remains the single printed and bound object that acts as authoritative pedagogic guideline for what should be learned and for how it should be taught and assessed; (b) although the textbook is assumed to provide devices for actively involving the students in examples and exercises, the studies focus on engagements with textbooks that are reserved mostly for the teachers; (c) new textbooks that attempt to adopt a somewhat less authoritative tone create a didactic challenge and are not likely to be fully adopted by experienced teachers; and (d) teaching does not depend on a single textbook: approximately 30% is

accomplished by teachers using other teaching materials, mainly previous textbooks that the teachers feel are more likely to achieve their goals. Technology has been designed as an important resource in several textbooks attempting reform. Nevertheless, such technological resources as the Open Education Resources, which accompany the newer books, are reported to be considered as enrichment while the textbook remains the core external authority. As the flow of information around the world results in a broad sharing of goals and instructional materials, syllabi and teaching strategies, the diverse evidence also appears to be shared worldwide (Leung & Li, 2010; Valverde et. al 2002).

## **E-TEXTBOOKS: ENGAGEMENT AND AUTHORITY**

The first generation of digital textbooks may be considered to be old wine in new wineskin (Gould, 2011), being merely digitized versions of their written counterparts and supporting limited interactivity by means of search and navigation of the digital document. The most noticeable feature of digital textbooks is the change in the *object*. Digital textbooks were developed and marketed primarily for higher education courses, but lately they have been making their appearance in schools as well. An increasing number of school textbooks are now supplemented by continuously upgraded digital resources that can be found on the Web (<http://classroomaid.tigblog.org/post/492241>). Publishers offer to allow teachers to personalize digital textbooks for their courses, emphasizing flexibility and inexpensive dynamic changes attempting to allow school teachers to personalize the textbook by selecting from existing chapters and content and even individualizing the book to the student. Hardware manufacturers have encouraged developers to use high-level development tools, and central authorities in India and in some African countries tout the digital textbook as a unique opportunity for delivering textbooks to distant rural schools. Korea, considered to be one of the leading countries in math and science achievements, became a leading innovator in the area of eTextbooks, especially in school math and sciences. Korea holds an integrative view in which textbooks remain the central learning resource, surrounded by other types of facilitating media (<http://blogs.worldbank.org/edutech/korea-digital-textbooks>). Other educational systems, including those of Japan, and some European countries, are adopting a similar view of the new textbook (Taizan et. al. 2012.) The Israeli education system requires that each textbook appear in at least one of three formats: a digitized textbook, a digitized textbook that is enriched with external links and multimodal materials, or a textbook that is especially designed to work in a digital environment and includes online tools for authoring, learning, and management. The assumption of the integrative view is that the digital textbook can function as printed text and thus assume the traditional format of the textbook. Thus, the change of the *object* prompted changes in *processes*, mainly the publishing process (editions, price, attractiveness) and the ways of arranging digital add-on materials as additional instructional materials. To challenge this integrative view, the textbooks described below use the unique affordances of the digital media to change the traditionally assumed features of textbooks and challenge their accepted functions.

### **External authority and readers' engagement**

Textbook authors have long been seeking less formal control structures that would better reflect student-centered teaching and support teaching as guided scientific inquiry. Although there are ample examples of eTextbooks, I find only few research reports that discuss the design rationales and analyze the use of such books. Here, I would extend on a model of non-ordered multimodal digital textbook that adopts what I and the coauthors define as the *museum image* (Yerushalmy, 2013). Yerushalmy, Shternberg, and Katriel (2002) designed a textbook characterized by the extensive roles assigned to visual semiotic means, to interactivity between the reader and the visual mathematical objects and processes, and to a conceptual order of digital pages that could be



rearranged to serve a variety of instructional paths. The design of the VisualMath is situated in the larger view field of mathematical guided inquiry. It contains the foundation of the “what” and the means for the “how” to teach and learn a full school-algebra course. Tasks (problems, exercises, and expositions) are organized around interactive diagrams. An interactive diagram (ID) is a relatively small and simple software application (applet) built around a pre-constructed example that serves as a basis for change. Multiple representations and linked interactively create mirror feedback that leads to conjectures and supports arguments. Indeed, the challenge in constructing a task around an ID is to design opportunities for action that the learner can manage in such a way that fits in with the structures the learner has already formed. Whereas static diagrams present information and a point of view (implicitly aiming at engaging the viewer in meaningful interpretations), IDs explicitly invite the viewer to take action, that is, to activate or change the diagram within given limitations (Yerushalmy, 2005).

In the museum view interactive textbook the central unit is a “gallery” that presents opportunities for students to focus on a concept and practice related skills, making the objectives of the learning apparent. Each gallery supports guided engagement. The interaction can be guided by the tasks, by the tools, by the feedback of the interactive diagrams, and by the problems and exercises. The views, borrowed from a museum setting and used to form our guided inquiry vision, were consistent with the distinction Kress and van Leeuwen (1996) made to describe linear and nonlinear texts. They compared linear texts to “an exhibition in which the paintings are hung in long corridors through which the visitors must move, following signs, to eventually end up at the exit,” and nonlinear texts to an “exhibition in a large room which visitors can traverse any way they like... It will not be random that a particular major sculpture is placed in the center of the room, or that a particular major painting has been hung on the wall opposite the entrance, to be noticed first by all visitors entering the room” (p. 223). To turn an unordered textbook into instructional material teachers must: design the visit to the gallery and personalize the tours. A long-term research that examined students’ engagements with the VisualMath tasks (Yerushalmy & Naftaliev, 2011; Naftaliev & Yerushalmy, 2011) revealed another aspect of noticeable personalization of the text: personalization by students. We identified three patterns of personalization of the text: (a) attention to details in the given example, resulting in the construction of additional details to the original example; (b) changing the given example to generate new examples; and (c) rephrasing the question of the task to ask a new question.

The greatest challenge of the design, however, was not in the structuring of the rich-content galleries that shapes the visit of the student but addressing the other reader, the teacher, for whom the non-ordered VisualMath book had to be designed so as to provide guidance while meeting the expectations of the external authority. For the semi-ordered materials and multimodal digital “pages,” which to a certain extent stand on their own, to be considered a textbook, the deep structure of the concepts and the inter-relations between them must be simple and clear. Our first decision was to organize the content along a core concept: the function is represented by an unusually large and diverse collection of activities, interactions, and representations. The second was to organize the materials along a relatively small number of mathematical objects and operations. When the two conditions, described in details in Yerushalmy (2013), are turned into design principles, they should offer the reader, first and foremost the teacher, a conceptual map of a relatively small number of objects and operations between them.

### **Internal authority: Evolving textbooks in open educational communities**

Technological advancements, mainly of Web2 participatory tools and norms that seem to be consistent with constructivist pedagogies, pose a challenge to the accepted function of the textbook

as a message completed in the past by a recognized external authority that must be unpacked by learners and teachers. Teachers everywhere create teaching materials, search through professional materials available on the Web, and share their repositories and ideas through on-line social networks. Although some teachers call for greater participation in choosing and authoring textbooks, it is unclear whether and how teachers and schools could assume an important role in designing and developing curriculum materials, and how this would change the way they use textbooks in a sustained way. Known successful collaborative projects can help study future developments in this area. I focus on properties of two such projects: the open source programming movement and the Wikipedia project.

If we regard the textbook as a message about which content should be taught and how, *coherence* becomes an important requirement. *Quality* and quality control are important challenges to any open source created by contributing communities, certainly to the open educational community and to the evolving collaborative textbook. The third challenge to evolving collaborative textbooks designed by teachers is *sustainability*. Would teachers commit themselves to long-term participation in an evolving project? What form would the sustained collaboration take and from where would the required leadership come that are needed to provide extended guidance for teachers and school systems? Revisions of digital textbooks are already taking place, and are likely to spread, although it is difficult to imagine that open communities creating evolving textbooks will assume a central role in this enterprise without some fundamental changes in expectations and norms.

### **BOOKS ARE BECOMING "SOMETHING ELSE" AS AUTHORITY TAKES PLACE "ELSEWHERE"**

I addressed two types of e-book that can be considered to be textbooks, although both violate some key functions of traditional textbooks while retaining others. One type, the unordered e-book, challenges the external authoritarian role of the textbook as a message from the past delivered in an orderly manner. The other, the evolving book, invites authority to reside within the textbook, based on open educational communities. The question remains whether the textbook will undergo a significant innovation to the point where the idea of a single document representing external authority will no longer be considered a foundational idea in teaching and learning. To conclude, I would point out two already noticeable directions related to advancements in technology that may be viewed as complementary: the attempt at standardization followed by high-stake centralized digital assessment, and the personal learning and tutoring systems.

This is the time to examine whether the disappearance of the traditional object and of traditional authority will result in the freedom to support inquiry-based student-centered learning, or in an anarchy that the educational system cannot afford. The latter scenario will produce a new tyranny in which traditional textbook authority is replaced by massive standardization driven by uniform technology.

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# **Oral Presentations**



# PEDAGOGICAL STRATEGIES TO TEACH AND LEARN MATHEMATICS WITH THE USE OF GEOGEBRA

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*This paper describes, in the researches carried out between 2009 and 2011 in the Postgraduate Studies Program - Mathematics Education using GeoGebra, which theoretical and methodological supports were used, GeoGebra's role in each of them, the considerations about its use, the goals of each author, and the results obtained. Considering aspects of the thematic analysis, it was found that out of all nineteen papers, nine of them used different theories and are analyzed in this study. The use of theoretical supports can be considered as an alternative to find ways to develop situations using GeoGebra that help to overcome common difficulties in teaching and learning mathematics. The conclusions show that the authors could search, throughout the entire strategy, the best solution for the activities proposed in a context that accepts the inclusion of traditional and digital media.*

## INTRODUCTION

The aim of this paper is to present, considering aspects of the thematic analysis, the researches carried out between 2009 and 2011 in the Postgraduate Studies Program - Mathematics Education using GeoGebra, in order to check the theoretical and methodological supports were used, GeoGebra's role in each of them, the considerations about its use, the goals of each author, and the results obtained.

Thus, the following questions guided the analysis performed: Which mathematical object treats the research? What is the proposal presented? Which are the theoretical supports used? What are the conclusions drawn?

In this study, rather than the thematic analysis as a whole, some of its aspects were used to support the work developed, since according to Minayano (2004), regarding its operation, the thematic analysis can be separated into three stages: Pre-Analysis; Exploration of Material; Treatment of Results Obtained and Interpretation.

Zanella (2009) highlights that:

In thematic analysis, you must understand the author's message, but not interfere in the ideas he/she professes. That means you have to listen to what the author wants to say without judging or criticizing it. Then, at first, try to identify the theme, reread the text and try to understand the reasons, the difficulties, i.e., the determination of the problem that led the author to write about such topic. In this regard, it is important that you ask yourself some questions in order to identify the problem, such as: "Which difficulty will be overcome? What is the problem to be solved?" Identifying the problem reveals the main idea defended by the author. The core idea in the text is always a statement, a thesis, and expresses the line of reasoning used to transmit the message, i.e., the author's process of logical reasoning. After showing the logical structure of the text, you may plot and develop a map based on the (primary and secondary) ideas presented in the text. (p.31)

Thus, following some of the steps suggested by the thematic analysis, we have found that, from 2009 to 2011, nineteen papers of the Postgraduate Studies Program - Mathematics Education of

PUC/SP faced the challenge of using GeoGebra. We tried to identify which aspects and dimensions were highlighted, and how and in what conditions such researches were carried out. For the analysis desired, each paper was read as a whole, in order to obtain a global view and understand the goals, procedures, and results of each research.

## **JUSTIFICATION OF THE STUDY**

Even though other papers in the field of Mathematics Education have already been written in our program using other software applications that enable interaction, what drives us to carry out this study is revealing the knowledge of the researches carried out, specifically those using GeoGebra, thinking about the postgraduate production obtained, which aspects and dimensions were highlighted and favored, and justifying the significance of using GeoGebra in Mathematics Education.

In his research, Soares (1989) states that:

Such understanding of the status of understanding of a theme, at a certain moment, is required in science's process of evolution, in order to periodically sort all the information and results that have already been obtained. Such sorting enables to point out possibilities to integrate different perspectives - which were apparently independent - the identification of duplications or contradictions, and the determination of gaps and deviations. ( p. 3)

On the other hand, according to Ferreira (2002), knowing what has already been built and produced allows to point out further research possibilities, towards what has not been done yet, in an attempt to handle an amount of knowledge that grows more and more very fast. Ferreira (2002) also highlights that:

In the last twenty years, thanks to the strengthening of the theoretical and scientific production - with researches carried out in several programs and postgraduate courses around the country - a movement has become an effort of several entities (colleges and research funding associations) to define a publishing policy for their scientific papers. (p.260)

All researches shown herein have the educational choice to use GeoGebra in Mathematics Education in common, were carried out by professors while working as such and, therefore, provide answers to a class of professionals - elementary school teachers - who see the university as a service provider which can assist their teaching and whose production is not restricted to the shelves of university libraries.

## **METHODOLOGICAL PROCEDURES**

In order to analyze the papers and after reading their titles, all papers were initially grouped according to the mathematical content explored. The mathematical content explored was initially identified based on the title of each paper; however, as Severino (2001) states:

The title of a unit does not always reveal its theme accurately. Sometimes, it merely suggests it by association or analogy; other times, it has nothing to do with the theme. ... it is necessary to identify the author's approach perspective: such perspective defines the scope within which the theme is treated, restricting it to specific limits. (p.54)

Thus, at a second moment, we read the respective abstracts in order to identify the research question, the goals, and the types of researches carried out: whether quantitative or qualitative, the respective audiences involved, and the theoretical and methodological support used. Reading and analyzing the abstracts enabled us to identify the underlying mathematical object of the research, the theoretical and methodological support, and the target audience. However, it was necessary to read



each research more carefully to accomplish what was intended for this study, analyzing the role played by GeoGebra in each of them, what considerations were made regarding its use, the purposes of each author, and the results obtained.

## **THEORETICAL SUPPORT OF THE REMAINING STUDIES**

Out of all nineteen studies, nine used different theories as support and will be analyzed in this study. The studies used strategies supported by theoretical alternatives that also enabled them to reach their respective goals. It can be seen that they played a key role in creating the proposals presented with the mediation of GeoGebra.

### **Argumentation as a teaching method: case - concepts of area and perimeter of plane figures**

This research analyzes what extent argumentation may be a method that promotes the comprehension of mathematical concepts. It treats the practice of the argumentation as teaching method, focusing the concepts of area perimeter of plane figures. Studies in national and international levels have already broached the subject, many times using the practice of the argumentation as method, not proposing, however, ways that demonstrate the functionality of that method. So this work answers the following question: in what measure the practice of the argumentation can present itself as method that contributes to the comprehension of concepts in mathematics taking as reference the case of the area and the perimeter of plane figures? To answer our question, we propose a didactic sequence modeled and analyzed with basis in the phases that compose the argumentative process, according to Toulmin (1996). The methodology of the study have been supported in Didactic Engineering purposes, the intervention have been effectuated with pupils at the fifth grade (students aged 10-11), using two argumentative institutions: the classroom and the informatics laboratory where we used the Geogebra software. The analysis of the activities have evidenced that the practice of the argumentation contribute to the comprehension of the concepts of area and perimeter of plane figures, habilitating this practice as teaching method.

### **A study of the demonstration in the context of the Licentiate Degree in Mathematics: using different types of proof and the levels of geometric reasoning**

The main purpose of this research was to investigate the influence of dynamic geometry environments in building up arguments by teaching graduating students in Mathematics. Also searched a probable articulation between the student's geometrical development levels and the types of tests he makes. The research done distinguishes itself as qualitative with aspects from a study case. The procedures for collecting the data were the students' written records, their geometrical constructions taped in Geogebra software, dialogues audio-recorded, and interviews semistructured. The bibliographical review indicated a need for studies about the learning process of demonstrations in Mathematics teaching initial formation courses. The results analysis permitted to observe that the dynamic geometrical environment has little influence on the arguments construction by the students. The research subjects were not familiar with the environment tools and the "to drag" provided by the software became much more a way of confirming the empirical suppositions. The results also permit to infer the existence of an intermediate level between the existent levels designed by *spatio-grafique* geometry (G1) and *protoaxiomática* geometry (G2) (Parzisz 2001, 2006) that welcomes the transition moment between them. This intermediate level would have as characteristics the instability in the type of invoked object (physical and theoretical) and in the type and validation (perceptive or theoretical). It was observed that the types of tests naïf empirism and crucial experience came up as a result of geometrical thinking in level G1, while the type of test mental experience appeared associated to geometrical thinking in level G2. Such observations also cooperated for the certain need of an intermediate geometrical thinking level between G1 and G2.

### **A hypothetical learning trajectory for trigonometric functions from a Constructivist perspective**

This work aims to verify: as compatible constructivist perspectives of learning with the planning of Trigonometric Functions; teaching them as researches in mathematics education field, which brings important results on the learning process may contribute to the organization of the Trigonometric Functions teaching that leverage best learning situations for students, as the performance of teachers of mathematics is revealed, with regard to planning activities in the teaching of Trigonometric Functions, consistent with a constructivist view of learning. With was developed a qualitative study with two teachers and 70 students from the 2nd Grade of high school in a public school in the State of São Paulo. Its theoretical work of Simon (1995) on the use of HLT in teaching mathematics to formulate models of teaching based on constructivism. As a component of the Mathematics Teaching Cycle developed by Simon, the elaborate HLT was made use of the research findings for the development of Trigonometric Functions through activities and solve problems involving: constructions with ruler and compass, manipulative material, scientific calculator, construct graphs using software GeoGebra and paper and pencil. The results led us to conclude that the use of research contributes to the education organization of Trigonometric Functions; however you must provide access to such teachers to such research. Although the HLT are potentially rich, complex is the task of developing activities to accomplish a constructivist learning perspective.

### **Pedagogical strategies using technologies to teach trigonometry with circles**

The effective learning of the student is the main goal of a reflective teacher, so it is not enough simply to have all the technical knowledge, you need something more, namely mediation, have a clear and objective, to mobilize material resources for the success of this process. Just thought, this work was to build a meaningful learning the basics of trigonometry in circumference. Use this research to teaching engineering, a research methodology that is considered as an experimental scheme based on achievements in teaching classroom. The research has two analytical tools, giving great importance to the training analysis of the error (Brousseau's 1983, 2001). The first instrument directs for construction of the trigonometric circle, using ruler, protractor and pencil, the second instrument construction is done in dynamic geometry software GeoGebra. The activities of this research tool, as applied to twelve students in the 2nd grade of high school. The trial shows that prior knowledge have been mobilized to carry out these activities, and information technology as a pedagogical resource, ie, activity with GeoGebra aroused the interest of students because they were more focused and performance was better.

### **Conical Sections: activities with Dynamic Geometry based on the school program of the State of Sao Paulo**

The present work proposes, based on the Mathematics curriculum of São Paulo State Educational Secretary (SEE-SP), complementary activities to the material supplied by the SEE-SP, seeking to approach less discussed aspects of the curriculum on the conic sections. The activities were elaborated according to the instruction to the material designated to the teachers for the use of digital technologies and software of dynamic geometry (Almouloud, 2007). Such activities were presented to the state public network teachers, in a formation course offered by the conic section classes. During the course for the teachers' formation audio records were taken of the spontaneous manifestations of the educators. Such manifestations were analyzed in order to answer the following questions: what activities could be recommended for the teacher work based on the São Paulo State curriculum? Which aspects should be taken in consideration by the public network teachers of São Paulo State facing the challenge of creating complementary activities to the pedagogical proposal in the teacher's guide book? The records of the teachers' manifestations show the importance of

continuing formations, the interest for certain approaches and possible obstacles for the implementation of such activities in the classroom.

### **Isometric transformations on GeoGebra with Ethnomathematical motivation.**

The research described reports on a qualitative research had the purpose to enable high school students in a public school in the Metropolitan Region of São Paulo, implement and develop knowledge of mathematical object Isometric Transformations by Rotation, Translation and Reflection. Was used in this research, as motivating factors, the Ethnomatematics with Sona Geometry of African ethnic group called Cokwe and Dynamic Geometry using the software GeoGebra. The methodology, Design Experiment, enabled the improvement of a sequence of activities and created the final product of research. Levels of development psychogenetic Piaget and Garcia (1983), intrafigural, interfigural and transfigural possibility to observe the relationships between students identify geometric figures, their properties and structures. The development of this study revealed, made after the analysis of the protocols of the proposed activities, which supported the GeoGebra and Ethnomatematics favored the learning of Isometric Transformations.

### **The Mathematics School Program of the State of Sao Paulo: suggestions of activities using GeoGebra**

This research presents the analysis of the learning situations in Geometryas investigation perspective and shows a proposal to articulate the situations of learning found in the Mathematics Teacher's Notebook for the 4<sup>o</sup> two months period of the first grade of the secondary school, published by the Educational State Secretary from the State of São Paulo. The hypothesis of investigation which has guided this survey is that the use of computational technology, with the aid of GeoGebra, could encourage the proposal of articulating the learning. The question of research is: In which way it is possible to create a dynamic approach with the software GeoGebra for the Secondary School content of Plain Geometry based on the State of São Paulo's curriculum? This paper has been elaborated following the methodological orientations of Bardin (1977), about the Analysis of Content, which consists in treat the information by the specific plain of analysis: pre-analysis, exploitation of the material, treatment of the results, inferences and interpretations.. At the end of this paper it has been observed that it is possible to articulate the learning situations with the use of the GeoGebra software in a simple and significant way.

### **Dynamic Geometry and Differential and Integral Calculus**

The aim of this work is to present fundamental ideas of differential and integral calculus and its applications in solving problems. Every teacher develops along its trajectory ways to represent the ideas you want to convey and that is the essence of pedagogical reasoning. In that sense, it is possible to understood that every idea must be transformed to be taught and it was this aspect that directed this work. Inspired by the possibility of using software in the teaching of Mathematics and didactically based on "Dialectic Tool-Object" and "Game Tables" by Régine Douady, apud Almouloud (2007), this work consists of a sequence of activities where basic ideas about derivative, integral and optimization functions are presented by means of software GeoGebra. With this work it is expanding the idea that most students have the Calculus and its applications, besides stimulating the use of technological resources as tools for large capacity in interpreting and solving problems.

### **Difficulties and possibilities Math teachers encounter when they use GeoGebra in activities that involve the Thales' Theorem.**

The aim of this research was to verify which are the difficulties and possibilities of Mathematics teachers when they use the Geogebra software on activities related to Thales Theorem. Four teachers of public schools of São Paulo State, Fundamental Level, took part of this investigation. The teachers participated in a workshop related to teaching math subject in focus in this work, and

which proposed the development of teaching strategies using the software GeoGebra. In the analysis of the proposed activities for teachers, resorted to the study of seizures proposed by Duval (1998) as well as to the work of "pedagogical transposition" proposed by Chevallard (1991). From the descriptions of the teachers, different profiles of use of technologies and educational performance, in relation to teaching strategies, were identified, highlighting the connection between the degree of reference mathematical knowledge and the proposed approaches with mediation by computational interface.

## CONCLUSION

The researches show that, in order for a teacher to use a technological tool as support to teach mathematics, it is first recommended to determine what goals he/she intends to reach, which type of mathematical knowledge he/she intends to provide his/her students with, and which technological tool can help him/her to reach such goal. Another piece of evidence shows that resources such as paper, pencil, ruler, among others, played a complementary role and may not be completely discarded. Likewise, the researchers, using institutionalization and playing an active role in the respective processes (but without robbing subjects of their freedom), enabled subjects to develop hypotheses, experiment, question, and use GeoGebra independently to the point they developed their own strategies.

The detailed analysis and interpretation of all nine studies presented herein showed the subjects of the researches, in their interaction with GeoGebra, developed mathematical knowledge, collaborating with the process of learning the respective mathematical object explored and revealing that the use of technologies as mediators in the learning process may subsidize pedagogical strategies to teach and learn Mathematics.

Today teachers have many other resources and tools with the new educational technologies, but they must reinvent or even rediscover themselves in a new role. Although the final objective is the same, since Mathematics is the mastery of abstractions, the technologies indicate other types of relationships with the students and new dynamics in the classes. Now, teachers and students all play an active role in this new scenario. The didactic material is interactive and can be modified dynamically through GeoGebra.

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# MATHEMATICS EDUCATION AND ELEARNING: MEANINGFUL USE OF THE QUIZ-MODULES

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*In this paper we discuss the use of close-ended questions in self-training and formative self-assessment in mathematics e-learning. Besides the undeniable value of automatic feedback and assessment, we argue how to avoid drawbacks of this tool offered by e-learning platforms, and focus on how to exploit the potential of this tool also by taking into account some key features of mathematics learning, such as multisemioticity. We give some indications on how to construct effective close-ended questions apt to actually evaluate competencies and not only contents.*

## EDUCATION AND SELF-ASSESSMENT

The spreading of e-learning has brought in vogue the use of online assessment, which consists in computer-assisted assessment (CAA) delivered on a local or distant server and gives powerful tools to automate an important part of the learning process. In this paper we discuss the use of module *Quiz*, available in any e-learning platform. It allows to set up various questions with close-answers (multiple choice, true/false, fill-in blank) and to have automatic evaluation. It is undoubted that they have some huge drawbacks: for instance, in the case of mathematics learning, they do not stake all the competencies, such as to construct a strategy or a text, and they allow the use of improper strategies, such as to take a wild guess of the right answer or to deduce clues from the form of the given distractors. Anyway, in this paper we argue that work can be done in order to limit such drawbacks and make the use of quizzes become a helpful learning resource for the student.

First of all, we assume that our use of quizzes aims to self-training and *formative* self-assessment, that is we want to use assessment not to measure the learning outcomes (summative assessment) but to improve the learning process during its progress. So it is not proposed at the end of a learning path, but it is integral part of the process and it is of use to guide, to develop, to make adjustments and to improve it. To this aim, the role of the feedback given to the student put it on a key position, because it becomes the main tool for making concrete a proper teacher's guide to improve the learning process. This is why we talk about "formative" assessment (Schulze & O'Keefe, 2002, in Ibabe & Jauregizar, 2008), which provides the student with those information such as hints, recovering materials, suggestions for further work, institutionalization of learning acquisitions, and so on. Therefore, to make the feedback effective through the quizzes, it is not sufficient to set up the questions so that the computer just check whether the answers are correct or not, but it is necessary that who is in charge of creating the quizzes should pay much attention to provide feedbacks helpful to explain the errors or to foster the students to think about why their choice is wrong or to suggest further work in order to fix the required knowledge. Then the design of a quiz requires an in-depth analysis of the computer-provided feedbacks in case of wrong answers. Finally we note that for our purposes it is better that the assessment does not give back any overall marks, in order to focus the student's attention on his strengths and weaknesses.

## ADVANTAGES

### Self-sufficiency

The possibility of automatic assessment and feedback offered by the selected-response items makes them particularly suitable to their implementation in e-learning platforms. This feature seems to be

very helpful when the assessment involves large groups of students, such as undergraduates, because they allow to equip the students with learning experiences which should be unaffordable in traditional settings. For instance, it is possible to assign weekly tests to provide the students with feedback on their comprehension of the face-to-face lectures or of materials suitably set for delivering in the previous week, or recurring quizzes with feedback on their comprehension of key concepts introduced during face-to-face courses (Jenkins, 2004-05). It is evident that the chance of automatic assessment and immediate feedback are indispensable requisites of the feasibility and the sustainability of such activities. Anyway we want to underline the fundamental role of the tutor, which remains indispensable, even if she changes the place and the time of her intervention. She is no more the one who intervenes when the individual student interacts with the activities, but she is the one who is in charge of suitably designing the items of the quizzes and her tutorial activity is made concrete in planning convenient proper feedbacks according each possible answer (correct or not). The great importance of the formative feedback in order to improve the learning makes the new role of the online tutor fundamental even when closed quizzes are used.

### **Flexibility**

The students can tailor training activities according a chosen level of dimension and difficulties. For instance, to make a quiz with a little number of random questions, or it is possible to construct a sequence of quiz of growing difficulties so to allow the students to face the difficulties gradually. This impacts on the affective aspects too, as the student has the possibility of doing activities adapt to his avoiding the frustration which can be derived from too many failures.

### **Challenge**

From the viewpoint of the self-assessment, the chance to repeat more and more the quiz appears fundamental because this can be useful in order to improve student's knowledge and at the same time to reduce the need of tutoring (Valenti, in Bartalini, 2008). On the other hand, the chance of getting immediate feedback can motivate the student to go on and improve his marks, repeating more times the quizzes. Moreover, to this end, the CAA offers the student reporting tools able and useful to monitor his progress, as an added value with respect to the paper and pencil assessment: each student should have the opportunity to look at his reports and to make comparison between his outcomes and the ones' of his classmates (in observance of the privacy laws) (Bartalini, 2008).

## **AN OVERVIEW**

### **The features of the quizzes**

The construction of a quiz requires first of all to have well defined its aim. In our case, as already said, our aim is self-training and self-assessment. The classical parameters to evaluate a quiz are validity, reliability, practicality, washbacks.

The *validity* is the property of a quiz (or even a single item) which actually stakes the competencies which it is designed to stake. In the next section we will see that also the format of the single item (not only its reference content or its structure) can break validity. The *reliability* is the property of a quiz whose outcome is not influenced by external or occasional factors. The *practicality* is the possibility to construct, use and evaluate a quiz without an excessive waste of resources of all the people involved in the whole process. Finally the *washbacks* derive from the fact that inevitably any form of assessment influences and steers teaching and learning. This is why also the quizzes should avoid to steer (even indirectly and unintentionally) towards kinds of dubious teaching/learning practices, such as focusing on contents only or neglecting the aspects related to semiotic representations and language.

In the case we are interested in, that is the self-training and self-assessment use, the four parameters just illustrated are all important, since it is needed to avoid to address the users' efforts towards wrong directions, to supply with warped information on their preparation and to steer their subsequent study in an improper mode.

### **The limits of the close-ended questions**

Many doubts are placed on the actual capability that the close-ended questions have to assess learning outcomes. Anyway there is a growing number of studies confirming that such capability can be more improved as more care is posed for the analysis and the development of the items (Conole & Warburton, 2005). To this aim, it is strongly recommended that the ones who develop CAA quizzes are properly trained and that the items are peer-reviewed and tested before the students' use (Boyle & O'Hare, in Conole & Warburton, 2005). Anyway it is beyond argument that the close-ended item do not stake the capability of setting out a solving strategy for a problem based on the only reading of the text, nor the capability of producing a text to describe or motivate the raised strategy.

The difference between a closed-ended question (for instance, multiple choice question) and the corresponding open-ended question (for instance, brief essay) highlights how poor and unproductive is the setting of one who identifies an item of a quiz with its content. Let us see an example.

#### ***Multiple choice version***

The initial value of a quantity whose value became 40 after an increment of 25% was:

- A) 30
- B) 31
- C) 32
- D) 33

#### ***Open-ended version***

Which was the initial value of a quantity whose value became 40 after an increment of 25%?

It is evident that the two items, even though they refer to the same content, involve completely different competencies. The multiple choice version supplies implicitly with various information (for instance, that the answer is an integer positive number) and requires at most some check, whilst the open-ended version requires anyway some modeling of the problem. Moreover, there is general evidence of the fact that the selected-response items leave more opportunities to random answers (possibly after the exclusion of the less plausible cases) or to answers that make use of inferences on the distribution of the distractors.

Since the close-ended items (or, at most, numerical ones) have the great advantage of the practicality, as they allow the student to get immediate feedback and assessment without the need for a tutor, then the key point consists in finding ways to construct close-ended items which require to stake competencies and forbid or point out improper strategies.

### **SOME PROPOSALS**

The CAA offers the possibility to construct types of items more complex of those which can be made with paper and pencil, such as for instance the inclusion of audio-visual materials or the use of various systems of semiotic representation or various registers of verbal language. The literature shows that the simple straight translation of the paper-base assessment into the corresponding online version is absolutely inadequate and it is fundamental to think about the pedagogical implications

that the online assessment requires in terms of formulation of the questions, feedback provided to the student, storage and reports of the data related to the student's interactions (Conole & Warburton, 2005).

### Structure of the items

From the above discussion it comes out the need for an accurate analysis of what an item actually requires, beyond the classification of its content. It is evident that items which just require the application of an algorithm for selecting one out of a set of options, actually do not necessarily require the competencies apparently involved. Then there is the need for designing items which: (i) require the careful reading of the text; (ii) require the modeling of the problem situation; (iii) discourage improper strategies. All this can be realized by means of continual variations of the assignment, with careful choice of the distractors, and with the systematic insertion of the option "other", which should be the right one in an appropriate number of occurrences. It should be borne in mind that, in contexts where a simple algorithm allowing to get an outcome by direct application to a set of data is available, the request of reconstructing the data given the outcome trigger processes much more complex than the simple request of directly finding the outcome from the data.

Examples of a multiple-choice problems fulfilling these conditions are the following:

- (1) "Which of the following values of  $x$  make the relation  $G.L.B.(x, 12) = 6$  true? [6, 9, 12, 18, none of them], or also:
- (2) "Which of the following values of  $x$  make the relation  $G.L.B.(x, 12) = 6$  false? [6, 9, 12, 18, none of them], or even:
- (3) "Which of the following values of  $x$  make the relation  $G.L.B.(x, 12) = 6$  true? [3, 9, 12, 24, none of them]

All of the problems require a careful reading of the text and the modeling of the problem situation, as the number of appropriate answers might change (0, 1, more than one), also related to the wording of the problem ("true", "false"). The solver has to know and use the definition of G.L.B. and cannot simply apply an algorithm. Problems of this kinds, although they involve very simple ideas and procedures, might prove tricky even for undergraduates.

### Multisemioticity

The quizzes allow multisemiotic activities, even if the multisemioticity of the texts should be explicitly planned, and it can require some supplementary efforts. According to O'Halloran (2005), we consider three groups of semiotic systems: the verbal language, the symbolic notations, the figural representations.

The role of the verbal language is central for various reasons: it is reflexive (that is, it is able to talk about itself), it is able to classify the reality, even in approximate and informal way, it constructs the human experience and makes it communicable, it articulates the various voices of a culture, it allows the use of a broad range of linguistic varieties.

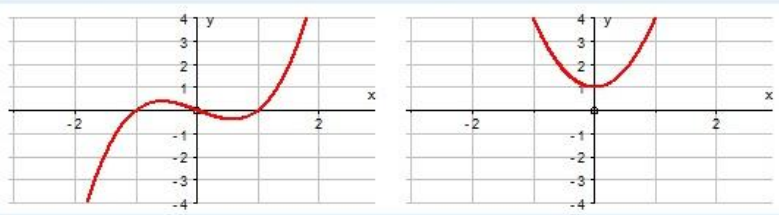
The role of the symbolic notations in mathematics is subtler and more debated, whose fundamental function consists in describing systematically the corpus of the mathematics knowledge, supporting the decidability of the concepts and the calculability of the processes, rather than in being guarantee of an improbably rigorous.

Also the role of the figural representation is much discussed, which stake processes, cognitive and not, non simply modeled.



Each item of a quiz can include verbal texts, symbolic expressions, images (Fig. 1). It is possible create proper items in order to assess and construct the capability of using at best the plurality of semiotic systems now available, according to that point of view which Duval (1995) names the *coordination of semiotic systems*.

Considera la funzione  $f$  il cui grafico è tracciato sotto a sinistra. Sappiamo già che  $f$  è derivabile e che la funzione  $g$  rappresentata sotto a destra non corrisponde alla derivata di  $f$ .



Quale o quali fra le frasi che seguono motivano correttamente il fatto che  $g$  non corrisponde alla derivata di  $f$ ?

Scegli una o più alternative:

- a. Perché il grafico di  $f$  passa per l'origine e il grafico di  $g$  non ci passa.
- b. Perché  $g$  è a valori positivi, mentre  $f$  non è crescente su tutto l'intervallo visualizzato.
- c. Perché  $g$  è a valori positivi, mentre  $f$  non è a valori positivi su tutto l'intervallo visualizzato.
- d. Perché  $g(0)=1$ , ed  $f$  è decrescente per  $x=0$ .
- e. Perché la derivata di una funzione dispari deve essere dispari.
- f. Nessuna delle motivazioni precedenti è adeguata.

**Figure 1: sample of item using various semiotic systems.**

## ISSUES FOR FUTURE WORK

Finally, we want just propose some recommendations regarding important issues to be tackled in order to seriously start to think about the usage of quizzes in learning mathematics:

- *Usage of CAS.* Even if many examples of questions not requiring mathematical engine can be found, it is natural to consider the possibility to use a CAS for generating on the fly different instances of the same question as well as for checking the equivalence of different symbolic answers. Moreover it allows to create new kinds of question, such as those requiring to construct an object satisfying some mathematical constraints (Sanwing, 2003, in Bartalini, 2008). Finally a CAS can be useful to distinguish different kinds of errors: logical inconsistency, theoretical gap, bad computation (Albano et al., 2003). The main drawback of the usage of CAS concerns the need for the student of knowing the CAS syntax.
- *Dynamicity of the quizzes.* For multiple choice and multiple answers quizzes the randomization of the choices can be introduced. So for instance each time the quiz runs the system randomly chooses 3 options among 20 stored ones, allowing to generate different instances of the same question. This way the student can exercise with the same questions more times and the risk of plagiarism is reduced if two students contemporarily exercise with the same question. If a CAS is used, then template of exercises (open questions) can be set up in order to generate on the fly different instances of the exercise/question data (Albano et al., 2003).
- *Usage of quizzes to personalize the learning path.* Tailored resources can be delivered to the students according to the quizzes' outcomes. A first example can be the Moodle activity 'Lesson', which allows to insert some check questions at the end of contents. If the student is successful in quizzes, she is addressed to further contents, otherwise to some recovery

activities. More sophisticated personalization can be allowed if a Knowledge and a Student models are available: in this case quizzes guide the selection of the next resources to be delivered, with respect information on the student's knowledge and learning preferences.

- *Analysis on quizzes' reports.* Reports on quizzes' outcomes provide valuable intake into students understanding of the matter assessed, offering the teacher new cues for further learning activities (Setzu et al., 2011). For instance they allow to identify the questions which most of students have troubled with, that are those with the minimum score, thus proper recovering activities can be suggested, such as forum to collaboratively face the problem. At the same time, in a context of community of practice, students who are usually successful can naturally become peer-tutors.

It is worthwhile to underline the need for teachers' training in order to foster the required "cultural shift" so that they spent time in preparing effective quizzes, pertained to competencies instead contents. Its use as self-assessment make it a fundamental tool in the view of the formative assessment, as it can help the student to monitor his learning progress (Ibabe & Jauregizar, 2005, in Ibabe & Jauregizar, 2008, Bartalini, 2008).

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## WAYS OF MANIPULATION TOUCHSCREEN IN ONE GEOMETRICAL DYNAMIC SOFTWARE

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*Touchscreen devices are spreading and becoming familiar for many students. This research identifies ways of screen touching during the process of solving problems using the Geometric Constructor (GC) software. We designed a case study with some Italian High School students, who used tablets to solve geometric problems. Since manipulation on tablet is different from that with mouse clicking, this kind of research investigates a new aspect of students' behaviours when using dynamic geometry software. Based on researches by Yook (2009) and Park (2011), we provide a new schema focused on geometrical thinking and strategies used by students to solve the proposed tasks. We observed singularities in the way students rotate (using one or more than one finger) and a different way of dragging: the dragging to approach.*

### INTRODUCTION

The emergence of multi-touch devices - such as iPods, iPhones and iPads - will promote new impact and challenges in learning and instruction in general, and in mathematics in particular. Although in Mathematics Education some touch devices have been developed (for instance, Fingu, Geometer Sketchpad Explorer, Geometric Constructor and Math Tappers) research is still scarce concerning mathematical learning through touchscreen (Toennies et al. 2011, Barendregt et al. 2012, Ladel and Kortenkamp 2012).

In our previous research we have stressed that mathematics learners use gestures, looks, gazing, words, sketches and productions on the interactive white boards or on a software (Arzarello et al. 2011). In one of our current project we are interested in the way of manipulation of tablet resources as iPad. Particularly, how ways of manipulation can enrich the multimodality of communication in instruction that touch devices provide. In this paper we address issues regarding cognitive features basing on touchscreen manipulation. Specifically, we point out some specific modalities of touchscreen in some Italian high school students, who solve problems using the Geometric Constructor (GC) software. Since students are becoming increasingly familiar with multi-touch technology and manipulation, this kind of research focus on a new aspect of learning mathematics within ICT technologies environments.

### FROM CLICK TO TOUCHSCREEN: NEW CHALLENGES OF INTERACTION AND LEARNING PROCESSES

Tablet manipulation is not the same as mouse clicks. Most current tabletop interaction techniques rely on a three state model: contact-down, contact-move, and contact-up—more akin to mouse dragging (Tang 2010). Touchscreen user interface is a type of graphical user interface that uses an electronic touchscreen display for input and output (Doyun 2011).

Interaction through current mobile touchscreens basically occurs with the computer recognizing and tracking the location of the user's input within the display area. Regarding their usage, environment mobile touchscreen user interfaces employ a specialized interaction model. A study by Yook (2009), as quoted by Park (2011), presents a framework on this interaction model. This interaction model is divided into finger action from the user and motion feedback from the interface. Finger action

consists of basic action and active action. Basic action refers to tap and hold, which are the basic ways of interacting with a touch interface. Active action is a combination of this basic action and the performed finger action which includes drag, flick, free, or rotate. According to Yook (2009), motion feedback that occurs in reaction to this finger action consists of closed motion and open motion. Closed motion refers to a designated motion that occurs in response to the user's input, such as scale, flip, move, or push. Open motion occurs in relation to the user's input by reflecting the spatial and time quality of the finger action.

So far, we have found few studies in Mathematics Education analyzing learning or instruction based on touchscreen devices. Two of them focus on geometrical concepts. Robidoux and colleagues (2012) implemented activities in two touch interfaces: The Geometer's Sketchpad on the iPad and the PHANTOM Omni<sup>h</sup> haptic device. The research hypothesized that mathematical activities within these kinds of multi-modal environments mediate multiple meaning-making tracks, as well as elicit discourse regarding conjectures and refutations in early learners' perceptions of complex geometric shapes. They allow users not only to see and manipulate geometric figures on a screen but also to feel, through force-feedback, and touch via direct contact with a screen, the result of interacting with such objects. Even though the implemented activities with the devices were not designed to instruct students, the researchers observed that they can yield an expressive language intimately connected to the induced mathematical concepts; moreover they supported students in using and enriching their prior experiences while making sense of a mathematical task within a visual-haptic environment. Toennies et al. (2011) studied the motivation to assist teachers in educating visually impaired children: they described initial feasibility studies using an available haptic touchscreen (Android tablet) to display grids, points, lines, and shapes with impaired students. They developed user studies to evaluate perception of these geometrical objects, and their early results showed that it was possible for most users to find specified locations on a grid, determine the locations of displayed points, and differentiate between lines and shapes, with haptic feedback, auditory feedback, and combinations of the two.

The research of Doyun (2011) and Park et al. (2011) focused on motion feedback as an element of interactivity that can be designed to enhance the user's experience of touchscreen user interfaces. Our research relates to that of Park since it considers the classification of the different typologies of manipulation as a strategy to analyze geometric cognitive processes on touchscreen software. By now we are interested in motion feedback as a powerful strategy to improve interaction, discovering and thinking in mathematics education.

## **THE GEOMETRIC CONSTRUCTOR (GC) SOFTWARE AND THE PROPOSED TASKS**

GC/html5 (Geometric Constructor) is a free dynamic geometry software developed in Japan by Yasuyuki Iijima at Aichi University of Education. One can use GC/html5 with PC (Internet Explorer 9, FireFox, Safari, Opera, Chrome) and tablets. The GC allows constructing basic geometrical objects (points, segments, lines, circles), measuring them, dragging and making traces of geometrical objects and so on. It is also possible to edit the construction (using different colors, names etc.) and save the constructed shape on server or on your machine (web storage). Below we show one implemented task.

**Varignon Theorem:** In the quadrilateral ABCD (Fig. 1), the middle points (E, F, G and H) on each side have been drawn, forming quadrilateral EFGH. What characteristics does EFGH have? What happens if ABCD is a rectangle? What if it is a square? What if it is any quadrilateral? Make your conjectures and prove them.

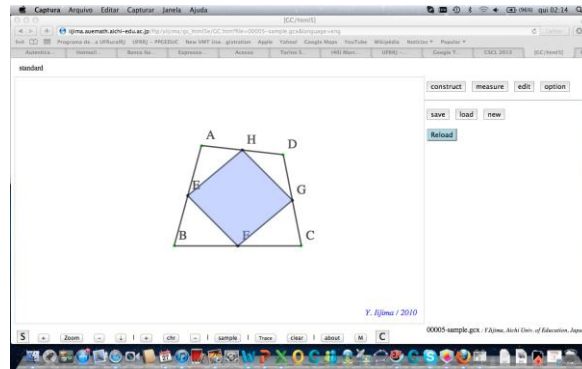


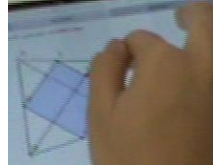


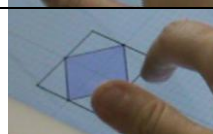
Figure 1


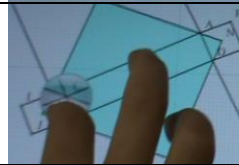
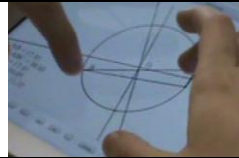
### Methodological aspects of the study

We developed a case study. Five High School Students (16-17 years old) at Liceo Volta (Turin, Italy) were enrolled on the three-research sessions. All of them had previous experience with Cabri Géomètre. Students worked in two groups: a pair (1 tablet) and a triple (2 tablets). Each session was 2 hours long and in each one the students did three activities like the one illustrated above. The analysis process was mainly based on the videotapes of students working on the software and written answers for each task. We identified type of actions (Yooks' quoted by Park 2011): basic (tap and hold) and active (drag, flick, free, rotate).

### Results: basic and active typologies of touchscreen

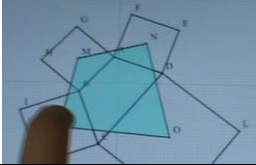
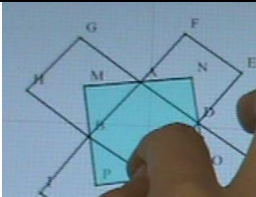

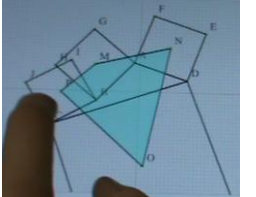
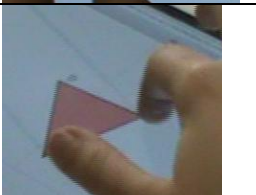
In the table below we illustrate example of students' basic action using GC and describe some aspect regarding their geometric thinking dealing with the software. We observed all the students' manipulations on the screen. Although in order to make a construction (point, line, angle, circle etc.) the user has to use the software icons, we did not consider such touches. Rather, we could observe more than one type of different touches: we selected some ones, in which the exemplified type has predominance.


| Basic action                    | Example   | Geometric process  |
|---------------------------------|---|--|
| Tap ( <i>single or double</i> ) |  | <i>Activity 2:</i> student constructing angle to observe relation among diagonals and the side of quadrilateral ABCD |
| <i>Flip</i>                     |  | <i>Activity 1:</i> student observing the figure, moving it on the screen and getting to know the software            |
| <i>Move</i>                     |  | <i>Activity 1:</i> student trying to move the picture on the screen  |
| <i>Push</i>                     |  | <i>Activity 2:</i> student pushing the point and trying to find the free one   |

|               |   |  |
|---------------|---|--|
| Scale         |  | <i>Activity 1:</i> student getting to know the environment and trying do amplify or reduce some piece f the screen |
| Hold (single) |  | <i>Activity 3:</i> student making a zoom at one point <sup>2</sup>   |
| Hold (multi)  |  | <i>Activity 1:</i> student moving randomly with more than one finger and observing the objects motions             |

**Table 1: Example of students' basic action using GC**

Although we observed the types of touchscreen in all implemented activities, the examples presented here were randomly selected. Since touching implies motion, it is difficult to perceive the movement using a static picture. Some types of manipulation as move, scale, flip etc., occurred few times. It can be due to software characteristics or to the nature of the task. We observed that push action could be just at one point or on the screen in a random way. In Table 2 we illustrate examples of students' active fingers action using GC software.

| Active action           | Example   | Geometric process   |
|-------------------------|---|---|
| Drag free               |  | <i>Activity 3:</i> after having constructed the last square on each side of the quadrilateral ABCD, a student drags freely point P to see what happen with the shapes |
| Drag to approach        |  | <i>Activity 3:</i> student approaching MNOP to a rectangle to analyze how the shapes become constructed on the side   |
| Flick                   |  | <i>Activity 1:</i> student getting to know the software and flicking the empty screen   |
| Free                    |  | <i>Activity 3:</i> student moving freely point C <sup>3</sup>   |
| Rotate using one finger |  | <i>Activity 1:</i> student making construction and moving the selected point with one finger  |

|  |   |   |
|--|---|---|
| Rotate using two fingers but with one fixed finger |  | <i>Activity 1:</i> student keeps one finger fixed (from the left), moves the middle and observes what happens |
| Rotate with two fingers (both in movement)         | Example of students' active action using GC                                       | <i>Activity 1:</i> student selects and rotates the shape in two points  |

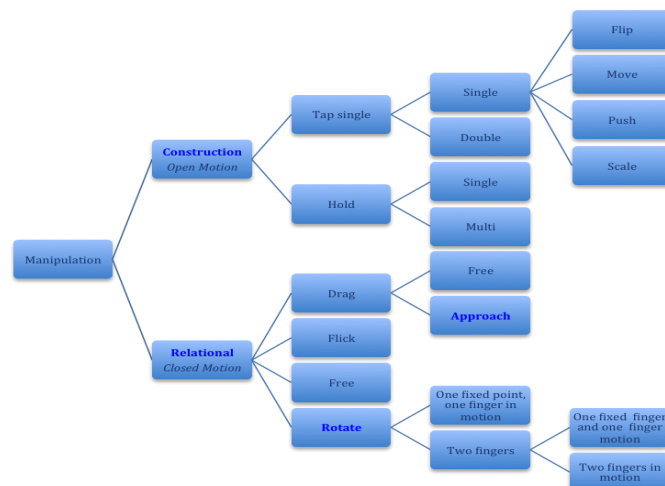
**Table 2: Example of students' active action using GC**

We found singularities on active actions, particularly, on drag and rotate type. Firstly, in drag and free types students manipulated with one or two fingers. In a few cases they used three fingers. Regarding drag we found two types: free and the one we called to approach. Due to the fact that students had to meet some request on the proposed task they wanted to go further on the way of dragging. So, in this sense, the dragging to approach<sup>4</sup> appeared. It is a kind of manipulation that students do when dealing with some specific geometric propriety, shape or construction.

Regarding rotate typologies, we observed three possibilities: rotation using one finger; rotation using two fingers, but with one fixed finger, and rotation with two fingers, with both in movement. Although the first two seem the same mathematically, we think cognitively they can provide different insights in terms of the use of the fingers. Conceptually, in order to rotate one shape we need to determine before in each point (the center of rotation) and with the use of two fingers the decision could have not been done beforehand. Or, at least this was not explicit for touch users. In that sense it can bring new conceptual aspects for the way we deal with rotation and, so far, in the same direction, the last observed way of rotating (two fingers in movement).

## FINAL REMARKS

In this paper we point out and classify different touchscreen typologies, which are produced during problem solving processes using the GC software. So far, regarding the usage of single or multi touch fingers we observed that students had been manipulating the figures using mainly one or two fingers only (Tang et al. 2010). Since they worked in pairs sometimes they also “shared fingers” (for instance, 1 finger each) to manipulate some figure, especially when the shape had more geometric objects or constructions. After identifying and interpretation each type of touchscreen related with geometrical thinking through the proposed tasks, we provide a picture that includes one more possibility to drag (approach) and three ways for the rotate action (Fig. 2).



**Figure 2: Structure of touchscreens on GC**

Due to the nature of the geometrical task we identified the predominance of touchscreen types on the relational domain and touch such drag free, flick or rotate occurred few times. The usage of drag to approach was dominant. Although rotate appeared few times, those appearances allow us to observe three different ways of rotate, as illustrated in Fig. 2. We observed students doing rotation with some shape: we believe that looking for the types of manipulation can provide new epistemological insights for geometrical conceptualizing in touchscreen devices. Identifying in which geometric construction the manipulation with more than 2 fingers occurs may be an interesting issue in future study.

Finally, besides cognitive challenges and constraints of the GC, we observed that the use of touchscreen devices can provide new pedagogical hints for mathematical instruction.

## NOTES

1. This study is part of the research carried out by the second author of this paper during his stay as visiting scholar (granted by Ministry of Education/Capes, Brazil – BEX 8845-11/5) at Torino University (Department of Mathematics) with the supervision of the first author during the second semester of 2012.
2. As far we understood Yook (2009) framework, this type of action also can be considered as a kind of scale. We had a lot of single actions to exemplify with, but we prefer to use this one to underline that some actions can be very closed in terms of cognitive understanding.
3. The picture illustrated student free manipulation after finishing the construction asked on the task.
4. This approaching can be understood in the same sense of maintaining dragging.

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## THE PLANIMETER AS A REAL AND VIRTUAL INSTRUMENT THAT MEDIATES THE INTRODUCTION OF AREA

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*The paper describes a particular approach to the notion of area in the secondary school as “swept area”, using a specific old professional tool, the planimeter. It allows measuring the area of a plane domain basing exactly on the idea of swept area. Hence students are introduced to a fresh notion of area, which is intrinsic of a figure and is not strongly based on the idea prevailing in the school, which founds the notion of area on algorithms, possibly confusing these with the concept itself. Using the instrument, the students are introduced into a didactical cycle that structure their learning processes. A semiotic lens is used to describe it.*

### INTRODUCTION AND THEORETICAL FRAMEWORK

The current research, which has been carried out as a M.Sc. Dissertation in Mathematics Education, considers the use of the planimeter as a tool for providing an alternative understanding to the concept of area making use of the “swept” area interpretation.

Let us frame the setting of our approach basing on Hasan’s definition (Hasan 2002) of mediation. In our case the planimeter plays the role of the *mediator*, in his different use, tangible or technologic (respectively with Lego bricks and GeoGebra software). The *content of mediation* is the concept of area and the *mediatee* are the students that have attended the set of lectures introducing the area by means of the planimeter.

The mediation has been performed principally with the use of the planimeter in order to exploit the close relationship between the bodily experience and the learning process (Arzarello, 2006; Lakoff & Nunez, 2000/2005). Since we have used a tool that generates the concept that we want to convey we are within the framework of *semiotic mediation* which studies how “within the social use of artefacts in the accomplishment of a task (that involves both the mediator and the mediatees) shared signs are generated.” (Bartolini Bussi & Mariotti 2008).

In the following we will distinguish between *artefact* and *instrument*, according to Rabardel’s point of view (Rabardel 1995). In fact we consider artefact (a material with its own physical and structural characteristic made for specific tasks) distinct from instrument (an artefact with a specific utilization scheme).

Another point of view we will use based on Rasmussen and co-workers (Rasmussen, Zandieh & Wawro, 2009) researches: they introduce the concept of *boundary object* and of *brokering*, i.e. the process of appropriation of meaning, which changes according to the community one belongs to. This is performed by the *broker*, that is a person being part of more than one categories, e.g. a teacher (part of the mathematic community and the class group) or the students (intended as belonging to the class and to the single work group).

We used this didactical tools to build up a transition path between two different concepts of area known by students in their school career. In fact, a major didactical aim of our project is to support the transition from an elementary concept of area, based on formulas and possibly on the equidecomposability notion (learnt by the students at the beginning of secondary school), towards the more elaborate concept of area defined as a definite integral (topic that they will learn in the last year of secondary school).

## AMSLER'S PLANIMETER

The planimeter is a tool for measuring the areas of flat shapes. Along centuries several kind of planimeters were conceived (Care 2004). For our research we have used the polar planimeter, built by Amsler in 1854 (Amsler, 1856).

Amsler planimeter is an artefact made of:

- Two joined arms whose constraint allows only reciprocal rotation,
- One of the arms is attached to another constraint (fixed point) allowing only rotation, right bottom in fig.1,
- A lens, called tracer, that follows the contour of the figure that we want to measure,
- A wheel, top right in fig.1, physically constrained to rotate only perpendicularly to the second arm,
- A counter that keeps track of distance travelled by the wheel.



Fig. 1

The utilization scheme for the measurement is the following: the fixed point is chosen outside the figure, the tracer is placed on a point on the contour of the figure, the counter is reset and the contour is followed for an entire cycle. The planimeter will return a number proportional to the area of the figure.

The functioning of the planimeter relies on two key ideas:

1. The distance covered by the wheel is proportional to the area swept by the second arm. In fact it is possible to split every possible movement of the arm into a translation and a rotation and it is known that:
  - a. Motion in the direction of the arm does not sweep any area,
  - b. The area swept by a translation is proportional to the height of the swept parallelogram, i.e. the distance covered by the arm,
  - c. The algebraic sum of all rotation is null because the planimeter must return in the initial position.
2. The area swept by the arm corresponds to the area of the figure, provided a concept of swept area with sign is introduced: positive or negative according to the direction of movement of the arm. For example in fig. 2, the area outside the figure, which is swept twice but with opposite directions will have a null value and only the area inside the closed curve will be evaluated.

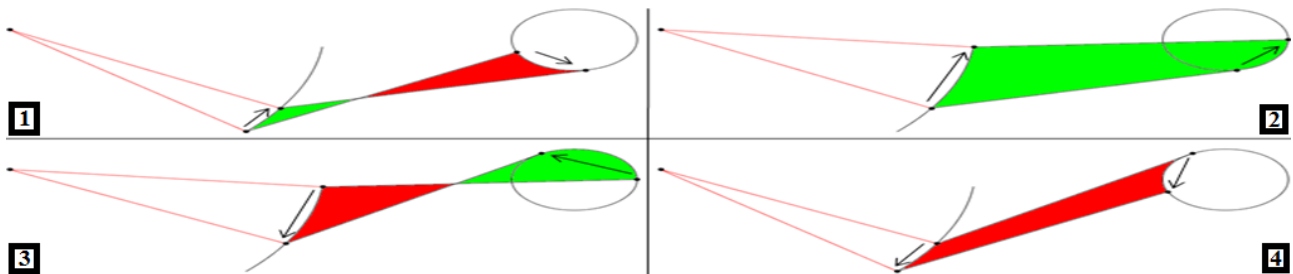


Fig. 2

## METHODOLOGY AND ACTIVITY

The activity we consider here is based on two teaching experiments lasted about a month each one. The experiment consists of a pilot lesson developed in a secondary school in Cremona (Italy) held in

January and repeated, after a careful analysis of the first results, in another secondary school in Torino in April. Both schools are Liceo Scientifico (science focused high-school) and both experiments consist of 8 hours lecturing and a 1 hours testing. At the time of the experiment, the students in Cremona were attending the third year (grade 11), and their previous knowledge about the concept of area was limited to the usual algorithmic definition, whereas the students in Torino were attending the second year (grade 10) and they knew the theory of equidecomposability. All the activities were carried out during regular mathematics lessons, and designed by the authors.

A further lecture for M. Sc students of Mathematics attending an advance course in Mathematical Education is currently being planned and will take place at the end of May 2013. In such lecture the planimeter will be used as an example of semiotic mediation for proving the Stokes-Green theorem. Since the lecture is yet to be given we restrict to shortly pointing at the planned activities without giving the details.

In the secondary class lessons, the students spent a lot of time working in groups of 2 or 3, and in individual tasks, being always required to explain their reasoning. The researcher also directed some general discussions that permitted the students to communicate and compare their different solutions.



Fig.3

Before the beginning of lectures the students received a package containing Lego bricks with an assembly manual for the planimeter (fig.3). This has been done in order to stress the embodied construction of mathematical meaning. Indeed as claimed by Arzarello (Arzarello, 2006) along the line of cognitive studies about embodiment (Lakoff & Nunez, 2000/2005), the mind construction of mathematical concepts strongly benefits from the actual use of the body and concrete tools. The lectures had been built on the concept of *didactical cycle* (Bartolini Bussi & Mariotti 2008). To this aim *Activities with artefacts* were scheduled for pairs or small group promoting the emergence of specific signs in relation to the use particular artefacts/tools, followed by *Individual production of signs*, to write individual reports on their own experience and reflections including doubts and questions related to the previous activities with artefacts. Finally concluded by *Collective production of signs*, in particular based on mathematical discussion in which the various solutions are discussed collectively and converge to shared mathematical signs.

In the first lecture we showed a video made by the authors explaining what a planimeter is and illustrating its use as an actual measurement tool for some figures without explicitly saying that the measurement is proportional to the area. Afterwards, each group of students used an industrially made planimeter on a sample figure and the planimeter they had built themselves on a different set of figures. Finally we lead a mathematical discussion in which we pushed the students to conjecture the proportionality between area and the measures got using the planimeter.

In the successive lectures we proved the statement. During this proof we lead, with some school works, the students' group to carry out independently some pivotal point of the proof. In the pilot teaching experiment we gave a worksheet to the students with the task of finding the ratio of a rectangle to the parabolic inscribed segment: the students were asked to use the Lego planimeter to measure and to explain (see *Individual production of Signs*) how the planimeter works and how they used it. Finally they measured the area of an "amoebic" figure, therefore experiencing the possibility of measuring areas without knowing a formula (specifically, when no such formula exists).

Another lecture was entirely carried out with the computer and was based on a model of planimeter done by the authors within the dynamical geometry environment of GeoGebra. In this lecture we added a further mediation instrument: the software. It was possible to highlight in a mathematical

discussion the great advantage in terms of usage simplicity and result precision provided by the computer technology.

The last lecture started with the discussion of the previous lectures, in which every student was invited to share his/her consideration with the classmates, according to the mentioned methodology of *brokering* (Rasmussen, Zandieh & Wawro, 2009). The aim was to *institutionalize* the concept of swept area, in the sense of Brousseau (Brousseau, 1997), also making reference to the historical background that led to the formalization of such a concept. For this reason, in designing the teaching experiment we considered also the cultural context, see the definition of *CAC* (Boero & Guala, 2008): in fact we used the planimeter context to introduce the theory of planets motion around the sun, highlighting the relations with the shape of orbits and Kepler's 2<sup>nd</sup> law. For this, we based our lectures on the studies of Kepler in "*Astronomia nova*" (Kepler 1609) and on the well known lecture by Feynman about Newton's explanation of Kepler's laws, as collected by Goodstein and Goodstein (Goodstein & Goodstein, 1996/1997).

In the final assessment we reserved a prominent role to the actual measurement of area and to the abstraction of the concepts presented in class when explaining the functioning of the planimeter. We asked to prove the functioning of the linear planimeter and to explain the well known algorithm for the computation of the rectangle area basing on the concept of swept area.

## ANALYSIS AND DISCUSSION

Since the research is still ongoing, it is not yet possible to present a definitive analysis of the lectures. Its analysis will possibly allow us to highlight the acquisition of the skills that we wanted to build through our lectures and to give more details about the *didactical cycle* (Bartolini Bussi & Mariotti 2008), into which the students enter. To this aim we will compare the given lectures, fully video-recorded, and the protocols produced by the students at the end of the cycle. The complete results will be illustrated during the presentation at the conference.

For the analysis we adopt three lenses: the mathematical approach, based on the *CAC* analysis (Boero & Guala, 2008), the educational consideration of cognitive construction of the concept, based on the *didactical cycle* by Bartolini & Mariotti, and the role of gestures. The last level refers to fresh studies in psychology and in education, in particular in mathematical instruction (Arzarello, 2006; Edwards, 2003; Goldin-Meadow, 2003; McNeill, 1992): it focuses on the multimodal construction of mathematical concepts, extending the semiotic analysis of signs produced in the activity beyond the usual verbal register (see the construct of *semiotic bundle* in Arzarello, 2006).

Another element is the use of *cognitive pivots*, such as words, gestures, symbols or signs that lead to the construction of a new concept (see the notion of *ZPD* in Vygotsky, 1998, that of cognitive pivot in Arzarello, 2000 and the idea of *semiotic node* in Radford et al., 2003).

A complete analysis will be illustrated during the presentation at the conference: here we sketch only an example, to give an idea of our methodology. We present the analysis of cognitive path to convey the concept of swept area: the movement of planimeter over a semi-circumference centred in the constrained point joining the two arms and with radius equal to the length of the second arm. The use of planimeter in this semi-circumference is of considerable significance because it manages to overtake the cognitive obstacle due to the sign of the swept area.

During the lectures we showed a video of the industrial made planimeter and the GeoGebra simulation, in order to support its understanding by means of graphic visualization (i.e. the possibility to paint the swept area). Finally we suggested to check this situation with the Lego planimeter.

The effects of this procedure were readily noticeable on the students: they interiorized the concept of swept area as the set of consecutive position of the moving arm. This result can be deduced by analyzing the final test protocol, where they explain the concept of swept area using the aforementioned semi-circumference. In fact, we can mention a student's intervention (Luca). As the other students in the final test, he refers to the semi-circumference to explain swept area. We can notice, in particular, the value of gesture: he waves his hand to draw the swept area. It happens twice: when he is sitting at his desk (fig.4) and when he is standing at the blackboard (fig.5).

He supports his gestures with some words:

Luca: Moving along the semi-circumference the planimeter draws this arch, to which the area of the circular sector is associated.

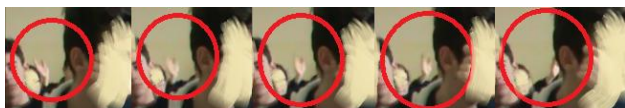


Fig. 4

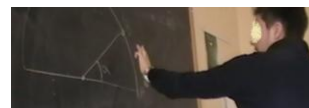


Fig.5

## CONCLUDING REMARKS

In this research, we show the effectiveness of the use of instruments to build mathematical meaning. The use of tools in the teaching overcomes the idea of lecturing in mathematics; in fact it allows students: to build mathematical meaning by their selves, by using of the tools, and to share it with the rest of the class; to attend lectures based on a new way of teaching, closer to cognitive construction of concept; to understand that mathematic has a deep meaning and it isn't just an algorithmic issue; to highlight the cultural value of mathematical themes; to get their view of mathematic closer to the scientists' one, i.e. making experiences, formulating conjectures and proving them; to emphasize their active role; to give a higher value to the building of skills than to the pure memorizing of knowledge. The development lines of our activity are numerous and interdisciplinary. Our cycle of lectures can continue: in geometry, calculating the area of the figures; in physics, dealing deeply Kepler's and Newton's law, or analyzing areal velocity, or discern constrained movement (holonomic or nonholonomic constraints); in analysis, building integral starting from the swept area, or talking about infinitesimal (we have used it in the intuitive proof of the main theorem); in history, introducing mathematical machines from a historical and cultural point of view, and showing/discussing some excerpts from the original texts (e.g. "Astronomia Nova" by Kepler, in particular his drawing, which are very important for stressing the notion of swept area); in computer science, proposing to program other simulations with GeoGebra; in numerical Analysis, calculating the error of the approximation done in the formula used on GeoGebra.

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## **ESTABLISHING CONDITIONS FOR COMPUTERALGEBRA AS AN ADDITIONAL VALUE IN MATHEMATICS TEACHING**

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*There is an ongoing debate about chances and burdens of implementing Computeralgebra in the classroom. Schools, school administration and especially mathematicians at universities are engaged in this open debate and still a lot of countries do not involve Computeralgebra in their examinations and mathematics classrooms. On one side the main reasons for excluding Computeralgebra from classrooms can be summarized as apprehension that students loose paper-and-pencil-skills such as solving an equation by hand. On the other side the reasons for including Computeralgebra into the classrooms are shown by a lot of studies which describe the potential of Computeralgebra in teaching and learning mathematics such as learning a better algebraic insight (Heid & Blume 2008; Zbiek et al. 2007). Against this background it is a big challenge for school administrations to decide in a correct and sustainable way to give schools proper and clear principles for their teaching and learning at schools. This challenge was the reason that in 2010 the Ministry of Thuringia in Germany commissions a meta-study to find out conditions that ensure a successful use of Computeralgebra in teaching mathematics and in assessment. This meta-study, written for the ministry, is the content of this presentation and paper (Barzel 2012).*

### **RESEARCH QUESTIONS AND METHODOLOGY**

According to the aim of the ministry the meta-study tried to find answers to the questions:

- Which conditions and criteria have to be fulfilled to make an integration of Computeralgebra successful and an additional benefit for learning and teaching mathematics?
- Is it worthwhile to integrate Computeralgebra in the final central examination (Abitur)? If yes, what are the parameters of a successful integration?

We have reviewed the international research field that dealt with the integration of CAS in learning, teaching and assessing. Papers in English and German have been taken into consideration. We only mentioned papers about findings of studies related to the implementation of CAS and especially with statements concerning the conditions of success in the implementation of CAS. Only current results (younger than 2000) were considered. Another criteria was to avoid redundancy - so we only have included central papers of one study in case there exist several papers of the study. Of course we did this reduction after checking whether we will lose important details. Eventually, more than 300 publications have been included in the review. Finally we did a close inspection of 163 sources taken out of magazines, (e.g.: Educational studies (EM), International Journal on Mathematics Education (ZDM), International Journal for Technology in Mathematics Education, International Journal of Computers for Mathematical Learning, Journal for Didactics of Mathematics (JMD)); Collected editions and meta studies; Monographs and dissertations; Conference proceedings.

The core ideas of these sources have been excerpted and categorized. This process can be described as a cycle of reading, excerpting and categorizing and especially as periodic change between individual and group work interaction of three persons (Raja Herold and Matthias Zeller have been involved beside the author).

## RESULTS: ASSUMPTIONS CONCERNING THE CURRENT RESEARCH

All the papers investigated in this study are scientific papers combining theoretical ideas and empirical results. Nevertheless the main focus of the single papers allows to structure into empirical papers about single studies and theoretical papers such as meta-studies or theory papers such as papers about “instrumental genesis” (Aldon 2010; Drivers & Trouche 2008; Heid 2005). The empirical papers use a theoretical background for the design of the study and the interpretation of the results and the theory papers use a lot of empirical results to foster their theoretical ideas and concepts. Nearly half of the empirical papers are about qualitative studies in single or few cases and the other half large-scale- studies with a mixed-method design combining qualitative and quantitative investigations (e.g. CAS-CAT, Australia, Ball 2004; CALIMERO, Germany, Bruder & Ingelmann 2009; MUKI, Germany, Barzel 2007, CAS Pilot Programme, New Zealand, Neill 2009).

The results and main aspects in all papers of the whole review have been categorized and structured into three fields CAS while learning, CAS while teaching and CAS in assessment.

In every field key points which can be found in several papers have been pointed out as assumptions. These assumptions do not follow the demand of being disjunct.

### Assumptions concerning CAS while learning

*The use of CAS can foster conceptual knowledge especially in the field of algebra.*

The facility that CAS offers manipulations with variables enables investigations of the given output by the machine. This output has to be reflected and interpreted in the frame of the specific task. Together with the direct adjustment of input and output and to understand the equivalence of input and output (Zeller & Barzel 2010) it is necessary to have a deeper view on the algebra behind it. Cuoco & Levasseur (2003) speak of a mathematical laboratory, which can foster the “algebraic insight” (Ball und Stacey 2005c, S. 128) and with this the conceptual knowledge in the field of algebra. An example which shows the potentiality to combine techniques with theory is the investigation of the results when the expression  $x^n-1$  is factorized (Kieran & Drivers 2006). This example forces not only to realize the equivalence of input and output but also to anticipate general patterns and structures between different expressions.

*Paper-and Pencil-skills can also be acquired in CAS lessons.*

Most often teachers are afraid, that students loose paper-and-pencil-skills when using CAS. A first argument against this fear is the interplay between procedural and conceptual knowledge:

"There seems to be an interplay between the algebraic considerations of the students and the procedural knowledge that can be left to the technological tool. A CAS might even provoke the development of procedural skills." (Abdullah 2007, S. 15).

CAS can be used to control paper-and-pencil-manipulations and on the other side it is important to initiate reflections on the results given by CAS. Therefore you need as well technical skills to use the tool and mathematical knowledge – both is being mixed up:

"So they [the students] need to have built some instrumental knowledge, which is a complex mixture of technological and mathematical knowledge." (Artigue 2004, S. 219) .

Beside these interrelations between conceptual knowledge and procedural manipulation skills it is important to require paper-and-skills explicitly in the learning process (Lagrange 2003, Ingelmann 2009). But it has to be clear to the students “how the use of technological tools relates to the required paper-and-pencil skills.” (Kieran & Drijvers 2006, S. 205).



### *The use of mathematical language can be supported by CAS*

Learning mathematics includes reading and writing of mathematical symbols and language and the development of different mathematical representations (Arzarello & Robutti 2010). Using CAS requires learning an additional language, the language of the tool. This new language is needed during input but also when interpreting the output of the machine. This additional translation causes problems and possible difficulties (Zehavi & Mann 2003, S. 189), but can also be used in a productive way to foster learning and to stimulate conceptual aspects (vgl. Greefrath 2007, S. 58).

An example for such a constructive way is the reflection on the special role of variables when using CAS. It is necessary to make explicit the specific variable when solving an equation or calculate a derivative. This is an additional request which you do not need when doing the calculation by hand but it can foster reflection to deepen the understanding (Heid 2003, S. 44).

### *Technical skills can purposefully supplement mathematical aims*

A lot of studies point out that it is important to learn how to use the CAS in a proper way and to use the interrelation between technical skills and mathematical aspects (Weigand 2006, S. 100–101; Lagrange 2003, S. 271).

## **Assumptions concerning CAS while teaching**

### *CAS can support the teaching contents' widespread and genetic structure.*

Teaching can be structured along a systematic, deductive way where mathematical knowledge is separated in units and presented to the students as an instruction. But teaching can also be structured in a social-constructive way following a more genetic idea beginning with an open problem leading to the new bit of mathematics. The way how to involve CAS seems to depend on these different paradigms. Kendal & Stacey (2001) have observed that teachers who focus on symbolic manipulations and rules stress more the systematic way of teaching. They use CAS more following the White-Box-Black-Box-principle (Heugl et al. 1998), where CAS is only used after having learned the procedural skills by hand. Only if a mathematical aspect is clear ("white") it can be done automatically ("black") and then it can be given to the machine, students have to „earn“ the use of CAS. These teachers use CAS at the beginning of a learning unit quite seldom and allow CAS at the end of the unit more often, they "move towards" CAS (Kendal & Stacey 2001, S.143). In contradiction to that approach is the way to begin with an open problem and CAS is involved in the learning process from the very beginning in a more conceptual approach, following the Black-Box-White-Box-principle. Students use CAS for unknown subject areas and they use it quite often at the beginning and less at the end.

"The teacher who stressed understanding moved away from using CAS, whilst the teacher who stressed rules, adopted it more." (Kendal und Stacey 2001, S. 143).

"Introductory algebra students can [...] learn to use and interpret the results of symbolic manipulation in their reasoning and problem solving without having learned to perform those symbolic manipulation procedures by hand." (Heid & Blume 2008, S. 92).

But this more constructivist approach is unfamiliar for many teachers who are used to follow a systematic instruction route (Kieran & Damboise 2007, S. 111)

### *The integration of open tasks is supported by the use of CAS*

Open tasks in mathematics teaching are tasks for modeling or problem solving. CAS can influence the solving process because it helps to focus on the interpretation of calculated results and enables

to follow the modeling cycle more than one time. Together with the possibility to use real data, it supports open tasks (Drijvers 2003, S. 241)

*With the use of CAS new opportunities concerning a variation of teaching methods can be possible.*

When teachers meet the challenge of a more investigative or constructivist approach they are forced to reflect on their teaching practices (Neill 2009, S. 17) and how to organize the learning process to enable open tasks allowing different approaches. They have much more to moderate instead of leading the teaching process (Drijvers & Trouche 2008) and they have to fulfill a conscious designing and changing of teacher centred and learner centred phases (Aldon 2010; Barzel 2007).

### **Assumptions concerning CAS in assessment**

*New task and examination formats are necessary to assess additional competences*

Quite often there are only small differences between CAS and traditional examination tasks (Brown 2003, Weigand 2006). A reason for this phenomenon is that examination tasks are less open than exercise tasks. Exercise tasks with CAS usually possess a higher complexity, higher language demands, more approaches and ways of modelling. All these characteristics do not fit the demand for examination tasks (Pallack 2007). A lot of studies about CAS in assessment point out the importance to improve existing examinations by adding new tasks to assess instrumental knowledge and mathematical understanding (Neill 2009, S. 25). A possible way to improve these tasks is to integrate CAS directly in the assignment of the task (Kieran & Drijvers 2006). There is a variety of proposals about new examination formats (Thomas 2004), such as splitting the examination in a technology-part and a technology-free-part or news formats like oral or group examinations.

*The use of CAS can increase the amount of text within solutions*

The increase of amount of text as well the variety of solution strategies makes it much more difficult to correct examinations with CAS than without CAS (Weigand 2006, S. 107). It is important for teachers and students to have clear standards for documenting solutions and to make clear if the focus is on the solution or on the process (Brown 2003) and which steps should be documented.

### **CONCLUSIONS: CONDITIONS AS RECOMMENDATION TO THE MINISTRY**

The question if students' competencies which are described in the national standards will be improve automatically and principally by using CAS cannot be answered in a simple way on the basis of the current state of the art in research. Regarding single details with a differentiated view you can conclude that students' competencies according to the national standards can be influenced in a positive way by using CAS, if CAS is used as catalyst towards a student-centred and understanding-oriented teaching. These conclusions can especially be drawn by looking at large-scale-studies with a theory based intervention (Neill 2009; Leng 2003; Weigand 2006; Barzel 2007). The advantages and benefits of CAS can only be developed if the integration of CAS comes along with an explicit reflection and a change of tasks and teaching style.

To foster such a process with the aim to improve the competencies requested in the curriculum and to ensure a positive development, certain institutional conditions are essential. It is important that the structural and financial frame is clear for teachers as basis for rethinking and reflection their teaching traditions. Therefore we recommend to the ministry that CAS should be obligatory in curricula and assessment and to divide the final examination into a part with and a part without technology to ensure that also paper-and-pencils-skills will be learned. It is also important to build up communities of practice for teachers to increase exchange and teamwork and a good system of

ongoing professional development. A network of information for parents and students could support teachers in their every day work to convince parents and students of the value of integration CAS.

Based on the expertise (Barzel 2012) the ministry of Thuringia has decided to make the use of CAS obligatory in the centralized final examination in Thuringia from 2014 onwards.

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## THE HUMAN CATAPULT AND OTHER STORIES – ADVENTURES WITH TECHNOLOGY IN MATHEMATICS EDUCATION

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*This paper reports on-going research into how the affordances of off-the-shelf technologies can be aligned with relevant mathematics pedagogy, to create transformative learning experiences with the potential to overcome some of the well-known impediments to mathematics teaching and learning. From a systematic analysis of recent literature on digital technologies and mathematics education, a set of guidelines has been formulated by two of the authors for the design of innovative and engaging interventions in mathematics education. In this paper the guidelines are presented, along with results from experiences with two such interventions. An exploratory case study methodology is employed, and the paper reports on an initial pilot study, the results of which suggest that the interventions are pragmatic to implement and may improve affective engagement and motivation.*

### INTRODUCTION

There is ongoing international debate about the quality of mathematics education at post-primary level. Research suggests that, while the capacity to use mathematics constructively will be fundamental to the economies of the future, many graduates of the secondary-school system have a fragmented and de-contextualised view of the subject, leading to issues with engagement and motivation (Gross, Hudson, & Price, 2009; Grossman, 2001). Digital technologies have the capacity to facilitate realistic, problem-solving and collaborative approaches to teaching and learning, providing coherency and context for the mathematics. However ICT is frequently used in a more traditional manner, with didactic teaching methods, and an emphasis on de-contextualised procedure as an end in itself.

Tools such as Dynamic Graphical Systems (Hoyles & Noss, 2003) (for example, GeoGebra, Cabri and Geometer's Sketchpad), Computer Algebra Systems (for example, Maple and WolframAlpha), tablet/smartphone apps, and educational websites all provide mathematics teachers with readily accessible, and often free, tools to help their students overcome the challenges in becoming mathematically proficient. However, teachers can be overwhelmed when faced with such an array of technologies and pedagogical theories, and may benefit from a framework to guide the integration of technology into their teaching so that it is not used in such a way as merely to re-instantiate aspects of traditional mathematics teaching.

Following a systematic review and analysis of technology interventions in recent literature such a set of guidelines is under development by two of the authors (Bray & Tangney, 2013). In the present paper, a brief background to the development of the guidelines to date is outlined, and the design and delivery of two activities planned in accordance with them are described. The paper reports on an initial pilot study in an out-of-school setting in which 40 mixed-ability students, ranging in age from 15 to 18, took part. An exploratory case study methodology is employed in order to gauge the feasibility of the approach and the potential for further research.

## BACKGROUND

The development of the guidelines is based on an ongoing, systematic review of recent literature in which technology interventions in mathematics education are described. The electronic databases in use for the review were chosen for their relevance to education, information technology and mathematics, and include: ERIC (Education Resources Information Center); Science Direct, and Academic Search Complete. To date, 25 papers in which specific interventions are discussed have been classified according to the lenses of technology, learning theory and level of technology adoption (Bray & Tangney, 2013). The review process and subsequent development of the guidelines is ongoing and iterative: while the guidelines and activities are informed by the reviewed literature, the results of the activities along with emerging research will continue to inform and refine their development.

Technology is classified as: Dynamic Graphical Systems (DGSs); Outsourcing of Processing Power (Computer Algebra Systems, Graphics Calculators); Purposefully Collaborative Tools (Google Docs, Knowledge Fora); Simulations/Programming (Microworlds); and Toolkits (technologies designed in accordance with a specific pedagogic approach, with support for student and teacher). The learning theories considered are divided into two main camps - behaviourist and cognitive - with the latter further broken down into constructivist, social constructivist, and constructionist.

The SAMR hierarchy (Puentedura, 2006) is used to describe the levels of technology adoption (Figure 1). This hierarchy classifies interventions according to two main categories – Enhancement and Transformation – with subcategories of substitution and augmentation (enhancement), and modification and redefinition (transformation).

To date, the review has revealed a marked prevalence in the literature of studies that involve DGSs and Outsourcing of Processing Power. Some collaborative technologies, programming/simulation tools, and toolkits are also in evidence. There is a strong socially constructivist direction evident in the interventions that have been considered. This pedagogical theory has its foundations in the learning theories of Kolb, Vygotsky and Bruner. The interventions deemed most successful, according to the review, are those that are classified as being within the transformation space in the SAMR hierarchy, that is, those that achieve significant task redesign or the creation of new, previously inconceivable tasks, through appropriate use of technology.

Many of the empirical studies reviewed are somewhat limited in that they concentrate on the implementation of a single technology without focusing on the more pragmatic issues around technology interventions that teachers may desire. These areas of consideration however, are addressed in a number of papers that do not examine specific interventions, but rather consider the circumstances under which learning has the potential to be enhanced by the use of technology (Means, 2010). These papers highlight the necessity for the use of a variety of appropriate technologies, implemented in meaningful, interesting and realistic scenarios.

## THE GUIDELINES

The literature review and subsequent classification of papers has informed the development of a set of guiding principles resonant with a view both of mathematics as a problem-solving activity and of

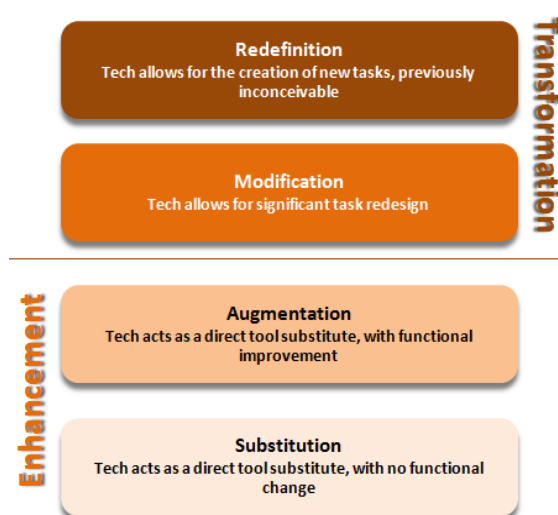


Figure 1 : The SAMR hierarchy

mathematics education as involving students in constructing their knowledge via the formulation and solution of problems. Moreover, it seeks to counteract beliefs – unfortunately prevalent – that mathematics is a collection of unrelated facts, rules, and ‘tricks’, and that mathematics education is about memorisation and execution of procedures that should lead to unique and unquestioned right answers (Ernest, 1997). The guidelines describe an approach to the design of learning experiences that aim to combine the educational potential of off-the-shelf technology with appropriate pedagogy. According to these principles, an appropriate and innovative technology intervention in mathematics education should exhibit the following properties (Bray & Tangney, 2013):

1. *Be collaborative and team-based in accordance with a socially constructivist approach to learning;*
2. *Exploit the transformative as well as the computational capabilities of the technology;*
3. *Involve problem solving, investigation and sense-making, moving from concrete to abstract concepts;*
4. *Make the learning experience interesting and immersive/real wherever possible, adapting the environment and class routine as appropriate;*
5. *Use a variety of technologies (digital and traditional) suited to the task, in particular, non-specialist technology such as mobile phones and digital cameras that students have to hand;*
6. *Utilise the formative and/or summative assessment potential of the technology intervention.*

## TWO INTERVENTIONS

In order to refine the guidelines and validate their usefulness, the authors are using them to create and evaluate a suite of technology-mediated learning experiences that aim to address some of the issues in mathematics education. These learning experiences are being initially piloted in an experimental learning space in the authors’ institution before being evaluated in mainstream classrooms. The design and delivery of two such interventions are described below. The longer term goal of this research is to create a community of teachers who are using the guidelines as part of their own classroom practice, and to determine if the learning activities so designed do in fact help address some of the concerns associated with the teaching and learning of mathematics.

### Scale Activity

In this activity, the students work collaboratively, in groups of 3 or 4, to develop a presentation about scale, orders of magnitude and scientific notation. The learning objectives include developing an understanding of how to recognise appropriate technological and mathematical techniques for measuring and estimating. The students, working actively and collaboratively, are required to select objects to measure and to make sense of their information, figuring out how to measure objects of diverse size, and to present their results.

In our implementation, smartphones are used to gather information, take measurements and perform some trigonometric calculations. Instruments utilised include tools for measuring distance and angles of elevation from the MobiMaths app (Tangney et al., 2010). The students are required to determine into which ‘Power of 10’ each measurement fits and have the option of further populating their collection using Google Earth, Google Maps, and research on the internet. The target for each group is to have two or three objects within each band

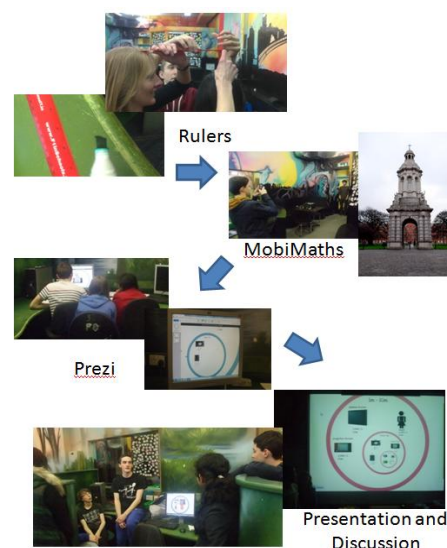


Figure 2 : Steps in the "scale" activity

of measurement and to cover at least five consecutive orders of magnitude. Prezi ([www.prezi.com](http://www.prezi.com)) is suggested as appropriate for creating the presentations, as it is straightforward to use and facilitates a zooming effect to simulate a perception of increasing and decreasing size.

The adoption of technology to mediate this activity has permitted significant task redesign, placing it in the level of 'modification' within the SAMR hierarchy. The students use smartphones for scientific calculation, capturing images, and a variety of measurements. Online mapping tools permit the measurement of greater distances than would be practical to calculate by traditional means.

This activity is designed to help students develop a sense of when and why different mathematical approaches and notations are required, and to acquire a realistic idea of scale and estimation, based on concrete examples. The final presentations and discussion allow for formative assessment of these learning goals, and for the scaffolding of deeper engagement with the topic.

### Slingshot Activity

Once again, a collaborative approach is adopted for this activity, in which students work in groups of 4 or 5 to investigate the properties of projectile motion. Particular emphasis is placed on functions relating height, horizontal distance and time; angles; rates of change; and velocity. Students use an oversized slingshot along with readily available, free software to conduct their investigations, moving from a concrete exploration of trajectory, to mathematical modelling of the activity, with verification of the results using a projectile motion simulation.

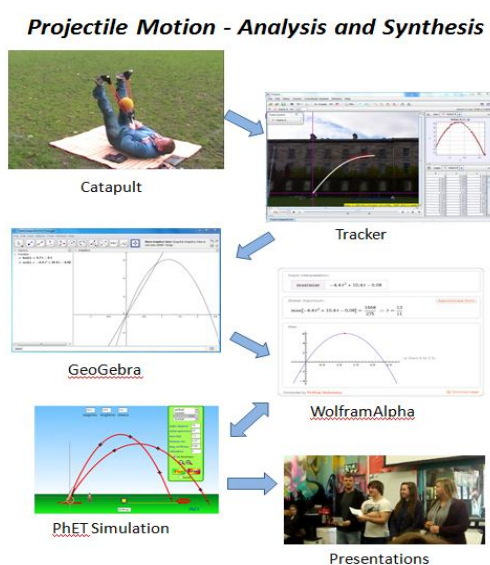


Figure 3 : Steps in the "slingshot" activity

Initially the students record videos of their team using the catapult to fire a foam ball. The trajectory of the ball is analysed using the free software Tracker (<http://www.cabrillo.edu/~dbrown/tracker/>) to trace the flight path, and also to generate functions relating height to time, horizontal distance to time, and height to horizontal distance. GeoGebra ([www.geogebra.org](http://www.geogebra.org)) is used for further analysis of the functions, enabling the students to estimate the angle of projection and initial velocity of the projectile. The investigations are guided and scaffolded by an instruction sheet, with suggested explorations provided. The computational website [www.wolframalpha.com](http://www.wolframalpha.com) can be used for routine calculations and for checking answers. Once the students have calculated the data required, they can use the simulation on [phet.colorado.edu](http://phet.colorado.edu) to check the validity of their results. Group presentations and whole class discussion conclude the activity, providing scope for

formative assessment as well as an opportunity for the students to consolidate and demonstrate their learning (Figure 3).

The tasks involved in this exercise would not have been possible without the use of technology, leading to its classification as 'redefinition' in terms of the SAMR hierarchy. The students are required to make extensive use of the computational facility afforded by the technology in a task that is designed to be engaging and immersive.

## METHODOLOGY & RESULTS

Our initial research is employing an exploratory case study approach, with use of direct observation, semi-structured interviews and questionnaires to gather data. This methodology was chosen to



address issues around feasibility of the approach and the suitability of the instruments. The Mathematics and Technology Attitudes Scale (MTAS) (Pierce, Stacey, & Barkatsas, 2007) is utilised to measure confidence, behavioural engagement, affective engagement, and attitude to using technology in mathematics.

In the pilot phase described here, three day-long sessions have been conducted with groups of students drawn from various schools. The setting is an experimental classroom environment in the authors' institution. A total of 40 boys and girls of mixed ability and socio-economic background took part in the trials. All the participants had previously taken part in group-based learning activities in the authors' institution.

### **Scale Activity**

Two day-long sessions were conducted based on the scale activity described above, with 13 participants on day one and 11 on day two. The participants were aged between 15 and 16. The day was broken down into an initial plenary session in which the main concepts were introduced; a planning phase, which gave the groups time to decide what to measure, estimate the sizes of the objects, and decide how to go about collecting the measurements with the tools available to them; an indoor and outdoor measuring session, which included estimation, generalisation and trigonometry; and a phase in which the students prepared their presentations. Each group was required to present their Prezi in a final plenary session, commenting on the accuracy of their estimates and measurements, and giving a short explanation of scientific notation, its relevance and how to manipulate it. Further questions, comments and corrections were posed by the coordinator.

The duration of the intervention is too short and the sample size too small for statistical significance to be of interest, but on each day the overall mean scores on the MTAS inventory rose, going from 73.9 to 79.4 out of 100 on the first day and from 68.8 to 74.8 on the second. Cognitive engagement with the material is suggested from student comments collected during the two activities. Quotes such as: "I found it a little bit hard to understand, but by the end of it, I sort of understood it"; "I learned how to look at maths in a different way"; "I would love to learn about other areas of mathematics using the things I did today"; and "I enjoyed the use of technology in maths. It makes maths fun and interesting", all go to paint a positive picture of the approach to integrating technology in mathematics education proposed in this research.

### **Slingshot Activity**

Sixteen upper second level students, aged between 16 and 18, working in groups of 4, engaged with the "catapult" activity which took place over a five-hour period, on a single day. This included an initial group session in which the tasks were discussed; an outdoor data gathering session; analysis using the various tools; and a plenary session at the end of the day, in which each team presented their findings. The day was punctuated with 'breakout' sessions in which one member from each team and the activity coordinator collaborated to discuss emerging difficulties with the tasks.

Once again, although the duration of the activity is too short to draw any substantive conclusions, informal interviews reveal that the students found the experience interesting and engaging; they particularly enjoyed the collaborative aspect and the immersive experience of the initial experiment. The use of readily available technology juxtaposed with the specialist software allowed them to experience mathematics at a realistic level, with numbers generated from the data they collected. Again, quotes such as the following suggest that the majority of students enjoyed this approach to mathematics education, believing it helped them to engage with the subject in a meaningful way: "Playing with catapults was enjoyable and using technology was a better way of learning and teaching maths"; "I found myself trying stuff and exploring lots of different things. Very fun."

## CONCLUSIONS AND FUTURE WORK

The purpose of this study is to assess the potential and feasibility of the design and implementation of activities in accordance with a set of guidelines under iterative development by two of the authors. The guidelines aim to assist in the generation of transformative activities that facilitate appropriate learning of mathematics at post-primary level, aided by technology. The present paper describes an exploratory case study examining two interventions designed in accordance with these guidelines. While the results of the pilot are limited in scope, they do suggest that the two initial activities are pragmatic, and warrant further investigation in real classroom settings. The results indicate that they may have the potential to engage learners cognitively, increase motivation and help contextualise mathematics. It bears noting however, that the participants' prior experience of working in groups, the small numbers of participants, and the novelty of the experience will have impacted positively on the results.

Running the interventions in the experimental setting has helped us refine the learning activities and has highlighted issues that may arise in a classroom setting. In particular, the duration of the activities extends over many single class periods, which is not usually feasible in traditional classrooms. In the next phase of our research we intend to cooperate with a network of schools that are currently engaged with our institution on an associated research project aiming to implement 21<sup>st</sup> century learning practices in mainstream classrooms (Tangney, Oldham, Conneely, Barrett, & Lawlor, 2010). These schools are already favourably disposed towards a collaborative, technology-mediated approach. In the longer term we also aim to engage with more traditional schools.

Further learning activities based on our guidelines are under development. The underlying literature review will continue to be expanded and reviewed on an ongoing basis in order to refine the results, keep the system of classification up to date, and further inform the guiding principles.

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## **THE MATHEXPLORER SYSTEM : STUDENT EXPLORATION WITH A MATLAB-BASED SYSTEM**

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*This article describes an approach currently being carried out at the University of Manchester where students are encouraged to explore and appreciate the mathematics as a complement to the traditional-style lectures. 1<sup>st</sup> and 2<sup>nd</sup> year students in Electrical Engineering use various 'notebooks' making use of Mupad in order to explore some mathematical topics, working faster than can be done when every calculation is by hand and gain an appreciation of the wider concepts surrounding the topics and how they fit into the appropriate topics within Electrical Engineering. The notebooks can be used during individual student study, or in teams, or can be used as part of lecture demonstrations.*

### **MOTIVATION**

The school of Electrical and Electronic Engineering (EEE) exists within the Faculty of Engineering and Physical Sciences at the University of Manchester and admits about 180 students to the first year of Undergraduate study. Of the 120 credits of study in year 1, 20 credits (10 in each semester) of mathematics courses are taken along with 100 credits of Electrical Engineering courses, some of which e.g. Circuit Theory, Electromagnetic Fields, make extensive use of the mathematics.

The material taught in the mathematics courses to EEE students has been of high-quality; it has enabled students to carry out routine (and sometimes less routine) calculations of the type encountered in Electrical Engineering. However, there are some aspects where it was felt that an additional tool or emphasis would be appreciated.

- Students could often carry out a calculation involving specific parameters e.g. analysing a circuit with particular values of resistance, capacitance etc. but did not see a more general picture of how the solution changed when the parameters were varied
- Students often struggled to use the mathematics in an Electrical Engineering situation despite the context often being shown. It was almost as if they struggled to comprehend that it was indeed the 'same' mathematics.
- Students often had difficulty visualising some geometrical situations, particularly in three dimensions, or properties of functions.
- Students were in danger of taking certain principles and properties for granted rather than exploring them properly.

In response to those points, it was decided to look for/create a tool which would allow students to experiment with functions, find solutions for various parameters, visualise situations etc. Initially two tools were considered i.e. Geogebra and Mupad. Mupad was chosen, partly due to its relationship with Matlab which the students were due to meet later in the course. Clearly, merely allowing the students to carry out unstructured exploration would not give meaningful results other than for the very brightest of students. Instead, it would be necessary to produce a structured series

of materials or worksheets each giving the students the option to explore particular mathematical topics within particular parameter and function ranges.

## CONSTRUCTION OF NOTEBOOKS

In Summer 2011, the school of EEE awarded £ 4 000 to the first author to produce a series of workbooks for use in academic year 2011-12 and beyond. The plan was for two students to carry out authoring of a series of notebooks covering various topics from year 1 of the mathematics course for EEE students. However, the preliminary stages involved extensive interviews with members of staff in both the school of EEE and the school of mathematics, in the former case to ensure that any applications were indeed relevant and in the latter case to ensure that the pattern of the notebooks followed the syllabus and that matters of notation etc. were consistent; this was indeed a joint project between the two schools. The intention was always that the notebooks would complement existing courses and materials rather than replace them. For example, students should be able to carry out the basic calculations by hand in full detail but should be able to use the notebooks to accelerate the process.

For two months in Summer 2011, Mustafa Ali and Jahangir Saif were employed as part of the project. Following a period of consultation and planning, they wrote 3-4 notebooks per week. Generally each notebook would undergo a time of authoring, followed a few days later by a time of checking and testing. Each student was generally working on several notebooks at a given time, with the relevant notebooks being at different stages.

## A TYPICAL NOTEBOOK

Generally, the notebooks would all follow the same structure with there being differences in the detail given the subject matter. An introduction is short as the topic is covered in the lectures and backup materials e.g. Harrison et. al (2004). The introduction section also allows students to define any functions that they wish to use and to define horizontal and vertical extents for graphs.

Figure 1 shows the relevant part of a typical notebook (on Maxima and Minima – one variable). The three central lines (in green on the screen) allow the student to control the functions and parameters. The left part contains the functions and parameters actually used (students can change these) with the central and right parts (effectively there as comments) giving the description and the default values. Students are not required to do any Matlab coding but are required to enter simple functions.

**Notebook's global variables** (*editable, remember "Notebook - Evaluate All"*)

```
f := x^3-6*x^2+9*x+3: // Function to evaluate (f := 2*x^3-9*x^2+12*x-3:)
xRange := 0..4: // x-Range for plot (xRange := 0..3:)
yRange := -6..9: // y-Range for plot (yRange := -2..6:)
```

### 2 Minima and Maxima

Figure 1 : The part of a notebook where functions and parameters are defined.

The middle parts of a notebook cover various sub-topics and reflect the choices that the student has made in the introductory section. For example, staying with maxima and minima, students can see the effect of their choice of function. The system calculates the derivative of the function and the position of any critical points as well as plotting the function and the derivative (see Figure 2) drawing attention to the fact that the derivative is zero at critical points.

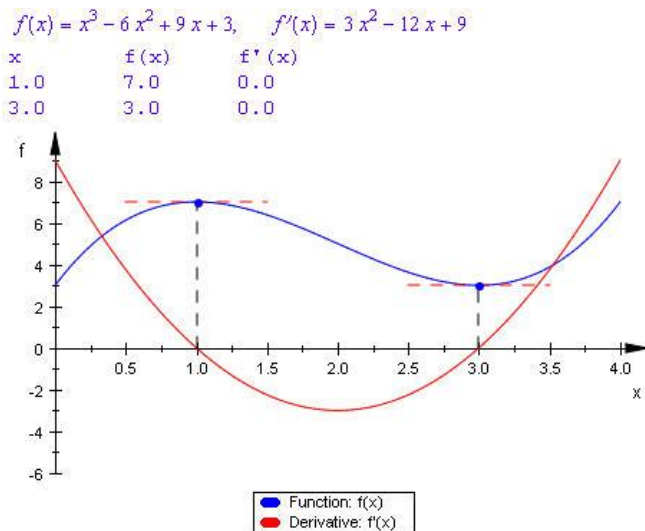


Figure 2 : Students choose a function and this is plotted along with its derivative and information about critical points.

The notebook on maxima and minima, has a third section which distinguishes between the two extrema. Once more, it uses the function provided in section 1 and plots the function and the second derivative along with the local quadratic approximation (second order Taylor series, although this terminology has not been developed by this point of the course) at each of the critical points, thus demonstrating the role of the second derivative on the nature of the critical point.

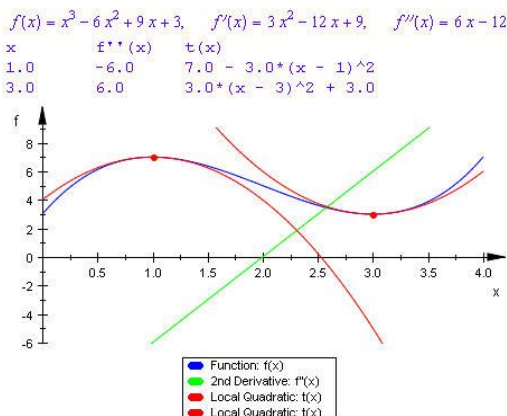


Figure 3 : The second derivative and the local quadratic approximation at each critical point.

The final section of each notebook is a series of exercises that students should carry out (see Figure 4). Students are encouraged to explore further and find out other interesting facts for themselves.

Q1) For the default function, describe the difference between the two points of inflection at  $x = 1$  and  $2$ .

Q2) For the following functions:

- $f := 2x^2 - 8x + 8$ ; xRange := 0..4; yRange := -4..8;
- $f := x^3 - 6x^2 + 9x + 10$ ; xRange := 0..4; yRange := -10..20;
- $f := x^2 + \cos(2x)$ ; xRange := -2..2; yRange := -2..4;

\* Write down the expression for the function's derivative (example (a)).

\* Plot the function and list the critical points (example (a)).

\* For each critical point, state whether the point is a maximum or minimum, by calculating the value of the 2nd derivative at that point (example (b)).

Figure 4 : Exercises on Maxima and Minima

## USE OF NOTEBOOKS

The notebooks form an integral part of the 1<sup>st</sup> year for EEE students and are a vital link between the mathematics course and the core EEE activities. At the beginning of the academic year, students are introduced to the notebooks and invited to download them using a wordpress site (Brown (2011)). The introduction takes place in a supervised session and students can then start work on the first notebook, on exponentials. This is a topic that the great majority will have met previously (or which others may experience on a diagnostic followup (Steele, 2009)) so that the emphasis can be on getting used to the notebooks. Each week subsequently, students are able to see new notebooks relevant to the studies at that point.

The notebooks can be used in many different forms. For example, in a lecture, the notebook can help to reinforce the relevant topic; in a small-group tutorial or examples class, the notebook can form the basis of a discussion e.g. an example is gone through in some detail by traditional methods, verified using a notebook and students are then invited to suggest subtle changes which are then explored using the system.

Students can also use the system away from classes, either individually or in small groups. During academic year 2011/12 students handed-in printouts or files detailing their explorations and these formed the basis of parts of a class given by a staff member to around six students.

Semester 1 notebooks include exponentials, vectors, cross products, complex numbers, complex arithmetic, differentiation, Newton Raphson, Maxima and Minima and Impedence. Semester 2 notebooks include indefinite integrals, definite integrals, Taylor series, multivariable functions & partial differentiation, multivariable Taylor series, multivariable stationary points, introduction to ordinary differential equations, unforced ODEs & complementary functions, Step-response ODEs & Particular integrals, Sinusoidal response ODEs & Particular integrals and applications to LRC circuits.

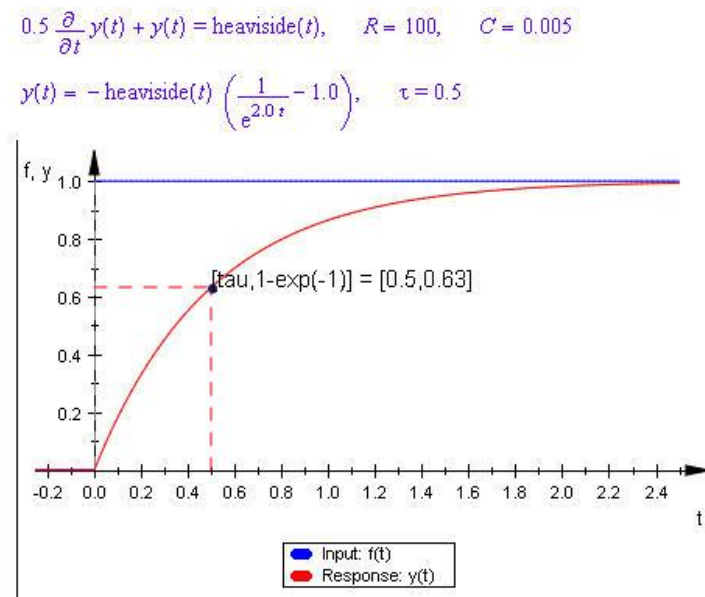


Figure 5 : Voltage across a capacitor subject to a unit-step voltage input.

Several notebooks e.g. Impedence, LRC Circuits, tend towards applications in Electrical Engineering. For an RC circuit, students are invited to enter values for R and C and are presented with the relevant differential equation and solution for the voltage across the capacitor (Figure 5)

## EXTENSION TO SECOND YEAR

During 2012, notebooks were written for the second year courses delivered by the school of mathematics to students in the school of EEE concentrating on such topics as Laplace transforms, Vector Calculus, Linear Algebra (matrices, eigenvalues, Gaussian Elimination). This work was funded (£ 7 000) by the TESS fund within the Faculty of Engineering and Physical Sciences and was used to employ the PhD student Ebtihal Gismalla to create the appropriate notebooks.

Second year notebooks were created using the same principles as the first year notebooks and can be used in the same manner. It should be noted that the calculations in the year 2 courses are often longer and the effort required to do a thorough exploration by hand could be prohibitive whereas an exploration using MathExplorer is fairly routine and allows many more aspects to be explored.

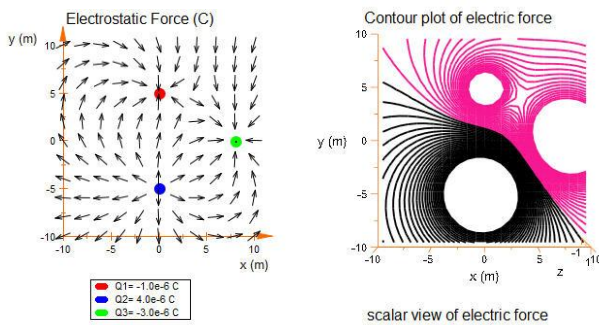


Figure 6 : Electric forces near three point charges (vector calculus).

## CONCLUSIONS

Notebooks have been produced for the majority of topics in years 1 and 2; these are an integral part of the syllabus. The majority of students have engaged with the notebooks and a number have carried out their own explorations.

## ACKNOWLEDGEMENTS

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# **FURTHER PURE MATHEMATICS WITH TECHNOLOGY: A POST-16 UNIT OF STUDY THAT USES TECHNOLOGY IN THE TEACHING, LEARNING AND ASSESSMENT**

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*Further Pure Mathematics with Technology is a new optional A level Mathematics unit that can be taken by pre-university students in England. The unit has been developed by Mathematics in Education and Industry, a mathematics education charity. It requires students to have access to technology, in the form of a graph-plotter, spreadsheet, programming language and computer algebra system (CAS) for the teaching, learning and assessment. This paper describes the development of the unit, including the rationale for the design decisions, and the implications for future developments of this type.*

## **BACKGROUND**

This first section is a background to Mathematics in Education and Industry (MEI) and gives some information about post-16 study in Mathematics in England so as to establish the context in which the Further Pure with Technology (FPT) unit was developed.

### **About MEI**

Mathematics in Education and Industry (MEI) is a UK-based mathematics education charity whose primary focus is curriculum development. MEI has been innovating in mathematics education for nearly 50 years, with landmark projects such as the creation of the Further Mathematics Support Programme, which has been a major factor in the large increase in participation in post-16 mathematics in England (Stripp, 2010) and substantial continuing development programmes such as the Teaching Advanced Mathematics CPD course (MEI, 2005).

MEI seeks to ensure that the teaching and learning of mathematics in schools and colleges continues to reflect techniques that are relevant and practised in industry and across Higher Education. One area that continues to develop at a fast pace is that of technology. MEI is keen to understand how software and hardware may be used to motivate learning in mathematics.

### **Post-16 study of Mathematics in England: A level Mathematics and Further Mathematics**

Education in England is currently compulsory up until the age of 16. Approximately half of all students continue to post-16 study of A levels and they will typically study three or four A levels over two years with the vast majority of these using this as preparation for university. There are no compulsory subjects at A level; however, about a quarter of students choose to study mathematics and of these students 16% also take a second, more demanding, A level in mathematics: A level Further Mathematics. In 2012 there were 78 951 students who took A level Mathematics and 12 688 who took A level Further Mathematics (JCQ, 2012a).

The study of A level Mathematics comprises of six units: four compulsory pure mathematics units plus two optional units that can be chosen from mechanics, statistics or discrete/decision mathematics. A level Further Mathematics also comprises of six units: two compulsory pure mathematics units plus four optional units that can be chosen from further pure mathematics, mechanics, statistics or discrete/decision mathematics. MEI has its own A level examination that is administered and assessed by OCR (Oxford, Cambridge and RSA examinations). Further Pure

Mathematics with Technology (FPT) has been developed as an optional unit that students can take as part of A level Further Mathematics.

## **DEVELOPING THE UNIT**

### **Process of development**

Although there have been a number of projects in England in the past 20-30 years to integrate the use of technology into the teaching and learning in mathematics it is not used effectively in many classrooms. This issue was raised by Ofsted (2008, p27):

Several years ago, inspection evidence showed that most pupils had some opportunities to use ICT as a tool to solve or explore mathematical problems. This is no longer the case ... despite technological advances, the potential of ICT to enhance the learning of mathematics is too rarely realised.

In the GCSE examinations that students take at age 16 they are allowed a scientific calculator, but not a graphical calculator, for some of the assessment. At A level students are allowed a graphical calculator in all but one of their examinations; however, these examinations are designed to be graphical calculator neutral, i.e. having a graphical calculator should offer no advantage to a student. It is not surprising that if the technology is expected to not offer an advantage in the examination then many teachers do not exploit its use for teaching and learning.

In addition to this there are no examinations where computer algebra systems (CAS) are allowed. The Joint Council for Qualifications requirements for conducting examinations (JCQ, 2012b) state:

Calculators must not ... be designed or adapted to offer any of these facilities: -

- symbolic algebra manipulation;
- symbolic differentiation or integration;

As a consequence of this there have previously not been any mathematics examinations in England that have allowed the use of a computer algebra system (CAS). As a consequence of this CAS are rarely used in the teaching and learning of mathematics in English schools. This is missing an opportunity to take advantage of the benefits of using CAS. Böhm et al (2004, p127) suggest these include making concepts easier to teach, supporting visualisations, saving time on routine calculations and improving students' perception of mathematics.

In 2008 MEI, in partnership with Texas Instruments, convened a seminar and invited leading experts to discuss 'Computer Algebra Systems in the Mathematics Curriculum'. The main findings of this event were (MEI, 2008, p17):

An ICT-based qualification where students have access to appropriate devices in the classroom and examinations would be useful. It could be a much more realistic qualification that allowed them to be better problem-solvers and mathematicians.

In this context MEI wanted to drive the debate forward by exploring the possibility of having part of the A level study involving the use of technology, including CAS, in a way that its use would be expected in the assessment and consequently this would drive its use in the teaching and learning. The aim of this is to aid the evolution of the role of technology in the curriculum from a computational tool to one that allows for observing and conjecturing as part of the mathematical process, as identified by Trouche (2004). MEI approached OCR (the examination board) who gave their full support to the development of a new unit in the A level Further Mathematics options.

## Mathematical Content of the unit

It was decided that the mathematical content of the unit would be taken from pure mathematics. This decision was taken because the current qualifications framework in England favours assessment by examination and it was believed that the study of pure mathematics offered more potential for examinations with technology than applied mathematics, where technology is more useful for project-based continuous assessment. This potential for using technology, and in particular CAS, for pure mathematics at A level was identified by Monaghan (2000) in highlighting its potential to increase the use of parameters and focus more on conceptual ideas and interpretation.

The requirements of regulation also impacted on the choice of topics: it was a requirement that the topics contained mathematics that wasn't assessed elsewhere in A level Mathematics or Further Mathematics. As there are a lot of optional units in A level Mathematics and Further Mathematics this limited the scope of topics for inclusion. The main criterion for inclusion was that they had to be areas of pure mathematics that benefitted from students being able to use tools to access a large number of results quickly and then to be able to analyse these results using their mathematical skills.

The topics chosen were:

- Investigation of curves;
- Functions of complex variables;
- Number Theory.

The investigation of curves can be greatly aided by access to a graph plotter which allows students to plot a family of related curves using parameters which can be changed dynamically. The use of CAS is also helpful in analysing the properties of the curves. Although students meet complex numbers in A level Further Mathematics they do not consider functions of complex variables, such as  $\sin(z)$ , or polynomials with complex coefficients. Using CAS allows them to investigate these and this is complemented by the use of a spreadsheet for iterations using complex numbers. Number theory is not studied at all in A level Mathematics or Further Mathematics; the use of a programming language, coupled with the access to CAS functions, makes this topic accessible to students at this level.

The full specification of the syllabus detailing the mathematical content can be found via the MEI website (<http://www.mei.org.uk/fpt>).

## Technology

The choice of the topics above suggested that it was necessary for the students to have access to:

- Graph-plotter
- Spreadsheet
- Computer Algebra System
- Programming language

There are different pieces of software that these applications; however, as they will be used together, especially with CAS being used in all the topics, it is advantageous to use a piece of software that features all of these in a way that objects, such as functions, can be used across multiple applications.

There was a conflict in considering which software to use. The examination board were reluctant to require a specific piece of technology as they perceived this ran counter to their aims. Similarly an overt focus on one specific piece of software could result in the unit being perceived as training in

using the software as opposed to learning the mathematics. In contrast to this is the concern that many teachers will be unfamiliar with some of the applications used and will desire teaching and learning materials that inform them how to use the software as well as covering the mathematics. It is also essential when designing both the syllabus and the assessment to know the capabilities of the software. Artigue (2002) identifies the complexity of the instrumentation process in which the technology becomes a tool that students can use effectively in mathematical tasks. This process is necessary; however, the goal is for students to learn the mathematics so having a single technology to use reduces the time required for students to go through the instrumentation process.

The compromise reached was that the specifications and features of the software allowed in the examination would be stated by the examination board and schools would be encouraged to seek advice from MEI about which software to use. TI-Nspire was chosen as the piece of software in which to write the teaching and learning resources as it features all the applications in a linked way, it is affordable for schools, and Texas Instruments have a strong record in supporting the use of technology in school mathematics. Schools are likely to be using other software in the teaching and learning and they may choose to prepare students for the examination using other software in future years.

### **Assessment**

To fit in with the current qualifications framework it was felt that a unit assessed by examination was more likely to be approved by the regulators. Consequently the assessment will be an unseen, timed, written examination in line with the assessment of other units in A level Mathematics and Further Mathematics. The students will have access to software in the examination and the questions will typically start with an investigation into a problem using the technology with some follow-up questions in which they analyse their results mathematically. The unit will carry the same weight as other units; however, the examination will be slightly longer, two hours as opposed to one and half hours, to take into account that in the programming question students may need to trial and amend the program to ensure that it works. The students will submit a written response to the questions but there is the possibility that, in future years, electronic submission of some or all of their work will be allowed.

As the topics are those in which students will generate results using the technology and then make inferences or deductions based on these this has resulted in questions that are longer than in other units. Monaghan (2000) suggests that problems with the use of CAS are less acute when longer questions are used but raises the concern that, by moving from questions which examine skills to those of a more conceptual nature, the questions may become more difficult. This is countered by questioning the assumption that standard routine questions are a lifeline to students who obtain low pass grades and offers alternative view that they may be 'good thinkers' who perform less well at reproducing algorithms. It will be interesting to observe how students perform on this style of questions and whether the longer questions impact disproportionately on students working at different grades.

The specification was accepted by the regulators and the first examination is scheduled for June 2013. The expectation is that around fifty students from approximately ten schools will take the unit in the first year.

An example question from the specimen paper is shown in figure 1. The full specimen paper can be found via the MEI website (<http://www.mei.org.uk/fpt>).

- 3 (i)** Create a program to find all the positive integer solutions to  $x^2 - y^2 = 1$ , with  $x \leq 100$ ,  $y \leq 100$ . Write out your program in full and list the solutions it gives. **[10marks]**
- (ii)** Show how the other solutions can be derived from the solution with the smallest  $x$ -value. Use each solution to give a rational approximation to  $\sqrt{3}$ . **[5 marks]**
- (iii)** Edit your program so that it will find solutions to  $x^2 - ny^2 = 1$ , where  $n$  is a positive integer. Write out the lines of your program that you have changed. Use the edited program to find a rational approximation to  $\sqrt{5}$  that is accurate to within 0.1%. **[6marks]**
- (iv)** Explain why the edited program will not give any results if  $n$  is a square number. **[2marks]**

**Figure 1: Example examination question from the specimen paper**

### Supporting the teachers

A key factor in developing the unit was providing sufficient support for the teachers involved. Goulding and Kyriacou (2007) highlight the importance of the teachers being confident users of the technology and the need to make the links between the outputs explicit to the students. As this is an optional unit teachers could choose whether to offer it to their students and it has been evident in working with them that they are teachers who feel positive about the use of technology in mathematics. However, they may not have used the software before or be familiar with the mathematics in the unit and consequently a support package has been designed and implemented with them.

This support has included teacher meetings and training days plus additional online sessions using a Blackboard Collaborate online classroom. These sessions and meetings have covered both the use of the software and the mathematical content. In addition teaching and learning materials are being provided free, to teachers and students, through MEI's online learning environment: Integral Online Resources for Mathematics (<http://integralmaths.org/>).

### IMPLICATIONS FOR FUTURE DEVELOPMENTS

This unit has been developed to promote the effective use of technology in the teaching and learning of mathematics and provide concrete examples of the challenges and choices needed to be made when designing technology-based mathematics curricula and assessment.

Through developing the unit the following have been observed:

- It is possible to develop technology-based mathematics courses within existing qualifications frameworks.
- The decisions made about the choice of topics demonstrate that technology is effective when it allows learners to access mathematical results quickly and efficiently so they can make inferences and deductions based on these.

- When designing curricula and assessments that allow the use of CAS it is important that the emphasis is on learners selecting and using techniques and not the mechanics of implementing them.
- Although different pieces of software have different qualities it is possible to identify the features that the software should have instead of requiring a specific piece of software: e.g. the graph-plotter should have the ability to control parameters with a slider or scroll-bar.

All of these demonstrate that there are requirements/constraints when developing curricula, but that suitable innovation can produce interesting, exciting and useful developments.

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## CULTURAL AFFORDANCES OF DIGITAL ARTIFACTS IN THE TEACHING AND LEARNING OF MATHEMATICS

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*The last two decades have seen a great development of research on the use of technology in teaching and learning of mathematics but, to date, these studies have not had a significant influence on teaching practice because research was unable to take sufficiently into account the challenges that digital technology poses in this context of use (Joubert, 2013). These challenges mainly concern the use of technology to facilitate reasoning in mathematics and to facilitate the construction of meanings, principles, values that are rooted in the historical, cultural development of this discipline. This contribution aims to develop a methodology centred on the concepts of affordance and narrative for describing and evaluating the potential of educational mediation of an artefact for the indicated objectives. The article is based on a previous work by Chiappini (2012) and the observations to this work carried out by Monaghan and Mason (2012)*

### THE CONCEPT OF AFFORDANCE

The concept of *affordance* was introduced by Gibson in perceptual psychology studies to denote the relation between the organism and its environment (Gibson, 1979). Gibson's affordances are very basic kinds of relationships. Examples of Gibson's affordances are surface that provide support, objects that can be manipulated, or that can be grasped, or that can be cut and scraped. Gibson stated that an affordance is equally a fact of the environment and a fact of the behavior and it may be detected and used without explicit awareness of doing so. Successively, Norman used the concept of affordance in Human Computer Interaction for the evaluation of system usability (Norman, 1988), but in his view, differently from the Gibson's view, affordance is conceptualized as *perceptive affordance*, since it includes information specifying how the object can be used. The distinction between these two meanings for affordance notion allowed McGrenere & Ho (2000) to discriminate between *usability* of a system and its *usefulness*. Usability refers to how well the task that is tackled by means of the perceived affordance provided by the interface of a system is completed in practice. Goal achievement is the criterion used to perform the usability evaluation of the perceived affordance of a system, considering the user as an expert both of the task to be accomplished and of the domain where the task is set. Usefulness refers to the affordance (beyond user interface) that is provided by the system software with respect to their functional use in the work domain (McGrenere & Ho, 2000). The usefulness of an artefact is more difficult to evaluate. To this end, Turner (2005) referred to the Gibson's affordance concept and to the notion of usefulness and distinguished between *simple affordances* (according to Gibson's view) and *complex affordances*. He stated that the latter embody such things as history and practice. Moreover, Turner connected the concept of complex affordance to the concept of *boundary object*. For Turner boundary objects are artifacts which support the work of separate communities such as different departments within an organization or even between very different communities of practice (Turner 2005). To be useful by these different communities they must be sufficiently flexible to be used in different ways, by different people for different purpose in a range of contexts. For Turner, the term boundary object represents culturally emergent affordances.

Turner & Turner (2002) designed a framework centred on a three-layer articulation for the evaluation of complex affordance. The basic level (*perceived affordance*) concerns the evaluation of basic usability aspects of controls that mediate interaction with the system (low-level properties of buttons or sliders; physical affordance of the mouse to move among environments...). The middle

level (*ergonomic affordance*) concerns affordance for the embodied actions involved in solutions of tasks and sub-tasks peculiar to the context where the system is used. At this level affordance concerns the usefulness of the environment through the embodiment of action possibilities for task objectives that users can detect and pursue. The third level concerns *cultural affordance*. For the concept of cultural affordance the authors makes reference to Ilyenco's (1977) work where he points out that human purposive activity endows artefacts with value and meaning, namely with *ideal properties* (Turner, 2005). At this level affordance evaluation refers to the cultural teaching/learning objectives underlying the system use. Evaluation of cultural affordances can be carried out through the analysis of how meanings, values and principles underlying the action mediated by the use of the ergonomic affordances, get to be known through the artefact-mediated activity.

The Turners used this model to evaluate a collaborative virtual environments (CVE) used to develop critical safety training simulations in maritime and offshore work practices (2002). Chiappini (2012) introduced the Turners' constructs within the context of mathematics education to evaluate the software *AlNuSet* he designed for algebra teaching and learning

### **THE INTERNALIZATION OF CULTURAL AFFORDANCE**

The present context concerns the design and the use of *AlNuSet* for the teaching and learning context of mathematics where three communities of practice interact. The community of those who designed the artifact and have incorporated into it affordances for mathematical activity; the community of teachers who intend to pursue specific learning goals in algebra (skills, meanings, values and principles of algebra) involving students in solving tasks mediated by the use of the ergonomic affordances available with the artifact; the community of students who use (or not) the affordances of the artifact to solve the tasks proposed by their teachers and to build their mathematical knowledge. In creating *AlNuSet* the designers of the artifact embedded into it affordances that are an expression of mathematical culture mediated by the new operative and representative possibilities of digital technology. The mathematical content embedded in the digital technology is characterized therefore by ideal properties (in the sense of Ilyenco) that are expressed through a digital medium. The significance of the ideal properties embedded in the artifact by the designer can be known by the other two communities of practice through the process of the artifact incorporation into activities to pursue their respective objectives. In this context students and teachers have different skills and objectives. The usefulness of the affordances of the artifact must be put in relation with possibility to promote the emergence of the objectives for the solution of the task which the students are engaged in, and with the development of the teachers' mathematics cultural goals (development of knowledge, meaning, principles and values of the discipline) that may also transcend those of the task in which students are involved. So that this may happen, the artifact must have characteristics of boundary object able to link the work of the different communities involved in its design and in its use. How is it possible to describe and evaluate this? To answer this question Chiappini (2012) has applied the framework centered on a three-layer articulation for the evaluation of the complex affordance elaborated by Turner to evaluate the usefulness of affordances provided by the *AlNuSet* system for the teaching and learning of algebra. In particular, in that work, his analysis has focused on the ergonomic and cultural affordances provided by two environments of *AlNuSet* - the Algebraic Line environment and the Algebraic Manipulato (see Chiappini, 2012).

Chiappini noted that the ergonomic affordances of the algebraic line and of the algebraic manipulator of *AlNuSet* have been created as embodiments of purposes to be incorporated into an algebraic activity to support the solution of algebraic tasks and acquire significance through it. He observed that they can be very effective in the solution of algebraic tasks by the students but they are not sufficient in themselves to allow students to master the meaning, values and principles of the



cultural domain which has inspired the creation of these ergonomic affordances. He evidenced that only through specific condition of use of these ergonomic affordance within an activity, these ergonomic affordance can be internalized as cultural affordance and can assume the values of the culture from which they arise. To describe the conditions of the activity development that can allow meanings, principles and values of the cultural domain to be internalized by the students, Chiappini refers to Activity Theory (AT) and adopts Engeström's notion of the *cycle of expansive learning* (Engeström and Sannino, 2010). The notion of cycle of expansive learning has been created by Engeström to describe the activity transformation processes which could determine a re-definition of objects, tools, and structures of the activity by participants able to promote conceptualization among them according to a innovative cultural perspective. The evolution of the activity able to promote this occurs in a given number of phases. Chiappini worked out some experimentations of AlNuSet with students in different context and he noted that the cycle of expansive learning mediated by the use of AlNuSet was able to promote a cultural enhancement articulated by four phases with the following specific characterization (Chiappini, 2012):

The first phase of the expansive learning cycle is characterized by the assignment of an open problem on an important issue of algebra learning and concerning an obstacle of epistemological nature. [...] a conflict emerges in terms of an unexpected representative event as retroaction of the system software to the student action that sounds surprising to their consciousness. Students should become aware that the surprise is determined by a conflict of epistemic nature that they are not able to solve by available means. In this phase the teacher helps students in the process of consciousness raising.

In the second phase students are requested to face tasks that broaden problematic areas of the knowledge in question. [...] Tasks of this phase are designed in order to allow students to explore the conditions, causes and explicative mechanisms of conflicts concerning problematic situations. [...] The practices of this phase are modelled to submit the experience to a common reflection and discussion in the class and aim to reach a shared interpretation of the meaning to be attributed to the actions performed. These meanings are expressed through terms that identify them. In this phase the teacher's crucial task consists in the introduction of terms and algebraic notions and the support to students in consciousness raising about the double bind situation they are in.

In the third phase the use of the algebraic right line is integrated with the axiomatic algebraic one incorporated into the AlNuSet algebraic manipulator. The two solution models are compared in relation to tasks of the preceding phase and with the new tasks assigned to students. Actions performed in this phase intend to favour the interpretation in symbolic terms of what students learnt through the experience with the algebraic line. [...] Practices of meaning negotiation are mediated by the different role that teachers and students play in the activity. [...] In this phase the teacher encourages both the establishment of the algebraic axiomatic model in the student's practice and the development of meta-cognitive processes involved in the re-configuration in symbolic terms of the algebraic meanings expressed beforehand in visuospatial and deictic terms.

In the fourth phase participants – students and teachers – become aware of the fact that the transformation of the activity system occurred through the use of the two models has changed respectively the state of their knowledge of algebraic activity and of the teaching of algebra. This phase is characterized by a teacher-guided assessment discussion on classroom activity meant to foster a full awareness by students of the developed knowledge through the comparison with the memory of their knowledge before the beginning of the cycle.

The reported description highlights that expansive learning starts from the individual action and evolves towards a form of collective action through cooperative actions and new types of practices.

Through the development of these practices, the ergonomic affordances supplied by AlNuSet assume the values of the culture from which they arise and are internalized as cultural affordances.

### **REACTION BY MONAGHAN AND MASON TO THE CHIAPPINI'S WORK**

With a contribution in two voices illustrated at a BSRLM conference, Monaghan and Mason (2012) discussed the work of Chiappini (2012) and presented some critical questions concerning the notion of cultural affordances. Monaghan showed, first, that of the two constructs, 'ergonomic affordance' appears relatively unproblematic compared to 'cultural affordance'. He argued that Engeström's version is a 'systems' approach and it could be argued that it is too big to capture the nitty-gritty details of students' actions in doing and learning mathematics. He suspected the AT approach of Luis Radford, who looks at nitty-gritty student details and pays close attention to gestures (which could be called ergonomic actions), might be a more suitable AT approach to looking at how the affordances of a mathematics SW system can assume "the values of the culture from which they arise". Finally, Monaghan wondered whether the construct 'affordance' has been stretched too far from very basic vision of Gibson's affordance and whether Gibsons would recognize the animal-environment relation in the visuo-spatial and deictic ergonomic affordance of AlNuSet.

Mason found the notions of *affordances*, *constraints* and *attunements* powerful triggers to direct attention to important aspects of tasks generally, and software in particular, but only by seeing them as evolving and developing during activity. He observed that the affordances perceived at the beginning are usually a subset of the affordances recognized later and that trying to encompass all of the affordances as basic, absolute affordances fails to take into account the user and their evolving attunement. Mason stated that one approach to a finer grained analysis of enculturation into, exploitation and evolution of affordances could be through *activity theory* as used by Engstrom or Radford. He evidenced that another could be through abstraction of Bruner's trio of modes of (re)presentation *Enactive-Iconic-Symbolic*. He then, through an example, compared the approach of Turner and Chiappini with that of Bruner.

In a common conclusions Monaghan and Mason posed the following questions. Might not the adjective 'cultural' mislead attention away from the personal, the psyche of the individual (their awareness, enactive potential and affective states) as an important component? The cultural and indeed the historical play a role in the genesis, but the condensation-contraction is likely to be personal. If the person's state were dominantly social, wouldn't everyone in the class give the same response to a teacher's probe, or at least the group would agree on a response?

### **TOWARDS A METHODOLOGY TO STUDY THE INTERNALIZATION OF THE CULTURAL AFFORDANCE OF A DIGITAL ARTIFACT**

The constructive contribution of Monaghan and Mason highlights a fundamental problem regarding the type of analysis that must be made to study the internalization of the cultural affordance of a digital artifact. The problem can be expressed in the following terms: the internalization of cultural affordances has to be studied considering the qualitative change in the activity or rather the process of condensation-contraction that occurs in the individual? Expressed in these terms the question takes a dichotomous character since it orients towards two different units of analysis: the analysis of the activity as a system (in the sense of AT) vs. the analysis of the individual behavior within specific practices. I think it is possible to find a unit of analysis that makes possible to consider and enhance the complementary nature of these two approaches and to overcome the dichotomy highlighted. The benchmark for this type of analysis is the notion of *narrative* developed by Bruner in its interconnection with the development of the *narrative thought* and of the *paradigmatic thought* (Bruner, 1997). The narrative is, as Bruner points out, the first device for understanding and interpretation of which man makes use. Through the narrative man gives meaning to his experience,

interprets events, actions, situations, builds forms of knowledge to orient his work, organizes and communicates the knowledge built. Bruner has produced a synthesis of the main properties of the narratives, the *nine universal principles of narrative realities*. These can be used to analyze the way in which the meaning of an experience is embedded in a narrative text. For more detailed analysis of these principles see (Bruner, 1997, p. 148).

The technological development makes available artifacts characterized by new ergonomic affordances for the mathematical activity able to provide a visuo-spatial, deictic and dynamic narrative of the function of specific mathematical objects in the solution of mathematical tasks. The narrative emerges through a *collaboration* between who uses the artifact and who has designed it incorporating in it ways of use of mathematical object mediated by the operative and representative potentialities of digital technology. This type of narrative can be primarily a useful support for the emergence of objectives for the task to be solved and for the development of mathematical reasoning. The principles of narrative realities developed by Bruner is an important reference for analyzing the collaboration between the user and the designer of the artifact that determine the rising of the narrative and the nature of the psychological support provided by the narrative.

Bruner also showed that the narrative thought lives together with another kind of thinking, which the author defines as paradigmatic, that is typical of the scientific reasoning and mathematical logic. Narrative thought is focused on the intentionality of the subject and the sensitivity of the environment whereas the paradigmatic thought requires a full mastery of the properties of (mathematical) objects with which the subject operates. In fact, such thought makes an extensional use of these properties to explore domains closed of mathematics and remove ambiguities. Only through the development of paradigmatic thought it is possible to understand the most profound aspects of the mathematical culture. According to Bruner the two modes of thought (the narrative one and the paradigmatic one) are complementary, and even if they are irreducible to each other (irreducible not incommensurable), they can never be completely separated. For example, the development of paradigmatic thought, by its nature decontextualized, needs of contexts and narrative modes so that it may arise. The transition to the paradigmatic thought cannot be seen as a simple translation of narrative thought into the formal language of mathematics. To explain the support that the narratives incorporated in the use of an artifact can provide for the development of paradigmatic thought it is important to observe that the narrative embeds ideal properties of algebraic objects that are reified in the functioning of the artifact, that is, the narrative incorporates aspects of the culture of mathematics of those who designed the artifact. The teacher, as an expert in the discipline, is able to recognize these ideal properties whereas a student may be able to exploit the embedded narrative to solve the proposed tasks without being able to recognize these properties. In fact, the recognition of these properties implies a cultural development that may not emerge only through the solution of the tasks. Chiappini (2012) showed that this cultural development can emerge only through the students' participation in the activity system orchestrated by the teacher. To study this process of acculturation, the concept of narrative can be very useful. The narrative embedded in the use of the artifact contributes to structure the activity system and the collaboration among the participants through which the cultural advancement of the student comes to life. These reflections make it possible to develop a methodology to study how the process of internalization of cultural affordances of an artifact can occur. This methodology:

- assumes that when students interact with an artifact to solve a mathematical task they exploit the ergonomic affordances implemented by its designer to investigate the solution and can have greater or lesser facility in detecting, understanding and using these affordances depending on the characteristics of the narrative that the affordance actualize through the subject action;
- assumes that in the narrative embedded in the use of the artifact there are meanings, values, principles of mathematical culture that have guided the design of ergonomic affordances, that

their comprehension is crucial for the development of a paradigmatic thought and this can occur through the development of activities and forms of collaboration through which the ergonomic affordances are internalized as cultural affordance;

- uses the nine principles of narrative discourse developed by Bruner to analyze the narratives that emerge through the use of ergonomic affordances and to understand their usefulness both for the solution of the task and for the appropriation of aspects of mathematical culture underlying (meanings, values and principles);
- realizes a dual analysis of these narratives:
  - i. the first analysis is focused on the solution of algebraic tasks with the artifact and uses the principles of narrative discourse elaborated by Bruner to identify the psychological support that narratives offer for the emergence of objectives in solving tasks;
  - ii. the second analysis is focused on the system of activity that the narrative contribute to structure, analyzes the contradictions that arise through its use and the forms of collaboration that determine transformation of the activity (cycle of expansive learning) that leads to a re-definition of the mathematical objects reified in these narratives;
- assumes that the integration of these dual analysis makes possible to study the process of internalization of cultural affordances of an artifact.

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## **CORNERSTONE MATHEMATICS: AN APPROACH TO TECHNOLOGY- ENHANCED CURRICULUM INNOVATION AT SCALE**

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*We report on a project, Cornerstone Mathematics<sup>1</sup>, that is researching the impact and scalability of an innovation for secondary mathematics, focused on embedding digital technology at points where its potential for enhancing the learning of mathematics is clear due to the availability of multiple representations that are visual, dynamic and inter-connected. The innovation combines four elements, each of which has been extensively researched: digital technology designed for mathematics learning; professional development; new curriculum 'replacement' materials; and explicit strategies established for scaling and sustainability. We will present the results of a pilot study in 19 English classrooms with students aged 11-13 years that reported positive outcomes, supported by the evidence from measured gains between pre- and post-testing, teachers' and students' feedback, and structured observations of experimental lessons.*

### **INTRODUCTION**

This paper describes the outcomes of the initial phase of a research project in England that is seeking to adopt a research-informed approach to the challenge of enabling a wide number of mathematics teachers in more than 100 schools to embed digital mathematics successfully within their lower secondary mathematics classrooms (11 – 14 years). We begin by describing the important ideas from research that have influenced the design of the processes and products within Cornerstone Maths, outline the outcomes of this initial phase and raise some important emergent themes for our research as we begin to implement the next phase of work. Cornerstone Maths is a collaboration between SRI International (USA) & the London Knowledge Lab, funded by the Li Ka Shing Foundation and the work being described in this paper took place between Summer 2011 and Autumn 2012.

### **THE CORNERSTONE MATHS APPROACH**

The Cornerstone Maths curriculum units, which include teachers' guides, student workbooks and bespoke software activities are derived from over 15 years work in the US and UK and they seek to exploit the dynamic and visual nature of ICT to stimulate students' engagement with mathematical ways of thinking. These units of work are in the form of replacement curriculum units, which means that the teachers can choose to substitute their 'normal' teaching approach to the topic at a time that the teachers deems to be appropriate for a particular class.

The aims of this first phase were to track the:

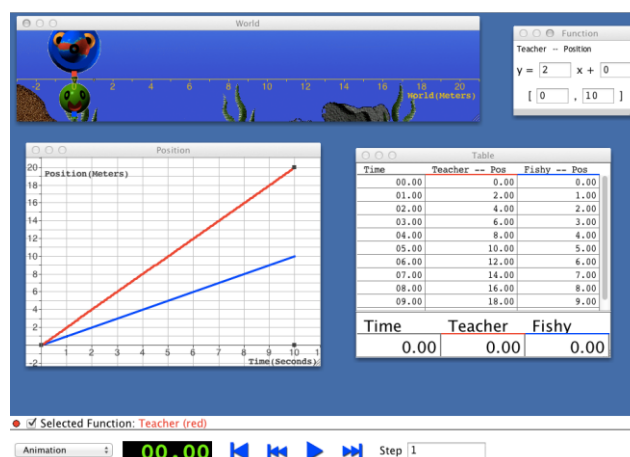
- implementation of the Cornerstone Maths innovation in terms of student learning and teacher engagement;
- implications of the work for teacher professional development and for material/software design in order to inform the larger-scale project;
- epistemological dimension of scalability - that is to build an *a priori* set of theories concerning the process and content of the expanding project.

The Cornerstone Maths approach comprises four elements: using dynamic representational technologies to teach 'big' mathematical ideas; teachers' professional development and community;

replacement curriculum units; and ‘designing’ for scale-up. The theoretical perspectives that underpin each of these elements will now be described.

### Using dynamic representation technology to teach ‘big’ mathematical ideas.

There is an extensive research base that documents how computer technology has been used to design ways of representing mathematical objects and to link these objects within multiple representations (Borba and Confrey 1996, Noss and Hoyles 1996, Papert 1996, diSessa 2000, Kaput, Hoyles et al. 2002, Heid and Blume 2008). The design of the first curriculum unit, *Designing Mobile Games: A module on linear functions* integrated the use of SimCalc software within an environment in which motion simulations are dynamically linked with graphical, algebraic and tabular representations of functions.



**Figure 4 SimCalc environment (Investigation 4.1)**

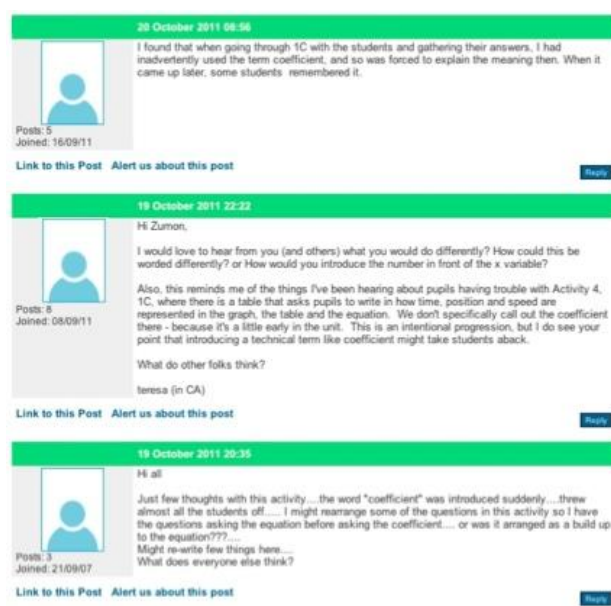
These lessons had been originally developed for the SimCalc project and there is an extensive research literature to support their design, implementation and subsequent evaluation within the US context. (For a summary, see (Hegedus and Roschelle 2013)).

### Teacher professional development and teacher community.

There is a consistent message from the research literature that better student outcomes are achieved where teachers experience professional development as part of their ongoing teaching practice.

Effective professional development is intensive, ongoing, and connected to practice; focuses on the teaching and learning of specific academic content; is connected to other school initiatives; and builds strong working relationships among teachers. (Darling-Hammond, Chung-Wei et al. 2009)

However, a number of English surveys and reports suggests that very little of our professional practice achieves this aspiration (Office for Standards in Education 2008, Pedder, Storey et al. 2008, Office for Standards in Education 2012). In order to build strong working relationships it is important to build such teacher communities through meetings and virtual communications. A number of past projects have brought together teachers of varying teaching experiences and personal mathematics confidence to focus on integrating digital technologies within their practices (Noss, Sutherland et al. 1991, Clark-Wilson 2008). A key element of these projects was to involve the participating teachers as co-designers and evaluators of digital learning objects and to encourage their deep evaluation and reflection of the classroom implementations as a cyclical process. Within Cornerstone Maths, the ongoing dialogue relating to these design cycles occurs within face to face meetings and through online collaborations within a bespoke community on the *National Centre for Excellence in Mathematics* portal ([www.ncetm.org.uk](http://www.ncetm.org.uk)).



**Figure 5 Extract from the Cornerstone Maths community online discussion**

Early in this phase of the project, the notion of ‘implementation’ began to be problematic as the difficulties associated with changing teachers’ practices emerged. This resonates with Levin’s concern that,

it has become clear that 'implementation' is the wrong word. Effective change in schools does not come from some kind of blind obedience to a central plan ... but from thoughtful application of effective practices in particular contexts. It requires the ability to achieve goals in new ways under differing - and changing – circumstances. (Levin 2008)

Hence, as we plan to scale the Cornerstone Maths project to 100 schools we are developing a toolkit of resources that will support the growth of Cornerstone Maths ‘hubs’, where a hub is a group of teachers and/or schools that have a central point of focus. Within the current English school system, these focus points might be a chain of Academies (privately sponsored state schools), a Local Authority advisor, a Teaching School or a highly motivated individual teacher. As these hubs emerge and evolve, they will inform the ongoing design research into scalability.

### **Replacement curriculum units.**

The Cornerstone Maths approach is based on four replacement curricula units of work that include a complete package of software, student workbook, lesson plans and teacher guidance. The replacement unit strategy is deemed important as one unit of work (approximately 10-12 hours teaching) is significant enough to observe the ‘hoped for’ classroom change and meaningful student learning, whilst small enough that teachers and schools will accept the risk associated with the experiment. It explicitly connects the innovation to schools’ priorities for both curriculum coverage and to improve student outcomes on hard-to-teach topics (Roschelle, Shechtman et al. 2010).

### **Designing for implementation at scale.**

The scalability of the Cornerstone Maths Project is the next step in the design research process and it is a critical demand of the project funders that we research how to make an impact ‘at scale’ that is more than just ‘delivering’ the resources to schools. This will inevitably involve a devolved model of scaling that takes account of:

- the need to grow the number of potential hub leaders and to work with them to develop a ‘toolkit’ of resources that will support them to engage schools and teachers with the project;

- the ongoing design objectives of the Cornerstone Maths units of work, which include widening access to the materials through a web-based offer that also facilitates access on a wide number of technology platforms (to include tablets and iPads) and the development of embedded collaborative formative assessment tasks;
- an awareness amongst the research team of the implications of the ‘implementation fidelity’ (Hegedus, Dalton et al. 2009) with respect to the innovation, which will necessarily require the careful development of evaluatory tools and techniques.

## METHODOLOGY

This paper reports on the outcomes of the first Cornerstone Maths unit on Linear functions. The research methodology followed that of the USA projects in order to ensure for a ‘control’ comparison with a randomised controlled trial that had been previously conducted in Texas (Roschelle and Shechtman 2013). Consequently, the English students participated in pre- and post-testing, in order to generate gain scores that were used for a quantitative analysis of learning gains.

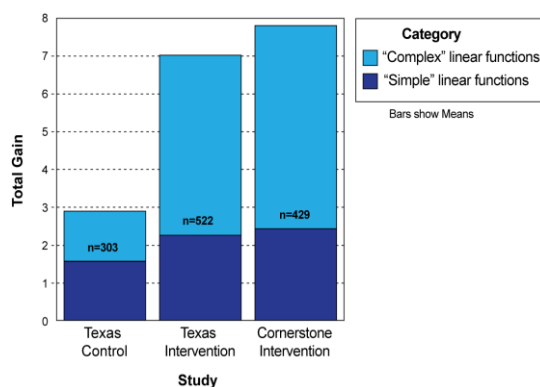
The unit of work highlighted: coordinating algebraic, graphical, and tabular representations;  $y=mx+c$  as a model of constant velocity motion; the meaning of  $m$  and  $c$  in the motion context; velocity as speed with direction; and the notion of average velocity and average rates. The unit also included some ‘transfer activities’ that required the students to use the mathematical concepts in less familiar contexts.

There was a total of 429 students (ages 11-13 years), taught by 19 teachers from 9 diverse schools involved in the study and the unit was taught over a 2-3 week period. Unusually for most schools, the majority of the lessons involved student use of computers. The intervention comprised: 2 days professional development for teachers; 0.5 day reflection/feedback session after classroom implementation; discussion through an online forum (see **Errore. L'origine riferimento non è stata trovata.**); pre-test and post-test of students’ understanding; and lesson observations and teacher and student interviews by the research team and external evaluator.

## RESULTS

The project was externally evaluated by the National Foundation for Education Research and the complete results are in their published report (Sturman and Cooper Jan 2012). In this section, we offer a brief summary of the quantitative and qualitative results.

The results of the English study ( $n=429$ ) showed similar learning gains between the pre- and post-tests and, by separating the questions to categorise them as testing ‘simple’ and ‘complex’ concepts, the outcomes are shown in **Errore. L'origine riferimento non è stata trovata.**



**Figure 6 Comparative data between the Texan and English studies.**



The results for student sub-groups, in terms of age, attainment level and socio-economic levels were also analysed (the latter only at whole-school level). This revealed that, although the higher attaining pupils and those with higher socio-economic status had higher pre- and post-test scores overall, an outcome that was consistent with the US studies was that the learning gains were spread evenly across the sub-groups, indicating that the intervention had positive outcomes for learning and did not widen or narrow the achievement gap.

The qualitative data, which emanated from the lesson observations and student and teacher interviews, revealed five themes that concerned: changes in the lesson style; the impact of the multiple representations; the pace of the teacher's implementation of the unit; the reorganisation of knowledge elements and the role of the context.

## CONCLUSIONS AND THE NEXT PHASE OF THE PROJECT

The purpose of the research that has been reported in this paper was predominantly to evaluate an existing curriculum innovation within the English context in order to both establish its validity and to inform it be redesigned to support it to be implemented in mathematics classrooms in over 100 schools. At the time of writing, 90 teachers from 45 schools will be engaging in professional development in June 2013 in preparation for teaching this unit in Autumn 2013. They will no longer be using the SimCalc software but a redesigned web-based environment (For an example, see **Errore. L'origine riferimento non è stata trovata.**) that does not require a software installation or any individual file management.

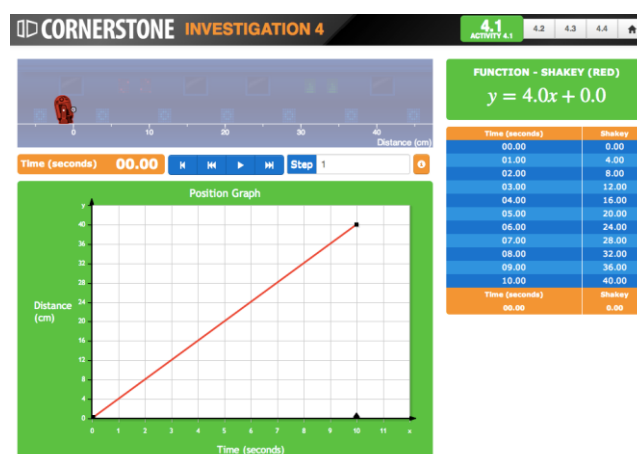


Figure 7 The redesigned Investigation 4.1

There will be a pivotal role for a 'mathematics specialist leader' who will support the school hubs to be active partners within the design research process, whilst the research team will concern itself with the design and provision of robust evaluatory tools to support the collection of data about the technical, mathematical, pedagogical and management aspects of the scale-up.

## NOTES

1 We gratefully acknowledge funding by the Li Ka Shing Foundation. The research has been an intensive collaboration with a team for researchers in the Center for Technology in Learning, SRI International, Menlo Park, USA: Jennifer Knudsen, Ken Rafanan, Teresa Lara-Meloy, Anna Werner, Gucci Estrella and Nicole Shechtman.

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## A PROGRAMMING ENVIRONMENT AS A METHODOLOGICAL TOOL FOR THE LEARNING OF MATHEMATICS

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*This work focuses on the use of technology in the classroom for the teaching-learning of mathematics. After a brief outline of the current situation in Italian schools, we address the issue of programming as an important teaching tool for its pedagogical and educational value. The central part of this study stresses the importance of choosing a programming environment suited to the teaching context. A practical teaching example and some conclusions round off the paper.*

### BACKGROUND

The recent Eurydice 2011 report warns of a serious delay in Italian schools in the use of technology in the classroom not just from a structural viewpoint, that is the setting up of school laboratories, but also with regard to the adoption of teaching practices based on the current trends. One reason for this delay might be the teachers' difficulty in keeping up with and making informed choices from the wide range of ever new hardware and software available on the market. Teachers, who are practically the sole agents of didactic change in schools, might easily feel disoriented when it comes to identifying the right tools to support the teaching-learning of mathematics.

When the use of computers became widespread in the 1970s the question of introducing computers in the Mathematics classroom became central in Italian schools. The cultural debate involving the use of computers in schools produced some practical results in higher secondary schools with the creation of various projects and experimentations (PNI, or National Program for Informatics, the Brocca Project, etc.) endorsed by the Ministry of Education. The object of these initiatives was the integration of information technology contents and methods into the teaching of Mathematics, which, albeit with different connotations, were common also to the field of Mathematics (variable, formal language, algorithm, problem-solving, etc.). The basic methodological assumption underlying the above-mentioned projects considered programming as a working method and included languages such as Pascal and Basic. Despite the official inclusion of computer technology in the curriculum of Italian lower and higher secondary schools, however, computers are still not widely used for educational purposes for a number of reasons- the lack of adequate computer laboratories and in-service teacher training to mention two. A first attempt to improve things was made with the introduction of LOGO (Papert, 1980), which represented a significant development in lower secondary education. Meanwhile, in higher secondary schools the situation was getting worse. Indeed the first CAS (Computer Algebra System) like Derive, Mathematica, etc., and the DGS (software packages for dynamic geometry) like Cabri-Geomètre, Cinderella, Geometer's Sketchpad, etc.), which became available on the market at the time, seriously neglected programming skills in favor of the immediate and automatic solution of problems. Moreover, these new technological tools which incorporated the use of graphics with the inclusion of icons, developed an imaginative and creative approach by apparently allowing the creation of dynamic images, while they were in fact 'ready-made' products of the software tools. In-depth analysis of this problem would require considerable time, but it would be easy to show how along with undeniable local technical and organizational motives, cultural and/or psycho-pedagogical reasons played a part in this (Reed, et al. 1988; Reggiani, 1992; Arzarello, et al. 1993; Costabile, 2000).

At the same time the new ICT tools, useful for the transmission and finding of information, were becoming popular but these have contributed in small ways to the development of the thinking

process and the acquisition of a working method.

The ever-changing, fast developing offer of always new technology has hindered rather than facilitated the solution of problems faced by the teacher, who struggles to effectively combine the traditional contents and methods of teaching with the use of computers through programming. The educational and training rationale behind the use of programming software has a strong innovative impact and requires a radical change involving both the teaching methods and the objectives of the teaching-learning of mathematics. That is why we believe that programming, albeit simplified and tailored to the learning context, should be an essential component in the teaching of mathematics. Below we present some arguments to support this thesis.

## **PROGRAMMING AS METHOD**

The pedagogical value of programming has been stressed on more than one occasion both at national and international level (Soloway, 1986; Burton et al., 1988; Oprea, 1988, Liao, 1991; Biehler et al., 1994), and should therefore be included in the setting of the teaching-learning of Mathematics. Indeed, programming is a constructive and cognitive activity that enables the learner to acquire skills, strategies and techniques for the solution of problems through the concepts of variable, procedure, repetition and recurrence, notions which are common to other school subjects (Linn, 1985). With regard to the cognitive and training aspects, programming has numerous advantages among which we would like to mention:

- a) the unpacking of a problem into sub-problems until the level of the initial data is reached;
- b) description (written/oral) of the necessary steps to reach the solution of the problem, in the appropriate sequence;
- c) the use - if necessary - of suitable methods for the representation of algorithms;
- d) formulating hypotheses or conjectures for the solving of problems (indicating the action to be performed to produce the desired effect; predicting the result of a given action);
- e) verifying and correcting errors (try out on the computer the formulated hypothesis; comparing the result obtained with the expected outcome; describing the suggested alterations to correct a possible error or make improvements).

We would also like to point out that a 'good' approach to the teaching of Mathematics today should definitely include the use of labs, the formulation of hypotheses, a creative attitude, the discussion of problem situations, the solution of these through pathways which are formally correct but may be non-canonical. The building up of a program as a learning strategy and its implementation on the computer in an easy and useful way contributes to creating a richly stimulating teaching-learning environment. Furthermore, a program which effectively includes and incorporates numbers, images and animation strongly motivates the learner also from an affective viewpoint, bridges the gap between reason and emotion, while also allowing an approach which is freer than the two traditional cognitive styles of visual-holistic and textual analytical.

Teachers of the subject for the reasons aforementioned initially welcomed the inclusion of computer programming in the Mathematics curriculum of Italian schools. However, the practical difficulties encountered by professionals in the classroom in the implementation of the new guidelines dampened the enthusiasm with which the novelty had been at first welcomed. The rather difficult programming language used (the Pascal language, whose syntax and declarative phase did not match the logical and linguistic abilities of the 14 to 18 year old target learners) together with unsolved logistic problems, generated a growing unease among the teachers involved. So we can say that the choice of programming language was definitely one of the causes for the partial failure of what was otherwise a valid attempt (Felleisen et al., 2004).

## THE MATCOS ENVIRONMENT

The MatCos programming environment, now in its 3<sup>rd</sup> version, was devised and created on the following pedagogic-educational paradigm:

- Introducing the learners to computer programming using mathematical concepts suitable for their age group;
- helping students to learn mathematical concepts and methods exploiting the potential of computers.

For reasons of space, for details concerning the educational aspects and technical requirements see (Costabile & Serpe, 2003; 2009, 2012, 2013).

## TEACHING STRATEGIES: AN INNOVATIVE EXAMPLE

The programming environment MatCos with its technical requirements can give a relevant contribution to the formation of mental pictures of real objects in the student's mind. It allows the teacher a choice of methodology:

- proceeding in an inductive way, guiding the students to the discovery of the mathematical properties of the object studied;
- proceeding in a deductive way, encouraging the learner to re-discover/verify/consolidate mathematical concepts etc.;
- solving problems with realistic data or meaningful examples from everyday life .

MatCos requirements enable the setting up of practical, workshop-like, lessons, which have as their aim the re-discovery of cognitive experiences in the field of mathematics. In this way the students acquire a really thorough knowledge of the topic both from the conceptual and the operative aspect. For obvious reasons of space we cannot include here examples of uses of the software in the three cycles of primary school and lower and higher secondary schools (see: CIRD website <http://cird.unical.it/CollanaDidatticaeDidatticheDisciplinari>).

### Introduction to the study of conics

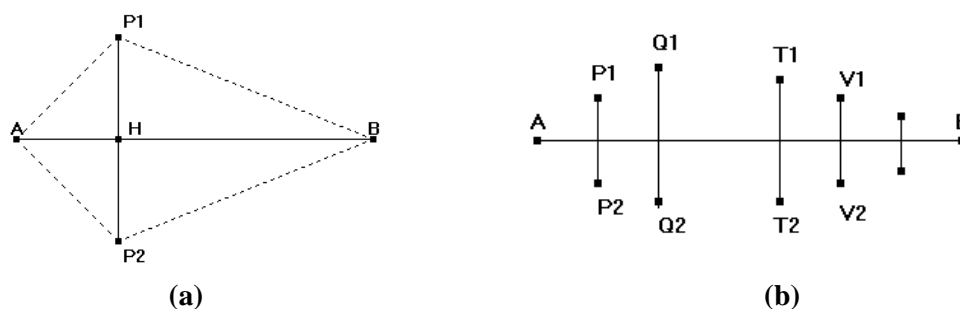
In the Italian school curriculum conics are studied in the third year of higher secondary education, and according to traditional teaching practices they are introduced on the Cartesian plane as a particular geometric place which involves the focuses or the focus and the directrix. Little or nothing is said on the actual construction or genesis of the plane. We present here an example of a laboratory activity different from the usual one which can be introduced, at least in part, during the first two years of secondary school. The idea behind this is not to consider the notion of focus on the Cartesian plane *per se*, but rather the concept of vertex also on the Euclidean plane, using Apollonius' 'symptom' (Costabile, 2013) on the plane. Before the necessary and indispensable formalization, the lesson could start with a constructive phase through points to be drawn either with ruler and dividing compass or through the use of a computer with an adequate programming environment, for example MatCos. The steps of the constructive phase on the Euclidean plane are as follows:

1. Given two points A and B, draw the segment AB;
2. Fix a number  $k > 0$  and choose a point H within the segment AB and construct [1] the point  $P_1$  on the straight line on the perpendicular straight line in H to the straight line AB so that:

$$\overline{P_1H} = \sqrt{k \times \overline{AH} \times \overline{HB}} \quad (1)$$

and let  $P_2$  the symmetrical of  $P_1$  to the straight line AB (Figure1a);

3. Repeat the previous operation a fixed number of times (like for example in Fig. 1(b)).



**Fig1: construction of the points  $P_1$  e  $P_2$  and other pairs of points  $Q_1, Q_2, T_1, T_2, V_1, V_2, \dots$**

The (algorithmic) procedure clearly leads to the construction of pairs of points  $(P_1, P_2)$ ;  $(Q_1, Q_2)$  ... and the set of all the pairs of points  $Q_1, Q_2, T_1, T_2, V_1, V_2, \dots$  consists of a symmetric closed curve with respect to the segment AB, called ellipse.

A particular case is obtained for  $k = 1$ , in this case the curve becomes a circle.

To construct the hyperbola it is sufficient to consider a point T on the straight line AB external to the segment AB, for example on the side of B, and choose a point H on the half straight line BT; for further details see (Costabile, 2013). For the parabola, instead, start from a half straight line in a fixed point A, choose on it a point at random H; let P be the point on the perpendicular in H to the straight line AB so that

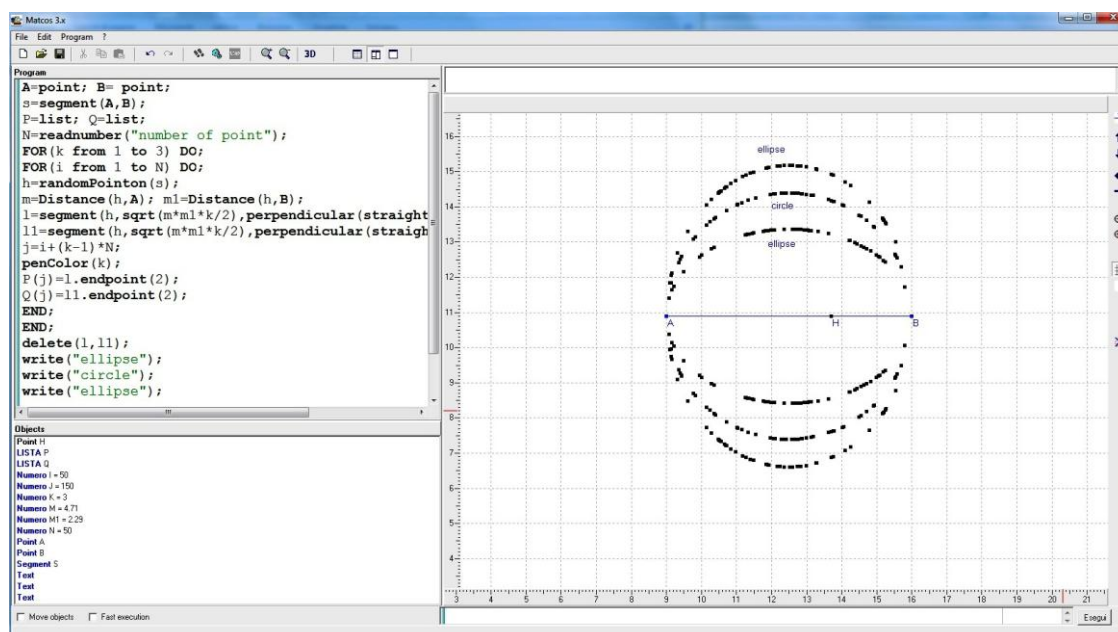
$$\overline{PH} = \sqrt{k \overline{HA}} \quad (2)$$

Let Q be the symmetric of P with respect to the straight line AB, repeating the operation you obtain a series of points  $P, P_1, P_2, \dots, Q, Q_1, Q_2, \dots$  and joining the points  $APP_1P_2, \dots$  from one end  $AQQ_1Q_2, \dots$  to the other you obtain a curve open to a branch called parabola of axis of symmetry AB and vertex A.

The previous algorithm can be easily implemented in a programming environment, such as MatCos for example, with the following MC1 code:

```
A=point; B= point; s=segment(A,B);
P=list; Q=list;
N=readnumber("number of point");
FOR(k from 1 to 3) DO;
  FOR(i from 1 to N) DO;
    h=randomPointon(s);
    m=Distance(h,A); m1=Distance(h,B);
    l=segment(h, sqrt(m*m1*k/2), perpendicular(straightline(A,B),h),1);
    l1=segment(h, sqrt(m*m1*k/2), perpendicular(straightline(A,B),h),2);
    j=i+(k-1)*N;
    penColor(k);
    P(j)=l.endpoint(2); Q(j)=l1.endpoint(2);
  END;
END;
delete(l,l1);
```

After this constructive/experimental phase one can formalize the rigorous definitions of curves, as the place of points which satisfy the Apollonius's "symptom" of respectively (1) e (2).



**Fig2: Output of the ellipse and the circle.**

The switch to the Cartesian plane will take place later and in the usual manner. The result will be a more unified procedure, also simplified from the point of view of calculations, compared to the traditional method. Obviously this innovative teaching practice implies an experimental phase with the computer, with the use of a DGS like Geogebra, for example. However, we would like to point out that DGS programs, even the most advanced of them, like Geogebra, are not entirely suitable for the simulations of problems which require a coded program within a programming environment. This is true for example for the simulation of probability experiments which, albeit easy, are very useful for teaching. (Costabile & Golemme, 2009).

For reasons of space we will not explain in detail the advantages of using MatCos instead of DGS, but just stress that the former is a programming environment, albeit best suited for use in secondary and primary schools, while the latter are not.

## CONCLUSIONS

The pedagogical-cognitive advantages of the outlined methodology seem evident:

- a constructive approach is adopted and the students become progressively more involved;
- formalization only takes place when the idea of the curve becomes the natural outcome of a process and not, as often happens, in a deductive, memorized or declarative way;
- the use of technology shows all the potential and the strength employed both from an operational and an educational and conceptual viewpoint;
- concepts are studied as part of a historical framework;
- students' curiosity, interest and participation are constantly stimulated and contribute to a harmonious balance of work.

## NOTES

1. The point  $P_1$  may be built by considering the segment  $k \times \overline{AH}$  and let  $H_1$  be the extreme different from H, then let us consider the circle with center at the midpoint of  $H_1B$ , and of radius  $O_1B$ ; the intersection of that circle with the perpendicular at H to the line AB is the searched point.

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## **PRE-SERVICE TEACHERS' PERCEPTIONS OF THE INTEGRATION OF ICT IN THE MATHEMATICS CLASSROOM**

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*This research examines middle school mathematics pre-service teachers' professional development during two years of their study at a teacher college regarding their perceptions of ICT use in their teaching of mathematics, specifically as a result of their preparation in ICT integration in the mathematics classroom in the frame of two didactic courses (in their second year of study) and in the frame of practicing teaching mathematics with technological tools (in their third year of study). The results indicated that generally, the pre-service teachers, as a result of their two years preparation, perceived the importance of integrating diverse technological tools in the mathematics classroom, probably because of the visual aspect of mathematics that the technological tools enable, which helps students with their learning of mathematics and encourage them to do so because they are part of their lives.*

### **INTRODUCTION**

Teacher colleges are a catalyst for pre-service teachers' change and professional development. This function of teacher colleges is especially important in the digital age, where technological tools are suggested for teachers' use all the time, and it is especially true for the mathematics teacher who is required to follow the new technological mathematical tools which make teachers mediate in a better way the mathematical knowledge associated with a specific mathematical topic, and, at the same time, help students independently discover mathematical knowledge or with the guide of the teacher. These suggestions are reflected in the literature, and specifically in conferences, as is the case with 17th ICMI study: Mathematics Education and Technology – Rethinking the Terrain (Hoyle & Lagrange, 2010). What is the influence of pre-service teachers' preparation as mathematics teachers by teacher colleges on their perceptions regarding the integrating the ICT (Information and communication technology) in their teaching and towards the role of the mathematics teacher in the digital age? The current research attempts to answer these questions.

### **LITERATURE REVIEW**

Researchers studied teachers' beliefs about and perceptions of the ICT in general, as well as towards specific ICT tools. Faggiano and Fasano (2008) studied in-service student teachers' and pre-service teachers' perceptions of ICT. They found that in-service student-teachers perceived technology as supporting their teaching only as a motivating tool that enables students to understand the subject matter. On the other hand, the pre-service teachers recognized that the knowledge of the instrument functionality is not enough for a teacher to use it as an effective tool to construct mathematical meanings. The in-service teachers did not recognize that technology could provide interesting and attractive learning environments, while, some of the pre-service teachers thought that the use of technological tools allows students to collaboratively solve intriguing problems.

Yuan and Lee (2012) studied the perceptions of 250 elementary school teachers in Taiwan regarding their use of Magic Board (an interactive web-based environment which provides a set of virtual manipulatives for elementary mathematics). The study revealed that teachers rated high scores on perceived teaching assistance, perceived learning assistance, and perceived competence of technology integration.

Mathematics teachers competencies is related to their preparation as teachers in the teacher college (Wu, 1999), where this preparation is considered as their professional development in the teacher colleges. This professional development is especially important when talking about the use of ICT in teaching, for preparing pre-service teachers for ICT use in teaching is preparing them for the new digital age which is full of ICT tools, especially for mathematics teachers. This issue of developing professionally pre-service teachers in ICT use in teaching is acknowledged in the literature, for example Chai, Koh, and Tsai (2010) point at the centeredness of the role that pre-service education plays in shaping teacher use of ICT in the classroom. Emperically, it has also been reported that pre-service teachers who received ICT training possessed a stronger sense of self-efficacy with respect to computer use (Brown & Warschauer, 2006). Furthermore, pre-service teachers who have acquired higher level of technological skills were more willing to use technology in the classroom (Paraskeva, Bouta, & Papagianna, 2008).

## **RESEACH RATIONALE AND GOALS**

Chai, Koh, and Tsai (2010) say that preparing pre-service teachers for ICT integration in the classrooms is a key focus for many teacher education institutes. This makes evaluating the preparation of pre-service teachers for ICT use in the classrooms required in order to understand the influence of the preparation program on pre-service teachers' professional development as mathematics teachers in the digital era. In this research we wanted to examine middle school mathematics pre-service teachers' professional development during two years of their study at a teacher college regarding their perceptions of ICT use in their teaching of mathematics, specifically as a result of their preparation in ICT integration in the mathematics classroom in the frame of two didactic courses (in their second year of study) and in the frame of practicing teaching mathematics with technological tools (in their third year of study). This examination would enlighten us regarding the main influences in the preparation program on the pre-service teachers' perceptions of ICT as a tool in the mathematics classroom, as well as giving us grounds of the steps necessary to deepen these influences.

## **RESEARCH QUESTIONS**

1. Which roles do middle school mathematics pre-service teachers perceive as roles that mathematics teachers should have in the digital era?
2. How do middle school mathematics pre-service teachers perceive the necessity of ICT in the mathematics classroom?

## **METHODOLOGY**

### **Research setting and participants**

The participants were 29 mathematics pre-service teachers in a teacher college in Israel. These pre-service teachers took two didactical courses in their second year of study: the didactics of mathematics teaching and the didactics of computer teaching, where they were introduced to specific ICT tools for the teaching of mathematics (in the first course) and to general ICT tools for teaching (in the second course). They were required, as well, in their third year of study to integrate ICT in their practice as mathematics teacher trainees in the training schools. In this integration they were requested to use various ICT tools and technological pedagogical models such as: videos, presentations, digital worksheets, digital games, spreadsheets, applets, GeoGebra, applications of cellular phones, Wiki, Google Docs and Sites and social networking sites such as facebook. The pre-service teachers were trained as well to use visual dynamic tools to investigate with students questions that encourage higher order thinking skills, such as: "Would the three perpendiculars in a

triangle meet at the same point? If so, what could you say about the location of that point?" They helped their students raise conjectures and discuss their correctness using mathematical reasoning.

### Data gathering tools

The data gathering tools were three forums in which they discussed issues related to ICT use in the mathematics classroom. One forum was given at the beginning of the second year (before the pre-service teachers took the two didactic courses), the second was given at the end of the second year (after participating in the two didactic courses), while the third was given at the end of the third year (after the pre-service teachers practiced as teachers with ICT in the classrooms).

### Data analysis tools

The first two stages of the constant comparison method (Glaser & Strauss, 1967) were followed to arrive at categories of pre-service teachers' perceptions of issues related to ICT use in the mathematics classroom. These stages were:

- Categorizing data: putting together data expressions or sentences that imply a category of pre-service teachers' perceptions of ICT use in the mathematics classroom.
- Comparing data: comparing expressions or sentences within each previously built category. This gave rise to sub-categories.

## FINDINGS

| Pre-service teachers' perceptions   | Percentages         |                      |                     |
|---|---------------------|----------------------|---------------------|
|   | First forum<br>N=80 | Second forum<br>N=84 | Third forum<br>N=71 |
| <b>The mathematics teacher should face the challenge</b>                        | <b>25.0%</b>        | <b>39.3%</b>         | <b>25.4%</b>        |
| • The challenge of the new technology   | 5.0%                | 3.6%                 | 0%                  |
| • The challenge of exploiting the computer potentialities                       | 13.8%               | 19.1%                | 11.3%               |
| • The challenge of developing his/her technological horizons                    | 6.2%                | 7.1%                 | 8.5%                |
| • The challenge to be innovative and constructive                               | 0%                  | 9.5%                 | 5.6%                |
| <b>The mathematics teacher should use diverse technological tools</b>           | <b>13.8%</b>        | <b>22.7%</b>         | <b>21.1%</b>        |
| • Using new software and tools  | 7.5%                | 13.1%                | 14.1%               |
| • Using diverse software and tools  | 6.3%                | 9.6%                 | 7.0%                |
| <b>The mathematics teacher should attend to students' needs</b>                 | <b>33.7%</b>        | <b>33.4%</b>         | <b>32.3%</b>        |
| • Using software tools appropriate for the students' needs                      | 8.7%                | 4.8%                 | 5.6%                |
| • Providing the students with research skills                                   | 11.3%               | 15.5%                | 2.8%                |
| • Helping the students get to know and using appropriately new tools            | 7.5%                | 6.0%                 | 7.0%                |
| • Helping students develop mathematical ideas                                   | 2.5%                | 0%                   | 0%                  |
| • Helping students develop their thinking skills                                | 3.7%                | 7.1%                 | 16.9%               |
| <b>The mathematics teacher should be good manager of teaching and learning:</b> | <b>27.5%</b>        | <b>4.8%</b>          | <b>21.1%</b>        |
| • A director and a guide for the pupils   | 11.3%               | 2.4%                 | 13.5%               |
| • Planning well the mathematics lesson  | 10%                 | 2.4%                 | 4.6%                |
| • Encouraging students to learn mathematics                                     | 6.2%                | 0%                   | 3.0%                |

**Table 1: pre-service teachers' perceptions of the mathematics teacher role in the digital era**

The results will be described in tables according to the issues in which this research is interested: the pre-service teachers' perceptions of (1) the roles that mathematics teachers should have in the digital era, and (2) the necessity of ICT use in the mathematics classroom.

Table 1 describes the development over two academic years of pre-service teachers' perceptions of the mathematics teacher role in the digital era.

**Table 2: pre-service teachers' perceptions of the necessity of using ICT in teaching mathematics**

| Pre-service teachers' perceptions  | Percentages         |                      |                     |
|--|---------------------|----------------------|---------------------|
|  | First forum<br>N=78 | Second forum<br>N=97 | Third forum<br>N=79 |
| <b>Technological tools and software are indispensable</b>                        | <b>17.9%</b>        | <b>19.6%</b>         | <b>22.8%</b>        |
| • The computer is indispensable in daily lives                                   | 14.1%               | 16.5%                | 13.9%               |
| • Pupils spend long hours on the computer and its numerous software              | 3.8%                | 3.1%                 | 8.9%                |
| <b>The computer makes learning the subject matter easier</b>                     | <b>67.9%</b>        | <b>62.8%</b>         | <b>57.0%</b>        |
| • Making students' absorption of the material easier                             | 21.8%               | 16.5%                | 22.8%               |
| • Helping the teacher give the material more clearly                             | 19.2%               | 13.4%                | 16.5%               |
| • Making the students learn the material more interestingly                      | 26.9%               | 32.9%                | 17.7%               |
| <b>The computer helps students perform assignments that need higher thinking</b> | <b>6.4%</b>         | <b>4.1%</b>          | <b>5.1 %</b>        |
| • Helping students perform investigations using higher order thinking            | 6.4%                | 4.1%                 | 5.1 %               |
| <b>The computer is a rich tool for teaching and learning mathematics</b>         | <b>7.7%</b>         | <b>13.4%</b>         | <b>15.2%</b>        |
| • Numerous applications fit the teaching and learning of mathematics             | 7.7%                | 13.4%                | 15.2 %              |

As can be seen from Table 2, two issues were perceived by the pre-service teachers as important in an increasing process along the two years: The indispensability of technological and computerized tools and software and the richness of the computer as a rich tool for teaching and learning mathematics.

## DISCUSSION

Teacher colleges are expected to lead to the professional development of pre-service teachers, especially in pedagogical content knowledge and technological pedagogical content knowledge (Tondeur et al., 2012). One of the pre-service teachers' professional development issues related to their technological pedagogic content knowledge is their perceptions of ICT as an influential factor in the mathematics classroom. The current research wanted to verify this issue, specifically how they professionally develop through taking two didactic courses and through integrating the ICT in their teaching in the training schools.

The first issue treated by the current research and related to the pre-service teachers' professional development is the development of their perceptions of the roles that mathematics teachers should have in the digital era. Some pre-service teachers said that mathematics teachers should face the challenge of new technologies, where the pre-service teachers' rate increased after participating in the two courses but decreased after the actual use of ICT in the mathematics classroom. It could be said that being exposed to new technologies in the two pedagogic courses made the pre-service teachers more aware of the need to keep in pace with the technologies and face the challenge of teaching with them, but after they used the new technologies in their teaching they became acquainted with the new technologies, so they felt less challenge to teach with them and keep in pace with them. Regarding the mathematics teacher's attending to students' needs, the pre-service teachers' perceptions of this issue is not the same when looking at the various themes. One of the important themes of this category is teacher's role to develop the thinking skills of the students, where the pre-service teachers increased their attention to this issue throughout their preparation as

teachers in the college. The reason for this increase could be the pre-service teachers' exposure to the technological tools potentialities as encouraging students' thinking (McNamara & Lynch, 2011) in the two didactic courses as well as their own experience with such tools.

The second issue treated by the current research was the pre-service teachers' perceptions of the necessity of ICT in the mathematics classroom. We have here three directions regarding the development of the pre-service teachers' perceptions of this necessity: increasing twice ('Technological and computerized tools and software are indispensable', or 'The computer is a rich tool for teaching and learning mathematics'), decreasing and then increasing (The computer helps students perform assignments that need higher thinking), and decreasing twice (The computer makes learning the subject matter easier). The pre-service teachers' experiences in the didactic courses and in the schools sometimes supported each other, but sometimes did not, which resulted in the phenomena that we described. The pre-service teachers were introduced to different technological tools in the two didactic courses, as well as being presented to different technological tools in the training schools, so their perceptions of the need to use the ICT in the mathematics classroom increased as a result of their experience. On the other hand, the students' perceptions of the need of some aspects of ICT use in the mathematics classroom lessened after taking the two didactic courses, but increased after teaching in the training schools, as for example the perception that 'the computer helps students perform assignments that need higher thinking'. It could be that the pre-service teachers solved difficult mathematical problems or problems that needed higher order thinking with the use of technology (Daher, 2009), but did not notice that this use made the solution process much easier for some reason that does not have relation with the technology, like the difficulty of the mathematical problem. It could be also that the technology did not help them justify informally their reasoning (Stols, 2012). From the other side, when using technological tools in the training schools they noticed that it helped the students understand mathematical ideas and relations (Daher, 2010), which changed positively their perception of the contribution of technology to the easiness of learning, especially when this learning need higher order thinking. In the third direction regarding the pre-service teachers' perception of the need for ICT in the mathematics classroom, this perception decreased after taking the two didactic courses, as well as decreasing after the training in the third year in schools. What influenced this double decreasing is the decrease, after participating in the two didactic courses, in pre-service teachers' perceptions regarding 'the ability of the computer to make students' absorption of the material easier' and 'the support given by the computer to the teacher regarding giving the material more clearly'. The double decrease was also influenced by the decrease, after practicing in the training schools, in the pre-service teachers' perceptions of 'the ability of the computer to make the students learn the material more interestingly'. We have explained the first decrease above, where the reason for the second decrease could be the pre-service teachers' noticing that the tools do not make the students learn mathematics more interestingly (due to other aspects, for example the difficulty to operate them).

## **CONCLUSIONS AND RECOMMENDATIONS**

In this research we wanted to examine middle school mathematics pre-service teachers' professional development during two years of their study at a teacher college regarding their perceptions of ICT use in their teaching of mathematics, specifically as a result of their preparation in ICT integration in the mathematics classroom in the frame of two didactic courses (in their second year of study) and in the frame of practicing teaching mathematics with technological tools (in their third year of study). The current research's results show that generally, the pre-service teachers, after taking the two didactic courses and after experiencing teaching mathematics with technological tools, perceived the importance of integrating diverse technological tools in the mathematics classroom, probably because of the visual aspect of mathematics that the technological tools enable, and which

help students grasp the mathematical concepts and relations more easily. Thus, the technological tools help students with their learning of mathematics and encourage them to do so because they are part of their lives. Further research is needed to examine mathematics pre-service teachers' preparation in teacher colleges, especially through in-depth interviews which enable to verify the main influence of each preparation factor on the pre-service teachers' perceptions and behavior in the mathematics classroom

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# A NEW LEARNING TRAJECTORY FOR TRIGONOMETRIC FUNCTIONS

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*Educational research has shown that many secondary school students consider the subject of trigonometric functions as difficult and only develop shallow and fragmented understanding. It is unclear which of the two popular approaches to introducing trigonometry, namely the ratio method and the unit circle method, works best. In this study we propose a new framework for trigonometric understanding and a new, dynamic geometry supported trajectory for learning trigonometric functions. We also report on the results of a classroom case study in which the new approach has been implemented and researched. We discuss the task-related difficulties that students faced in their concept development and we describe their trigonometric understanding in terms of our framework.*

## BACKGROUND

In the triangular geometry, the sine and cosine of an acute angle are defined as ratios of pairs of sides of a right triangle. This is referred to as the ratio method of introducing trigonometric functions. The right triangle is often embedded in the unit circle, but then the notion of angle actually gets the meaning of rotation angle. The sine and cosine of a particular angle are now defined as the horizontal and vertical coordinate of a point obtained by rotating the point (1,0) about the origin over the angle. This is called the unit circle method. It is unclear which approach works best. In practice, a combination of these approaches is often applied. In both approaches, the trigonometric functions are functions of an angle and not of a real number. This is more or less repaired by introducing the radian. It is important to make clear to students that the notion of angle differs in the two approaches: it is in the ratio method an angle of a triangle with values between 0 and 90 degrees, whereas it is in the unit circle method a rotation angle which has both a magnitude and a direction.

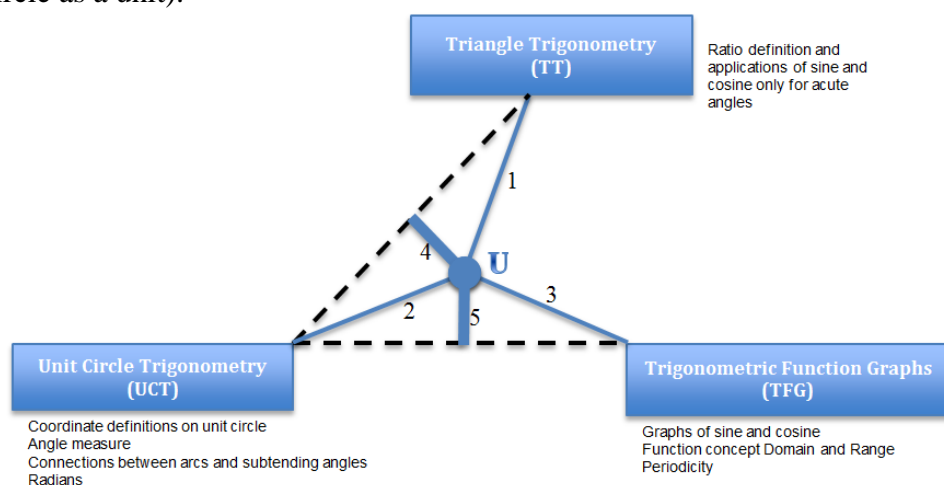
The research literature on students' understanding of trigonometric functions is sparse, but most studies conclude that students develop in the aforementioned approaches a shallow and disconnected understanding of trigonometric functions and underlying concepts, and have difficulty using sine and cosine functions defined over the domain of real numbers (Challenger, 2009; Moore, 2010, in press; Weber, 2005, 2008). The sine and cosine functions may have been defined, but the graphs of these real functions remain mysterious or merely diagrams produced by a graphing calculator or mathematics software. The complex nature of trigonometry makes it challenging for students to understand the topic deeply and conceptually.

In this paper we present a model of trigonometric understanding. It played an important role in the design of a new instructional approach to sine and cosine functions and in the analysis of the data regarding students' understanding of these functions that were collected in a classroom case study.

## A MODEL OF TRIGONOMETRIC UNDERSTANDING

Our model of trigonometric understanding is based on a conceptual analysis of mathematical ideas within and among three contexts of trigonometry, namely triangle trigonometry, unit circle trigonometry, and trigonometric function graphs (Figure 1). The conceptual analysis of trigonometry based on angle measure by Thompson (2008) was a source of inspiration: we also strive for coherence between mathematical meanings at various levels of trigonometry. But our approach differs in two aspects: (1) we use the functional relationship between arc length and corresponding vertical position for the case of sine in the unit circle prior to angle measure; (2) our model is broader than Thompson's example using angle measure in the sense that it includes trigonometric functions in

the domain of real numbers. We presume that it is easier for students to study first covariation of quantities having the same unit for measurement — in our approach a relationship between path length and displacement along coordinate axes — than to start with a function from angle measure (degrees or radians) to length measure (using the length of the hypotenuse of a right triangle or the radius of a circle as a unit).



**Figure 1: A model of trigonometric understanding**

The contexts TT, UCT, and TFG represent three contexts in which trigonometry can be partially understood, while the central point U represents the desired trigonometric understanding of students. The numbered line segments indicate that trigonometric understanding should entail aspects in the three contexts and the connections among them. It is important to keep in mind that point U should not be considered static. It may have different places in between the three contexts with respect to the quality of different students' understanding because different tasks may require different aspects of trigonometric understanding. The line segments labelled 1, 2 and 3 represent understanding of different aspects within three different contexts. These aspects are not only about factual knowledge, but also concern the students' ability to elaborate on them. A deeper level of understanding is represented by the thicker line segments 4 and 5 in Figure 1, which represent understanding the connections among the contexts represented by the dashed lines. Various aspects were hypothesized as important for students' trigonometric understanding: they are listed in Table 1 and formed the basis of the design of a hypothetical learning trajectory of trigonometric functions.

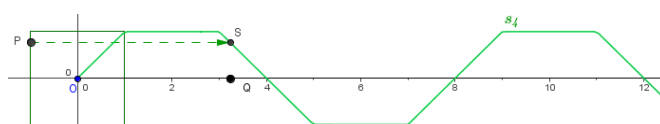
| <i>segment</i> | <i>context or connection</i> | <i>Aspects of trigonometric understanding</i>  |
|----------------|------------------------------|--|
| 1              | Triangle (TT)                | Ratio definition of sine and cosine in a right triangle, and applications  |
| 2              | Unit circle (UCT)            | Coordinate definitions of sine and cosine, and applications with angles or real numbers  |
|                |                              | Application of the coordinate definitions to evaluate sine and cosine for certain inputs   |
|                |                              | Trigonometric relationships deducible from the unit circle, e.g. sine is an odd function   |
|                |                              | Relationship between arcs and subtended angles through the notion of radian  |
|                |                              | Converting between degrees and radians via proportionality   |
| 3              | Graphs (TFG)                 | Connection of the sine and cosine graph with real functions  |
|                |                              | Functional properties of trigonometric graphs, e.g., domain and range of functions   |
|                |                              | Trigonometric relation revealed by the graphs, e.g., even and odd property of functions  |
| 4              | Connection TT-UCT            | Integration of the ratio method with the unit circle method via reference triangles, affording trigonometric values of angles larger than 90 degrees |
| 5              | Connection UCT-TFG           | Construction and interpretation of trigonometric graphs  |
|                |                              | Explanation of properties of trigonometric functions   |

**Table 1: Aspects of the new model of trigonometric understanding**



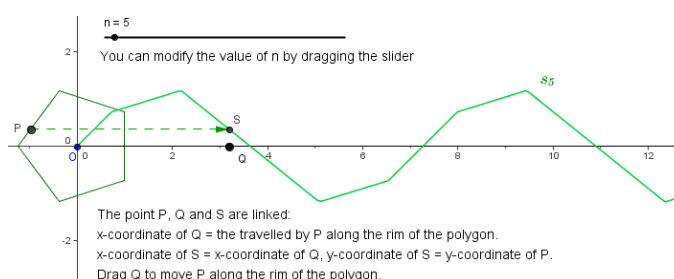
## A NEW LEARNING TRAJECTORY FOR TRIGONOMETRY

The basic idea in the new approach is to avoid an early introduction of radians as angle measure, but instead make the winding function of the real line onto the unit circle the principal concept and use the concept of arc and arc length to introduce sine and cosine as real functions. We presume that, when the sine function is introduced, it is easier for secondary students to consider a function with distance as input and height as output than to consider a functional relationship between angle measure and height. The nature and unit of the involved quantities are in this case the same. Also it is important in our approach that students can first practice with pencil and paper and develop understanding of the geometric construction of the sine and cosine function. Our reasoning is similar to that of Weber (2005) and Moore (2010): students better experience the application of a particular process and reflect on it before they mentally apply the same process. To get hands-on experience with the process of applying a winding function onto the unit circle and with coordinate functions, students first explore the winding function defined on a regular polygon that circumscribes the unit circle and is oriented in the Cartesian plane such that the point  $(1,0)$  is a midpoint of a vertical edge. A point  $P$  moves counter-clockwise along the rim of the regular polygon and the covariation of the travelled distance and the vertical position is studied. In case of an  $n$ -gon, this leads to a sine-like function  $s_n$ . Students begin with the construction of the function  $s_4$  by pencil and paper. Next they explore the same construction by a dynamic geometry package; see Figure 2 for a screenshot of the GeoGebra activity used in class. The purpose of this applet is to promote student construction of knowledge: it is meant to function as a didactic object (Thompson, 2002), that is, as an object to talk about in a way that enables and supports reflective mathematical discourse.

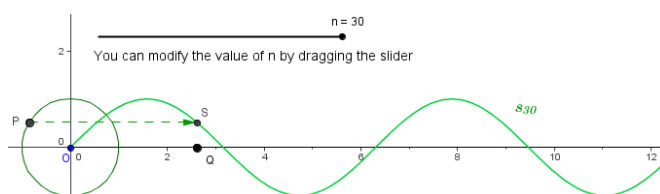


**Figure 2: Screenshot of a GeoGebra activity on the graph of the sine-like function  $s_4$ .**

The next step in the trajectory is to extend the graph over the negative horizontal axis and to make a similar construction of a cosine-like function  $c_4$ . Various trigonometric-like properties of the functions  $s_4$  and  $c_4$  can be explored such as  $s_4(x+8) = s_4(x)$ ,  $s_4(-x) = -s_4(x)$ , and  $s_4(2-x) = c_4(x)$ , for all  $x$ . Also note that by construction  $s_4(x) = x$  for small values of  $x$ . What can be done for a square can also be done for a pentagon, hexagon, and so on. With great effort, students can explore cases for small values of  $n$  by pencil and paper, but for most regular  $n$ -gons dynamic geometry software is helpful: see the screenshot of a GeoGebra activity in Figure 3 and 4.



**Figure 3: Screenshot of a GeoGebra activity on the graph of the sine-like function  $s_5$ .**



**Figure 4: Screenshot of a GeoGebra activity on the graph of the sine-like function  $s_{30}$ .**

The graph of  $s_{30}$  is very smooth and it can hardly be distinguished with the naked eye from the sine graph. Functions  $s$  and  $c$  can now be introduced by taking limits:  $s = \lim_{n \rightarrow \infty} s_n$  and  $c = \lim_{n \rightarrow \infty} c_n$ . Of course, this is not done in a formal way at secondary school level. It is only important that the students realize that the winding function for the unit circle can be defined in a way similar to the construction using regular  $n$ -gons and that the graphs of  $s_n$  and  $c_n$  look for large  $n$  almost the same as the graphs in case the unit circle is used instead of a regular polygon with many edges. We hope and expect that students develop in this way a process view of the sine- and cosine-like mathematical functions that helps them to understand the mental construction of the sine and cosine functions.

Until this point on the learning trajectory no attention has been paid to rotation angles and no link has been laid with the geometric definition of sine and cosine. Presuming that the students already know that the circumference of the unit circle equals  $2\pi$  and that a point on the rim of the unit circle is mapped by a rotation of 360 degrees about the origin onto itself, they can find out that a counter-clockwise walk along the rim of the unit circle starting from (1,0) over a distance of  $x$  units corresponds with a rotation of  $x \cdot 180^\circ/\pi$  about the origin. By drawing a right triangle for a point on the unit circle and using the ratio definition of sine, students can find out that  $s(x) = \sin(x \cdot 180^\circ/\pi)$ . This formula allows them to compute function values such as  $s(\pi)$  and  $s(\pi/3)$ . The introduction of radian finally boils down to the understanding that an arc of length 1 corresponds with a rotation angle of  $180^\circ/\pi$  and that this leads to  $s(x) = \sin(x \text{ rad})$ . We are close to calling  $s$  the sine function and denoting it by  $\sin$ , too. The introduction of the cosine function can similarly be realized. This finalizes the linking of the unit circle geometry with the triangle geometry.

### CLASSROOM CASE STUDY

Based on the model of trigonometric understanding (Figure 1 and Table 1) and the new approach outlined in the previous section, we examined students' concept development and understanding of sine and cosine functions through a classroom case study. Demir (2012) has written a detailed report about this research study. Here we only outline the work and some of the results.

The classroom study was conducted at a secondary school in Amsterdam with a class of 24 pre-university students (17 female and 7 male; age 16-17 yr.), who were classified by their teacher as a high achievement group in mathematics. The first author designed and taught five lessons in class. The cooperating teacher wrote observational notes, translated English mathematical terminology into Dutch when needed, and also helped students during their work. Students' readiness for the instruction was evaluated one week before the start of the instructional sequence through a half-hour diagnostic test. In the lessons, students were encouraged to construct their knowledge through interactions with peers, the researcher (acting as teacher) and the regular teacher. Working in pairs and whole classroom discussions were important elements of the lessons. The grouping of the students in dyads was done on the basis of advice of the cooperating teacher. GeoGebra applets were used as didactic objects in combination with tasks given to students through worksheets.

The research was conducted to find answers to two descriptive research questions: (1) What task-related difficulties do students face in their concept development within the designed instructional sequence based on the new hypothetical learning trajectory of trigonometric functions? (2) What characteristics relating to students' understanding of sine and cosine can be found in the data resulting from the intervention based on the new model of trigonometric understanding

We applied classical research methods for data collection. Participatory classroom observation was an important data source. We held interviews with the cooperating teacher after each lesson to record her impressions of the activities and of how the instructional materials and the ICT tools had functioned. Four semi-structured audio-recorded interviews were held with students after the instructional sequence for the purpose of getting an impression of their understanding of the key

mathematical points underlying the instructional sequence. Audio recordings of discussions of students on worksheet tasks gave an impression of their concept development. In each lesson, four dyads were recorded. We collected all completed worksheets of pairs of students. We administered a 50-minutes trigonometry test after the instructional sequence. It was based on our model of trigonometric understanding and designed to assess students' understanding.

Concerning the first research question, the analysis of student's responses to worksheet tasks and the audio recordings of the related group discussions revealed that in general the students were quite successful when working on most tasks. They only faced difficulties in the following tasks regarding their concept development within the instructional sequence: (1) drawing the graph of the vertical position of a moving point along the rim of the unit square against the travelled distance; (2) deriving the formula  $s(x) = \sin(x \cdot 180^\circ/\pi)$  of the graph of the vertical position of a moving point on the unit circle plotted against arc length; (3) converting  $180^\circ/\pi$  to radians and using it in the transition to function on real numbers; and (4) calculating the sine and cosine of  $210^\circ$ .

It was clear that the first task was very uncommon to the students. They were only familiar with drawing the graph of a function for which a formula has been given. Once they understood the task, they knew what to do and could determine points on the graph. But then it was difficult to decide how to connect these points: by line segments or curved segments? Having seen many smooth graphs in their school career, students tended to do the same in this task. The difficulties in deriving the formula of  $s(x)$  had to do with proportional reasoning to connect arcs and subtended angles, and with the required movement from specific cases toward a general case expressed via a mathematical formula that involves variables. Group discussions revealed that few students could link  $180^\circ/\pi$  with the notion of radian as angle measure. The radian concept was probably at this stage in the instructional sequence not fully understood yet and the task was therefore too challenging for the students. Later on in the learning sequence, many obstacles with the conversion between degrees and radians disappeared. The difficulty of calculating  $\sin(210^\circ)$  and  $\cos(210^\circ)$  had two sources: (1) some students did not recall the related values of  $\sin(30^\circ)$  or the method how to compute them; and (2) many students could not figure out how to link triangle trigonometry with unit circle trigonometry. Whole classroom discussed helped students overcome these difficulties.

Concerning the second research question about the students' integrated understanding of trigonometry after the instructional sequence, several findings could be derived from the interviews with students and the trigonometry test. Some of them are discussed below.

In general, the students developed a good level of understanding of aspects in the unit circle context. They were able to evaluate trigonometric functions of real numbers by associating them to an arc on the unit circle and they understood the notion of radian well enough for the construction of knowledge of trigonometric functions based on the fact that the value of the angle measured in radians is equal to the arc length on the unit circle. Although many students understood trigonometric relationship for specific angles, most students were not able not prove a trigonometric equality when expressed in algebraic format.

For the context of trigonometric function graphs, we found that the students conceptualized sine and cosine as functions of real numbers, and that they grasped how to interpret the graphs in terms of domain, range, periodicity, and symmetry properties.

From the trigonometry test and the interviews with students, we concluded that the students showed good understanding of how to integrate right triangle definitions of sine and cosine with the unit circle method in order to calculate trigonometric values of angles larger than  $90^\circ$ . However, most students could not calculate trigonometric values of well-known angles like  $30^\circ$ ,  $45^\circ$ , and  $60^\circ$  because

they had not memorized these special values, nor the methods to find the values. Nevertheless, they seemed to understand the connections between triangle and unit circle contexts

Students also developed a deep understanding between the unit circle context and the graph context. The most remarkable finding was that the students continued to base their understanding of such connections on arcs. This was found on many occasions. Concerning the construction and comprehension of trigonometric graphs, it was found that students conceptualized trigonometric graphs through the arc length on the unit circle. They explained coordinates of points on the graph with a journey metaphor based on arc length as travelled distances and they related the direction of the movement with the sign of an angle or the sign of a real number. Furthermore, students could explain the shape of trigonometric graphs.

## CONCLUSION

We proposed the use of arcs of a unit circle to serve as glue between the unit circle trigonometry and trigonometric function graphs, and the use of the metaphor of travelling along the rim of a geometric object like a regular polygon or a circle to help students develop coherent meanings based on arcs of a unit circle. Angle measure was addressed in our learning trajectory only after most connections between the trigonometric function graphs and the unit circle trigonometry have been set. We examined our new approach in a classroom case study. It provided evidence of the effectiveness in promoting: (1) integrated understanding of trigonometric functions in such way that students do not have as many difficulties and misconceptions as reported before in the research literature on trigonometry; and (2) connected understanding of trigonometric functions as functions defined for angles and as functions defined for the domain of real numbers. Students developed good understanding of trigonometric graphs and related properties of the trigonometric functions.

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## **USING INTERACTIVE WHITEBOARDS TO ENHANCE MATHS TEACHING: HOW, WHEN AND WHY?**

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*This paper focuses on how interactive whiteboards (IWBs) can be used to enhance maths teaching and learning, the attention being especially concentrated on the role and fundamental qualities of teachers. Special attention is also paid to the skills of teachers so as to make maximum use of these tools. It has also been stressed how a correct use of IWBs can be functional, i.e. how useful these tools can be when students approach, develop and strengthen mathematical concepts.*

### **INTRODUCTION**

Cognitive studies relating to the mental process involved in maths learning have recently shown how direct manipulation, signs and gestures are important when mathematical meanings are explained and represented: see Radford et al., (2003), Lakoff and Núñez, (2000). In some of these studies, beside the traditions of teaching and epistemological origin linked to the use of tools, special attention has been shown in signs, such as words, actions, gestures, looks, meant as values which give an idea of what thought is like (Goldin-Meadow, 2003), this attention having especially been paid to the interaction between students and technologies.

Using technologies in general, and IWBs in particular, can nowadays be useful and effective if the attention is focused on their teaching aspects and on the opportunities they give to improve the learning process.

As far as mathematics is concerned, as Arzarello et al. (2006) have stressed, technological instruments can have a crucial role in the teaching and learning process. And yet, it is extremely important to understand how and when technologies can influence, support and mould the way in which students learn mathematics. A large-scale survey (Arzarello, 2005) reveals that it is rare for the many papers about the teaching advantages that result from the use of technologies to comment on their real value added in detail and so they do not allow to assess the impact of their use on the cultural quality of both teaching and learning.

On the other hand, national policies have lately aimed at increasing the number of IWBs in all kinds of Italian schools. In parallel with them some “blended” training courses have been done all over the national territory for a lot of teachers.

After a nationwide campaign to raise public awareness of IWBs use, scenarios have been diverse: in some schools IWBs are put in rooms distinct from classrooms so that students can use them according to strict school timetables; classrooms where old slate blackboards and IWBs coexist, so they are different from each other but make a good combination. There are also classrooms where the classic blackboard has been considered as no longer useful or relevant to modern teaching in favour of IWBs.

We think that it is extremely important to tell exactly how a right use of IWBs can be functional and improve the way in which students learn mathematics if the aim is to understand how IWBs can become a useful instrument in teaching methods.

Some questions are therefore to be put:

- What is, if any, the news (i.e. technology, impact on the learning process, cognitive approach of students, the management of teachers, and so on) about the use of IWBs in teaching methods?
- What are the technological and learning skills which we believe to be necessary for teachers to use effectively IWBs in their classrooms? What are the teaching methods they must be able to implement?

New proposals for integrating IWBs with the teaching practice can be put forward, this means that it will be necessary to well define how technological (and non-technological) tools must be used in maths teaching (i.e. in a well-established field of research) referring in particular to the “maths lab” (UMI, 2004) and to the teaching strategies and methods connected with it.

### **THEORETICAL FRAMEWORK**

Over the past few decades all schools have been concerned by many Ministerial reforms and by attempts to connect with the Recommendations of both the European Parliament and the Council of Europe. Behind reforms there has been a redefinition, or simply, a variety of methodological approaches and teaching strategies that can promote the development of skills. In particular, special attention has been focused on one of the main mathematical concepts, that is to say the lab approach, meant as a teaching-learning methodology.

As everybody knows the maths lab is designed as “a structured set of activities aimed at representing the meaning of objects [...], which can some way or other be comparable to that of the Renaissance workshop where apprentices learn by doing and observing, communicating with each other and with experts” (UMI, 2004). The lab suggests involvement of both body and mind; the lesson evokes an exclusively intellectual participation. The craft activity or job that is made in the lab goes on a long time. It is also important to draw attention to meanings that are based on the semiotic mediation of suitable tools and on the interaction of all people involved in the lab activity. On the other hand, dealing successfully with complex requests and tasks involves not only having knowledge and skills, but also using strategies and routines required by the application of such knowledge and skills, as well as appropriate emotions and attitudes and an effective management of these components.

Moreover, several studies have adopted a “semiotic” perspective, the attention being focused on the role of signs, symbols, and their uses and interpretations. The notion of “semiotic mediation” introduced by Vygotsky and recently taken and further developed (see for example Mariotti, 2002, Bartolini Bussi and Mariotti, 2008), turns out to be particularly useful in studying the dynamics related to the integration of the technologies in maths teaching.

According to this point of view, the new technologies provide a great potential in terms of semiotic mediation since they can promote a process of internalization that changes – thus contributing to define new meanings – the use of a tool, that is to say an outside-oriented means whose function is to describe how men can have an influence on the activity, on the use of a sign, or an inside-oriented means to control themselves.

As part of the instrumental approach developed from the studies of VÉrillon and Rabardel (1995), the “instrumental genesis” expression has, on the contrary, be coined to show the long process during which a student makes a tool from an artefact developing techniques and mental patterns that allow him to use the artefact for a well-defined purpose. It’s a complex process, which is at the same time social and individual, connected to the limits and potential of the artefact and to the student qualities. This process is divided into two aspects: “instrumentation” and “instrumentalization”. The first concerns the use of the artefact (these patterns of use are aimed to

solve particular kinds of problems) and refers to the way in which the artefact has an influence on the student behaviour and thinking, while the second one concerns the way in which the student knowledge influences the artefact and brings him to take possession of the schemes of use and potential.

As widely emphasized by Artigue, the student educational process is strongly stimulated by the leading role of the teacher who must promote the creation of meanings through an orchestration process (Lagrange et al., 2003).

According to Trouche, the “instrumental orchestration” is defined as “the teacher’s intentional and systematic organization and use of the various artefacts available in a learning environment – in this case a computerized one - in a given mathematical task situation, in order to guide students’ instrumental genesis” (Trouche, 2003).

The instrumental orchestration is based on the combined action of three elements:

- “didactic configuration”: arranging artefacts according to the teaching purposes fixed in advance;
- “exploitation mode”: deciding on the roles that artefacts, teachers and students should play and choosing the technologies and procedures to develop as regards the didactic configuration;
- “didactic performance”: assessing all the choices that a teacher should make during their implementation and envisaging possible inputs from students and any consequent choices to adopt.

It is clear that the instrumental orchestration can not be handled in full before the educational project starts, while both “exploitation mode” and “didactic performance” can, on the contrary, change easily and adapt to different conditions because of the feedback got during the course of the project. That’s the reason why this pattern has inevitably a global dimension, strongly characterized by all teaching and experiential strategies that teachers have, but also a local dimension as to the way strategies can adapt to a particular teaching environment.

In particular Trouche emphasizes that teachers who want to integrate certain artefacts in their teaching experience, so that they can become for students tools to develop meanings, must keep in mind that integration has three different levels of complexity: mathematical (new environments require new mathematical problems); technological (getting limits and potential of each artefact); psychological (getting and managing the instrumental genesis).

As a result, the mathematical knowledge relating to an artefact becomes accessible to the student especially thanks to the teacher involvement.

### **USING IWBS IN THE ENZA’S TEACHING EXPERIENCE**

IWBs can play a very important role in that they allow all above-mentioned aspects that cannot be renounced to be integrated. The potential of IWBs can be exploited to develop and implement some lab models, moments of exploration that involve and are shared by every student, moments used as a way of reorganizing all concepts dealt with earlier.

The aspect that is innovative compared to what can already be obtained with other technologies and educational software (which, if properly used, can stimulate exploration and search for solving paths or models) is given by the possibility of:

- sharing one’s own work with other fellows, which helps to make the ensuing discussion more interactive;
- analysing and drawing attention to a lot of ideas in parallel;

- allowing students to continually express their thoughts and opinions when they are comparing their work to that of other students;
- keeping and re-employing the entire cognitive and metacognitive process in temporal and spatial contexts.

We think that it is important to draw attention to the focal role of the teacher because it represents the real, crucial and problematic factor of the integration of technologies into the teaching methods (Drijvers et al, 2010).

The work carried out by Drijvers has emphasized how the variety of strategies that teachers have at their disposal, the wealth of experience gained over the years can unintentionally lead to a “stability” of teaching methods that are difficult to change. We can therefore say that teachers need a new way of thinking and acting, rather than new technologies, to have a teaching-learning environment and educational activities that are really useful for students

We think that, if IWBs are used enhancing their potential and minimizing the risks due to their overuse, they can give the possibility of exploring situations, motivating students to query about the whys and wherefores of what they observe and ultimately facilitating the transition from an implicit and tacit knowledge to an explicit and conscious knowledge.

Students may, for example, be initiated into a theoretical study and, in particular, into the acquisition of the sense and meaning of the notions of demonstration and theory thus encouraging/promoting the transition from facts to demonstration and from empirical objects to generic objects, paying special attention to the role of semiotic mediation played by the Dynamic Geometry environment. Making activities can be a very important experience for the development of a theoretical way of thinking about geometry. In educational methods that offer exploration and observation activities, which encourage conjectures and motivate their validity, IWBs can play a supporting role in the crucial phase of recording the activities carried out by children in order to push them to describe and comment on the solution to problems and finally follow them in their process of logical and deductive growth.

A remarkable example is, in our opinion, the teaching experience related to the notion of plane circumference carried out in two first-grade classes of the Liceo Scientifico Cirillo (a high school with scientific orientation) in Bari. In this experience two technological tools have been used, that is GeoGebra and the IWB, strongly relevant to the lab procedure and to the teaching purposes fixed in advance.

Enza, the teacher who has carried out the circumference-related experience, is a secondary-school maths teacher with a good knowledge and skill in the use of technology for maths teaching and learning. The topic, suitable for being dealt with the chosen technology, would have given the opportunity to verify the effectiveness of the experience when bringing the matter up again after the school summer holidays.

To introduce the notion of circumference Enza has decided to “orchestrate” the teaching action using from the very beginning some exercises designed to do geometrical objects, by means of slides projected onto the IWB. These slides were made up of easy-to-understand exercises and other exercises aimed to lead students to discover in an empirical way both characteristics and properties of the circumference. Once all the exercises suggested are solved, students with the help of their teacher have formalized the notion of circumference and its properties by creating a “map” on the IWB.

This “didactic configuration” has allowed an obvious choice of the “exploitation mode” in which the IWB has played a very important role. The students, divided into groups, have suggested their



solutions working directly on the IWB, debating each time with their classmates and getting to the solution with the contribution of all. The teacher has played both the role of an observer who studies the dynamics triggered by students and, where needed, the role of a critic ready to reintroduce the problem in the stalemate.

The roles played by each component of this “instrumental orchestration” have obviously changed as the teaching activity has developed thus allowing the “observer” teacher to assess all the choices made during the implementation phase and compare the students’ input and their consequent choices to those fixed beforehand, in the “didactic performance” perspective.

Activities succeeded in bringing the lesson alive, (“yes, I can see. It is clear”). At the end of the experience students demonstrated to have understood the notion and properties of the circumference, however, after the school summer holidays, they proved they had not in the least dealt with the subject. For this reason, Enza now says that she will repeat the experience in a new class paying particular attention to this aspect:

“the introduction of the circumference and its properties can be done with the use of GeoGebra and IWB, discussing with students their solution to some tasks, but it is extremely important to take care of the students’ meaningful understanding. Technology is a means to achieve the discovery of a property, and to implement an interactive process in which students can have a voice, but knowledge thus obtained also needs to be transferred to the paper and pencil environment. The map created on the IWB by all students has also allowed to “let see” the conceptual problems come out, but it may also be necessary to find a way to keep track of the “collective history”, without ever having to exclude a viewpoint compared to another one, but at worst “suspending” it temporarily”.

The effectiveness of an instrumental orchestration seems therefore to be strictly related to many aspects. It depends on the whole class as well as on the single student learning style, on the time and resources as well as on the teaching and learning of a specific subject.

The described experience of helped Enza to verify potentialities and constraints of her orchestration and focus on the opportunity to use technology within an integrated learning environment in which a central aspect is the continuous alternation between technology and paper and pencil.

## CONCLUSIONS

Despite the results of surveys and monitoring activities have detected a high level of satisfaction from teachers who use IWBs, some studies (see for example Moss et al., 2007) must not be neglected because they have pointed out that the regular use of IWBs is not a guarantee of better learning results.

It is a well-known fact that the use of any kind of tools in a classroom, although they can help some students find an explanation, is not enough to guarantee a permanent explanation, least of all to promote a conscious and thoughtful learning.

To put it in pragmatic terms, a “sensible” use of the new technologies requires teachers to be initiated into them, this means that teachers must first of all be prepared to put themselves on the line, even before a suitable and careful orchestration.

Consider for example how the IWB can let concepts “burst out” allowing all aspects that are obvious or are not certain to be shown and reducing misconceptions. The IWB can lead to the creation of “maps” that become really interactive but they can turn out to be spatial rather than two-dimensional and open compared to the initial project of the teacher. Thanks to the graphic impact the maps are really eye-catching and allow the several socio-cultural and technological aspects inherent in a given mathematical concept to be seized. It is clear that there is the autonomous and unplanned attempt to devise a process of orchestration without attaching great importance to the

three elements of Trouche orchestration. There are a lot of risks in these approaches, they are not all predictable and often they cannot be checked during their implementation.

The decision to use IWBs (how, when and why) for teaching and learning maths in a “traditional” situation is therefore an important responsibility for teachers whose duty is to have a responsibility to find out learning environments, activities, ways and tools that allow students to both benefit from fields of experience that are important and useful and promote socialization, a process by which students are made to learn maths.

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## INVENTIVE MOMENTS TO MOBILISE SINUSOIDAL FUNCTIONS

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*This paper presents ways of animating the concept of function at primary school, through the aid of a motion detector that works with bi-dimensional motion. The technological environment favours a graphical approach to function, engaging the children in bodily activities. Starting from the new materialist perspective offered by de Freitas and Sinclair, we want to discuss ways of moving, doing and knowing in the classroom as pedagogical possibilities of in(ter)vention and inventiveness to mobilise the mathematical concepts at play. Particular focus will be on the instance of sinusoidal functions and their relative properties through reference to circular motion.*

### INTRODUCTION AND THEORETICAL BACKGROUND

Much recent research on embodiment in mathematics education has well recognized the role of the human body in learning mathematics. However, mathematical concepts are still thought of as immaterial and inert abstractions that are out of reach by the human body. In so doing, this research fails to recognize that mathematics also pertains of a body, as well as the materiality and agency of non-humans (de Freitas & Sinclair, 2013). In a complementary line of thought, de Freitas and Sinclair offer a new materialist perspective that also entails a new theorizing of embodiment in mathematics education. Their attempt is to recast “the relationship between matter and meaning (body and concept) in such a way that the one is not the foreclosure of the other.” (p. 468). Drawing on the work of Gilles Châtelet (2000), the authors propose to rethink the nature of mathematical concepts in a way in which the boundaries between the body of mathematics and the human bodies are always shifting: “Human bodies are constantly encountering, engaging and indeed amalgamating with other objects; the limits of our body are extended through these encounters” (de Freitas & Sinclair, 2013, p. 458). It is a view that makes room for in(ter)vention and inventiveness in the mathematics classroom and curriculum, which animate the mathematical concepts (Sinclair, de Freitas & Ferrara, 2013). As a result, it may change the ways we can think of and design mathematical activities in order to make learners engage with the mobility of matter, and their mobility itself in the processes of becoming mathematical subjects.

In this paper, we position ourselves according to de Freitas and Sinclair’s materialist perspective, to discuss pedagogical possibilities of mobilising sinusoidal functions. To this aim, we present ideas from a longitudinal study carried out at primary school (Ferrara, in press; Ferrara & Savioli, 2009; 2011). The study centred on the concept of function and was designed using motion detectors for a visual approach based on graphs. In particular, software for decomposing motion on a plane according to the two motion directions was used: it is called Motion Visualizer. The idea is that the use of the Motion Visualizer can enable ways of knowing and doing in the mathematics classroom, which animate graphical representations associated to specific bi-dimensional motion trajectories. Animating the graphs entails mobilising the functions that these graphs describe. In particular, sinusoidal functions might be animated through reference to a circular movement in the plane and to the ways this may occur.

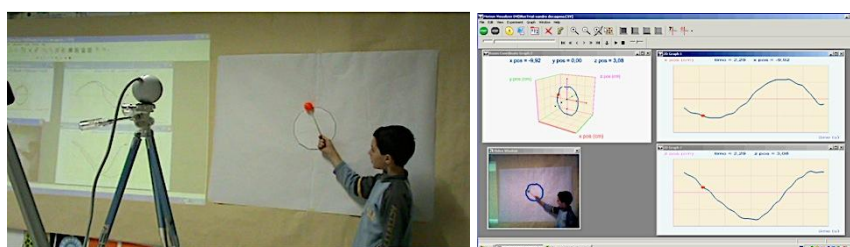
### THE STUDY AND THE MOTION VISUALIZER

The study took place in a primary school in Northern Italy and involved, since grade 2, a group of 16 children and their teacher in activities concerning the use of motion detectors to introduce the concept of function. One of the authors (Ferrara) was in the classroom as a researcher and led all the activities, while the teacher had the role of an active observer. This kind of tools enables a graphical

approach to function in that they model motion phenomena through graphs. Our focus in this paper is centred on a specific environment called Motion Visualizer. The experiences with the Motion Visualizer were carried out from grade 3 to 5 to work with bi-dimensional motion and its graphical representations. It captures the movement of a coloured object in two dimensions and displays two graphs of position versus time, according to the motion directions, through a webcam/camera connected to the computer. The environment was used in the classroom in real time, furnishing online feedback to the children's actions in terms of graphical representations. So, the qualities of a given movement could be associated to the shapes of the graphs, while motion is occurring. The children also worked on the opposite task, that is: producing movements to obtain given graphs. Starting from the interpretation of graphs, the study aimed to develop in the long period mathematical literacy about graph sense, relations between variables and the concept of function.

The whole class (8 males and 8 females) took part in the activities working as a group, most of the time, but there were also occasions for individual and pair work. Both oral interaction and written tasks were part of the study. A classroom discussion, led by the research, was always the final stage of a meeting or the starting point of the following meeting.

The use of the Motion Visualizer involved two lands: Movilandia, where the children experimented motion trajectories, and Cartesiolandia, in which position vs. time graphs related to the components of motion were displayed (Figure 1). Movilandia was created with a big paper on the wall and movements were performed on it, using coloured objects: an orange glove and an orange ball attached to a stick. Cartesiolandia was given by the projection of the computer screen with the graphs. It also showed a cube for the room with the trajectory, and a digital video of the movement, again with the trajectory superimposed. If the plane where the movement occurs is the vertical plane  $xz$ , the two graphs of  $x(t)$  and  $z(t)$  are displayed (being  $t$  the variable for time).  $x$  position and  $z$  position were presented to the children as two secret agents (Mister  $x$  and Mister  $z$ ), hidden in the glove/the ball and speaking "Cartesiolandese" (the language of the graphs). The challenge for the children was to understand this language through the interpretation of the graphs in terms of motion.



**Figure 1: (a) Setting in the classroom; (b) Interface of the Motion Visualizer**

The Motion Visualizer was crucial to create possibilities for in(ter)vention and inventiveness about the mathematical concepts at play. On the one hand, it was appealing for the children, who were attracted to searching for relationships between the phenomenon of motion and the graphs produced by the computer. On the other hand, thanks to real time feedback, perceptions and sensations, as well as mobility, could become part of a more conscious way of conceptualizing the graphs and understanding the covariance of variables. In the next sections, we first highlight possibilities for inventiveness in the general instance of functions that describe motion in two dimensions, then in the specific case of sinusoidal functions concerning movement along circular trajectories.

## FAVOURING INVENTIVE MOMENTS

The children experimented many kinds of movements during the three years in which they used the Motion Visualizer. Absence of motion and movement along trajectories with various geometrical shapes (starting from horizontal, vertical and slanted segment lines to squares and regular polygons,

with an increasing number of sides) were explored in grades 3 and 4. In grade 5, focus was centred on circular movement, with the aim to introduce the children to sinusoidal functions. The idea at the basis of these activities was that of approaching sinusoidal functions step by step passing through their piecewise function approximation (thinking of having as motion trajectory a regular polygon with a more and more increasing number of sides). For the teacher and the researcher, the way the environment works gave a way of looking at the two components of position and reasoning on their behaviour, in order to conceptualize the decomposition of motion in mathematical terms. The main difficulty here lies in the fact that, in Cartesiolandia, both components of position are drawn on the vertical axis. Considering a centre in Movilandia, the vertical component can be thought of as contributing to motion in terms of being above/below the centre, and the horizontal component in terms of being on the right/on the left of the centre. When the components are transferred on the Cartesian plane, the vertical position has a natural correspondence with its vertical representation: so, being above/below (the centre) in Movilandia still means being above/below (the origin) in Cartesiolandia. Instead, for the horizontal position, being on the right/on the left (of the centre) in Movilandia corresponds to being above/below (the origin) in Cartesiolandia.

The Motion Visualizer practically works detecting the movement of each component ( $x$  position,  $z$  position) in Movilandia and then displaying its values versus time on the Cartesian plane. For the teacher and the researcher, this was the starting point to favour and create inventive moments for understanding the graphs in relation to certain motion trajectories. Bodily activities were arising naturally from the mobility of function, with which the Motion Visualizer was engaging the class. Inventive moments were experienced by pairs of children asked *to pretend* to be  $x$  position and  $z$  position in Movilandia. As some trajectory was being produced in the air (by another child or by the researcher), the two children could mime together the movement of the components (Figure 2).



**Figure 2: Pretending to be  $x$  position and  $z$  position in Movilandia**

In this way, the children were imagining the situation in Movilandia, materializing the components of position. The perception of their change in time also became evident thanks to the moving hands. The shift to Cartesiolandia requires a conceptual change, though: thinking of the changing positions as placed on (moving along) the vertical axis of the Cartesian plane. This could be realised through two contemporary bodily actions: transferring the movement of each component along the vertical direction; and adding time as *pulling* position in the positive direction of the horizontal axis. A new hand could act for stretching the position components, bodily mobilising the graphs that the Motion Visualizer was producing. The idea of time that “pulls” position derived from a class discussion of which we report a passage below. Here, the researcher (Res.) was making the children think of the way things work in Cartesiolandia, trying to focus attention on the conceptualization of the graphs.

Res.: If I match the change of positions with time, what do I find?

Benny: A graph.

Res.: A graph or what it is, a graph, that is, a drawing. Let's see what the software does. Mister  $z$  goes up and time goes on. What does time make the position of Mister  $z$  do? (*miming  $z$  position, while Francesco 'makes' the vertical axis; Figure 3a*)

- Benny: Time pulls it. (*miming the action of time*; Figure 3a)
- Res.: But Mister  $z$  is already vertical in Movilandia, so the fact that we put it vertically in Cartesiolandia doesn't change anything. The problem exists for Mister  $x$ .
- Benny: You have to turn it.



**Figure 3: (a) Time pulling position in Cartesiolandia; (b) Turning horizontal movement vertically**

When the idea of *turning* was introduced, new occasions for mobilising the relationship between Movilandia and Cartesiolandia arose: the turning of position and the pulling of time needed to be joined. To this aim, the researcher went back to Movilandia:

- Res.: In Movilandia, Mister  $x$  doesn't move vertically but horizontally, this way; so its position changes. The problem is that the Motion Visualizer, which is smarter than me, says: In Cartesiolandia, I cannot put it horizontally, because horizontally it's the place of what?
- Children: Time.
- Res.: So, time cannot be put horizontally at a time, vertically another time. Otherwise, we don't understand anything. We are constrained to put the position of Mister  $x$  vertically. The position of Mister  $x$  is put vertically, but in Movilandia it moves horizontally, so what do I do with that horizontal movement?
- Benny: I turn it.
- Res.: Do it!
- Benny: This way. (*turning Res.'s right arm vertically*; Figure 3b)
- Res.: Now that I turned it, how does it move?

Once it is evident that the turned movement determines the graph thanks to the pulling of time, Benny went back to the "turn" of the horizontal movement to also think of the opposite passage:

- Benny: If I'm in Cartesiolandia, how can I return to the previous situation? (*meaning in Movilandia*) It's enough to remove time, 'cause this way the graph gets squashed on the vertical axis (*miming it*), and I get all the positions of Mister  $x$  again.
- Res.: When it's got squashed, I pretend that it's dynamic. It's changed from here to here during movement, so I pretend that it's dynamic. Then, I turn it, what do I find?
- Children: The position of Mister  $x$ .
- Marco: To redo the graph...
- Res.: To redo the graph, I repeat the same thing.
- Benny: I put it vertically again.
- Res.: I think of the way the position of Mister  $x$  changes and then I add time.

These inventive moments were shared by the children as ways of moving to animate the graphs with which they were working and to mobilise their corresponding functions.

## WORKING WITH SINUSOIDAL FUNCTIONS

Sinusoidal functions were introduced in grade 5. Using the inventive strategy above, the children quite easily associated the shape of the graphs related to circular movements to mountains. Then, the most interesting thing was to work with these functions, through their mobility, exploring some of their characteristics (for example, the fact that each is translated with respect to the other, as well as their relative stationary points). In this perspective, during one of the last meetings, the discussion focussed on the challenge of thinking of a movement for having the *same* “mountains” (a certain number) for Mister  $x$  and Mister  $z$  ( $x(t)$  and  $z(t)$ ). Some children considered the possibility of moving along a circle. The discussion went on reasoning on this issue, with the children miming the movements of the components, their spatial intervals, as well as the mountain-like shapes of the two expected graphs (Figure 4: on top, working in Movilandia; at bottom, working in Cartesiolandia).



**Figure 4: The mobility of the sinusoidal functions**

The researcher then asked the children to draw on paper the two graphs of  $x(t)$  and  $z(t)$ . After this work, she involved them in a new discussion about the differences in shape.

- Res.: How did you draw your mountains, each as a copy of the other?
- Enrica: No, 'cause... Arianna and I drew them, they're similar, 'cause Mister  $z$  starts a bit higher than the middle and Mister  $x$  starts from the middle. Then, Mister  $z$  doesn't finish, makes two mountains but it doesn't finish exactly below, but in the middle.
- Res.: Is the same for Mister  $x$ , is the height from which I start the same as the final one?
- Children: Yeah.
- Res.: But if we compare Mister  $x$  and Mister  $z$ , the height from which I start and that at which it arrives aren't the same.
- Teacher: You said that these mountains are similar (*referring to Enrica*). In which sense? What do you mean with similar?
- Enrica: Hm, they are equal, only the start changes and... hm... 'cause Mister  $x$  starts from the middle while Mister  $z$  starts from a bit higher than the middle... (*long pause*) hm, the first mountain comes, the peaks of the mountains come equal but the end of Mister  $z$  goes, finishes in the middle.
- Res.: What does it mean that the peaks of the mountains come equal? The peaks of which mountains?

- Enrica: The tops.
- Res.: But the tops for only one graph or all the tops? Are you referring to a single graph? Are you comparing them? Or are you saying that this happens for both?
- Enrica: I'm comparing them.
- Res.: Then, both for the position of Mister  $x$  over time and for the position of Mister  $z$  over time... Is the mountaintop, which is reached by the mountain, at the same height for both the Misters? Like for the bottom, say, the valley of the mountains, is it at the same height for both the Misters? Is this what you're saying?
- Enrica: Yeah.
- Res.: But, why?
- Enrica: 'Cause... 'cause when they go circle, the circle is always equal. Only it comes different 'cause Mister  $z$  starts from a different position.

## CONCLUSIONS

We have no more room here to further develop the discussion. However, it appears from the speech of the children that they were in the phase of conceptualizing the translation of each sinusoidal function with respect to the other. In this phase, the children just used their imagination by means of that “didactics of gestures” that they acquired in the inventive moments discussed in the paper.

The use of the Motion Visualizer was not at all secondary for these kinds of activities in the primary classroom. Indeed, it was really effective in favouring inventiveness for working on the introduction of the concept of function, for finding ways to animate the mathematical concepts, and for driving the children to a sensible and bodily engagement in doing and knowing this mathematics. It would probably have been different an approach possibly created in the classroom without the cues and insights given by the same technological environment.

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## USING DGS TO INTRODUCE ALGEBRA PROPERTIES

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*This paper describes a summary of an experimental study that involved teachers and students in 10 high-schools in Sicily together with a team of University Professors. The topic of the study was on symbolic algebra. The goal was to investigate the impact on teachers and learners of an “innovative” way of introducing algebra inspired by a modern reinterpretation of the second book of Euclid’s Elements, and based on the use of a dynamic geometry system. The activity has been carried out in 15 classes of the first years of high-school (14-15 year-old students). Results of the experimentation are provided.*

### INTRODUCTION

“One of the main difficulties that secondary school students have to face in Algebra is the insufficient mastering of syntactic rules as often pointed out by teachers as well as researchers” (Kouki & Chellougui, 2013). On the other hand there is a quite wide international researchers’ community that work on questions concerning problems of students’ approach to school algebra (Carraher & Schilieman, 2007).

Without appropriate basis algebra is presented to the student "like a forest of signs without meaning" (retrived in Paola's home page [http://www.matematica.it/paola/materiali\\_vari.htm](http://www.matematica.it/paola/materiali_vari.htm)), "a useless game of manipulation of symbols just to solve complicated equations or simplify expressions" (Anzalone, Margarone & Micale, 2003). Also according to Paola, it is good to create and develop the language of algebra and its formulas in a concrete context as a tool for describing, modeling and solving problems. Courses and activities offered to students should be designed as processes which, starting from concrete, intuitive, perceptive, empirical aspects, gradually lead students to theory. In such a way it is possible to direct students, trough the understanding and interpretation of simple mathematical formalisms, to that abstraction that will be conquered step by step in the following years. It is needed, after all, a "wise teaching of algebra" (Arzarello, 1994).

In this context we realized an activity addressed to secondary school students. It was proposed a modern reinterpretation of certain propositions of Book II of the Euclid’s Elements. In these propositions, geometric properties reveal algebraic properties, and, not surprisingly, this book is considered the geometric algebra book. Such an approach allows students to see mathematics as a whole rather than as a collection of separate chapters, creating a link between algebra and geometry: the use of the language and the methods of geometry facilitates the understanding and the acquisition of algebraic properties. Moreover for once it is no longer algebra used for geometry (the only possible collaboration between the two branches from the point of view of the students) but at last geometry in used for algebra.

Finally, this proposal is original because based on the use of a *dynamic geometry system* (DGS). For these features, our proposal is part of the vast and complex issue on integration of digital technology in mathematics education (Drijver, 2012).

In this paper, after some theoretical considerations, we describe the contents and the modalities of the experimental activity that we carried out and some general results obtained in its experimentation.

## INTEGRATING TECHNOLOGY IN A DIDACTIC LABORATORY OF MATHEMATICS

The whole activity we are about to present has been carried out within a didactic laboratory of mathematics. Such a laboratory is intended as “a phenomenological space to teach and learn mathematics developed by means of specific technological tools and structured negotiation processes in which math knowledge is subjected to a new representative, operative and social order to become object of investigation again and be efficaciously taught and learnt” (Chiappini, 2007). In a laboratory students would work like in a “Renaissance workshop, in which the apprentices learned by doing, seeing, imitating, communicating with each other, in a word: practising” (MPI, 2003).

Moreover in our activity the laboratory of mathematics was carried out in a computer laboratory. The activity in fact was based on the use of a (DGS), Cabri Géomètre.

DGS are of great help in mathematics teaching, not only because they are quite popular among students’ (they look forward to go to a computer lab...), but also because they allow to have an

Explore-conjecture-verify-proof approach

that otherwise would have been hard to get. In the activity (see next paragraph) we present here we actually have an

Explore-conjecture-verify-“translate” to algebra approach.

DGS is used to see algebraic properties shaped as geometric ones. DGS are nowadays quite well known by the teachers’ community. Most of the teachers would know what a DGS is. Only few of them actually use DGS in their classes practices. There are several difficulties encountered by teachers using DGS or technology in general: 1. there are few computer labs in schools; 2. using technologies implies spending more time for teaching a topic; 3. how to integrate technologies when teaching.

As for the first difficulty, this is a problem we have to face, but that we can not handle. As for the second difficulty, on the one hand it is true that it takes more time teaching with technologies but on the other the benefits for the students are a good “payback”. As for “how” to integrate technology, this is, to us, the real problem of teachers. Teaching with technologies does not mean just giving a task to the students: it implies integrating Content Knowledge, with Pedagogical Knowledge, with the most “suitable” technology (Koehler and Mishara, 2008). This is what we concentrate our attention as well.

### CONTENT OF THE PROPOSAL

In our proposal we considered four plane geometry propositions taken from the second book of Euclid’s Elements, that we denoted by  $P_1$ ,  $P_2$ ,  $P_3$  and  $P_4$ . These propositions express algebraic properties through geometry. Precisely: the distributive property of multiplication over addition, the square of a binomial (sum and difference), the difference of two squares.

Propositions have been formulated according to Euclid’s original content, but in a language more congenial to us. They are on equivalent polygons. Privileged and recurring figures in these properties are the rectangle and the square, and it is this fact that makes such propositions very easy to understand.

In addition to these propositions of Book II were considered four propositions of geometry of space, that we denoted by  $S_1$ ,  $S_2$ ,  $S_3$  and  $S_4$ . They can be considered as geometric algebra propositions too, as they highlight the algebraic formulas concerning the cube of a binomial sum, the cube of the binomial difference, the sum of two cubes and the difference of two cubes, respectively.

The algebraic interpretation of the geometrical propositions is based on the following correspondence between geometric entities and arithmetic entities:

| GEOMETRIC ENTITIES             | ARITHMETIC ENTITIES           |
|--------------------------------|-------------------------------|
| Segments                       | Numbers                       |
| Sum and difference of segments | Sum and difference of numbers |
| Area of a rectangle            | Product of two numbers        |
| Volume of a parallelepiped     | Product of three numbers      |

**Table 1: Correspondence between entities**

In the following table we collected the geometric properties and the relative algebraic properties.

| GEOMETRIC PROPERTIES   | ALGEBRAIC PROPERTIES  |
|--|---|
| <b>P<sub>1</sub></b> : If there are two straight lines, and one of them is cut into any number of segments whatever, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by the uncut straight line and each of the segments.  | $a(b + c + d) = ab + ac + ad$   |
| <b>P<sub>2</sub></b> : If a straight line is cut at random, the square on the whole equals the squares on the segments plus twice the rectangle contained by the segments.   | $(a + b)^2 = a^2 + b^2 + 2ab$   |
| <b>P<sub>3</sub></b> : If a straight line is cut at random, then the sum of the square on the whole and that on one of the segments equals twice the rectangle contained by the whole and the said segment plus the square on the remaining segment.   | $a^2 + b^2 = 2ab + (a - b)^2$<br>from which<br>$(a - b)^2 = a^2 + b^2 - 2ab$                  |
| <b>P<sub>4</sub></b> : If a straight line is cut into equal and unequal segments, then the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section equals the square on the half.   | $(a + b)(a - b) + b^2 = a^2$<br>from which<br>$a^2 - b^2 = (a + b)(a - b)$                    |
| <b>S<sub>1</sub></b> : If there are two straight lines a and b, then the cube over the segment a+b equals the cube over a, plus the cube over b, plus three times the parallelepiped with base the square over a and with height b, plus three times the parallelepiped with base the square over b and with height a. | $(a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$   |
| <b>S<sub>2</sub></b> : If there are two unequal straight lines a and b, with a>b, the cube over a equals the sum of the cube over the difference of the two segments, plus three times the parallelepiped with dimensions the two segments and their difference, plus the cube over b.                                 | $a^3 = (a - b)^3 + 3ab(a - b) + b^3$<br>from which<br>$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ |
| <b>S<sub>3</sub></b> : If there are two straight lines a and b, then the sum of the parallelepipeds with height a+b and with basis the square over each segment equals the sum of the  | $a^2(a + b) + b^2(a + b) = a^3 + b^3 + a^2b + ab^2$<br>from which                             |

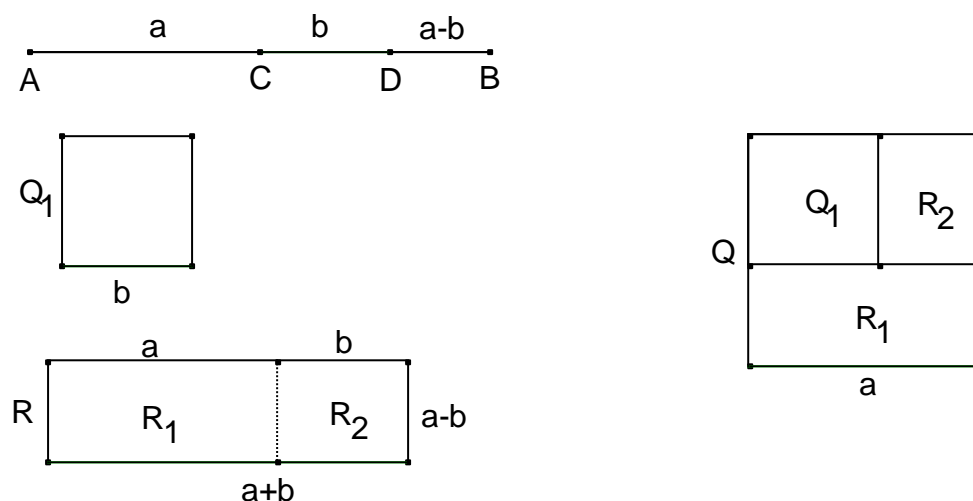
|  |  |
|--|--|
| cubes over the two segments plus the parallelepipeds with basis the square over each segment and with height the other segment.  | $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$  |
| <b>S<sub>4</sub>:</b> If there are two unequal straight lines a and b, with $a > b$ , the cube over a equals the sum of the three parallelepipeds with height the difference between the two segments and with basis the square over the first one, the square over the second one and the rectangle contained by a and b, plus the cube over b. | $a^3 = a^2(a - b) + b^2(a - b) + ab(a - b) + b^3$<br>from which<br>$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ |

**Table 2: Table of content**

The translation from geometry to algebra is not always immediate. It follows, for example, the transposition of proposition P<sub>4</sub>.

**Proposition P<sub>4</sub>.** Consider a segment AB and let C be its midpoint. Consider a point D of the segment, D different from C, and assume  $AD > AC$  (Fig. 1). Let  $\overline{AC} = a$  and  $\overline{CD} = b$ , it is  $\overline{AD} = a + b$  and  $\overline{DB} = a - b$ , therefore the proposition state the following property:  $(a + b)(a - b) + b^2 = a^2$  from which it follows the formula for the difference of two squares:  $a^2 - b^2 = (a + b)(a - b)$ .

The property is achieved by the students with the use of the Cabri Géomètre: students are asked to construct a rectangle R of sides AD and DB and of two squares Q and Q<sub>1</sub> of side AC and CD respectively and to explore about the possibility of covering Q with Q<sub>1</sub> and R by dragging figures; then they are led, if not successful, to cut R in two rectangles R<sub>1</sub> and R<sub>2</sub> so to properly cover Q. The student is also invited to provide conjectures (expressing in natural language the observed property) and by means of translation to algebra (writing final formulas).



**Fig.1: Cutting R and moving Q<sub>1</sub>, R<sub>1</sub>, R<sub>2</sub> to cover Q in worksheet P4**

In general, in all worksheets prepared for the students (see next section) there is a construction of the objects that help to translate the property from algebra to geometry. The objects are built, and explored, with Cabri Géomètre.

Because of space, we cannot report here all the worksheets. You can e-mail the authors for more info.

## THE EXPERIMENTAL ACTIVITY

The activity has been planned by the authors of this paper, who wrote some notes on the contents for involved teachers, planned and guided the meetings with teachers, followed the activities in schools, organized the final meetings with all the students. The activity has involved 15 high-school teachers together with the students of their classes of first and second year of high-school.

We prepared worksheets (Ferrarello & Mammana, 2012), for class activities together with teachers. Most of the teachers had never used technology in the classrooms and we wanted to make them understand not only “how to use computers, but also how to incorporate computer when teaching” (M. Dogan 2011); moreover we wanted teachers to be aware of how to prepare worksheets themselves. “The integration of technology in mathematics education is not a panacea that reduces the importance of the teacher. Rather, the teacher has to orchestrate learning. To be able to do so, a process of professional development is required” (Drijvers, 2012).

One worksheet has been elaborated for each property of Table 2. Each worksheet started with the statement of the geometrical property that students were afterwards guided to represent graphically with the help of the DGS. In particular students were asked to draw squares, rectangles, cubes and parallelepipeds. In order to facilitate the construction of these objects, especially of cubes and parallelepipeds, two “macros” were provided to the students: one for the constructions of squares and rectangles and the other for the construction of cubes and parallelepipeds of given dimensions. Then students had to explore the figure they obtained, to verify some assumptions they made and to give the algebraic interpretation of the property. The worksheets relative to properties  $P_1$  and  $P_2$ , moreover, asked for some more investigation that led students to some applications (formulas for the binomial product and the square of a trinomial).

## SOME GENERAL RESULTS

In this section we present some general results concerning: a) the construction and exploration of figures; b) the precise statement of a formula in a natural language; c) the determination of the algebraic formula. We get these results from an analysis of the forms compiled by the students and from teachers’ log-books.

### Construction and exploration of the figures

All students, through the guided exploration, were able to make the correct constructions (with the DGS) of the figures required and to explore the figure itself so to get to the algebraic property. The fact that for a lot of students it was the first time that they were using a DGS was not a problem.

### Statement of a formula in a natural language

Some difficulties arose in this phase. For example looking at the first worksheet about 80% of the students have not been able to correctly formulate the algebraic property (“the product of a number and a sum equals the product of the first number and all other sums” – Antonio and Daniel, worksheet 1P). This percentage actually fell to 41%, 21% going towards the activity down to 17% in worksheet 4 (“The difference of two cubes equals the product of the difference between the first and the second number times the square of the first number plus the square of the second number plus the product of the first and the second number” Antonio and Daniel, Worksheet 4S). Where else in worksheets on space properties something strange happened: students reacted quite well at the beginning (about 15% of wrong answers) but did not perform well in the following worksheets with 61% of wrong answers in the last worksheet.

## Writing algebraic formulas

Both in plane and in space, i.e. in the first four worksheets and in the second four, students were able to write the algebraic formula that they had found, with almost no wrong answers in the last worksheet!

## CONCLUSIONS

Lack of space does not allow us to go into the details of the activity and of the obtained results. However, we briefly want to report here some comment of teachers that took part in the activity.

The main problem arising when dealing with algebra topics is the abstractness and the lack of meanings conferred to formulas. The main advantage of this activity was, according to the teachers, the fact that students have been able to "see" algebraic properties through geometry with the aid of the DGS, and this has led to the involvement of students, who have shown interest and curiosity. This has led to a better understanding of the topic and, consequently, to an effective memorization, i.e. the "placement" of the topics in the long-term memory; this happens every time a learner gives a meaning to a topic.

All this was realized thanks to the use of DGS, which enabled students to develop intuitive abilities, relieved them from drawing each case, allowing an immediate generalization, gave a boost to creativity and improved their open-mindedness.

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## AN EXTENDED DESIGN EXPERIMENT CONNECTING SOFTWARE DEVELOPMENT WITH CURRICULUM DEVELOPMENT: THE CASE OF THE CORE-PLUS MATHEMATICS PROJECT

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*In the United States and internationally, design and development of mathematical and statistical software most often occurs independently of the development of school mathematics curricula. In this chapter we describe an extended design experiment that has pursued an integrated design approach that connects software development with the development of a curriculum spanning the full range of contemporary high school mathematics. The resulting software, CPMP-Tools, is an open-source suite of linked Java-based software tools that include general purpose tools—a spreadsheet, a computer algebra system, dynamic geometry, data analysis, simulation tools, and discrete mathematical modeling tools—together with topic-focused custom apps.*

### BACKGROUND

Since 1992, the Core-Plus Mathematics Project (CPMP), with funding from the National Science Foundation, has been involved in research and development of a problem-based, inquiry-oriented, technology-rich four-year high school curriculum that interprets and implements curriculum, teaching, and assessment recommendations in national school mathematics standards documents (cf. Fey & Hirsch, 2007). Because of concerns for access and equity, the first-edition materials were based on a modest technology assumption—students would have access to graphing calculators in class and outside school. As planning began for the second edition, surveys by the Pew Internet and American Life Project (2009) and our own pre-planning surveys revealed that students were increasingly using the Internet and, indeed, had greater access to the Internet than to graphing calculators outside of school. In addition, the new contextual and mathematical problems that the curriculum was being organized around, and the learning expectations for students, were such that it was essential to augment graphing calculator technology with the integrated use of computer technologies.

The power and potential of computer technologies for enhancing student learning and understanding of mathematics has long been recognized (cf. Fey et al., 1984; Zbiek & Heid, 2012; Zbiek, Heid, Blume, & Dick, 2007). However, in spite of the considerable promise that computer technologies provide for the improvement of school mathematics, the fulfillment of that promise has been stymied by issues of finance, access, and equity, among others (Heid, 2005). These intermingled issues have been exacerbated by the economic recession in the U.S. that began in 2008 and that brought about often repeated and deep cuts in state-level funding of public schools. In this chapter, we will consider the following question:

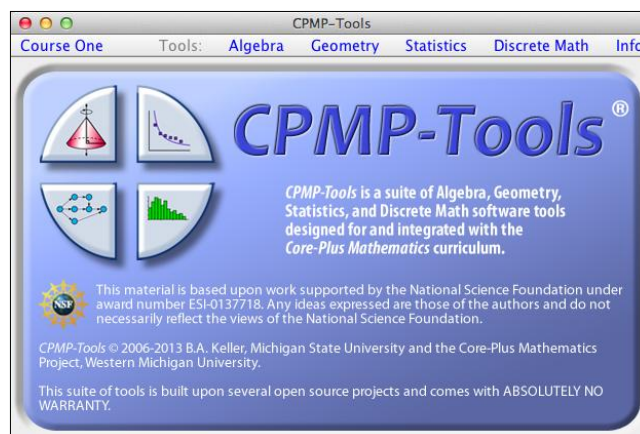
*What would it take for computer technologies like spreadsheets, computer algebra systems (CAS), and dynamic geometry, statistics, simulation and discrete mathematics tools to become an integral part of student learning and work?*

Among the answers to this question as reported in Heid (2005), and which we address below, are:

- universal availability
- potential to be tailored for particular purposes
- curricula that incorporate these tools as an integral part of the development of mathematics.

## DESIGN CONSIDERATIONS AND FEATURES

To meet this challenge and maintain our commitment to access and equity, the Core-Plus Mathematics Project systematically explored development of Java-based software concurrent with the creation of the second-edition four-year curriculum through successive iterations of design, development, and testing of the curriculum and support software (Figure 1).



**Figure 1: CPMP-Tools opening screen**

*CPMP-Tools* is a suite of both general purpose and custom tools whose development continues to be guided by, and integrated with, the development of the curriculum materials, currently being refined to align with the Common Core State Standards for Mathematics (CCSSI, 2010).

Design features of *CPMP-Tools* include:

- Tools are organized by courses and strand so that the software learning curve is not steep.
- Tool functionality is responsive to needs of the curriculum and increases with course level and students' understanding of mathematics and statistics.
- Tools share similar menu screens and interfaces promoting learning transfer among tools.
- Tools are built using Java WebStart, which permits safe, easy, and reliable distribution of software and software updates across different types of computers. It permits teachers familiar with Java programming to add additional custom tools.
- The Java-based software, used online or downloaded to the user's computer, can be freely accessed in mathematics classrooms, libraries, at home, or anywhere there is Internet access.
- The software is self-updating whenever the user has a browser open.
- The software includes four linked families of tools, briefly described and illustrated below, and a collection of custom apps focusing on specific topics within the curriculum.
- As a linked suite of curriculum-embedded tools, their selection and use supports student development of the important mathematical practice of using appropriate tools strategically.

*Algebra* tools include an electronic spreadsheet (also serving as a data entry sheet) and a computer algebra system (CAS) that produces tables and graphs of functions (in two and three variables), manipulates algebraic expressions and matrices, and solves equations and inequalities (Figure 2).

*Geometry* tools include an interactive drawing program for constructing, measuring, manipulating, and transforming geometric figures in a coordinate, synthetic, or vector environment (Figure 3). The software supports work with two- and three-dimensional representations. Also included is a simple object-oriented programming language for creating animation effects.



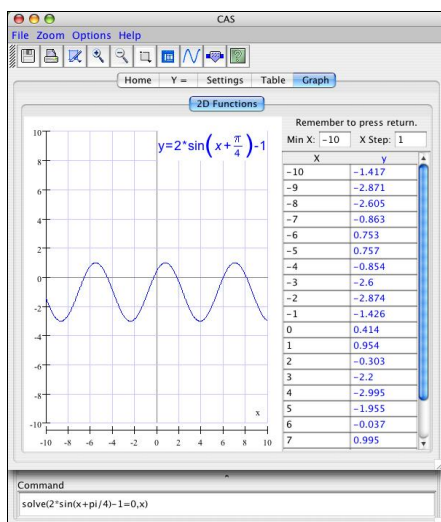


Figure 2: Linked function representations

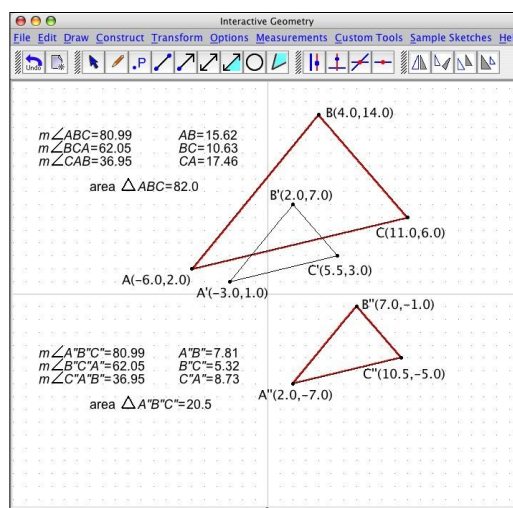


Figure 3: Composite of a dilation and a translation

*Statistics* tools include tools for graphic display and analysis of univariate and bivariate data, including finding function models for bivariate data, simulation of probabilistic situations (Figure 4) and mathematical modeling of quantitative relationship. Figure 5 shows a dynamic modeling line that students can manipulate by moving three control points. A least squares regression line with equation can also be fitted to the data. The software includes pre-loaded datasets that are used as problem contexts in CPMP Courses 1-4. Spreadsheets allow easy insertion of class-collected data or data from other sources.

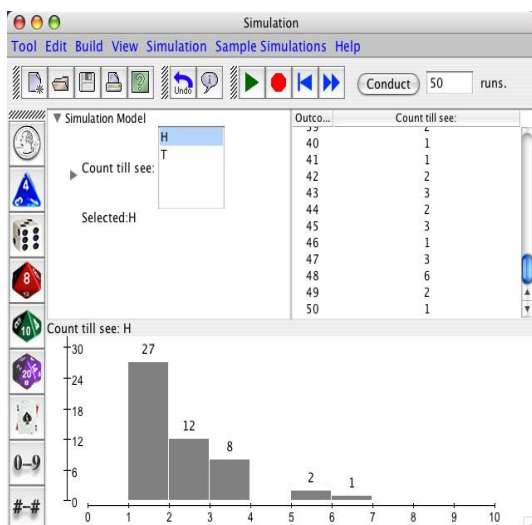


Figure 4: A waiting time distribution

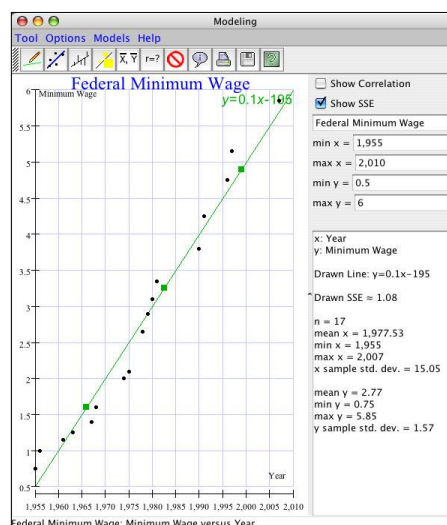


Figure 5: Moveable modeling line with control points

*Discrete Math* tools include software for discrete mathematical modeling involving constructing, manipulating, and analyzing vertex-edge graphs and custom apps for investigating function iteration and cobweb diagrams, vote analysis methods, and topics from cryptography (Figures 6 and 7).

Experiences with such tools in learning and doing mathematics can make important contributions to college and career readiness. For example, spreadsheets are ubiquitous in their use across the disciplines in college (Ganter & Barker 2004). One of the custom statistical tools supports students solving problems involving statistical process control as widely used today in a range of industries.

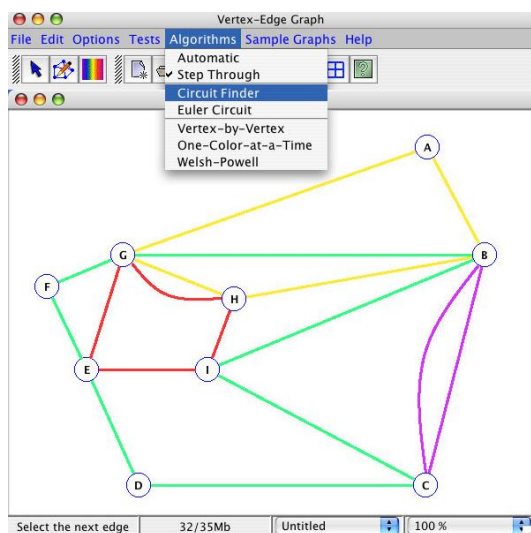


Figure 6: Finding a circuit in a vertex-edge graph

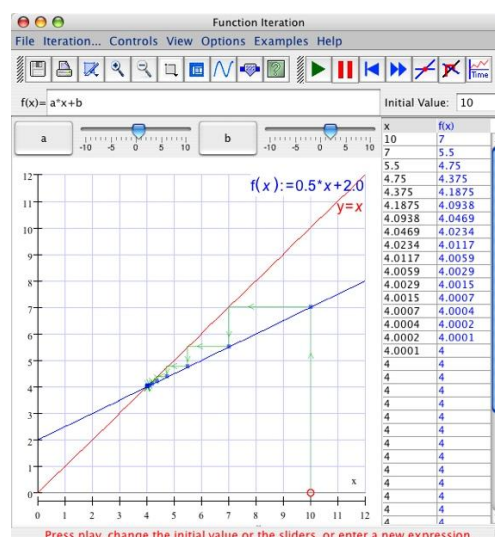


Figure 7: Function iteration and fixed point, a sample custom app

## CURRICULUM-EMBEDDED SOFTWARE

The design of the interactive geometry software illustrates how the functionality of *CPMP-Tools* is informed by its intended use in the curriculum. For example, to promote deeper student understanding of the connections between coordinate and matrix representations of geometric ideas, and to support student learning of applications of the mathematics to animation, the interactive geometry software was designed to perform matrix calculations (both automatically and by command) and to offer a simple, easily understood programming language (Figure 8). The capability for the learner (and software) to move flexibly between coordinate and matrix representations of figures and transformations is an important goal of the curriculum. The programming capability further supports the curriculum goals of focusing on contemporary applications of mathematics and developing deeper student understanding of logic, variables, functions, recursion, iteration, and algorithmic problem solving.

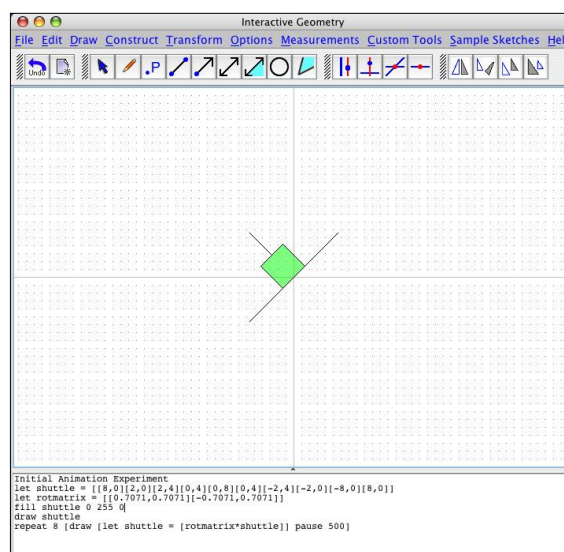


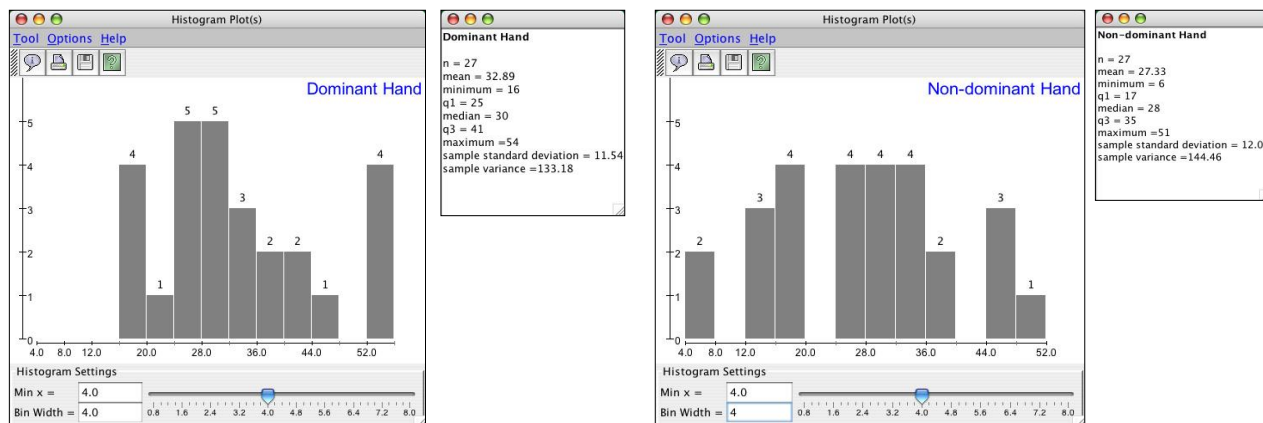
Figure 8: Coordinates, matrices, and animation

As a second example of how curriculum needs influenced the nature of *CPMP-Tools*, consider the following stochastic problem in *Core-Plus Mathematics Course 3* (Fey et al., 2009). It highlights the design of the statistics and probability strand around the unifying theme of constructing, interpreting, and making inferences from distributions of data and from probability distributions.

*A teacher in Traverse City, MI was interested in whether eye-hand coordination is better when students use their dominant hand than when they use their nondominant hand. She asked students in her first two classes to perform a penny-stacking experiment.*

Each class was randomly divided into two groups of about equal size. In one group, each student stacked pennies using only their dominant hand. The students in the other group used their nondominant hand. Each student's score was the number of pennies stacked before a penny falls.

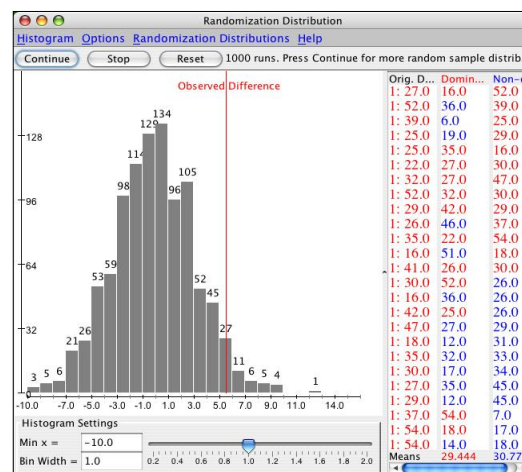
Once the data for the two classes were collected and pooled, students produced and interpreted graphical displays and summary statistics, as shown in Figure 9.



**Figure 9: Number of pennies stacked by dominant and nondominant hand**

The displays show considerable overlap in these two distributions. The mean number of pennies stacked by those using their dominant hand is 32.89, whereas the mean for those using their nondominant hand is 27.33. The difference of the dominant hand mean minus the nondominant hand mean is 5.56. Does use of dominant hand or nondominant hand make a difference?

Suppose the numbers of pennies stacked by the students in the two classes were written on identical slips of paper, mixed up well, and then half of them are drawn out to represent the students who used their dominant hand. Compute the mean number of pennies stacked for the dominant hand based on these slips of paper. Then compute the mean number of pennies stacked for the nondominant hand, using the remaining slips of paper. Subtract the *nondominant hand mean* from the *dominant hand mean*. If this process is repeated, say, 100 times, you can examine how many times a difference as large or larger than 5.56 is obtained. The result of that simulation can be used to estimate the answer to the original question posed. To get a better estimate, you could increase the number of trials, as shown in Figure 10.



**Figure 10: Randomization distribution based on 1000 run**

Because this method (called a *randomization test*) is tedious, it is seldom taught. Computing technology provides access to this simple and powerful idea and makes it an integral part of how students think about statistical inference and statistical significance. The Randomization Distribution software enables students to control the number of runs, observe the samples as they are generated, and see “in the mind’s eye” how, as the number of runs increases, the distribution begins to stabilize.

## CLOSING COMMENTS

The team of mathematics educators working on the iterative design and development of the CPMP curriculum and software tools viewed the process as an extended design experiment (cf. Design-Based Research Collective, 2003) that included cycles of concurrent design, development, field testing, and revision of the materials and supporting software. This process and the decision to fully incorporate software use in the instructional materials led to a richer, more engaging, and realistic problem-based curriculum. Embedding *CPMP-Tools* in the curriculum and making it available over the Web and free to users of the curriculum (and of other curricula) permits these tools to become an integral part of student learning and work. For further information on the design and development of *Core-Plus Mathematics* and *CPMP-Tools*, visit [www.wmich.edu/cpmp/](http://www.wmich.edu/cpmp/)

## ACKNOWLEDGMENT

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## TOUCH & MULTITOUCH IN DYNAMIC GEOMETRY: SKETCHPAD EXPLORER AND “DIGITAL” MATHEMATICS

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*This paper documents potential impacts of the novel multitouch tablet screen, popularized by Apple’s iPad, on Dynamic Geometry software use, design, and research. Work with Sketchpad Explorer relates multitouch technology to the representation and control of multiple variables within the mathematical environment; to the conditions and expressions of embodied forms of cognition at the juncture between the environment and its user; and to the possibilities of multi-user technological and pedagogic interaction within the large social context of use.*

### INTRODUCTION

*Sketchpad Explorer*, an interactive viewer for documents created by *The Geometer’s Sketchpad*, is the first multitouch-capable Dynamic Geometry environment for the Apple iPad (Jackiw & Brooks, 2011; Jackiw, 1991). While sharing document compatibility, *Sketchpad Explorer* and desktop *Sketchpad* differ in fundamental ways. The former has no construction tools, the latter many. As a free “app” accessible to the global community of iPad users, *Sketchpad Explorer* is designed to have little or no “learning curve,” where *The Geometer’s Sketchpad* has a well-scaffolded one intentionally designed to carry users from early primary school through to undergraduate mathematics. From the perspective of new opportunities and forms of technological engagement with mathematics, however, the most significant difference between *Sketchpad Explorer* and earlier Dynamic Geometry systems is its full embrace of multitouch input technology, whereby users can directly manipulate multiple on-screen mathematical objects both simultaneously and

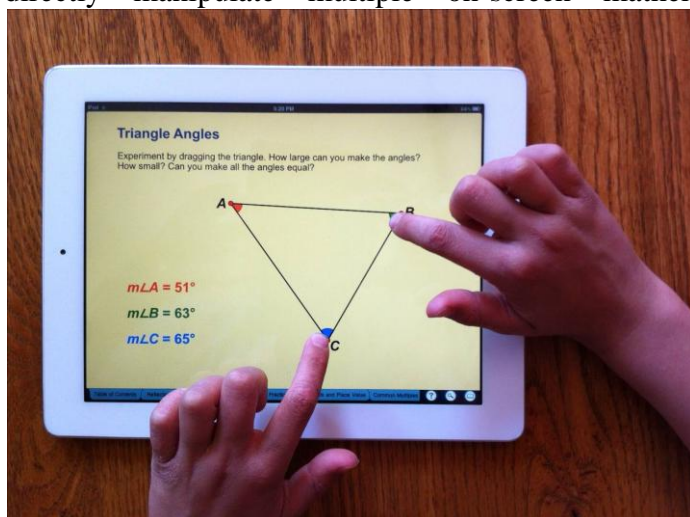


Figure 8: Using a typical sketch in Sketchpad Explorer

independently. If we imagine the glass tablet screen as some sort of conceptual border between the Platonic realm of geometric abstraction (on the computer’s side of the glass) and the tactile empire of sense experience (on the user’s side), this paper explores the ramifications of multitouch Dynamic Geometry from both sides of that glass, asking—on one side—“how do multitouch ideas shape mathematical ones?” and —on the other—“how do they impact learners’ interactions?”

### BACKGROUND: MULTITOUCH & SKETCHPAD

As an input technology, multitouch itself is far older than Apple’s media claims to its iPhone-era “origin.” Preliminary technology was developed at the University of Toronto in the early 1980s, capitalizing on earlier capacitance-sensing touchscreen work from IBM and CERN. Multitouch technology identifies hardware/software combinations capable of detecting multiple on-screen touch-locations simultaneously, and (in the case of *continuous* rather than *discrete* multitouch applications) tracking subsequent movement of those touches as they are dragged across the screen.

Apple's iPhone (2007) and iPad (2010) have greatly popularized the technology, and the hardware industry has adopted it in forms ranging from the mobile phone screen up to table-top and wall-sized installations. (For an excellent historical summary, see Buxton (2007).) In iPad's multitouch implementation, electrodes coating the screen detect changes in electrostatic capacity as a user's finger approaches the glass. Combined software/hardware logic then computes that finger's location by interpolating ratios of these electrodes' currents. The current iPad driver stack is capable of tracking up to eleven touches simultaneously, sufficient for all the user's fingers and one more besides—in case curiosity introduces a toe or a nose!

*Sketchpad's* history is almost as long, with early software development dating to the late 1980s, when the advent of the computer mouse and bitmapped display in commodity hardware combined to form generalized 2-D input and output capabilities, and thus a more direct “access” to the mathematics of the plane than was possible with previous input/output technologies. Similar ideas evolved in France in response to the same hardware factors, with the development of *Cabri* (Laborde et al., 1988). *Sketchpad* continues to colonize new hardware developments and to exploit them for their Dynamic Geometry potential, and pioneered geometry applications in early PC GUI operating systems, pen-based handheld devices, browser-based web interactivity, and other technology contexts. Across this span, *Sketchpad* design has pursued a single vision of Dynamic Geometry as “the milieu in which the individual ‘touches’ raw mathematics, where personal volition and physical exertion can make a seismic impact on disembodied abstractions” (Jackiw, 2006, p. 155). *Sketchpad Explorer* on the iPad is thus significant in this tradition of innovation, in that the touchscreen, for the first time, offers far more “direct” manipulation—less interpretively or symbolically-mediated manipulation—of mathematics than the mouse device, which itself defined the paradigmatic exemplar of the “direct manipulation” conceit for the past 25 years. From the perspective of mathematical experience, to an unprecedented degree, *touch* is *direct*; and *multi-touch*, multiplicatively so.

## PREMATHEMATICAL MULTITOUCH

New users, on first encountering an iPad or similar multitouch screen, are often drawn first to the almost haptic pleasure by which specific finger “gestures” can express unique actions to the operative software system. Swiping one's hand right-to-left across the screen might dramatically scroll its virtual content with a corresponding velocity. This gesture, though perhaps involving multiple fingers, are singular in their semantic identity—multiple touches express *one* action—and exist in a variety of relationships to the actions they communicate. We might deploy a Peircean hierarchy (Peirce, 1893-1913) in analyzing these gestures, and call “iconic” those gestures that analogically reference in their manual form similar gestures with similar consequences in similar physical (rather than virtual) domains. Thus scrolling one's virtual document onwards uses a digital gesture akin to that used to turn a physical book's physical page. “Indexical” gestures invoke or correlate to actions in the physical realm, often through a sensory analogy. Thus while pinching things in the physical realm rarely makes them smaller, the decreasing distance between fingers pinching implies a rate of reduction—and we hold up two pinched fingers to signify a minuscule quantity—and thus on the iPad, “pinch” becomes a natural gesture for “making smaller” or “zooming out.” Finally there are the entirely “symbolic” gestures whose relation to action is arbitrary or, in Saussure's sense, conventional, as in the “press-and-hold-for-two-seconds-to-delete” gesture. Thus the “gesture space” of a multitouch device establishes a full semiotic system, in which the conditions for *mathematical* structure or meaning become possible.

## MATHEMATICAL MULTITOUCH

While *Sketchpad Explorer* defines a small variety of these semiotic gestures, deploying them for interface actions like the ones already mentioned (scroll, zoom, delete), it focuses its *mathematical* interpretation of multitouch on the iPad's more general identification of a user's finger with an idealized point in the virtual landscape of the operative software. Where fingerpresses are in some mechanical sense fleshy ovoid planar regions animated by three-dimensional force vectors, the iPad drivers communicate them to application software as idealized points of no dimension located in an idealized 2D plane. In other words, when a finger is *not* in the process of participating in the action choreography of certain canonical semiotic gestures, the iPad treats it as a mathematical point—a foundational object in both Dynamic and static Geometry's mathematical domain.

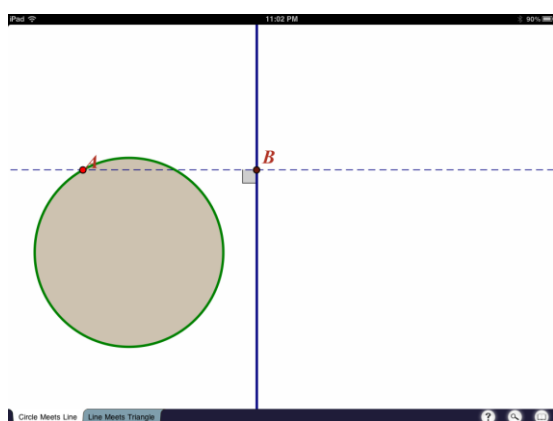
For a *single* touch, this of course requires no great stretch of the imagination: the iPad simply maps a verb (“to point”) to its noun (“the point”). But our foundational claim applies to *multi-touch* as well, with an early formulation appearing in Euclid's first postulate, where “*two distinct points uniquely determine a line.*” If a finger is both our pointer and our embodiment of a point, then lines in their foundational construction require two fingers to locate, not one. Thus it is in Euclid, not on an iPad, that mathematical multitouch is born! From the perspective of traditional mouse-based technologies, drawing a virtual line or line segment with a (single-touch) mouse pointer has previously been a multi-step transaction. You point-and-click once to establish the first endpoint of your imagined line, then point-and-click again, elsewhere, to establish the second one. Thus you realize the postulate's explicit two points only through the temporalized redeployment of a single-point technology. And to demonstrate the dynamic generality of your constructed line  $AB$ , where  $A$  and  $B$  are both dynamic proxies for *any* possible point rather than only the specific points of their original construction, again a mouse forces you into temporalized choice. You drag point  $A$  saying “*this point can be any point*”—but demonstrating only the variation in lines passing through our fixed (undragged) point  $B$ . Then you *stop* dragging  $A$  and *start* dragging  $B$ —“and so is can one!”—while again demonstrating a pencil of lines through a fixed point. Multitouch fundamentally de-sequentializes these sorts of transaction. The line defined through two points can be summoned into existence, or dragged into new configurations, by two *simultaneous* finger-touches, with each finger exerting an equivalent influence on its shared dependent, the line. In this alternate interaction regime, there is no obligatory precedence of  $A$  over  $B$  and no suggested causal or functional priority of one point to the other. (The line no longer extends—in time as well as space—from one point *toward* the other; instead, it *connects* or *passes through* both.) The autonomy of simultaneous, independent finger touches implies a symmetry, and clarifies a mathematical equivalence, between the two points of the line's definition.<sup>2</sup> The multitouch idea generalizes this symmetry and equivalence to any number of points—even if our available fingers limit us to ten.

## MULTITOUCH DYNAMIC GEOMETRY

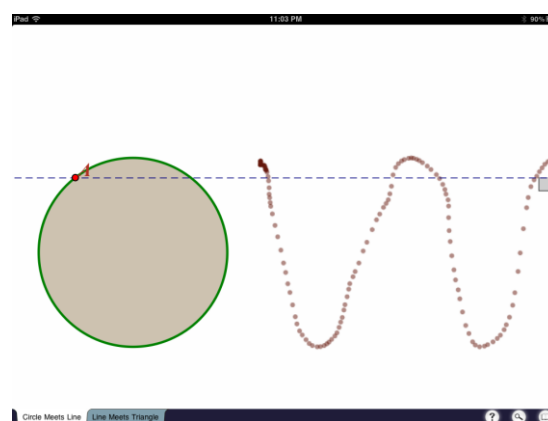
For a broader framework in which to locate multitouch mathematics, we can generalize the touchscreen identification of multiple human fingers with multiple (moving) mathematical points through two additional insights that are well-understood from prior Dynamic Geometry experience:

1. that a moving point is the central agent through which Dynamic Geometry imports into the traditionally-static domain of geometry powerful mathematical ideas of variation, continuity, temporalization, and generalization drawn from algebra, calculus, and real analysis;
2. that a moving point's motion is itself a mathematical object, a trajectory through space-time that we traditionally formalize as producing a “curve” through mathematical means such as locus definition or parametric equation.

These dynamic principles take on new dimensions when the experience of point motion is driven through touch and multitouch mechanics. The first interprets moving points as tangible models of formal variables; and so *multitouch* generalizes Dynamic Geometry to mathematical situations governed or motivated by multiple independent variables, and to regimes of co-variation rather than only strictly of functional causation. The second principle subordinates the dull dimensionlessness of the point to the rich mathematics of its trace, which the greater “directness” of touchscreen interaction raises to a level of profound embodiment. If one’s finger *is* a mathematical point, then its gestural movement through space-time *is* a geometric curve. Older mouse actions do not take part of these embodied truths—it is impossible to “draw a circle freehand” or even “sketch a line” by dragging an unrestricted mouse pointer across the plane. Even the gesture of mouse-dragging a single point is perhaps more (indirectly) “body syntonic” than (viscerally) “body identic:” with a mouse, the “point” in question is ambiguously split between the mouse wheel and the screen pointer, and the reference “plane” in which this ambiguously twinned point is has a definition hovering somewhere amidst the screen, the desk, and the internal rotational axes of the mouse wheel itself, each of which impose separate mechanical constraints on the space of possible gesture. The touchscreen simplifies this dramatically: dragging your finger in a circle produces an *actual* (embodied) circle—regardless of whether it also produces a virtual circle (or a virtual point moving in a circular trajectory, or some other virtual construct).



**Figure 2: The circle and line.**



**Figure 3: The coordinated sinusoid.**

These new dynamics become apparent in an example of an extended multitouch interaction. The design of the *Sketchpad Explorer* activity of Fig. 2 encourages experimental combinations of linear and rotary motion. Consider a user’s finger dragging point *A* around and around the fixed circle. As her finger repeatedly traces (and repeatedly recreates as gesture) the circle, the projection of that point—the projection of her finger!—onto the nearby vertical line *j* (where it appears projected at point *B*) moves with the height of her gesture, but not with its rotational dynamic. This in turn produces a somewhat startling choreography: her finger, travelling smoothly through the space of the circle gesture, is stalked by a horizontal line rising and falling with anything but a smooth periodicity. Instead, it *rushes* to the top of the screen—then pauses there, as if teetering on a cliff, before *plunging* back down to the bottom—where it reposes a moment, catching its breath before another rapid ascent. Thus we perceive the visceral dynamics of harmonic motion, counterposed to the (approximately uniform) velocity of the circle-finger gesture.

To investigate this unsettling phenomenon, let’s further ask her to continue dragging *A* around the circle while at the same time moving the vertical line *j*. The up/down motion of the slowly circulating finger on the left plots itself against the sideways lateral motion of the finger pushing *j* to the right, leaving behind a trace of its past positions. This trace has clear structure, and with a small amount of experimentation our learner discovers she can coordinate her two fingers’ autonomous



motions to produce a repeating curve depicted in Fig. 3<sup>3</sup>. The closer her two fingers come toward the same speed, the closer this rough oscillation approaches the mathematical sine wave.

Driven by the hard teleology of a curriculum sequence, this is perhaps only an outpost or side-effect of the march toward a more formal development of sine and its graph, with or without Dynamic Geometry facilitation. But it is worth pausing here to consider this informal material from other possible mathematical frames of reference. To review: two physical gestures separately embody the circle and the line. We have expressed and been impressed by these gestures well before our arrival in the mathematics classroom; they are already our fluent routines. But performing them together is new, and our first attempts are inevitably awkward, like the challenge of rubbing one's belly while simultaneously patting one's head. But slowly a gestural arithmetic happens in time: the two gestures stop requiring separated aspects of our attentional focus and autonomy of our fingers, and instead coalesce into a third gesture—the sinusoid—that coordinates and subordinates the other two. As in any dance, we may fall into and out of rhythm, but we know when we are in rhythm that our body is doing *one* thing, rather than desperately attempting to coordinate many.

Points of comparison between this act of production and more mathematically familiar actions or activities arise from the realization that though the circle and line are extrinsically *two*-dimensional, our experience of them as gestures is intrinsically *one*-dimensional. They are the first-order extruded trajectory of our dimensionless point/finger under motion, point-positions determined by time. So in our example we are working in the direction exactly opposite Descartes, whose coordinate system dimensionally separates a two-dimensional curve into its constituent one-dimensional gestures. We are instead building that curve out of those simpler vectors—an act of co-variate composition, rather than bivariate decomposition. Yet at the same time, our intentional effort to synchronize our two gestures is clearly the quest for a common clock which our activity shares with symbolic mathematics, where it finds form as the expression of separate equations in terms of a common independent parameter  $t$ . Thus while the mathematics of our multitouch experience travels temporarily opposite the analytic geometers', we are moving in the same direction as more modern modelers, experimenting with temporality and dimensionality as constructive (indeed, dynamically *manipulable*) components of higher-level mathematical representation and production.

## MULTITOUCH AND MULTIPLE LEARNERS

So far our discussion has focused only on a single learner, interacting with multiple touch-mediated “variables” in a Dynamic Geometry situation. But put your iPad into a classroom of children and you rapidly discover that multitouch also implies multiple learners, as two or more students struggle to participate on the same screen. This context will clearly grow in centrality, as popular multitouch implementations spread from the small-screen tablet to larger screen formats such as Microsoft's Pixel-Sense tabletop and SMART Board's interactive whiteboard displays. While their detailed analysis falls outside this paper, we can identify several central issues raised by the multi-user multitouch scenario here and chart out some research directions. In terms of activity structure, multiple learners can clearly cooperate, compete, or simply co-exist in the same virtual space. There is opportunity here to extend Hegedus & Kaput (2004)'s “democratically accessible” participatory and digitally-connected task structures, bringing multiple learners together not through the virtualized telepresence of network connectivity but into direct physical collaboration via the touch environment. This possibility further suggests retheorizing embodied cognition from a (socially) distributed vantage. (de Freitas and Sinclair (2013) appear to be working in this direction.) From a software design perspective, the premise of multiple users forces a ground-up rethinking of basic user interaction design, since while the multitouchscreen can gracefully track multiple fingers, it has no idea how many bodies are connected to those fingers! Software interactions that assume the continuity of a user's focus of attention across some multi-step transaction (e.g. *first* choose this

tool, *then* click that point) are now at risk of being interleaved, in time, with other steps of other interactions sponsored by other users' pursuing their own equally valid, and equally single-minded, aims. *Sketchpad Explorer* design responds to that risk by remapping user actions from multi-step expressions of intent into singular or instantaneous statements of volition, or by replacing temporally-proximate interactions with spatially-proximate ones instead, but more work is needed here to understand how to recraft rich software interfaces toward multi-user asynchronous usability.

## CONCLUSION

The touchscreen unifies the (formerly separate) technological domains of *input* and *output* in a device whose characteristics mirror many idealized attributes of the geometric plane. Multitouch further endows this technology with a mathematically advantageous identification of users' fingers with multiple moving points. *Sketchpad Explorer* brings these fortuitous ingredients together into a Dynamic Geometry environment that enables uniquely-embodied interactions with multivariate mathematics, and that serves as the physical interface at which a mathematical progression of point/curve/family meets the embodied progression of finger/gesture/routine. In the generative implications of this mapping, we see opportunities to both extend and rethink existing research on embodiment in Dynamic Geometry in both individual and social formations, as well—at a more general level— as a boldly-literal new meaning for “digital mathematics.”

## NOTES

1. Portions of this work were funded by the National Science Foundation (USA) under awards DRK12#0918733 and REC-0835395. The terms “Dynamic Geometry” and “The Geometer’s Sketchpad” are registered trademarks of KCP Technologies, Inc.
2. We see this exact transition—from single to multipoint perspectives—in the evolving language of the postulate itself. Euclid’s original Greek is perhaps haunted by a linguistic trace of the directed (or temporally asymmetric) nature of antiquity’s line-producing technologies. But by Hilbert’s reformulation for a modern audience, all such traces have vanished, and the points have been elevated to a symmetric peer relationship. Compare Euclid’s “to draw a line *from* any point *to* any point” (in Heath’s 1908 translation) to Hilbert’s “for *every two points* there exists a line that contains each of the points...” (in L. Unger’s 1971 translation, stress added).
3. The traces recording past locations of point *B* slowly fade in time, leaving a trace of your most recent manual competencies while muting or erasing the messy history of their experimental origins. Or, more violently, shake the iPad to erase immediately any accumulated traces.

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## **INVESTIGATING THE IMPACT OF A TECHNOLOGY-CENTERED TEACHER PROFESSIONAL DEPARTMENT PROGRAM**

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*This study investigated the impact of a technology-centered professional development program on high school geometry teachers' change and their students' geometry learning. 64 geometry teachers were randomly assigned to two groups. The teachers in the experimental group participated in a one-week summer institute followed by six half-day workshop sessions during the school year, in which they studied the critical features of the dynamic geometry (DG) approach and the DG-oriented teaching strategies they were expected to use in their classrooms. The teachers in the control group received workshop sessions of mathematics content and taught as before. Teachers in both groups were found to be faithful to the instructional approaches assigned to them. Teachers in the experimental group scored higher in a conjecturing and proving test than did teachers in the control group. The students of teachers in the experimental group significantly outperformed the students of teachers in the control group in a geometry achievement test.*

### **INTRODUCTION**

A four-year research project funded by NSF is conducting repeated randomized control trials of an approach to high school geometry that utilizes Dynamic Geometry (DG) software to supplement ordinary instructional practices. It compares effects of that intervention with standard instruction that does not make use of computer drawing tools. The basic hypothesis of the project is that use of DG software to engage students in constructing mathematical ideas results in better geometry learning for most students. The project tests that hypothesis by assessing student learning in 64 classrooms randomly assigned to experimental and control groups. This paper is to report a study conducted in project Year 2, whose purpose was to investigate the impact of the professional development designed and offered to the teachers in the experimental group. The study built upon related research studies on mathematics teachers' professional development (e.g., Carpenter et al., 1989), including those concentrating on technology-centered (and especially DG-centered) professional development (Stols et al., 2008; Meng & Sam, 2011).

### **THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS**

An integrative framework (Olive & Makar, 2009) drawing from Constructivism, Instrumentation Theory and Semiotic Mediation was used to guide the study.

This study addresses the following research questions:

- 1) Did teachers in the experimental group develop stronger conjecturing and proving abilities than did teachers in the control group?
- 2) Did the students of teachers in the experimental group perform significantly better in geometry learning over a full school year than did the students of teachers in the control group?
- 3) How did the instructors implement the DG approach in their classrooms with fidelity?

### **METHOD**

The population from which the participants of the study were sampled was the geometry teachers at high schools in Central Texas urban school districts. The study followed a randomized cluster

design. 64 teachers who were randomly assigned to two groups (an experimental group and a control group) received relevant professional development, implemented the instructional approaches respectively assigned to them, helped the project staff in administering the pre- and post-tests of the participating students, and participated in other data collection activities of the project.

### **The DG treatment**

Dynamic geometry is an active, exploratory study of geometry carried out with the aid of interactive computer software. DG environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena (CCSSI, 2010, p.73). With its main features such as dragging and measuring, DG software can be used to help students to engage in both constructive and deductive geometry (Schoenfeld, 1983) as they build, test and verify conjectures using easily constructible models. This instructional approach of using DG software is referred to as the dynamic geometry approach in this study.

In order to effectively implement the DG approach in their classrooms, teachers need to learn the approach first. Without professional development training, “teachers often fail to implement new approaches faithfully” (Clements et al., 2011, p. 133). So teachers’ professional development (PD) was an important component of the project. For our PD to be effective, it must be long enough, intensive enough, and relevant enough, with substantial support from the school districts. Based on these guiding ideas, a weeklong summer institute was offered to the participating teachers in the DG group, followed by 6 half-day Saturday PD sessions across the school year.

The nature of each PD session for the experimental group was interactive and emphasizing participating teachers’ active involvement and conceptual understanding of mathematics. Important geometric concepts, processes, and relationships were presented or revisited through challenging problem situations, which were explored with the DG software (The Geometer’s Sketchpad [GSP] for this study) as a tool. Teachers learned GSP skills in the process of using them to tackle the problems. They came to learn, first hand, as learners of mathematics, how GSP encourages mathematical investigations by allowing users to manipulate their geometric constructions to answer “why” and “what if” questions, by allowing them to backtrack easily to try different approaches, and by giving them visual feedback that encourages self-assessment. The method of leading the teachers to conduct investigations modeled what they were expected to do with students in their classrooms.

To further help teachers to consider changes in their instructional strategies, in the relation between them and their students, and in how they facilitate student learning, mathematical explorations are always followed by discussions on questions such as “How will you teach this content in your classrooms using DG software?” and “How will you lead your students in conjecturing and proving using DG software?” We realized the importance of teachers learning from each other and sharing ideas. Therefore, teachers were encouraged to give presentations on their important insights of DG implementation and successful stories, or to describe problems they anticipated with other teachers offering suggestions to address the concern. Teachers also worked in groups of 3 or 4 to prepare lesson plans to share with the entire group.

### **The control group**

The teachers in the control group taught geometry as they had done before. They also participated in a PD workshop, in which the same mathematical content covered in the DG group PD sessions was introduced to them in a non-DG environment. The PD facilitators used an approach that most teachers were currently using – a lecture based, though activity based instructional methodology – to conduct their training, mainly lecturing, heavily relying on the textbook, and participants’ problem solving exercises without using technology tools). The amount of instructional time spent on this regular workshop was the same as that for the DG group PD training. The purpose of holding this

non-DG workshop was to address a confounding variable. With this comparable amount of professional development, if differences appeared on the project's measures between the experimental and control groups, we would be able to rule out the possibility that the mathematics content presented in the DG group PD sessions could account for them rather than the interactive DG learning environment.

## **Measures**

### ***A measure for teachers' knowledge***

A conjecturing-proving test was developed by the project team to measure teacher knowledge. Our first draft of the test was reviewed by the advisory board (AB) members, and pilot-tested with 200 geometry students in a local school district. Based on the recommendations of the AB members and the analysis on the pilot-test data, the project team decided to not only revise and refine the conjecturing-proving test, but also design and conduct validity and reliability studies on the test.

After a series of intensive work (a thorough literature review, construct development, item development, the second AB review, more pilot-testing, several round of revision, etc.), a version of the test consisting of 26 multiple-choice items and two free-response proofs was produced. The instrument was administered to the teacher participants as both a pre-test and a post-test at the PD Summer Institute.

### ***Teachers' implementation fidelity and classroom observations***

The DG approach involves using dynamic software intensively in classroom teaching to facilitate students' geometric learning. The critical features of the DG approach include using the dynamic visualization to foster students' conjecturing spirit, their habit of testing conjectures, focusing on relationships, and explaining what is observed, their logical reasoning desire and abilities, as well as their conjecturing-investigating-proving oriented learning style in exploring problem situations. To determine how to capture these critical features of the DG approach, the Geometry Teaching Observation Protocol [GTOP] was adapted from the *Reformed Teaching Observation Protocol* (Sawada, et al., 2002), based on the critical features of the DG approach. The main part of GTOP consists of 25 items in four sections (*Description of intended dynamic geometry lesson, Description of implemented dynamic geometry lesson, Assessment of quality of teaching, Assessment of engagement and discourse*).

### ***Student level measures***

The instruments used for measuring students' geometry knowledge and skills were: (1) (for the pre-test) Entering Geometry Test (ENT) used by Usiskin (1981) and his colleagues at University of Chicago; and (2) (for the post-test) Geometry Achievement Test (GAT). GAT was developed by selecting items from California Standards Tests – Geometry (CSTG). The final version for GAT has 25 multiple-choice items. Based on the pilot data, the instrument has high reliability ( $\alpha = .875$ ). Factor analysis provided strong evidence that GAT corresponded to uni-dimensional scale. Item Response Theory (IRT) scoring routines were applied to the scored post-test to generate examinee 'abilities' and item parameters, which allowed the researchers to determine that collectively the items included in GAT provided a range of performance that holistically represented a well-functioning instrument. The adherence of the data to the three-parameter logistic IRT model provided evidence for the assessment's construct validity.

## RESULTS

### Findings from the Conjecturing and Proving Test

This instrument was administered to project participating teachers at the beginning and end of the weeklong Summer PD Institute. For the 26 multiple-choice items, Cronbach's alpha was .707 for the pre-test administration and .765 for the post-test.

A statistic for overall competence on the instrument was calculated by adding the number of correct multiple-choice responses with points from free-response items. Teachers averaged 20.49 on the pre-test and 21.86 on the post-test with an average gain of 1.37. A paired-sample t-test showed that this gain was statistically significant ( $p = .003$ ). This indicates that the professional development had an effect on teachers' capabilities. The average gain was greater for the experimental group (1.56) than the control group (1.18), but this difference was not statistically significant ( $p = .670$ ).

### Findings from the classroom observations

Two versions of the GTOPI instrument were developed. The GTOPI for the control group was a subscale of the GTOPI for the experimental group. The items removed for the control group were items in the implementation aspect that are related to the use of software functions (dragging, dynamic measuring, etc.). Reliability for each version was calculated with Cronbach's alphas of 0.957 and 0.952 for the DG and control groups, respectively.

The results for the observations in the DG group are summarized in Table 1. The main result that stands out is that teachers demonstrated an intention to teach a lesson with the DG approach and they also demonstrated, to some extent, the knowledge required to do so. These were the aspects with the highest scores (Good Lesson Design, Use of DG Features, and Teachers' Knowledge). Overall, teachers are implementing the DG approach at a medium level (2.28). Considering the challenges (the inaccessibility of a computer lab in the first several weeks, the pressure of the intensive state testing, etc.) that the teachers experienced during the school year, most of them should be regarded as being faithful to the DG approach.

**Table 1: Comparison between DG and control groups**

| Aspect                   | Sub-aspect                     | Mean DG | Mean Control | p-value |
|--------------------------|--------------------------------|---------|--------------|---------|
| Intended Dynamic Lesson  | Good Lesson Design             | 2.81    | 1.85         | .032*   |
|                          | Use of Dynamic Features        | 2.75    | 0.70         | .000*   |
| Implementation           | Actions beyond Use of Software | 2.06    | 1.33         | .095    |
| Quality of Teaching      | Cognitive Demand               | 2.30    | 1.78         | .113    |
|                          | Teachers' Knowledge            | 2.89    | 2.84         | .924    |
|                          | Conjecture/Proof               | 1.93    | 1.40         | .206    |
| Engagement and Discourse |                                | 2.37    | 2.29         | .735    |
| Overall GTOPI Score      |                                | 2.28    | 1.68         | .088    |

Table 1 also shows the comparison between the results for the two groups. The p-values test the significance of the treatment effect and were obtained using a mixed effect ANOVA. As expected, the results showed that the control group did not use dynamic features of the approach to teaching geometry. Further, the intended lessons for the DG group were significantly better designed than the control group. In particular, lesson plans for the DG group had in general appropriate objectives and they were designed to move students from initial conjecture, to investigation, to more thoughtful conjecture, to verification and proof. Although teachers in the DG group and teachers in the control

group did not differ significantly in other aspects, the RTOP scores of the DG teachers in these aspects are all higher than those of the control teachers.

### Findings from the student geometry achievement

Two-level hierarchical linear modeling (HLM) was employed to model the impact of the use of the DG approach on overall student achievement. The model was analysed using student pretest (*ENT*) scores as a covariate. The sample of classrooms studied included three different levels of Geometry: Regular, Pre-AP and Middle School (middle school students taking Pre-AP Geometry). Since the classroom expectation and quality of the students in each of these levels was very different, the factor *Class Level* was included in the model. Additionally, the years of classroom experience of the teachers in the sample varied a lot, ranging from 0 years all the way up to 35 years. It was possible that a more experienced teacher might have greater command of the classroom but be less able to implement the technology. For this reason, the covariate *Years Exp* (number of years of classroom experience) was included in the models.

The HLM model shown in Table 2 examines the effect of the DG intervention when taking into account Entering Geometry Test (*ENT*) as well as *Class Level* and *Years Exp*. To simplify the interpretation of the other coefficients, *ENT* was centered by subtracting the overall mean. The results indicated that the DG effect was strongly significant ( $p = .002$ ).

**Table 2: HLM Results with Pretest as a Covariate**

| Fixed Effect               | Coefficient | Standard Error | T-ratio | Approx d.f. | p-value |
|----------------------------|-------------|----------------|---------|-------------|---------|
| Intercept                  | 79.05       | 6.119          | 12.919  | 28          | .000*   |
| DG Effect                  | 5.62        | 1.678          | 3.352   | 43          | .002*   |
| Level                      |             |                |         |             |         |
| Regular                    | -20.10      | 6.119          | -3.284  | 21          | .004*   |
| Pre-AP                     | -13.23      | 6.293          | -2.102  | 20          | .049*   |
| Level*Years Exp            |             |                |         |             |         |
| Regular*Years Exp          | -.4137      | .1516          | -2.729  | 53          | .009*   |
| Pre AP * Years Exp         | .4451       | .1617          | 2.753   | 22          | .012*   |
| M. School*Years Exp        | 1.811       | 1.868          | 0.969   | 20          | .344    |
| <i>ENT</i> (Mean Centered) | .4114       | .0366          | 11.237  | 47          | .000*   |

As expected, *ENT* is a significant predictor of student performance on *GAT* ( $p = .000$ ). However, even controlling for the pretest, compared with Middle School students, on average Pre-AP students scored 13.2 points lower ( $p = .049$ ) and Regular students scored 20.1 points lower ( $p = .004$ ). The effect of experience was significant in the Pre-AP group ( $p = .012$ ), but not significant in the Middle School group ( $p = .344$ ). An increase in 10 years of experience raised the scores 4.5 points for the Pre-AP group and decreased the scores by 4.1 points for the Regular group. Comparing the means, the DG group outperformed the control group in each level of Geometry and the effect size (.45) was substantially larger at the Regular Geometry level.

## DISCUSSION

The HLM model taking pretest, class level, and teaching experience into account showed that the DG group students significantly outperformed the control group students in geometry achievement. Given that teachers were randomly assigned to the two groups and that both groups received professional development of the same duration on the same topics, the results from this study provide strong evidence to support the finding that the DG professional development did make a difference – it did positively impact the student geometry learning. Both DG and control group



teachers have obtained significant gains on the Conjecturing and Proving Test through the one-week summer PD institute. This result suggests that both the PD sessions designed for the DG group and those designed for the control group had an effect on teachers' conjecturing and proving. Although the DG and control teachers did not differ significantly on their mean gain scores, the DG teachers' mean gain score was still 32% higher than that of the control teachers. The classroom observation data revealed that lesson plans that the DG group teachers prepared were significantly better designed than the control group, aiming at facilitating students' conjecturing and proving abilities. The teacher GTOP scores (overall and in each sub-scale) consistently favored the DG group although most of the differences were not statistically significant.

In summary, the results of this study suggest that the DG professional development offered to the participating teachers has had a significant positive effect on the teachers' change on both mathematics content knowledge and teaching strategies. The teachers, in turn, helped their students achieve better geometry learning. This can be illustrated by the following comment made by one of the participating teachers: "*From the DG technology professional development I have gained new teaching strategies to engage my geometry students. I have found by using these learned strategies that my students are able to make their own conjectures about the figures they construct using Geometer's Sketchpad. Many students are both kinesthetic and visual learners; I am able to reach these students with the dynamic-ness of constructing using technology.*"

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## **MEDIATED MEANINGS FOR THE CONSTRUCTION OF THE CONE IN A 3D DIGITAL ENVIRONMENT**

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*In this paper, we explore the ways 10th grade students organize their mathematical knowledge and the meanings created in the construction of the right circular cone through their engagement in three different tasks. We want to study whether the resources of the medium, a 3d Logo / Turtle Geometry environment, 'Machine Lab Turtleworlds' (MaLT), which combines multiple representations of geometrical objects and their dynamic manipulation, mediate a construction by students who are taught solely the Euclidian constructions, having available only the compass and protractor. The results illustrate a progression in the process of the construction of new meanings in mathematics and development of student's focusing onto the objects within the setting, as different areas of mathematics are interwoven, marked by shifts in analyzing and interpreting the mathematical notions engaged in the tasks.*

### **INTRODUCTION**

The construction of the geometric figure, planar or solid, in the Greek school, is determined by the use of geometric tools such as compass, protractor, rectangle (set square) and materialized on a sheet of paper. On the one hand, the geometry of space is minimally taught in lower secondary school and at all in upper secondary school (Lyceum) since emphasis is placed more in Algebra rather than in planar Geometry. On the other hand, the growth of 3-D educational digital tools creates new challenges both for the teaching and for the research related to the difficulty of interpreting the 3-D shapes as they are projected onto the 2-D paper or on the digital screen (Parsys, 1988). Those challenges have acted as an incentive for our work.

### **THEORETICAL FRAMEWORK**

Our theoretical assumptions are based on Vygotsky's ideas associated with the tools and their cognitive function. These tools shape as well, for the students, the meaning of the geometric objects, since the instruments used by the students, as Vygotsky (1987) argues, constitute crucial cognitive units and mediate both human practical and our mental functioning. As Chassapis (1999, p. 276) reasons:

In this account it is assumed that the use of different types of mediatory means structures practical activities in different ways and hence has a differentiated impact on thinking and, consequently, on the generation of concept-developing processes.

Noss et al (1997) highlight that digital tools, especially those that provide opportunities for symbolic expression, contribute to meaning constructions different in quality than those that stem from the use of paper and pencil solely. They, moreover argue, that the key insight is that parts of the mathematical model of the situation are built into the fabric of the medium; they are not only available in the mind of the learner, as in the case of inert media, such as pencil and paper when at hand (pp. 230-213).

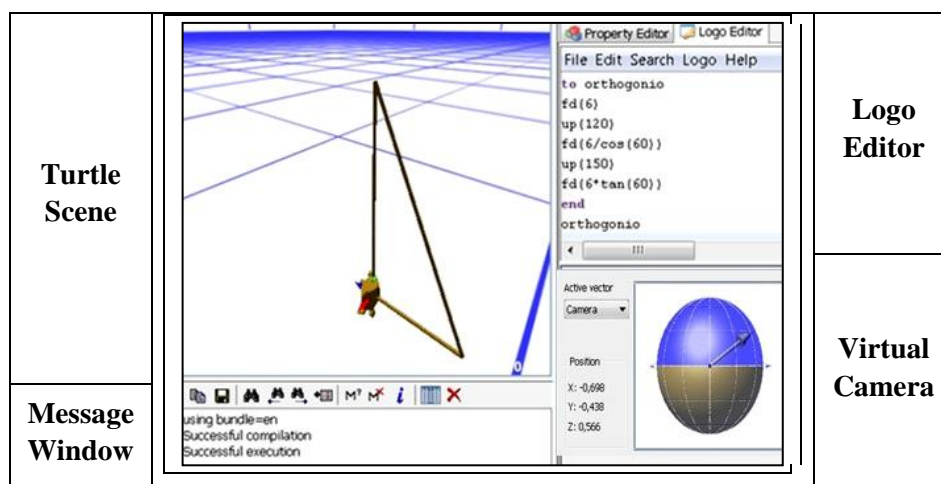
The digital environment of our research, MachineLab Turtleworlds program (MaLt), is a 3D Logo based computing environment that combines multiple representations of geometrical objects and

their dynamic manipulation. It has already been used in a series of researches that focus on various issues such as navigation and Cartesian Coordinates (Gavriliş, Keisoglou, & Kynigos, 2007), the notion of Angle (Kynigos, Latsi, 2007) and Differential Geometry (Zantzou, Kynigos, 2012).

We chose our research theme as supplement to the above issues, on the one hand, and on the other, the construction of the triangle and the cone are close to the Greek math Curriculum.

## THE DIGITAL ENVIRONMENT AND THE INDUCED GEOMETRY

The digital environment, MaLT, as a microworld can be seen from the perspective Papert (1980) ascribes to it, viz as an investigating learning environment of the properties of space and its objects. The software environment consists of 3 discrete contexts (scenes)-*turtle scene*, *logo editor*, *message window* and a tool that determines the user's perspective, the *virtual camera*.



**Fig.1: The MaLT Environment**

As shown in figure 1, the digital microworld provides the user with visualization possibilities of his/her construction described in the form of commands in Logo language. Moreover, he/she can change the view angle through a virtual camera. In order to construct a shape, the user navigates the turtle having in his/her quiver two types of commands: those referring to straight course *fd* (forward) and *bk* (backward) and those referring to turn consisting of three different command pairs a) *rt* or *lt* (right –left turn), b) *up* or *dp* (vertical turn, up pitch -down pitch) and c) *lr* or *rr* (rotation, left roll –right roll, about the turtle's axis)

The induced geometry materialized in such a digital environment has been characterized as turtle geometry by Abelson & DiSessa (1981), which in fact is determined as intrinsic geometry; namely a geometry referring to a particular shape autonomous and independent from any reference system. In turtle geometry the construction of a shape coincides with its description and thus with the recording of the navigation commands through which a turtle avatar will form the shape. It is exactly this approach to the construction of a shape that we were intrigued: to study, the way which the tools of MaLT mediate some construction by students who are taught solely the Euclidian method in which the mediating tools are mere the compass and the protractor. A crucial difference is that in the Euclidean construction of the shape the measures of some of the angles and sides would suffice, when in turtle geometry all the linear and angular elements should be calculated.

## METHODOLOGY AND RESEARCH QUESTIONS

In the conducted research, five 10<sup>th</sup> grade students took part, from a school that had previously undertaken a project with this learning environment, but in a different subject for investigation (Moustaki, Kynigos, 2012). Therefore, the students were reasonably experienced programmers; they

had a working knowledge of Logo and knew how to operate MaLT's functionalities. Furthermore, nothing was ruled out: they were free to adopt any method of construction. So, for example, during the process of completing their programs they could use *up* and *right* if they so wished.

The students' mathematical knowledge, regarding the triangle constructions, was determined by the way the Greek Lyceum teaches this particular subject; with the use of geometric tools in the relevant chapter that refers to equal triangles. The two groups formed, consisted of two and three students respectively, worked jointly for four hours on a pre-designed series of activities. The research was qualitative and involved the design of an experiment, as the learning environment was an innovation for the teaching of spatial geometry (Cobb, et. al., 2003). All researchers, in the role of participant observer, took down field notes referring to the negotiation of mathematical meanings, provided technical information-when needed, or addressed questions with the aim to strengthen their in between negotiation. The corpus of data consisted of: a) the researchers' field notes b) each group's worksheet, where they did mathematical calculations or drew geometric figures (4 altogether, two for each session) along with c) each group's saved files of the 3D MaLT program. The research questions were hinged on the prospect of mediation tools of the learning environment. In brief, we investigated the ways the students organize their mathematical knowledge (formal or informal) and the meanings created in the construction of the geometric solid of the cone using as unit construction-generator a right-angled triangle. The systematic study of research data revealed that the students were involved in three mathematical activities; the construction of a right triangle, generalization and approximation. During the analysis of the way they created meanings for these mathematical notions we opted for focusing on the distinction between institutional and personal meaning of the mathematical entities and procedures (Godino, Batanero 1998). Institutional meaning is a shared meaning and depends on school practices and curricula. The personal meaning depends on the subject and on time.

## PROCEDURE AND FINDINGS

Three tasks were handed out to the students during the research: first the construction of a fixed right-angled triangle in space, next the construction of a right-angled triangle, with sides dynamically changing and finally the construction of a dynamically changing cone.

### a) The mediated meaning of the construction

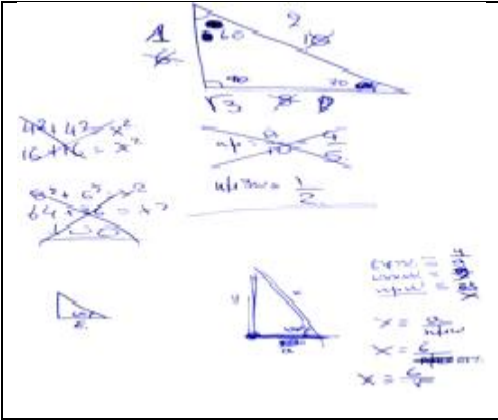
Initially, the interactions between the students and the computers appeared to be fragmented. The students' actions can be characterized from both as spontaneous, informal options of school mathematics (institutional meaning) as well as lacking of systematic mediation of the instrument, as the selected constructs were not compatible with the software's requirements. So, they chose to construct an isosceles right-angled triangle, in which one group supported that the sides should be of 10, 10 and 20 units, whereas the other sought for such whose measure of the sides would be rational numbers. Notable is a student's argument; to try to construct a triangle whose sides would be analogous to its angles. As an aside, all groups systematically overlooked the need to define the angles of the triangle under construction- which is necessary to build the notation code, and focused on the side lengths. The labeling of the researchers that the students give commands to the turtle to move was crucial, as since then, the students started discussing about the turn angle and the exact distance the turtle should cover. A first option was of a triangle with acute angles  $30^\circ$  and  $60^\circ$ , but the exact distance the turtle had to move was another obstacle that the students had to overcome, who continued as follows [1]:

Katerina: Why don't we make a triangle of 6, 8 and 10, since this comes out from the Pythagorean Theorem?

Marie: Here, how much will it turn?

Katerina: We should put an angle of....no we can't put whatever. Should we use trigonometry?

The use of trigonometry gave the students the possibility of describing the shape in the context of a digital environment.

|   |  |
|---|--|
|  | <p>to orthogonal<br/>fd(6)<br/>up(120)<br/>fd(6/cos(10))<br/>up(150)</p> |
| <p><b>Construction of the triangle using Trigonometry</b></p>                     |  |

**Table 1: from Pythagorean theorem to Trigonometry**

The notation code describes symbolically the construction of a right-angled triangle

#### b) The mediated meaning of generalization

Generalization in geometry, particularly in constructions by means of a ruler and compass, is defined by the fact that measures of geometric figures on which the construction is based, are not at all numerically determined. Such a problem in geometry textbook is posed as follows: “Construct a triangle if known a vertical side and an acute angle”. This expression implies that the triangle under construction is a representative of a whole class of triangles. So, from the teacher’s point of view the construction has as starting point the expression: “Let AB the side and  $w$  the angle...” The notion, thereby, of generalization is expressed through a random choice of measures of the geometric figures. In the digital environment the students were asked to transform the notation code they had constructed in the previous stage, so that the triangle was changing but always right-angled.

|  |   |
|--|---|
| <p>To parork<br/>fd(:a)<br/>up(180-(:b))<br/>fd((:a)/cos(:b))<br/>up(90+(:b))<br/>fd(((a)/tan(:b))<br/>end</p> | <p>Since the students had previously worked on a similar theme, in short time they intervened in the code and superseded the concrete numbers with parameters. Now the construction of the generalized triangle is materialized through the introduction of variables in the measures and consequently the generalized triangle has the meaning of a parametric one (personal meaning).</p> |
|--|---|

**Table 2: Parameterisation of the right-angled triangle**

#### c) The mediated meaning of approximation

When the students were asked to construct a cone, both groups proposed constructing a circle as the base of the cone, whereas its surface would result from the application of multiple triangles around the circle. The students seem to hazard such a conjecture based on the draft of the cone built in their worksheets. Noteworthy is a student’s view, with high performance in school mathematics:

Researcher: (to all groups) can you think of a way of constructing the cone?

Peter: I think ..., it is impossible!!

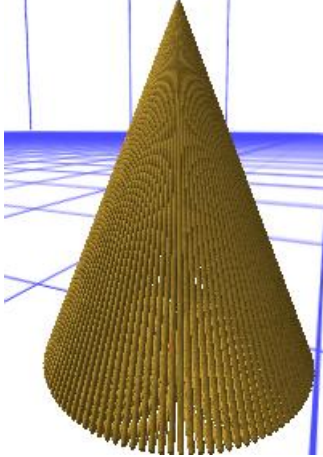
Researcher: Can you explain why?

Peter: As much as we try with the triangle, we will only construct lines (referring to the hypotenuses of the right triangle) and not the continuous surface.

Researcher: Hence, can't we either make a circle?

Georgia: No, just polygons with very small iterative segments all around!

Georgia's comment was decisive for all groups. They re-estimated the possibilities of the software to repeat a construct at will. The construction of the notation code of the cone was an object of negotiation among the group members and emerged otherwise. Group 1 suggested the construction of an isosceles triangle revolving about its vertical axis; was no straightforward. The difficulty of the task led the students—much to our surprise, to argue that they could construct a cone consisting of a set of circles lying on equidistant and parallel planes to the base cone –generator, of which the radius decreases in a constant rate until it becomes zero (0), but alas the code was never completed. The group that opted for making use of the right-angled triangle, presented codes containing a parameter (:n) through which they could repeat the triangle with rotation. That resulted in the appearance of only one triangle instead of: n, as one triangle overlapped the other. This failure in constructing: n discrete triangles led the second group to fix this bug by adding in the initial notation code an extra rotation command (rt)

|  |   |   |
|--|---|---|
| <pre>to parorth :a :b :c   fd(:a)   up(180-(:b))   fd((:a)/cos(:b))   up(90+(:b))   fd((:a)*tan(:b))   up(90)   rt(:c) end</pre> | <pre>to konos :a :b :n :c   repeat (:n)   [parorth(:a :b :c)] end</pre> |  |
| <p>Modified code of the construction of the Cone.</p>  | <p>Code of the iterative rotation of the Cone</p>                       | <p>Approximation of the surface of the Cone.</p>                                    |

**Table 3: Working with the Cone**

## CONCLUSIONS

During the research the possibilities and the constraints of the digital environment mediated the mathematical meanings related to three important activities, the construction of the right-angled triangle, generalization and approximation. The meaning of construction of a right-angled triangle shifted from the single use of linear elements (institutional meaning) towards the solution of an equation by means of trigonometry, i.e. the exact calculation of all its elements based on a given length and an acute angle (personal meaning). The construction of a right-angled triangle with varying sides was mediated by the use of the software's functionalities, that is the activation of the sliders, and hence the meaning of generalization was shifted to that of parameterization. Furthermore, the meaning of the smooth (continuous) curved surface of the cone was shifted to the area of the

approximation with arbitrarily large number of appropriately arranged triangles. Finally, students' suggestions for the use of an isosceles triangle as a generator along with the positioning the constantly diminishing circles (the circle as the generator), a striking facet to the task - if we yet disregard the feasibility and the misconception of the curvature of the sphere, are innovative in conception. Moreover, their referring to approximations of the cone is found neither in the Greek textbooks nor generally in Greek schools. We believe that the intimacy with the medium is what made progress through the task possible: using the microworld's resources in a different way forged a different – and, as it turns out, more productive – connection in different areas of school mathematics.

## Notes

1. For reasons of ethics, the names of the students referred to the following incidents are fictitious.

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## **MATHPEN: IDENTIFYING AND SOLVING THE PROBLEMS OF ONLINE COLLABORATIVE LEARNING FOR MATHEMATICS**

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*Combining the interactive communication power of Web 2.0 and social-constructivist theory in education research, online collaborative learning (OCL) has now become an area of intensive research and has generated many favourable results. Yet, the term online collaborative learning, or any other related terms, are seldom seen in mathematics education journals. This paper will, after a brief overview of OCL theory, describe the problems associated with OCL in mathematics education and offer MathPen (an online handwriting recognition system) as a potential solution.*

### **INTRODUCTION**

Many researchers are well acquainted with the benefits of student group work. Based on the social-constructivist theories, studies have shown that students' learning can be enhanced as they explore different ideas, challenge various assumptions, justify and defend their understanding and finally draw a well-informed conclusion on the subject matter. It is argued that such interactions can uncover and challenge underlying misconceptions, thus widening the students' perspectives and deepening their understanding (Mercer, 1995). Therefore, it is increasingly common to see the use of group work being promoted in schools, colleges and other educational institutions (Edwards, 2009).

Online collaborative learning (OCL) is a term used to describe similar learning methods, but with an emphasis on internet-based collaboration between learners (Harasim, 2002). As with its offline counterparts, a clear distinction is drawn between *collaborative* and *cooperative* learning: the former refers to a collective effort of mutual engagement in the exploration of a given problem, while the latter refers to an organised manner of work division between students for task accomplishments (Roschelle & Teasley, 1995).

Research in many subject areas has shown that collaborative learning can be implemented online, and similar (if not better) results can be obtained through OCL (Allen & Seaman, 2010). The most commonly cited benefits include: improved reasoning skills, increased awareness of assumptions, enhanced understanding of scope and limitations and improved ability to apply new found knowledge in unfamiliar circumstances, all of which are desirable (even essential) for mathematics learning. Besides, with the increasingly easy access to the internet, successful implantation of OCL affords students extra learning opportunities/ flexibility beyond the physical constraints of time and space.

However, despite the many successful reports of OCL in text-based subjects, investigations into OCL for mathematics education has remained largely off the research radar. To understand the underlying reasons, this study investigates: a) the effectiveness of the internet as a platform for mathematical discussions, b) the evidence for suggesting the challenge of electronically formatting mathematical expressions is a significant barrier to OCL adoption, and c) the evidence for proposing MathPen (an online handwriting recognition system) as a solution.

### **METHODOLOGY**

Since this study is primarily interested in the *interactive* nature of communication, an internet forum, where there is a large population of participants communicating mathematics in an



interactive manner, is a natural choice for investigation. For the purpose of this study, the pre-university forums at [www.mathhelpforum.com](http://www.mathhelpforum.com) (MHF) were selected primarily due to their size, popularity and their administrator's kind approval for observing site activities for research purposes.

Adopting Sande's (2011) method of studying online communications, this study began with examining one hundred threads of mathematical discussions from each of the five different mathematical topics (algebra, trigonometry, geometry, pre-calculus and statistics) from MHF. Special care was taken to note the entry methods used to represent mathematics online and to identify cases that exemplifies the typical use, pros and cons of each entry method observed. The problems identified from these observations were then cross-examined and further investigated with an online questionnaire completed by eighty participants including internet forum members, practicing online tutors, UK-based qualified mathematics classroom teachers and university professors/ lecturers. To further verify the research findings and identify a possible way forward, a new revised concept of online handwriting recognition system (now called MathPen) was constructed especially to address the identified issues. To obtain reviews and comments from experts directly involved in mathematics education, the concept of MathPen is summarised in a short video and was sent to seven experts for professional feedback. Through this mixed method approach, the problems associated with OCL for mathematics learning were identified and MathPen is proposed as a potential solution.

## RESULTS: EXAMINATION OF INTERNET FORUMS

The five hundred threads examined contained a total of 4819 mathematical statements or expressions, giving an average of 9-10 mathematical statements per thread/ discussion. With an average of 10-11 posts/ exchanges per discussions, it can be seen that these exchanges contain a mixture of mathematical statements and textual arguments/ explanations. Although the nature of these forums typically attracts a single standard textbook exercise per thread and therefore has little room for exploration, the potential for collaborative learning is strong.

|             |            | Topic   |          |             |            |              | Total |
|-------------|------------|---------|----------|-------------|------------|--------------|-------|
|             |            | Algebra | Geometry | PreCalculus | Statistics | Trigonometry |       |
| InputMethod | ASCII      | 580     | 348      | 580         | 336        | 288          | 2132  |
|             | Ext. Links | 19      | 37       | 16          | 32         | 18           | 122   |
|             | Latex      | 767     | 418      | 574         | 224        | 432          | 2415  |
|             | Pictures   | 19      | 76       | 19          | 10         | 26           | 150   |
| Total       |            | 1385    | 879      | 1189        | 602        | 764          | 4819  |




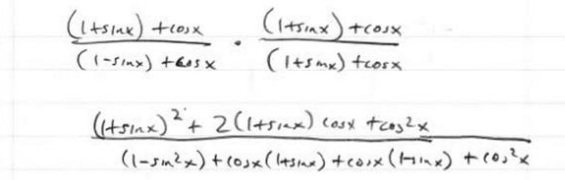
**Table 1: Input method usage per mathematical topics on math help forum**

The one hundred algebra-related threads contained a total of 1385 mathematical statements over just a mere 25 day period. Within the five topics studied (algebra, pre-calculus, statistics, trigonometry and geometry), 1385, 1189, 602, 764 and 879 mathematical statements were posted over a period of 25, 45, 49, 66 and 76 days respectively, giving an average of 55, 26, 12, 12 and 12 mathematical statements posted per day. The amount of mathematical statements posted within a short period of time prior to the start of the summer holiday further demonstrates the potential of the internet as a platform for online collaborative learning for mathematics.

The differing number of mathematical statements posted between each subject area is statistically significant (ANOVA,  $p < 0.001$ ), with Algebra being significantly more than Pre-Calculus, which is in turn significantly more than the rest. Although not significantly different from each other, both Trigonometry and Geometry are significantly more than Statistics. Possible reasons for the differences may include: A) higher interests in improving algebraic skills; B) some subject areas are more suitable to online communication; C) an imminent national exam affecting a significant number of students; and D) communication of algebra is less likely to be in text form.

Although most of the mathematical expressions were entered using Latex (50%) or ASCII (44%), establishing these as the main means of mathematical communications, the decision between using Latex or ASCII is subject dependent ( $\chi^2(12)=228.385$ ,  $p<0.001$ ). To highlight the inadequacy of ASCII, one of the threads studied shows a student asking for help with simplifying the fraction: “ $(\sin(n+1)A-\sin(n-1)A)/(\cos(n+1)A+2\cos(nA)+\cos(n-1)A)$ ”. Figure 1a shows the students’ subsequent attempt at solving the problem.

In order to help the student and work on the mathematics, one must first convert the mathematical statement into a recognisable form such as  $\frac{\sin(n+1)A-\sin(n-1)A}{\cos(n+1)A+2\cos(nA)+\cos(n-1)A}$ . Only then can one begin to interpret and comprehend the mathematics in the conventional way. Besides, did  $\sin(n+1)A$  mean  $A\sin(n+1)$  or  $\sin\{A(n+1)\}$ ? Studying the second statement  $2\cos\left[\frac{\{(n+1)A+(n-1)A\}}{2}\right]\sin\left[\frac{\{(n+1)A-(n-1)A\}}{2}\right]$  as shown in Figure 1a, and converting it into  $2\cos\left[\frac{\{(n+1)A+(n-1)A\}}{2}\right]\sin\left[\frac{\{(n+1)A-(n-1)A\}}{2}\right]$  indicated that  $\sin\{A(n+1)\}$  was meant, a conclusion that is not immediately obvious in the ASCII format. Given that this was the student’s 43<sup>rd</sup> post on this forum and having been a member for over a year, it is inferred that there is a certain measure of difficulty or reluctance to learn Latex. In this case, the complexity of the mathematics and the number of careful manipulations required to simplify the algebraic fraction shows that the use of ASCII would indeed distract the user from the mathematics itself, thus rendering its use unfit for online collaborative learning.

|   |  |
|---|--|
| <p>Numerator<br/> <math>N = \sin(n+1)A - \sin(n-1)A</math><br/> <math>= 2\cos\left[\frac{\{(n+1)A+(n-1)A\}}{2}\right]\sin\left[\frac{\{(n+1)A-(n-1)A\}}{2}\right]</math><br/> <math>= 2\cos(2nA/2)\sin(2A/2)</math><br/> <math>= 2\cos(nA)\sin(A)</math></p> <p>Denominator<br/> <math>D = \cos(n+1)A + 2\cos(nA) + \cos(n-1)A</math><br/> <math>= 2\cos(nA) + \cos(n+1)A + \cos(n-1)A</math><br/> <math>= 2\cos(nA) + 2\cos\left[\frac{\{(n+1)A+(n-1)A\}}{2}\right]\cos\left[\frac{\{(n+1)A-(n-1)A\}}{2}\right]</math><br/> <math>= 2\cos(nA) + 2\cos(nA)\cos A</math><br/>         So <math>D = 2\cos(nA)(1 + \cos A)</math></p> <p>N/D<br/> <math>= \frac{2\cos(nA)\sin(A)}{2\cos(nA)(1 + \cos A)}</math><br/> <math>= \frac{\sin A}{1 + \cos A}</math><br/> <math>= \frac{2\sin(A/2)\cos(A/2)}{2\cos^2(A/2)}</math><br/> <math>= \frac{\sin(A/2)}{\cos(A/2)}</math><br/> <math>= \tan(A/2)</math></p> | <p><b>skeeter</b><br/>         MHF Contributor<br/> <br/> <br/> <br/>         Joined: Jun 2008<br/>         From: North Texas<br/>         Posts: 11,631<br/>         Thanks: 422</p> <p><b>Re: trig identities....i need help needed solving????</b></p> <p>my fun meter pegged with this one ... I wasn't about to Latex it all, so hope the scan is satisfactory. I left the most important step to you.</p>  |
|---|--|

**Figure 1: a) ASCII as an ineffective input method, b) Latex as an ineffective input method**

That Latex can also be problematic is demonstrated in Figure 1b, which shows a post of an *expert helper*, despite his more than 11,000 posts, saying, “I wasn’t about to Latex it all”! The helper subsequently uploaded a full page of scanned handwritten work instead. These Latex-replacement uses of scanned-pictures further indicate that forum members do rely on pen-and-paper to perform their calculations as opposed to calculating while writing online, thereby supporting the idea that the use of complex notations can be a barrier to online communication. Secondly, there is a cost associated with the use of Latex. Consider this: the helper would have performed the calculations on paper, confirmed its correctness, gone to the scanner, performed the scanning procedures, saved the file to a specified location on the computer, gone to the forum, located the pictorial file again and finally uploaded the file with a brief comment. Yet significantly, performing all these steps were considered to be faster and easier than having to “Latex it all”, thus demonstrating Latex’s weaknesses.

By studying a small sample of this vibrant forum, it can be seen that interactive communication of mathematics, and hence online collaborative learning, is possible. However, the two most commonly used input methods are inadequate for current needs. Users are still required to laboriously transfer their pen-and-paper-based calculations one line at a time and, when the tedious process becomes unbearable, users often resort to scanned-images.

## RESULTS: ONLINE QUESTIONNAIRES

Of the 80 participants, 55% (44) are UK-based qualified mathematics teachers. There were also 18 university postgraduate students and 18 unqualified teachers. Amongst the qualified teachers, the average offline mathematics teaching experience is about 12 years, with the lower and upper quartile being 4 years and 19 years respectively.

|                         | N  | Confidence Level |     |      |         | Competence Level |     |      |         | Accuracy Level |     |      |         |
|-------------------------|----|------------------|-----|------|---------|------------------|-----|------|---------|----------------|-----|------|---------|
|                         |    | min              | max | mean | Std.dev | min              | max | mean | Std.dev | min            | max | mean | Std.dev |
| MS Math Editor          | 32 | 3                | 5   | 4.08 | 0.776   | 3                | 5   | 4.09 | 1.125   | ---            | --- | ---  | ---     |
| Latex                   | 50 | 1                | 5   | 3.96 | 1.224   | 1                | 5   | 3.54 | 1.503   | ---            | --- | ---  | ---     |
| MathML                  | 05 | 1                | 4   | 3.00 | 1.225   | 1                | 4   | 3.00 | 1.225   | ---            | --- | ---  | ---     |
| Handwriting Recognition | 28 | 2                | 4   | 2.67 | 0.707   | 2                | 4   | 2.75 | 0.707   | 2              | 4   | 2.67 | 0.866   |

**Table 2: Descriptive statistics of commonly used input method**

The questionnaire (Table 2) reveals that Latex is the most commonly used input method. Though incompatible with the web, Microsoft's Mathematics Editor (MS-ME) also proves to be very popular and has the highest score for user confidence level (4.08 on a scale of 1-5). The confidence and competence levels also show that MS-ME is the most user-friendly, followed by Latex, MathML and current freely available handwriting recognition technology (such as Interactive Whiteboard and Windows 7 Math Input Panel). Unlike its coding/ programming based counterparts such as Latex and MathML, which requires a substantial amount of learning, MS-ME's mouse-based point-and-click operations make the user interface very easy and intuitive to learn. Similarly, since current freely available handwriting recognition technology suffers from poor accuracy (scoring only 2.67 on a scale of 1-5), it is not surprising to see it at the bottom of the preference list.

It is also noted that amongst those who hold a UK-based mathematics teaching qualification, there is a strong preference towards MS-ME over the use of Latex ( $p < 0.001$ ). One possible reason could be the simplicity of MS-ME. Interestingly, university postgraduates, who can be expected to overcome the challenges of learning Latex, have a strong preference towards Latex over any other technologies ( $p = 0.028$ ). These results further verify that Latex, although one of the most commonly used input methods for online communication, is non-intuitive to use.

Regarding the use of handwriting recognition technology as a solution to the problems, 72% of the participants believe that the technology will prove to be useful and 69% indicated that they are likely to use it should it become widely available. 56% of the participants have also volunteered personal contact details to be informed of future developments. Despite the small number of participants in this survey, these figures give a strong indication that an intuitive, user-friendly and accurate handwriting recognition technology would be greatly appreciated.

## EXPERT REVIEWS

With the selection criteria of those who a) have been teaching mathematics for the last 10 years, b) have some experience of handwriting recognition technology and c) have not previously been known to the researcher, seven experts were identified and shown a video of MathPen. Their comments are:

*"It would let me concentrate on substance rather than formatting."* --- Expert Helper of a Free Math Site

*"If MathPen functions as shown in the video, then it will save me the time of having to look up LaTeX codes and syntax. I would be able to tutor more students in less time, if I were able to easily and accurately transfer my writing into bulletin-board posts. I do have experience with web sites that offer LaTeX symbol recognition by drawing the corresponding math symbol using*

the mouse. If MathPen works as shown in the video, then MathPen is a vast improvement over what I described above.” --- Experienced Mathematics Teacher, Community College, USA.

“MathPen would speed up making worksheets. It would allow me more freedom over giving out worked solutions. I have used the free Microsoft Mathematics 4 on a tablet PC and on an interactive whiteboard, although it is not very accurate and it is only one line at a time. The ability to convert multiple lines is particularly attractive.” --- Experienced Mathematics Teacher, Comprehensive Secondary School, UK

“For students, it would greatly facilitate their posing questions correctly. Less knowledge of math formatting is required.” --- Head of Department, Comprehensive Secondary School, UK

“Yes. A robust, reliable, scalable mathematical character recognition package compatible with the industry standard of Latex is long overdue and something that we have been saying should be developed for the past decade.” --- Senior Lecturer in Mathematics, UK

“It streamlines computer-mediated math communications; Learning LaTeX is tedious and will no longer be necessary.” --- Professor in Mathematics Education, a university in Finland.

“Students could present math expressions in their questions by simply writing it out. Student would not need to learn LaTeX or texting conventions. Scanning to jpeg goes partway; Using MathType can be slow but is easily edited. If editable markup is available, MathPen would be a superior choice.” --- Emeritus Professor of Mathematics Education, USA.

## PROPOSING MATHPEN

As the forum activity shows, the desire to engage in mathematical conversations online can be so strong that people are willing to overcome the challenge of painstakingly converting each line of handwritten mathematics into electronic format. Figure 2a shows a student’s attempt at seeking online help along with the Latex code required for such a post. Considering the complexity of the code, it is absolutely heart-warming and encouraging to see young school-aged children demonstrating such determination and passion. Yet, as the expert reviews suggested, should formatting mathematical expressions be a necessary evil? Should spontaneous responses to discussions (Figure 2a) be dampened by the strenuous efforts required to communicate? Should students not be able to write on tablet computers, as they would on a piece of paper? Should handwritten work (Figure 2b) not be automatically formatted into Latex (Figure 2c)?

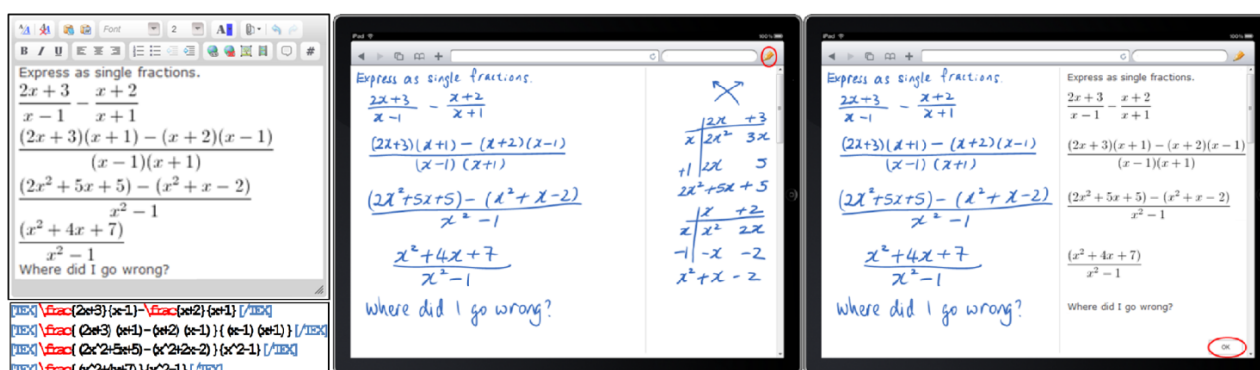


Figure 2: a) A forum message and its Latex code, b) handwritten work, c) automatic formatting

Although such recognition algorithms are freely accessible through research publications, commercial products are prohibitively expensive for many. Additionally, being *word-processing* orientated, every recognition system available assumes the user knows what each line of mathematics looks like before they start and that the users would be content with recognition one

line at a time. In reality, however, each line of mathematical statements evolves as different pieces of information are processed. At the end of each line, new insight is gained thus sparking off another line of evolving mathematical statement. Therefore, the current user assumptions are invalid when it comes to the *doing* of mathematics (as opposed to the *word-processing* of mathematics) and the *interactive communications* of mathematics. Consequently, it is proposed that the published handwriting recognition algorithms should be repackaged with appropriate user interface to facilitate the *doing* of mathematics online and open the way for OCL in mathematics education. In fact, MathPen is currently under development and is intended to be a free, open-source online handwriting recognition system specifically designed for mathematics education.

## SUMMARY

In agreement with other researchers (Catalin, Deyan, Kohlhase, & Corneli, 2010; Costello, 2010; Reba & Weaver, 2007), one of the problems associated with OCL in mathematics education is, as the forum analysis and online questionnaires showed, the lack of a natural and effective means of entering mathematical expressions online. Observations of forum discussions provided a glimpse of current practice and the challenges associated with entering mathematical expressions online. These were further verified with an online questionnaire, which provided further insight into the usability problems of current technologies. Based on these findings, the concept of MathPen was designed and sent to seven experts for professional feedback. All experts unanimously agreed on the potential benefits that MathPen could bring to mathematics education. Therefore, it is concluded that serious considerations should be given to online handwriting recognition systems as a means of opening the way to OCL for mathematics education.

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## THE RECONSTRUCTION OF MEANING FOR THE CONGRUENCE OF TRIANGLES WITH TURTLE GEOMETRY

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*In the present study, 'Turtleworlds' a programmable Turtle Geometry medium, is utilized by teachers and students as a means of exploring the congruence of triangles with the help of a half-baked microworld. This activity brought to light a relation which, somehow, brings order to chaos, creating categories (classes) of triangles, each one represented by the unique, constructed triangle. It was also shown that in order to arrive at the process of proof as documentation, it is essential to begin with a generalization of observations, formulation of arguments and their articulation in unified reasoning, so as to enable the student to understand and effectively formulate more formal proofs.*

### INTRODUCTION

Handling geometrical concepts with the 'Turtleworlds' involves an approach which is different from that of traditional teaching. A big difference between the Turtle Geometry and the Cartesian Geometry derives from the concept of intrinsic attributes of geometrical shapes. An intrinsic attribute is one that depends exclusively on the particular shape, not on the relation of the shape to a system of reference (Abelson & diSessa, 1980).

The books of Euclid rely on the three conditions for triangle congruency: the side-angle-side (SAS), the side-side-side (SSS) and the side and two angles (SAA). These three conditions for congruency (SAS, SSS, SAA) are subsequently used in the books of Euclid to prove many more propositions. (Jones & Fujita, 2013). As Freudenthal (1983) argues "this artificial system of congruent triangles has been canonised in the traditional school geometry" (Freudenthal, p. 342, 1983).

In the Euclidean geometry taught in Greek schools, the congruence of triangles is regarded as the possibility of coincidence by superposition, and is proved following a typical scheme of logical reasoning using the three given "criteria of equation" (Argyropoulos et al., 2003).

In 'Turtleworlds' the superposition of two different triangles can be replaced by the construction of a triangle, so that, students to be able to understand the equality of triangles as an equivalence class of all triangles of the level. So, "learning is not taken as a simple process of the incorporation of prescribed and given knowledge, but rather as the individual's (re)construction of geometry" (Laborde et al., p. 278, 2006)

### METHODOLOGY OF RESEARCH

We followed the above approach for all three criteria of triangles congruence with students of the first grade of senior high school, who had prior experience in the use of Logo (in Turtleworlds, the elements of a geometrical construction can be expressed in a Logo procedure), and also with teachers of mathematics during a teacher training programme on the use of digital technologies. We focused in the way students and teachers perceived an 'other' approach of congruence of triangles which was mediated by 'Turtleworlds'.

#### **In the classroom:**

The students worked on the computer in pairs for two teaching hours. They were asked to handle a half-baked microworld in the Logo environment.

In ‘Turtlewords’, what is manipulated is not the figure itself but the value of a variable of a procedure. This is done by means of a variation tool, which is activated upon clicking on the trace of the figure constructed by a variable procedure. (Kynigos, 2007).

The students’ actions were recorded (by saving files and images). The researcher coordinated the activities and recorded the students’ observations and the groups’ discussions as well.

### **In the teacher training programme:**

The teachers worked on the computer individually. They were asked to handle the same half-baked microworld in the Logo environment as the students did, and their actions were recorded (by saving files and images).

In both cases there followed a discussion, with the researchers acting as coordinators at each stage of the procedure.

## **PROCEDURE AND FINDINGS**

With ‘Turtleworlds’, we aimed at the construction of the triangle using the least possible data. That is, if we construct a triangle given the lengths of two of its sides and the angle contained in these sides, then this triangle will be unique. This observation is related to our initial reference to the intrinsic qualities of the geometrical shape. Initially, in order to construct the triangle, we used all of its main elements: the three sides  $a$ ,  $b$ ,  $c$ , and two angles,  $x$  and  $y$ , (obviously, the third angle can be calculated). The half-baked microworld we suggested is shown in Figure 1.

```
for triangle :a :b :c :x :y
  fd :a
  rt 180-:x
  fd :b
  rt 180-:y
  fd :c
end
```

**Fig. 1. Half-baked microworld ‘triangle’**

### **In the classroom:**

The students activated the variation tool and, correcting the occurring shape, they constructed a triangle; rather, they closed the triangle. In the discussion that followed they realized that the triangle was different for each group. The students observed that, playing with the variation tool, we can construct an infinite number of triangles, each one different from the rest. In this way, our set of reference was defined: it is the set of all the triangles of the plane.

### **In the teacher training programme:**

A question posed by the teachers led the teachers’ group to further investigation, discussion and experimentation with the software, so as to give an answer to the question: “How can we be sure that the triangle has been closed?” The suggestions always led to a dead-end like (Fig. 2).

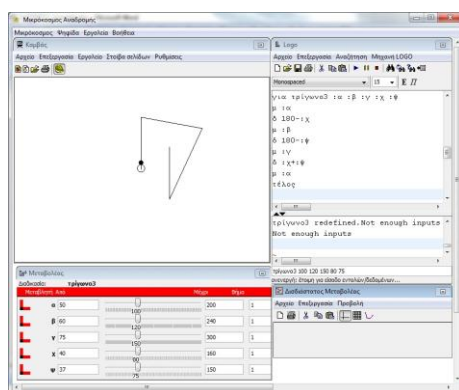


Fig. 2. Teacher suggestion 1

Others that use repetition (Figure 3):

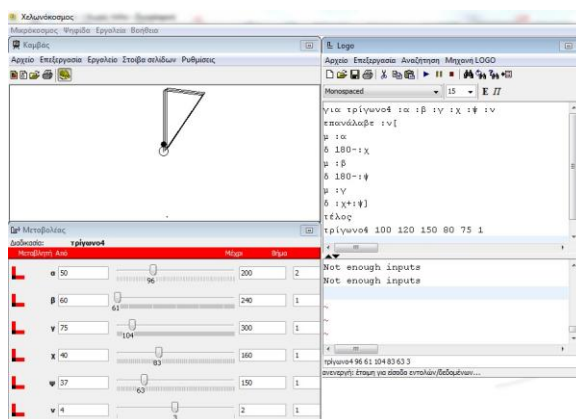


Fig. 3. Teacher suggestion 2

The solution was given by the researcher with the use of the commands (see Figure 4).

```
make "d distancetoxxy
if :d<0.5
[setheading 0 0]
```

Fig. 4. Solution

Next, we invited the students to modify the code in order to construct triangles with a specific (fixed) side length. In this way we limited both the use of variation tool and the number of possible triangles at the same time. The code that occurred is shown in Figure 5:

```
for 333 :a :b
fd 100
rt 180-:a
fd 150
rt 180-:b
fd 120
end
```

Fig. 5. Half-baked microworld 'SSS'

Each student or group of students constructed their own triangle, which will also be unique in the sense that any change in the angles a and b will result in the construction of the same triangle.



The measure of the two angles (53 and 41) that makes the triangle close became a new topic of discussion among the teachers, when the results differed from these measures even slightly.

In the next activity we once again asked the students to correct and to “rediscover” the initial code by giving fixed values on one side and the adjoining angles, and to “rediscover” the uniqueness of the triangle they can construct. The result is shown in Figure 6.

```
for ASA :a :c
  fd :a
  rt 180-40
  fd 150
  rt 180-60
  fd :c
end
```

**Fig. 6. Half-baked microworld ‘ASA’**

We discussed with the students the meaning of their previous observations. The observation that in the two previous constructions the occurring triangles were unique led them to fully understand the existence of triangles congruence: when we compare two triangles, it is always the same triangle that appears at two different instances. Following the above observations, we invited the students to experiment by developing the code that corresponds with the criterion SAS (SAS congruency theorem) (side-angle-side) and by playing with the triangle.

The question we posed next refers to the case when the equal angle is not the one contained in the two equal sides (SSA side-side-angle) and we invited the students to formulate a code which materializes the above condition, to apply it and to investigate the multitude and the kind of triangles that can be constructed playing with the the variation tool. During the discussion prior to the investigation with ‘Turtleworlds’, the students excluded the possibility of triangles congruence “since none of our familiar criteria is present.” During the discussion about the results, there occurred observations like: there are two triangles we can construct or the triangle is unique. These observations inspired a heated discussion that resulted in searching for a fourth criterion of equality of triangles, whose formulation and validity had to be explored. Zodik & Zaslavsky (2007) report a discussion among their students about the same topic:

Teacher: Are they congruent according to SAS?

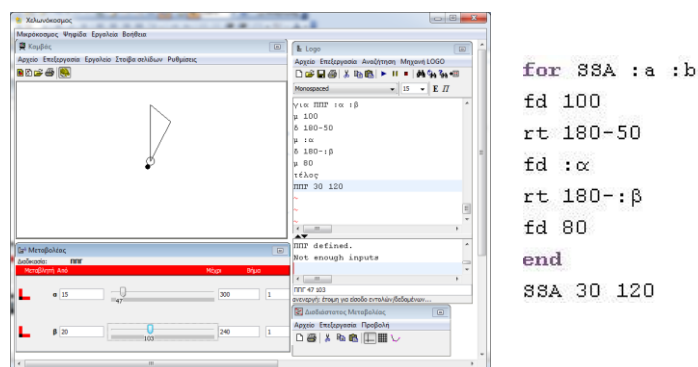
Student: In both of them there isn’t SAS.

Teacher: But are they congruent or not?

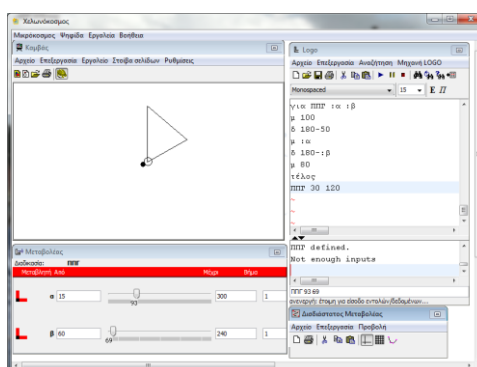
Student: Yes they are. (Zodik & Zaslavsky, p. 2031, 2007)

In order to investigate the condition, they developed the code shown in Figure 7. With the appropriate use of the variation tool, this code gave us two different triangles, thus two different categories (classes) of triangles (Fig. 7 and Fig. 8).

An issue of concern that was brought about by this activity is the issue regarding the use of rules and the primitive epistemology underlining or even promoted sometimes in the teaching of mathematics, that is, anything that does not satisfy the rule is wrong: The case of the criterion Side-Side-Angle, when the equal angle is not the contained one, does not exclude the congruence of the triangles.



**Fig. 7. Half-baked microworld ‘SSA’**



**Fig 8. Half-baked microworld ‘SSA’**

## CONCLUSIONS

Each individual must reconstruct knowledge. Of course, one does not necessarily do this alone. Everyone needs the help of other people and the support of a material environment, of a culture and society (Papert, 1990).

The reconstruction of knowledge mediated by the microworlds gives new meaning to the three criteria-congruency theorems.

The initial construction of triangles using their basic elements (three sides and three angles) freely changing introduces us to the concept of set of reference, that is, the set of all the triangles of the plane. The construction of triangles with less than the six basic elements, in some way, brings order to chaos, creating categories (classes) of triangles, each one represented by the unique, constructed triangle. ‘Turtleworlds’ a programmable Turtle Geometry medium, provides us with the means to investigate through modification, while it restricts us in a shape that contains its qualities and ultimately represents an infinite multitude of shapes.

Furthermore, the concept of mediation is appeared in this intervention. The initial use of ‘Turtleworlds’ transformed to a mediating role for understanding a mathematical content (Drijvers, et al., 2010). The mediating role of the tool gave the opportunity to both teachers and students to develop their own mathematical meanings. The transition from the personal perspective, to the developing of a framework which consciously approaching mathematical concepts, is a process that is neither spontaneous nor provided. The existence of a teaching cycle, which is constructed so as to include semiotic activities and a collective mathematical discourse, with the role of teacher-trainer to focus on facilitating and guiding the evolution of signs are particularly important (Bartolini Bussi & Mariotti, 2008).

The teachers in training, resisted in a way this approach, and were very skeptical about the closing of the triangle. The cause of this resistance is the demand (or necessity?) for a fixed procedure leading to the proof, which is usually applied at school. Proof is not only the typical one that the student will encounter in senior high school. The concept of proof as documentation begins with the generalization of observations, the formulation of arguments, and their articulation in unified reasoning. It is necessary for the students to proceed through all these stages, before they can start comprehending and become able to formulate more typical proofs. Otherwise, they will regard mathematical proofs as a meaningless ritual. Indicative of this is the behaviour of students who, not recognizing the characteristics of SAS, but encountering an ASS case, refuse to even speculate on the possibility of congruent triangles before they see them taking shape with the navigation of the turtle.

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## **THE NEW CHALKBOARD: THE ROLE OF DIGITAL PEN TECHNOLOGIES IN TERTIARY MATHEMATICS TEACHING**

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*Mathematics is a discipline with a distinctive pedagogy that reflects how knowledge is expressed and developed in symbolic and diagrammatic form. Pedagogical approaches have both influenced, and been influenced by, the architectural design and educational technologies of the environments in which they are used. In the tertiary sector in particular, traditional methods of teaching mathematics have been challenged by computing technologies that are based on keyboard and mouse interfaces and learning environments that emphasise digital displays. This Paper discusses how the use of pen-enabled Tablet PCs can build on the benefits of traditional pedagogical approaches while facilitating the development of new approaches.*

### **INTRODUCTION**

Shulman (2005) identified characteristic or ‘signature’ pedagogies that are pervasive within particular professions and implicit in the way discipline knowledge is defined, developed, and valued. The dependence of mathematics on the use of symbolic and diagrammatic form for the expression and creation of knowledge has resulted in the development of distinctive pedagogical approaches involving exposition using hand-written methods.

In classroom environments, the medium for this hand writing has traditionally been that of large boards (chalk or white boards). Artemeva and Fox (2011) examined tertiary mathematics teaching across a range of countries and identified a common multi-modal approach, “specific to the activity system of teaching undergraduate mathematics”, in which board writing was accompanied by commentary and meta-commentary. They termed this approach ‘chalk talk’.

While the term ‘chalk *and* talk’ is commonly used to describe a generic approach that has been criticised for being didactic and teacher centric, Fox and Artemeva (2011) maintain that ‘chalk talk’ can be “pedagogically interactive, meaningful, and engaging” in the context of mathematics education. The value of traditional approaches that incorporate lecturer modelling of problem-solving behaviour, particularly in the early years of tertiary study, has been recognised even by proponents of alternative approaches such as problem and project based learning (Perrenet, Bouhuijs, & Smits, 2000; Mills and Treagust, 2003). As Samuelsson (2010) noted, different teaching approaches, including traditional and problem solving approaches, can develop different areas of mathematical proficiency, and it may be that an ‘eclectic approach’ has advantages.

Shulman (2005) suggested that signature pedagogies influence, and are influenced by, the design of the learning spaces in which they are practiced and the educational technologies that they use. While Fox and Artemeva looked closely at the dynamics of lecturer teaching approaches, their study also revealed agreement among lecturers on the necessity of a large writing board as the medium for writing (Artemeva & Fox, 2011).

While the mathematics-specific method of multimodal handwritten and oral exposition has a particular manifestation as chalk talk in the lecture theatre environment, the essential components of the method are also found in many situations where mathematics is dynamically shared - even in the case of two individuals exploring a problem together on pen and paper. In this paper the focus is on the digital technologies that can support chalk talk in current tertiary learning and teaching environments, and potentially within a range of learning and teaching strategies.

## **HANDWRITING IN MATHEMATICS EDUCATION**

While the use of the term chalk talk implies a traditional *handwritten* approach, the method is not bound to traditional board technologies. However the way that the writing in chalk talk is interlinked with diagrams, commentary and gesture in a multimodal way means that the method is difficult to reproduce fluently using a keyboard and mouse interface in a standard digital environment.

It is not just for the lecturer that writing by hand provides a more fluent interface. Romney (2010) notes the difficulty for students in trying to take live notes in mathematics classes using keyboard and mouse interfaces. Studies by Anthony, Yang and Koedinger (2008) show that on computer devices that allow handwritten input, extraneous cognitive load is reduced when using handwriting over keyboard input, and that hand written and hand drawn input provides “better support for the two-dimensional spatial components of mathematics.”

While the use of digital technologies may be regarded as progressive, if dynamic handwritten methods are not supported, they may be disadvantageous in mathematical contexts.

## **TECHNOLOGIES TO SUPPORT HANDWRITTEN MATHEMATICS**

Early teaching technologies supported only handwritten input, and were used in common across all disciplines. The chalkboard became a standard starting in the 1800s, with a range of newer technologies being introduced progressively: the whiteboard (or dry erase board) began to supplant chalkboards in the 1980s; the overhead projector began to supplement writing boards from around the 1950s (Kidwell, Ackerberg-Hastings, & Roberts, 2008; Krause, 2000). More recent technologies that have been promoted are the document camera (Brooks-Young, 2007) and, particularly in primary and secondary classrooms, the interactive whiteboard, (Brown, 2003; Higgins, Beauchamp, & Miller, 2007). At the Auckland University of Technology (AUT), large monitors incorporating pen digitisers have been installed in some lecture theatres.

These different tools have different capabilities, and different limitations in particular learning environments. However a key issue for staff adoption of technology is whether they can rely on it to be available in all their timetabled teaching spaces (Spotts, 1999; Brill & Galloway, 2007). Rather than seek to provide standard devices that are room based, providing portable devices that are lecturer owned or managed and can be readily taken *to* rooms may be a better alternative.

While most computing technologies remain keyboard-mouse centric, the Tablet PC is a mature digital technology that supports precise hand-drawn pen input, with embedded digitisers able to provide the control required for detailed symbolic and diagrammatic mathematical writing. It should be noted that this pen technology uses a more precise technology that is available in devices that rely solely on capacitive touch, such as the iPad. Digitiser pen technology provides highly accurate positional resolution and a pressure-sensitive thickness response; the sensor technology detects the presence of the pen as it approaches the screen, so that the hand can be rested on the screen without producing unwanted inputs. This technology is able to provide a relatively natural writing experience that can be readily adopted by the new user without major difficulty. With integrated pen support provided by the Windows Operating System and ‘inking tools’ available in standard and specialised software, the Tablet PC has the capabilities to support fluent input of handwritten mathematics.

## **ORGANISATIONAL INFLUENCES ON TEACHING TECHNOLOGIES**

Moyle (2010) notes that attempts to replace existing technologies with new technologies has had limited success without an accompanying change of pedagogy. Krause (2000) suggests that success

in the effective introduction of technologies has come where the new technologies have enhanced existing accepted pedagogical approaches, rather than encouraged new approaches. However the introduction of new technologies has often been driven by changing physical and organisational structures, accompanied by institutional, rather than faculty, based initiatives for pedagogical change (Moyle, 2010).

An organisational goal of improving efficiency has led to centralisation of the timetabling of rooms and management of technology, and standardisation of the technology provided in teaching spaces. In the past, tertiary teaching spaces were often managed by departments structured on a discipline basis, and could be more readily customised to suit particular discipline needs.

### **THEN AND NOW: LEARNING SPACES AND TEACHING TECHNOLOGIES**

A typical example of such an institutional approach to standardise technologies in teaching rooms and encourage new forms of teaching and learning is described in a 1998 report from Heriot-Watt University (Marsland, Tomes, & McAndrew, 1998). It describes the refurbishment of a 100 seat lecture theatre, involving removing two existing roller chalkboards, retaining OHPs and a 35mm projector, and introducing a range of new digital technologies. Perhaps unsurprisingly, the report records the mathematics department as registering concern at what they saw as reduced support for hand writing; the report suggested that an alternative to adapting to the capabilities of the available technology might be “to designate rooms to best support the different styles of teaching.”

The issues of the interaction of new technology, learning spaces and pedagogical approach are ongoing. At AUT, a new building has just been opened, which includes a number of innovative types of classroom-scale learning spaces, but also includes a large (385 seat) traditional lecture theatre, a smaller lecture theatre and case-room. The technology setup of the theatres follows the trend towards emphasising digital technology, with multiple data projectors - but no whiteboards. The scale of these larger rooms is such that the limitations of a whiteboard on readability from a distance would make them an impractical option. Again, concerns were expressed by lecturers in mathematics based disciplines about the lack of writing capability; again, moving mathematics classes into different rooms where whiteboards were available was an institutional response.

Whiteboards are present in all smaller classrooms in some form, but their dimensions are variable and their placement and lighting is often a secondary consideration to the requirements of a data projector. While a number of theatres spaces are equipped with a document camera, and two have digitiser monitors, these are not universally available in all teaching spaces. In fact, the only common display technology now across all teaching spaces is the digital data projector connected to a standard PC.

While there are pedagogical arguments against the large class model, its use continues. Foreman (2003) suggests this is “mainly because it is cheap and pragmatically useful: the economies of scale generate a surplus that supports low teacher-student ratios in major classes”. The large class requires a large learning space, where the limitations of whiteboards are exposed.

In the absence of suitable whiteboards, or even document cameras, lecturers in mathematical disciplines timetabled in such spaces must abandon handwritten chalk talk approaches and resort to digital display of pre-prepared material (i.e. PowerPoint). In this case, a change (in availability) of technology forces (rather than motivates) a change in pedagogical method – but not a positive change. Lecturers in mathematical disciplines faced with this scenario have protested strongly.

## INTRODUCTION OF A NEW TECHNOLOGY - THE PEN ENABLED TABLET PC

At AUT a pilot project to use Tablet PCs was initiated in 2012, involving lecturers from the School of Engineering, using five HP2760P pen-enabled Tablet PCs in lecture and classroom sessions (Maclaren, Singamneni, & Wilson, 2013). Despite implementation delays allowing for limited training in the use of the device, staff were able to quickly adopt the technology. While there were some initial technical teething problems, staff were encouraged to work through those issues by the overwhelmingly positive response of the students; in one class fast-feedback survey, 74 of 77 students (96%) rated the presentation approach as superior to other methods that had been used with them. One lecturer, who was teaching a primarily non-mathematical subject, found the Tablet PC did not particularly suit his teaching approach, and this Tablet was passed on to another lecturer involved in teaching a mathematics intensive course.

The importance to both lecturers and students of mathematical processes being dynamically modelled is apparent from corresponding comments from the AUT pilot study and the Artemeva and Fox (2011) study (Table 1) involving board-writing. The value of the handwritten approach in pacing lessons and enabling adaptability and spontaneity was also acknowledged by lecturers and students. It is clear that the essential element is not the particular technology (whiteboard or Tablet PC), but the modality of handwriting and narration that is enabled by both. As Artemeva and Fox recognised *“it is important to stress that this consistent view of the usefulness of chalk talk in no way precludes the introduction of advanced technology to university mathematics classes.”*

| Student Comments (AUT 2012)   | Lecturer Comments (from Artemeva and Fox, 2011)   |
|---|---|
| This method is more effective in a way since we get to see all the steps required/executed in order to attain the final answer  | . . . you need to show . . . what you're thinking; you need to show the process in order to teach that  |
| The lecturer goes much slower in the lecture and covers the material one step at a time rather than displaying a PowerPoint slide with lots of words which can be hard to follow. | .. . with overheads and a computer [Ppt] presentation you can go way faster than the students or anybody can comprehend the mathematical stuff. |
| .it shows the step by step explanation before the materials reach the important part  | without visual aid, and the time line that comes from writing it all down, . . . it's impossible to appreciate [the logical constructions]      |
| Slows down lecture so notes can be taken. Because it is handwritten its easier to copy down.  | . . . the writing, the act of writing, keeps the pace   |

**Table 1: Comparison of Student Comments from a 2012 AUT Pilot Study Student with Lecturer Comments from Artemeva and Fox ( 2011)**

In the AUT pilot study, students also commented on functional improvements of the projected handwritten material over handwritten whiteboard material: the screen was much easier to see and read from anywhere in the room; the lecturer did not block the view; material was scrollable and correctable, and was not constantly being rubbed out; different colours could be used to highlight different aspects; pens didn't run out and writing was larger and clearer; notes were recorded and could be made available later. Particularly in larger rooms, this use of digital technology can enhance the student experience while maintaining a handwriting capability.

For the lecturers, the change from the use of large board technologies to writing on a small screen is not seamless, and requires adaptations which will take time to refine (e.g. how best to reproduce the ‘gestures’ from the large board setting requires development). Lecturers have explored the use of different software to provide the ‘paper’, different approaches to page layout and problem development, and variations on the physical positioning of the device. While there is an established model for a fluent chalk talk approach with large boards, best methods for utilising the small Tablet PC screen will continue to evolve.

The project is entering a second phase where other lecturers are taking up the technology on the recommendation of the pilot users and evidence of the enthusiasm of students. For some lecturers, faced with new teaching spaces where whiteboards are unavailable or would be unreadable, the digitiser technology embedded in the Tablet PC appears to be the best technology option to support their preferred pedagogical approach.

### **ADDITIONAL AFFORDANCES OF THE TABLET PC**

The Tablet PC provides not just the handwriting capabilities for chalk talk; the device also supports a full range of PC computer applications. The lecturer is able to switch instantly between a handwritten problem to run mathematical software, present video material, or connect to online course material and references, without requiring a change in lighting and room setup. A paperless process for digital pen marking of PDF assignments is also being trialled.

The Tablet PC technology has potential to be used in the context of a wide range of student centric and collaborative pedagogical approaches, particularly as the technology becomes affordable for students, and not just the lecturer (Loch, Galligan, Hobohm, & McDonald, 2011; Romney, 2011). Collaborative software that supports screen sharing allows groups of students and staff to work together in problem and project based approaches.

Material developed on a Tablet PC can be recorded as a video (a screencast) along with an audio commentary. This can provide access to the multimodal benefits of chalk talk, but in a format that can be accessed asynchronously from a resource repository (Loch, Gill & Croft, 2012).

### **OVERVIEW**

The successful experience with lecturer adoption of Tablet PCs is not unique to AUT, and has been reported elsewhere (Galligan, Loch, McDonald, & Taylor, 2010). These experiences suggest that Tablet PC technology can gain acceptance from faculty teaching in mathematical disciplines because it can provide access to the familiar handwritten pedagogical approach of chalk talk within an institutional digital environment. The wider capabilities of the technology provide opportunities for further development in contexts outside the classroom and where students also have personal access to the technology. While it is common to have positive reports from early adopters of technologies, and it is difficult to predict long term impacts with rapidly changing technologies, the pen-enabled digital tablet looks to have the potential to make a significant contribution as a tool in mathematics education.

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## **EXPLORATORY OBJECTS AND MICROWORLDS IN UNIVERSITY MATHEMATICS EDUCATION**

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*This paper is centred in university mathematics education. It draws parallels between the work of students who develop, program and use Exploratory Objects, a requirement in a core mathematics program at Brock University, and the work of students with microworlds as it is reported in the literature. In both of these computer environments our lens is on the students' activities as they develop and program them, and our focus is on students' potentially learned skills. This work highlights a fundamental gap between research and sustained implementation of student generated computer environments by university mathematics majors.*

### **INTRODUCTION**

At the university level the use of technology in undergraduate mathematics education has been mainly centred on the use of Computer Algebra Systems (CAS). A recent international survey of CAS use (Lavicza, 2010) has shown that their integration in mathematics courses is more prevalent amongst faculty who use them in their research. Whereas a considerable proportion of mathematics instructors seem to use CAS in teaching (Lavicza, 2010), the systemic integration of this technology, that is, by departmental decision tends to be rare (Buteau et al., 2010). A number of factors, such as academic freedom, set barriers for such departmental decisions (Buteau & Muller, in press). Instructors draw on the many mathematics concept programs that are integrated within existing software or that have been independently developed. There are different ways whereby the evolution of technology use in undergraduate mathematics education can be categorized. For the purpose of this paper we distinguish between computer environments that are 'expert' developed and programmed, such as Applets (e.g., MIT Mathlets, n.d.), with those that are 'student' generated, such as microworlds (e.g. Blikstein & Wilensky, 2009) and Exploratory Objects (Muller et al., 2009). Because the terminology keeps evolving we stress that for the category of interest we expect student self-sufficiency and ability to change the code (programming). In this paper our discussion is limited to drawing parallels between 'student' generated exploratory objects and microworlds in university mathematics education. Our lens is on the students' activities as they develop and program them, and our focus is on students' potentially learned skills.

Exploratory objects (EOs) were first characterized in 2001 as central components of a core mathematics program at Brock University. These student developed mathematical objects are part of a learning activity in an innovative undergraduate mathematics program called MICA (Mathematics Integrated with Computers and Applications) (Ralph, 2001). A description of the student generated tools was provided by Muller et al. (2009) stating that, "[a]n Exploratory Object is an interactive and dynamic computer-based model or tool that capitalises on visualisation and is developed to explore a mathematical concept or conjecture, or real-world situation." (p. 64). For the purpose of this paper we augment this description to emphasize that undergraduate students learn mathematics as they engage in designing and programming an interactive EO and then use it to investigate a mathematical concept, application or their own mathematical conjecture. This focus, beyond the digital tool that is an EO, is to include also the 'kinds of student activities emerging from their creation and use' which aligns with that of microworlds (Healy & Kynigos, 2010).

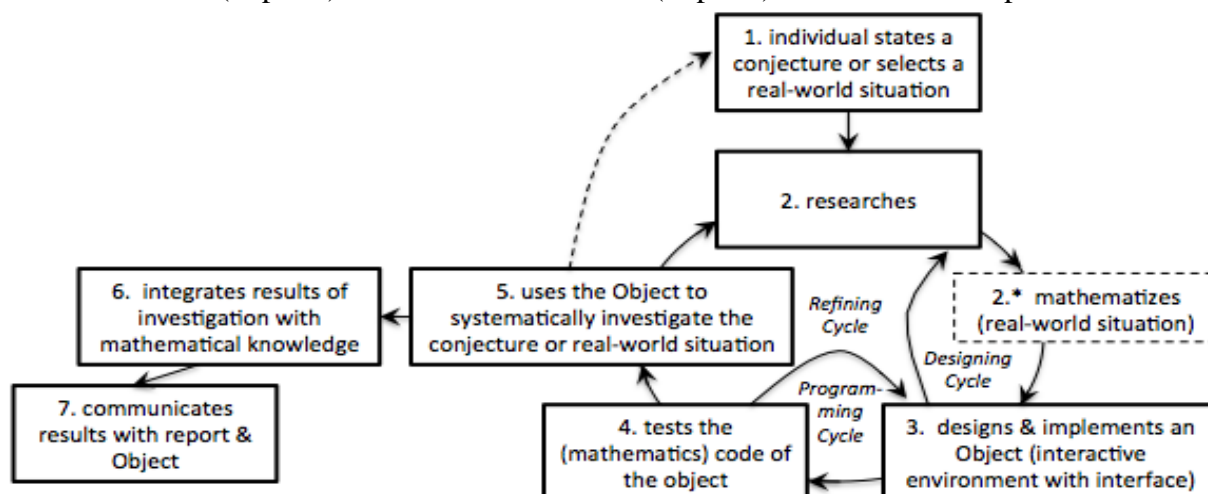
Healy and Kynigos (2010) describe the evolution of the descriptions of microworlds "[s]tarting from Papert's introduction of the concept during ICME 2 in 1972" (p.63) until the time of writing. Their focus is on "the development of theoretical ideas and constructs" (p.63), and while they

emphasize the student activities within microworlds, they note that microworlds "have been redescribed as computational environments embedding a coherent set of scientific concepts and relations designed so that with an appropriate set of tasks and pedagogy, students can engage in exploration and construction activity rich in the generation of meaning" (p.64). For example, Blikstein and Wilensky (2009) describe the use of "MaterialSim ... an agent-based set of constructionist ... *microworlds*" (p. 82) by university students enrolled in a materials science course.

As a result of a recent literature review, Marshall (2012) argues that "students engaged in designing, programming, and using a mathematics [EO] for the investigation of a conjecture, concept, or real-world situation, experience, in a context of experimental mathematics, inquiry-based learning and mathematics learning through programming and simulation" (p. 50). Furthermore, building on this literature review Marshall and Buteau (in press) propose that EOs and microworlds are comparable in the sense of the common student actions involved in both activities. In this paper, we strengthen the latter claim by exploring students' potentially learned skills that are common in EO and microworld activities. Whereas microworlds have long been discussed in the literature, the EO activity remains relatively unknown, that is why we first briefly elaborate on this activity.

### UNDERGRADUATE STUDENTS CREATING AND USING EXPLORATORY OBJECTS

We describe the EO activity by exemplifying it through the EO work done by a first-year undergraduate student in 2006. We structure the description according to the Student Development Model of an EO (Buteau & Muller, 2010; Marshall, 2012) that summarizes the student actions (Figure 1); see Buteau and Muller (in press) and Marshall & Buteau (in press) for detailed exemplification.



**Figure 9–Student Development Process Model of Creating and Using an EO**

Marshall created his *Encryption Pseudorandom-ness Explorer* EO (2006), now accessible from the website (MICA URL, n.d.) with many other students' EOs, for the investigation of how random ciphertext would appear using various encryption methods. He chose this topic based on his own interests (Step 1), and used online and textbook resources to research the selected topic (Step 2). After designing and programming his EO in Microsoft Visual Studio (Steps 2-5), he used it for a preliminary investigation (Step 6). He then tested his "initial conjecture by visually comparing the character distribution histograms created by his EO of the three encryption methods he had selected. Thereafter, Marshall extended his EO (Refining Cycle) by adding parameters" (Marshall & Buteau, in press, [p. 8]). He found that the results from his investigation (Step 6) were in contradiction with his conjecture and had to integrate it into his mathematical understanding (Step 7). Marshall handed in a written report detailing his investigation and results as well as his EO (Step 8).

## EOS AND MICROWORLDS – TWO SIMILAR WORLDS RICH IN LEARNING SKILLS

We mentioned previously that, building on a literature review, we proposed that EOs and microworlds are comparable in the sense of the common student actions involved in both activities (Marshall & Buteau, in press). The literature review involved 82 papers from 14 mathematics or science educational research journals and 2 conference proceedings and about four different areas in university education: 1) learning mathematics by programming; 2) learning mathematics by simulation; 3) learning experimental mathematics; and 4) inquiry-based learning of mathematics and science. The Excel template used for the review contained diverse categories, including every component of the Student Development Model (Figure 1). Among the results, we found that,

the learning activity involving an open source simulation (e.g. Blikstein & Wilensky 2009;...), referred to as *microworlds* by these authors, showed to be the closest activity in our literature review to that of creating and using an EO: all components of the Development Process Model were identified as activity of open source simulations. (Marshall & Buteau, in press, [p.19])

For example, Centola et al. (2000) describe how a student working with a microworld modeling cooperative evolution, researched the literature and decided to add a new parameter which "limited the range of movement of the cows, thus keeping the populations localized in their grazing habits. The change in the model behavior was dramatic." (p. 172). Another example is found in Blikstein and Wilensky (2009), who note that their student, after having modified his microworld, "was concerned with the correctness of his work. He generated a series of plots and screenshots as to match his data with the textbook plots." (p. 110). Whereas the former relates to Steps 2 to 4 (in Figure 1) about implementing a design cycle in the EO activity, the later relates to Step 5 about testing the correctness of the implemented mathematics. Marshall and Buteau (in press) have elaborated these parallels in more detail. However, they also stressed that two key aspects of the activities do not coincide: the student's initial step of the activity (exploring a topic in a microworld versus selecting a topic and conjecturing for the EO activity) and the digital environment initially provided to the student (a modifiable microworld versus none for the EO activity).

The literature review was part of a study that also aimed at identifying potentially learned skills that could be attained through the EO learning activity (Marshall, 2012). This involved theoretical and empirical data. For the theoretical data, a 'potentially learned skills/competencies' category was part of the literature review template and selected excerpts and engendered coding were recorded. The skills (coding) were thereafter regrouped and summarized according to the student actions in the discussed activity (of one of the four reviewed areas) that were common to the EO activity, i.e., identified by the Student Development Model components. The empirical data involved a posteriori self-reflection, recorded in a journal, of a MICA graduate, Marshall, about his learning experiences when creating three of his EOs, including *Encryption Pseudorandom-ness Explorer*. The journal was coded twice, 1) for the Student Development Model components and, 2) for the potentially learned skills, and mainly used for empirical verification. In what follows, we first summarize in Table 1 the potentially learned skills identified in our study in relation to the Student Development Model (Figure 1). Second, based on revisiting and focusing on the microworld literature data of our study, we discuss a selection of potentially learned skills in both contexts of microworlds and EOs.

Wilensky (1995) reports on a microworld activity wherein a student was attempting to resolve Bertrand's Paradox, where the number of "random" chords longer than an inscribed triangle is entirely dependent on how the random chords are chosen. Wilensky notes that, "It was not until [the student] programmed a simulation of the problem that she began to resolve the paradox" (p. 272). This exemplifies a student *reflecting carefully on a problem*, which was facilitated by the action of modifying, i.e., designing and implementing in the microworld environment. This is also observed

in EO activity. For example, Marshall noted in his journal, when reflecting about his experience of creating and using his *Encryption Pseudorandomness Explorer* EO,

I also decided that my original assumption of a baseline of a uniform distribution was probably a bad one, so I decided to generate a random string of equal length to the test text and run my analysis tool on it as a means for comparison. (p. 4)

It is in the context of accessibility to the digital tool code for possibly modifying parameters, initial conditions, and/or representations of results, that Marshall challenged his initial hypothesis and views on the problem. Furthermore, he commented,

I remembered that when the first code breakers started their tasks from a mathematical point of view, one of the things they looked for was the distribution of letters of an encrypted text [...] I decided that this might be a possible measure as to how random a message of encoded text could appear, and one that I saw how to measure easily. (p. 1)

| Student Development Model Components | Potentially Learned Skills/Competencies                                 |
|--------------------------------------|---|
| 1) States Conjecture                 | To self-motivate to learn/do mathematics                                |
|                                      | To engage in divergent thinking   |
| 2) Researches                        | To research mathematical topics   |
| 3) Mathematizes                      | To develop mathematical intuition                                       |
|                                      | To understand mathematical models                                       |
| 2-4) Designing Cycle                 | To closely reflect on problems  |
| 4-5) Programming Cycle               | To program mathematics (simulations, mathematical experimentation etc.) |
|                                      | To get a feel for inappropriate answers                                 |
|                                      | To work with abstraction  |
| 6) Investigates with Object          | To visualize mathematics  |
|                                      | To connect different representations of concepts                        |
|                                      | To interpret mathematical results                                       |
| 8) Communicates Results              | To communicate one's mathematical results                               |
| Overall                              | To engage in the process of mathematics research                        |
|                                      | To learn/do mathematics independently                                   |

**Table 1 - List of competencies related to Development Process Model (Figure 1).**

A related skill is *working with abstraction*. In Centola et al. (2000) a student worked with the researchers to develop a cooperative (behaviour) evolutionary model that “embodied complex interactions between individuals, groups and the environment” (p. 171), and modified the microworld simulations accordingly. Programming these simulations thus may enable students to construct abstract computer-based representations of real world phenomenon. Similarly, first-year student Phipps created an EO, *The Structure of the Hailstone Sequences* (MICA URL, n.d.), to explore these sequences by reversing their development. His exploratory work to identify patterns involved working with abstraction as can be seen through his communication of results implemented in his EO (see patterns A, B, C, and comments shown from the FACTS button).

Exploring with microworlds could help students *link different representations of concepts*, thus enable them to work with different representations. Blikstein and Wilensky (2009) report on their research study in which undergraduate students learn about Grain Growth, a major subject of Materials Science using simulations of a macroscopic system through simple, intuitive rules at the micro level rather than in a traditional equation lecture format. These simulations are provided to students as microworlds. The authors noted that,

Some of the students wanted to slow down the simulation [as a microworld] and use the “zoom” tool to see the process in more detail. But in doing that, students could only see the micro-level phenomenon... By zooming out again, they could observe the emergent behavior: curved surfaces disappearing as the Laplace-Young equation would predict ... not only are students observing an expected outcome, but they are able to see the process unfolding at various levels. (p. 103)

In the first-year MICA course, students are introduced to design, program, and use multiple representations by creating and using an EO for the graphical and numerical exploration of the behaviour of the discrete dynamical system based on the logistic function (Buteau & Muller, in press). For their subsequent EOs students are urged and guided to design and use multiple representations.

Finally, Blikstein and Wilensky (2009) also mention about the students *engaging in the process of mathematics (or scientific) research*: “coding their own models was a particularly valuable learning experience ... students had an opportunity to create fluency with the computational representations, by testing and debugging their theories, and reconciling them with previous knowledge—just as a scientist would do.” (p. 115). As a MICA graduate, Marshall points to a similar experience in Muller et al. (2011), when reflecting on his overall experience with EOs: “Conjecturing, designing mathematical experiments, running simulations, gathering data, recognizing patterns and then drawing conclusions are things many modern mathematicians do as part of their research.” (p.72).

## FINAL REMARKS

Building on a literature review we have summarized university students’ competencies when engaged in creating and using an EO (Marshall, 2012). The skills served as a basis to again bring together EOs and microworlds through student individual works. Whereas we used student EO work from our institution to exemplify and discuss a selection of identified skills, for microworlds, we used examples described in the literature. Our study didn’t involve an exhaustive literature review about microworlds in university mathematics, however the number of publications in this area seems to be relatively small. From the small microworld literature corpus we had (identified after revisiting the data), we could find evidence of some of the potentially learned skills listed in Table 1, but not all. Nevertheless since the microworld activity involves all the EO student actions identified by the Model (Figure 1), the same list could possibly be adequate for microworlds too.

Our work highlights a fundamental gap in university mathematics education. Though there is considerable education research about microworlds, some at the post-secondary level, there are few reports of sustained implementation in mathematics classrooms, and more specifically with university mathematics majors. In fact, Healy and Kynigos (2010) argue that, “[t]he ideas behind the microworld culture have not yet been presented in a form readily acceptable not only to school systems, but also to other stakeholders in education” (p. 68).

In contrast, our work with EOs provides an example of systemic integration of student generated computer environments for their learning of mathematics. The MICA program, which was instituted by departmental decision in 2001, is still ongoing. For example, Buteau and Muller (in press) describe the role of faculty in the introductory first-year MICA course by detailing the instrumental integration of programming technology to create and use EOs for mathematical investigations and applications. To understand the scope of the EO implementation in the MICA program, an

undergraduate student creates during his/her studies at least 12 EOs, three of which necessitate an original topic selected by the student. Last year (2011-12), 471 assigned EOs (i.e., topic and exploration questions provided to students in the assignment guidelines) and approximately 98 original EOs (i.e., topics selected by students) were created.

With the similarities we have identified in learning activities for EOs and microworlds we suggest that the sustained implementation of MICA could provide a model for integrating microworlds into the curriculum of university mathematics departments: Is it possible that EOs are an implementable alternative to microworlds in university mathematics education thereby bridging the gap and bringing some (or all) identified theoretical benefits of microworlds into the university mathematics classrooms? And why are there so few publications about sustained implementation of student generated computer environments in mathematics education at the postsecondary level?

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## THE BEGINNING OF THE ADVENTURE WITH PASCALINE AND E-PASCALINE

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*The paper presents the idea of “duo of artefacts”, constituted by the pascaline (i.e., the arithmetical machine Zero+1) and its digital version e-pascaline. This “duo of artefacts” is proposed here to support student’s learning about the position notation in base ten at primary school. It also represents an example in which the development in technology (Cabri Elem environment) allows the relationships between material and virtual manipulatives to be discussed.*

### INTRODUCTION

The use of manipulatives in mathematics education is wide spread, especially at primary school level. There are two main differences among manipulatives that are available as pedagogical supports. A first distinction concerns the roots of these manipulatives: some of them are related, more or less, to the history of mathematics, others are expressly created for educational purposes. For instance, mathematical machines (Maschietto & Bartolini Bussi 2011, [1]) are reconstructions of historical devices that are present in mathematical treatises; they can be considered historical manipulative materials. A second distinction concerns their nature: beside the manipulative materials, virtual manipulative is present. The latter is defined as “an interactive, Web-based, visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge” (Moyer *et al.* 2002, p.373). In this paper, we consider two kinds of manipulative: a material one that is linked to the history of mathematics (the arithmetical machine Zero+1, called pascaline, see Figure 1 on the left) and its digital simulation (called e-pascaline, see Figure 2), constructed in the Cabri Elem environment.

Manipulatives have been especially considered from a student’s perspective. Researchers have been paid attention to the relationships between the use of manipulatives and student’s learning (Durmus & Karakirik 2006). In the last years, they are also considered in teacher education (Hunt *et al.* 2011), with a focus on the analysis and awareness of specific epistemological character of manipulatives (Nührenböcker & Steinbring 2008) for teachers. This position is coherent with the use of manipulatives (for instance, mathematical machines) in terms of tools of semiotic mediation (Bartolini Bussi & Mariotti 2008) and cultural analysis of content.

Research works usually focus on one kind of manipulatives, i.e. material or virtual ones, even if virtual manipulatives are proposed as simulations of (historical) materials. Several research works consider the design and analysis of use of digital tools in mathematics learning (Schnotz & Lowe 2003). But there are very few studies in mathematics education that analyse their articulation, although that topic is present in educational debate (Bartolini Bussi & Borba 2010; Maschietto & Bartolini Bussi 2011). In this paper, we intend to contribute to that kind of research.

In general, the aim of our work is to study why and how the use of technology is an adding value to the use of other kinds of teaching tools, in particular materials. Our idea is to provide the teachers of a *duo of artefacts*, that is a physical pedagogical material and its digital counterpart. Our hypothesis is that such a *duo of artefacts* will enlarge the learning experience of the students. For this, we aim to study the articulation between the two kinds of manipulative in teaching and learning mathematics at primary school level (Maschietto & Soury-Lavergne submitted).

This paper intends to contribute to the conference themes of using technology to support students' learning and of developments in technology for learning and teaching mathematics by the discussion of the idea of the *duo of artefacts* (pascaline, e-pascaline).

## DUO OF ARTEFACTS

The starting point in the construction of the *duo of artefacts* was the research work (Maschietto 2011) on the use of the arithmetical machine Zero+1 (called pascaline by students and teachers that have used it; see Figure 1, on the left), a cultural material manipulative, within the mathematics laboratory methodology (Maschietto & Trouche 2010). The first step in the conception of the duo is the design of the virtual manipulative. The second step is the conception of tasks and scenario for the *duo of artefacts*. They are discussed in this section.

### The arithmetical machine “pascaline”

The arithmetical machine Zero+1 [2] is inspired by the mechanical calculators, like the *Pascaline* (Figure 1, on the right) designed by Blaise Pascal (1623-1662) in 1642.

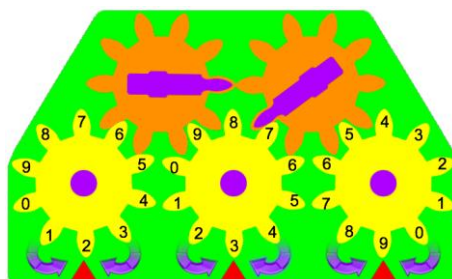


**Figure 1. The arithmetical machine Zero+1 (on the left); Pascaline by B. Pascal (on the right)**

Zero+1 is a small plastic tool (27 cm x 16 cm) and is composed of a base with a gear train of five wheels (each of them has ten teeth). This machine allows numbers to be written in decimal position notation. It works as a counter, because of the number to be carried in an automatic way. The wheels can be turned clockwise (with relation to make addition) and anticlockwise (with relation to make subtraction). For writing numbers and making operations two procedures are identified: iteration by the operator '+1' on units wheel, decomposition of numbers in hundreds, tens and units (Maschietto 2011).

### Design of the e-pascaline

The e-pascaline (Figure 2) has been created within the authoring Cabri Elem environment, developed by the Cabrilog society. This authoring environment enables to design activity books, which consist of a succession of pages incorporating some representations of objects and sequence of tasks (for more details about the design of books with the Cabri Elem technology see (Mackrell *et al.* forthcoming) or (Laborde & Laborde 2011)).



**Figure 2. The e-pascaline**

The e-pascaline has been created as a complex object very similar to the physical one. The shapes, the main constitutive elements and the colours are respected, but the e-pascaline is not a simple virtual reconstruction of the *Pascaline* (contrary to Bascoul's work [3]) or the Zero+1. The implementation of the e-pascaline has required additional design decisions, which are related to the authoring environment and to didactical choices (Maschietto & Soury-Lavergne submitted).

The design of the e-pascaline is mainly grounded on two theoretical frameworks, the instrumental approach (Rabardel & Bourmaud 2003) and the theory of semiotic mediation (Bartolini Bussi & Mariotti 2008). These frameworks are involved in our research methodology at two levels (Maschietto & Soury-Lavergne submitted): [1] they have been used to plan and analyse teaching experiments with the physical pascaline (teaching experiments with the pascaline have been carried out in Italian and French classes at primary school); [2] they support the choices for the design of the e-pascaline in terms of continuity and discontinuity between the material and digital artefacts. In particular, our choices in terms of continuity/discontinuity are based on the analysis of utilisation schemes interlinked with the analysis of the semiotic potentials of the artefacts. This kind of analysis allows characterising several components of the artefacts and their behaviours as a consequence of student's actions. For instance, the rotation of a tooth at a time (a discrete movement of wheels) is a fundamental feature of the pascaline that is kept in the e-pascaline. But each e-pascaline wheel turns by the use of two buttons that launch its animation (curved arrows at the bottom, see Figure 2 under each yellow wheel) in both directions. In such a way, wheel gestures of grasping a tooth, the upper arrow on the upper wheel or the whole wheel are not possible any more. They were eliminated because they contribute to the instrumental genesis by the students but they did not have a semiotic potential that contributes to didactical aims. This choice enacts a discontinuity in the instrumentation from the pascaline to the e-pascaline. On the other hand, the automatic turning of the wheels (for the number to be carried when making arithmetical operations) is preserved as it is a fundamental feature of the pascaline.

### **Interactive books with e-pascaline**

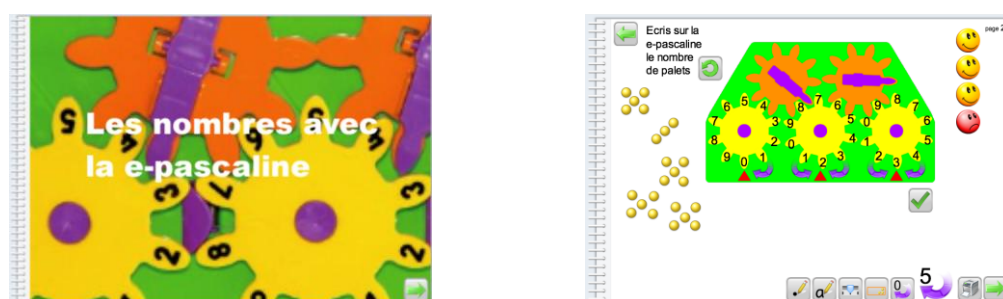
Many different interactive e-books can include the e-pascaline as the principal component (Figure 3). Each e-book can be composed of several pages with the possibility to change several elements and behaviours for each of them and to add new elements. In such a way, e-books can organize and structure tasks for students and feedback, i.e. the reaction of the objects to the action of the user.

Concerning feedback, Mackrell *et al.* (forthcoming) have identified three levels of feedback that can be provided in Cabri Elem e-books: (i) the direct manipulation feedback, that can be linked to the perceptive aspects of the objects behaviour, (ii) the evaluation feedback, (iii) the strategy feedback that results from an analysis of student's strategy. The third level of feedback is strictly related to another feature of an e-book. With respect to the educational aim and tasks, important elements in the pages represent didactical variables, that are parameters whose values are chosen in order to foster or to prevent students' resolution strategies. This kind of control represents an important point in the discussion about the articulation of material and virtual manipulatives. Indeed, a shared argument is that virtual manipulatives allow eliminating some constraints that manipulative materials impose (Durmus & Karakirik 2006, p.121). Here, in the e-books it is possible to constraint the use of e-pascaline with respect to the educational aim. For instance, if we want to make operations with the minimum number of clicks, to promote the evolution of procedures from iteration to decomposition, we can hide the arrows moving the wheels if the students has turned too many times. This example of strategy feedback is also an example of added value for the use of the e-pascaline with respect to the pascaline. Moreover, in the pages of an e-book other components that are not the e-pascaline can be included: for instance, the "tools box" (see Figure 3 on the right, at the bottom), with tools to write texts and numbers on the page, to mark points and to reset the e-

pascaline. Also this choice is based on the analysis of teaching experiments carried out with the pascaline revealing that the pascaline is rarely used alone, but it belongs to a system of instruments for students.

## Discussion

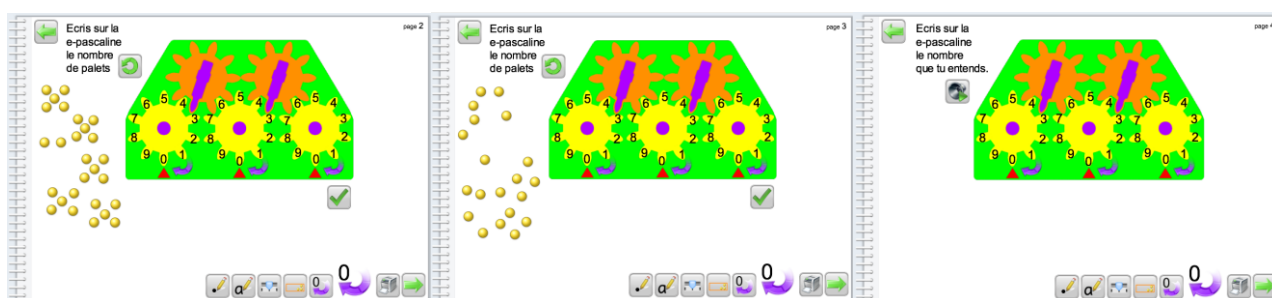
The e-pascaline is developed in the “Mallette” project, a French project supported by the Ministry of Education, directed by the IFÉ [4], in collaboration with the Laboratory of Mathematical Machines from the University of Modena e Reggio Emilia (collaboration between the two authors of the present contribution). It aims to improve the teaching and learning of mathematics at primary school by providing teachers with pedagogical devices that foster manipulation. Within the “Mallette” project, teaching experiments with the duo of artefacts is ongoing at grades 1 and 2 in France. In the first phase, the pascaline was introduced in the class with some tasks of writing numbers. The two main procedures - iteration and decomposition - have appeared, but in both cases, the students’ control has operated on the final display and not on the process to get it, which process relies on the number of “click” or the number of “tooth”. In the second phase, the e-pascaline has been introduced. For these teaching experiments, several e-books have been designed.



**Figure 3.** The cover of the e-book (on the left); the e-pascaline used by a student (on the right).

A first e-book is about the decimal system for writing number (Figure 3, on the left). The kind of task proposed to the students is the same from one page to another: to write a number with the e-pascaline. The difference consists in the way the target number is presented, by a collection of counters or by an oral message. The spatial organisation of the counters and the size of the numbers are didactical variables of the pages (Figure 4). Their choices will entail different strategies for students.

On each page, there are almost the same objects (see Figure 3, on the right): the e-pascaline, a reload button (to get a new number to be written with the e-pascaline), a set of tools (including a click counter), an evaluation button (to get a feedback from the system about the correctness of the number written on the e-pascaline) and arrows to move to the previous or next page.



**Figure 4.** Page 2; Page 3; Page 5

In the first pages of the e-book (Figure 4, on the left and middle), counters up to 30 are randomly determined and displayed. Students have to determine the number of counters in the collection and write it in the e-pascaline. At least two strategies are possible: 1) first the student counts the counters, then it uses the iteration or decomposition procedure to write the number on the e-pascaline; 2) the student matches each counter with a click on the right arrow of the units wheel, and he stops when the collection has been enumerated. At the end, the student can check his solution: a smiley appears when the evaluation button is pushed (it is an example of evaluation feedback, Figure 3 on the right).

In the last pages of the e-book, the target number is given orally by a message that can be repeated as many times as needed (Figure 4, on the right). Since those pages contains the crucial task of numerical transcoding from oral to written registers, the e-book takes into account the irregularities of French words for number (the 100 first numbers are irregular from 0 to 16 and from 70 to 99).

The aim of the first page is also the appropriation of the elements of the e-book by the students, especially the reload button and the evaluation button. It allows initiating the instrumental genesis of the main elements of the pages. On the other hand, in our idea of the *duo of artefact*, the appropriation of e-pascaline can be supported by the use of the pascaline. This element represents an essential point of continuity. For instance, iteration and decomposition strategies are discussed in the phase of work with the pascaline. Moreover, during experimentations, we have observed some students using the pascaline while solving task on the e-book with the e-pascaline.

The evolution of the constraints from one page to another should lead the student to adapt and/or change his/her strategy. A didactical aim is that students are able first to understand and explain the mathematical meanings behind their utilisation schemes, then to formulate them in a mathematical text. This e-book provides an example to discuss the added value of the digital artefact with respect to the physical one. Added value results from the differences and common points between the two artefacts. These differences and common points enable to control the instrumental genesis, to impede some gesture and to reinforce some others. But the added value does not concern just the functioning of the e-pascaline. It also results from the way the task is proposed, the range of numbers, the possibility for the student to repeat the task and to have a personal feedback. Indeed, e-books are very engaging for the students, especially because of the evaluation feedback, as far as we have observed from the recent experiments. All those elements are important for students' learning.

With the ongoing teaching experiments, we will analyse the *duo of artefact* from the point of view of the students' learning. But it is not the only element. The introduction of the duo of artefacts also concerns the teacher that uses it for teaching. In this sense, it will be analysed from the point of view of the teachers' appropriation. So, we can have elements for studying why and how the use of technology is an adding value to the use of other kinds of teaching tools. And we can participate to the educational debate on virtual and material manipulatives.

## NOTES

1. Laboratorio delle Macchine Matematiche <http://www.mmlab.unimore.it>; Associazione Macchine Matematiche [www.macchinematematiche.org/](http://www.macchinematematiche.org/)
2. It is produced and sold by the Italian company "Quercetti" (<http://www.quercetti.it>).
3. A virtual reconstruction of the Pascal's *Pascaline*: <http://christophe.bascoul.free.fr/spip.php?article20>
4. <http://educmath.ens-lyon.fr/Educmath/recherche/equipements-associees/mallette12-13/>

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## **LACK OF SENSE OF PURPOSE IN THE USE OF TECHNOLOGY FOR MATHEMATICAL TEACHING**

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*The use of technology as a support for teaching-learning processes has grown exponentially within educational systems worldwide, and Mexico has not been the exception. We report on how teachers, from several schools within three important high-school level educational systems in this country, perceive, understand and use digital technologies (DT) as a support in their practice, for what aims and in which ways. It has been proposed that adequate use of DT can achieve meaningful learning in students; however, in our research, we found that teachers, as well as institutions, lack clarity and knowledge on how to integrate digital tools to improve mathematical learning in students.*

### **INTEGRATION OF DIGITAL TECHNOLOGIES IN CLASSROOMS: INTRODUCTION AND THEORETICAL FRAMEWORK**

Around the world, in schools of all educational levels, mathematics teachers are now using digital technologies (DT) as an educational support affecting their practice in different ways, as part of the era of knowledge society that has transformed all productive processes of the different activities worldwide (Jara, 2008). Many international organizations have addressed –such as during the United Nations-sponsored “World Summits on Information Society” held in Geneva (2003) and Tunisia (2005)– the importance of this transformation, and been concerned with appropriate implementation of the new technologies and the way they enter classrooms which should be, as Jara (2008) points out, instruments to enhance, improve and even transform the teaching-learning processes. In fact, many proposals of integration of DT in schools promote constructivist approaches: As Sánchez (2000) has noted, it is important to relate constructivism with the technology-assisted teaching of mathematics, so that it becomes a supporting tool for developing superior cognitive skills in students and for integrating previous knowledge with the new, in a way that makes learning visible and technology invisible. As we present further below, constructivist paradigms are also included in the curricular reforms in Mexico, and we will present some of the specific aspects related to constructivism within the discussion of research results.

In terms of the use of digital tools in education, many authors (e.g. Coll, 2004) point out that these technologies can enrich learning contents and facilitate their understanding. But we consider that for this to be achieved in a school setting, the role of the teacher is crucial. However, the uptake of technologies by teachers is not straightforward. The British Educational Communications and Technology Agency (BECTA, 2004) reviewed research reports from 26 countries, during a period of 10 years (1993-2003), in which they identified the barriers to ICT uptake by teachers. In its report, BECTA (2004) organizes the barriers in categories, including: the teacher’s lack of confidence, lack of competence, lack of pedagogical use of technology training, lack of resources, lack of time during class, change resistance, among others. As we will partly attempt to show in this paper, in our investigation we have identified the existence of several of these categories.

The questions that we ask ourselves are: How do teachers perceive the curricular proposals regarding technological integration, how do they put them into practice, and how do these perceptions and the knowledge that they put into practice in the classroom contrasts with the proposals as well as with the knowledge that may be necessary to promote effective learning? Regarding the latter, in teachers’ professionalization, several elements have been identified as



relevant in teachers' practice, such as the Pedagogical Content Knowledge (PCK) proposed by Shulman (1987). More recently, Mishra and Koehler (2006) extended the Shulman model, by considering the technological changes in education and the necessary knowledge that teachers should have concerning technologies, thus adding in model the Technological Pedagogical Content Knowledge (TPCK) aspect, which we have found very useful for our research analysis (see below).

## METHODOLOGY

Our investigation has focused on observing high school mathematics teachers' knowledge (that is, in the PCK and TPCK categories) and the existing relation between teachers' beliefs and claims, their practice in the classroom and their use of DT. Our research has been carried out in eight renowned public schools, regulated by three different institutions in the country: (i) the *CCH* and the *Preparatorias*, both regulated by the National Autonomous University of Mexico (UNAM); (ii) the *Bachilleres* and *ICEL*, both regulated by the Ministry of Education (SEP); and (iii) the *CECyT* which depends from the National Polytechnic Institute (IPN). Some results derived from this research were previously presented at ICTMT 10 (see Sacristán, Parada & Miranda, 2011), where some of the factors that directly influence teachers' practice when they use DT, were mentioned.

In the first research stage, the various curricula of the different above-mentioned educational systems were reviewed. Two relevant aspects were observed in all of them: a) constant recommendations to use technologies as an educational support in the classroom; and b) the recommendation that teachers' practice should follow a constructivist paradigm. Since the mid-1980's, both of these aspects have increasingly been stressed, more so in the 1990's reforms; but it is in the 21<sup>st</sup> century, that a higher emphasis was placed on these aspects of teaching practice. This is why our investigation focuses on observing how teachers are using digital technologies in their classrooms and if they have a clear understanding of why and for which purpose to use DT.

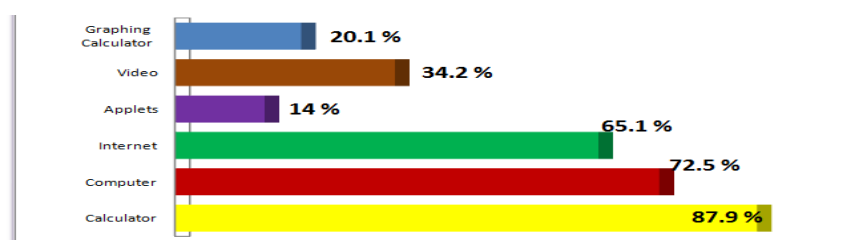
The instruments used include a survey, interviews and classroom observations. The survey was designed to get insights into how much teachers are acquainted and understand their curriculum and the global educational changes that have taken place; as well as of what kinds of DT they are familiar with and use in their practice. Thus, some of the survey questions focused on the internal and governmental modifications of the institutions, considering the educational changes around the world; other questions focused on teachers' current practice and use of DT. If teachers answered affirmatively to knowing the curriculum requirements and to accordingly engaging in the use of DT in their practice, an interview was made to expand on some of the given answers to the survey, and to explore these teachers' beliefs and use of DT as an educational support (see Table 1 below).

A total of 180 teachers, from different schools, belonging to the above-mentioned institutions, and located in different states of the Mexican republic, answered the survey. Two or three teachers from the eight different schools were interviewed, for a total of 20, and some of their mathematics classes were observed and recorded. Unfortunately, despite each claiming to use DT in their practice, only eight of them were observed doing so. The aim of the observations was mainly to get insights into how each of these teachers perceptions of their own practice differed or matched their actual practice, particularly when using DT; that is, get insights into the relation between what a teacher said when interviewed and what he/she does in the classroom. Finally, for the general analysis of the information collected, a methodological triangulation was made of the different results derived from the different research instruments, aimed at obtaining a structure of the inclusive relationships between the qualitative, the quantitative and the documentary (Bryman, 2007).

## SOME RESULTS

The following results were obtained from the survey of the 180 teachers: 92 % considers that there have been many educational changes in the last two decades, with 76% mentioning that the changes

have been primarily related to the use of DT in the classroom. However, only 19% said that these changes have affected teachers' practice, with 21% (38 teachers) explicitly saying that they modified their teaching practice because of the recommendations in the curricular reforms.



**Figure 1. Digital technologies that teachers say they use in their practice**

Figure 1 shows results related to the types of DT that the 180 teachers claim to use in their practice, which points to a very vague use of how DT are used, and a minority claiming to use specific tools for mathematical learning, with a predominance of information and presentation media (Internet, video, etc). In Table 1, we summarize the data from the 20 selected interviewed teachers that were also observed in their classrooms. We expand on each of the interview aspects in the next sections.

| Questions  | Teachers' written replies  | Observations  |
|--|--|---|
| 1. Do you know the constructivist theories that you have to use in your practice?              | <ul style="list-style-type: none"> <li>▪ <i>Affirmative answers:</i> 19 (yes / somewhat / or have heard of them)</li> <li>▪ <i>Negative answers:</i> 1</li> </ul>  | During their classroom practice very little understanding was observed regarding the implementation of constructivist theories.   |
| 2. Does the curriculum state that as part of your practice you should use DT in the classroom? | <ul style="list-style-type: none"> <li>▪ <i>Affirmative answers:</i> 18</li> <li>▪ <i>Negative answers:</i> 2</li> </ul>   | From the curricula review, it was found that all curricula promote the use of DT in classrooms.   |
| 3. What is the goal you pursue when you use DT in the classroom?                               | <ul style="list-style-type: none"> <li>▪ <i>Answers stating wanting to improve learning: Total 2</i> <ul style="list-style-type: none"> <li>▪ 1: to show applications on some mathematical content e.g. Differential Calculus.</li> <li>▪ 1: that students watch parametrical behavior when they create a graphic on the computer.</li> </ul> </li> <li>▪ <i>Other replies: Total 18</i> <ul style="list-style-type: none"> <li>▪ 5: use it to compare or verify, e.g. graphic behavior.</li> <li>▪ 3 use it to create an attractive class for the student (less boring)</li> <li>▪ 8: to optimize time (e.g. Power Point presentations to avoid writing on the blackboard)</li> <li>▪ 2: to be "ad hoc" with modern times, and so the students know DT</li> </ul> </li> </ul> | <p>Neither of the two teachers used DT, during the semester in which they were interviewed.</p> <p>None of them mentioned using DT with an aim of improving learning or achieving meaningful learning in the student.</p> |
| 4. Have you taken courses on the use DT?   | <ul style="list-style-type: none"> <li>▪ <i>Affirmative answers: Total 20</i> <ul style="list-style-type: none"> <li>▪ 8 have taken certification courses related to ICT or mathematics packages.</li> <li>▪ 4 say that they have designed their own material (web pages or interactive software)</li> </ul> </li> </ul>   | We were only able to access the designed material of one teacher.   |
| 5. How many times do you use DT with your students during a semester?                          | <ul style="list-style-type: none"> <li>▪ <i>Replies of more than once: Total 15</i> <ul style="list-style-type: none"> <li>▪ 4 said they use it more than 5 times</li> <li>▪ 11 said they use it between 1 and 3 times</li> </ul> </li> <li>▪ <i>None:</i> 6</li> </ul>  | On the survey they all had previously mentioned using DT with their students.   |
| 6. What kind of DT do you use?   | The most mentioned were: computers, projectors, plotters, mobile phones, web pages, and internet.  | In most of the classroom observations, computer,  |

|  |  |                                   |
|--|--|-----------------------------------|
|  | A few mentioned: video, movies, iPod, retro projector. | projector and plotters were used. |
|--|--|-----------------------------------|

**Table 1: Summary of the replies by the 20 interviewed teachers**

### Teachers' lack of understanding of constructivism

In relation to question 1, which refers to teachers' knowledge of constructivist theories, we contrasted teachers' replies and our classroom observations of their practice, with categories and principles derived from the literature describing constructivist paradigms. We observed a minimal teachers' use of constructivist theories in their practice and teaching planning, and more generally a lack of understanding of the use of these theories in the classroom. Below we discuss some of the replies given by teachers when asked to explain how they used constructivism:

- *They ask their students to consult the Internet.* The teacher who gave this reply considers that with this single activity, students can properly develop their knowledge. But, as Coll (2004) points out, teacher's mediation is often necessary to cognitively confront what the student researched and what it has to learn.
- *They get their students to work in teams.* However, the activity in which students are working on is about algorithmical exercises of repetitions without feedback among the teams or the whole class. Vygotsky (1978) mentions that learning is product of social interaction where higher psychological processes are acquired in a social context and then internalized.
- *They give some context to problems, considering that this can bring students closer to real life applications.* One teacher gave as example the following problem:

"A scientist is researching bacteria; bacteria have a distance between one and another, with coordinates  $(-1, 4)$  y  $(-3, 5)$ ; find the equation and make the graphic"

The teachers who feel that using context is being constructivist, probably feel that context is a way of engaging students, but this shows a lack of understanding of those theories. Moreover, the problem above is completely artificial; teachers need to consider what type of context might be motivating to students. Coll and Sole (1996) mention that constructivism in school learning must include explorations of complex situations designed by the facilitator (teacher).

- *They ask students "what's next" when solving algorithmical problems on the blackboard.* Most teachers assume that with this type of questioning they are involving students and thus helping them to "construct" knowledge. Piaget (1973) suggests that for effective learning, the learner has to actively manipulate the information, thinking and acting reflectively on it. Providing steps in an algorithmical procedure, possibly out of memorization, that is collectively resolved, does not necessarily imply any deliberation.

### Teachers' knowledge of the curriculum

We observed that many teachers are not familiar with the school curriculum; two of them didn't even know that the use of DT was recommended in the curriculum.

### Teachers' goals when using DT in the classroom.

It was observed that even teachers who do claim to use DT in their practice, do not seem to have a clear purpose on why use them. As mentioned before, all 20 teachers had replied on the survey that they used DT in their practice and 18 of them confirmed so during the interviews; however, we were only able to observe 8 of them doing so in the classrooms. Some of the justifications given by teachers have already been reported in Sacristán et al. (2011), including the lack of equipment and facilities in schools; only incorporating DT as part of homework assignments (e.g. asking for

Internet searches; asking to redo a graph using the computer; or simply just sending homework through email). There was the case of one institution that implemented a weekly session called “virtual hour” so that mathematics teachers could ask students to use technologies on their own:

it has been a year since a virtual hour was assigned. Virtual hour is designed so that the student is outside the classroom, using Internet or some electronic information media ... but it has not worked very well. ... I feel it was an administrative thing; the idea was that [the students] went somewhere else to investigate mathematics, or that we designed a class through the Internet and I could be on a server interacting with them.

The teacher who expressed this, didn't take advantage of this opportunity. We can see that even when the tool exists and the institution aims to comply with what is established in the institutional guidelines and curriculum, teachers may not have a clear idea of how to integrate DT in his practice with his students. Moreover, in this case, this teacher was the one that mentioned during the interview that he had as goal using DT to show applications of Differential Calculus; but we were unable to obtain any evidence of his use of DT in his practice.

Out of the 8 teachers that we were able to observe using DT, five took a projector and their own laptop to the classroom. One of these teachers had to place the projector on a stack of chairs, (Fig. 2); but this was simply used so that every student could give a final presentation on PowerPoint. By her own admission, this was the only class in which this teacher used DT in the term; her stated goal was to use DT to do presentations and optimize the time it takes for elaborating graphics, lacking a true didactical sense of DT use.



**Figure 2. Students projecting PowerPoint presentations**

One teacher showed more familiarity with DT in the classroom. The activity he developed was for students to observe and analyze the behavior of quadratic functions in application problems, showing well designed problems in which students had to understand for which values of the function domain, a solution makes sense or not.

The teacher seemed to have a clear understanding of the mathematical content and incorporated it to a computer-designed activity. However, he simply presented the activity himself in front of the class, instead of engaging the students could have used to construct and experiment with their own graphics (which he could have done since his school has a computer center). Thus, even though this teacher seems to have good Content Knowledge, he may still need to develop more the Pedagogical Knowledge of the digital technologies (a structuring part of the TPCK).

Three teachers took their class to school's computer laboratory. In a first case, a teacher filled a table with data from a statistical application problem, and taught her students to enter formulas in Excel and create graphs related to the data. Another teacher also used Excel to graph the sine and cosine functions, though the activity used was a paper-and-pencil traditional activity. The third teacher took her students to plot graphs of irrational functions in order to identify parametrical behavior and to define the domain and co-domain of each of them; it was mainly an algorithmical activity whose purpose was to optimizing time rather than promoting meaningful learning.

## **FINAL REMARKS**

The results of our research point to a use of digital technologies in classrooms that is scarce, and rarely do teachers have a clear understanding of how to take advantage of DT for promoting mathematical learning, with a predominance of DT being used for presentations or time optimization. Although teachers know that the educational needs are constantly changing and some are willing to participate in this process, aspects such as lack of resources, training in the use of

technology and pedagogy, and others (such as those mentioned in BECTA, 2004) are affecting the actual use of DT by teachers, who in most cases still need to develop TPCK. The three educational institutions that regulate the various curricula and their incorporated schools, are autonomous; nonetheless, many similarities were found concerning the way in which teachers are implementing curricular modifications; the format and design of their practice; and the challenges in integrating DT for mathematical teaching and learning, whether at all (due to lack of resources), or when resources are available, in a meaningful way with a clear sense of purpose. Moreover, as pointed out by Díaz (2010), curricular proposals in Mexico lack clear descriptions for developing classroom work. Delval (2012) stresses the need to pay attention to the way in which teachers implement curricular changes, because some factors such as beliefs and social and professional contexts, can determine the absence uptake of new proposals in their practice. Teachers have great responsibility in reflecting on the knowledge to be taught and used in their practice; however, institutions also need to promote the necessary conditions, such as providing elements for quality teachers' professionalization, so that teachers can develop TPCK and succeed in meaningfully integrating DT to their practice.

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## **A WEB-BASED LEARNING SYSTEM FOR CONGRUENCY-BASED PROOFS IN GEOMETRY IN LOWER SECONDARY SCHOOL**

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*International research confirms that many secondary school students can find it difficult to understand and construct mathematical proofs. In this paper we report on a research project in which we are developing a web-based learning support platform (available in Japanese, English and Chinese) for students who are just starting to tackle congruency-based in geometry in lower secondary school. In using the technology students can complete the congruency-based proofs by dragging sides, angles and triangles to on-screen cells and our system automatically translates the figural elements to their symbolic form. Using the notion of ‘conceptions of congruency’ as our framework, we compare the tasks provided in our web-based learning system with similar tasks in a typical textbook from Japan. Our analysis shows that the tasks provided in the web-based platform aim to help learners to develop a correspondence conception of triangle congruency.*

### **INTRODUCTION**

The discussion document for the ICMI study on *Digital Technologies and Mathematics Teaching and Learning* identified a key question for mathematics education research: “how can technology-integrated environments [in mathematics education] be designed so as to capture significant moments of learning?” (IPC, 2005, p. 356). This paper reports on aspects of the design of a web-based learning support platform (available in Japanese, English and Chinese) for students in lower secondary school who are just starting to tackle congruency-based proofs in geometry; see: [http://www.schoolmath.jp/flowchart\\_en/home.html](http://www.schoolmath.jp/flowchart_en/home.html)

When using this learning platform, students can tackle geometric problems by dragging sides, angles and triangles to on-screen cells. As this happens, our system automatically translates the figural elements to their symbolic form. When students complete their proof, the system identifies any errors and provides relevant feedback on-screen. Using the theoretical notion of ‘conceptions of congruency’ as our framework (see below), we set out in this paper to compare the tasks provided in our web-based learning system with similar tasks in a typical textbook from Japan. Our research question is ‘How do the tasks provided in our web-based learning system compare with similar tasks in a typical textbook from Japan?’ For more examples of the technology-based tasks that we have designed within the learning platform, see Miyazaki, Fujita, Murakami, Baba and Jones (2011).

### **WEB-BASED PROOF LEARNING SYSTEM IN GEOMETRY**

Building on the description of our proof learning support system in an earlier paper (Miyazaki, et al, 2011), we focus here in this section on how and why we designed the tasks that are available within our learning platform.

At this stage of our project, we have designed 20 tasks. The mathematical content is based on the Japanese geometry curriculum for 13-14 year-olds (Grade 8 in Japan). As such, our system is aimed at students who are starting to learn deductive proving through the use of properties of basic 2D objects (lines, angles, parallel lines, triangles and quadrilaterals). Our motivation for developing this

system is the need to improve geometry teaching as, from our classroom observations, we are aware that many Grade 8 students can find proofs with congruent triangles difficult (see Fujita et al, 2011).

To make proofs accessible to as many students as possible, we utilise a range of technological capabilities in the design of our system. For example, it is constructed so as to be available via the Internet. By using Flash-based technology (Adobe system), which enables interactive actions on the web, students can complete proofs by dragging sides, angles and triangles to on-screen cells and our system automatically transfers figural to symbolic elements illustrated in Figure 1 (left-hand illustration). Students also choose appropriate conditions for triangles by using drop-down menus. By this automatic translation, students can concentrate on making a proof without being distracted by how the conventions of how to ‘write’ their proof. In addition, to help learners construct a proof step by step, answers within the system are data-based so that if a learner constructs an incorrect proof, then the system gives relevant feedback by indicating where the proof needs to be corrected. This latter capability of the system is illustrated by the right-hand part of Figure 1. Decisions for giving what feedback would be provided are based on our theoretical ideas for learners’ structural understanding of proof (Miyazaki & Fujita, 2010).

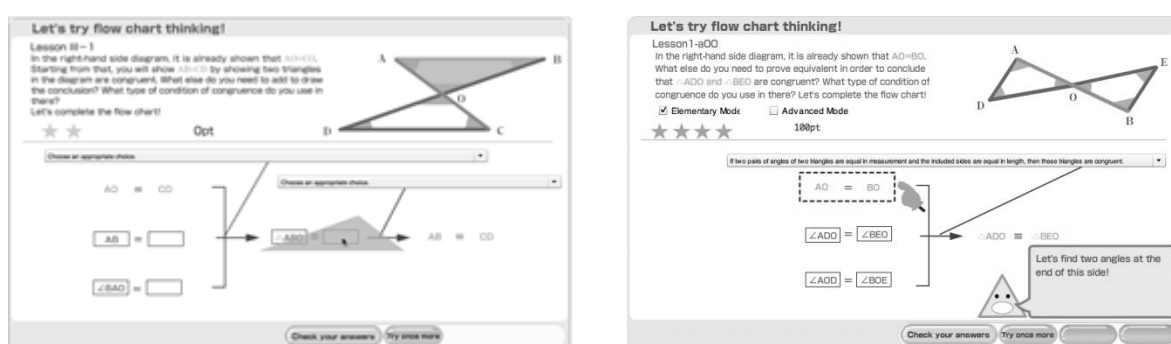


Figure 1: proof tasks within the web-based learning support platform

Overall, our interest is to investigate how and why our system can be an effective tool to promote students’ proof learning experience. So far, evidence from our pilot studies (e.g. Miyazaki, et al, 2011; Fujita et al, 2011) suggests that learners’ proving processes can be enriched when learners used our proof system. In this paper, we explore further the features of our system by characterising the tasks it contains.

## ANALYTIC FRAMEWORK AND METHOD

In the analysis we present in this paper we follow the approach of González and Herbst (2009, p. 154) in taking a ‘conception’ as being “the interaction between the cognizant subject and the milieu – those features of the environment that relate to the knowledge at stake”. In this approach, a conception comprises the following quadruplet (P, R, L,  $\Sigma$ ): P: a set of problems or tasks in which the conception is operational; R: a set of operations that the agent could use to solve problems in that set; L: a representation system within which those problems are posed and their solution expressed;  $\Sigma$ : a control structure (for example, a set of statements accepted as true). In their paper, González and Herbst (2009, pp. 155-156) propose the following four conceptions of congruency:

- The perceptual conception of congruency (PERC) “relies on visual perception to control the correctness of a solution to the problem of determining if two objects (or more) are congruent”.
- The measure-preserving conception of congruency (MeaP) “describes the sphere of practice in which a student establishes that two objects (e.g. segments or angles) are congruent by

way of checking that they have the same measure (as attested by a measurement instrument)”.

- The correspondence conception of congruency (CORR) is such that “two objects (segments or angles) are congruent if they are corresponding parts in two triangles that are known to be congruent”.
- The transformation conception of congruency (TRANS) “establishes that two objects are congruent if there is a geometric transformation, mapping one to the other, which preserves metric invariants”.

By using the above ideas as our analytic framework, we have analysed tasks which can be found in a commonly-used Grade 8 textbook in Japan; see Jones and Fujita (2013). What we found, in brief, is that the Japanese textbook contained a lesson progression from PERC or MeaP to CORR. Nevertheless, National Survey data from Japan has indicated that Japanese Grade 8 students struggle to solve geometrical problems. For example, a recent national survey in Japan reported that the proportion of Grade 9 students who could identify the pair of equal angles known to be equal by the SAS condition in a given proof was 48.8% (National Institute for Educational Policy Research, 2010). This indicates that many students in Japan have not fully developed their CORR conception of congruency despite studying congruent triangles and related proofs during Grade 8.

With our proof learning system, learners can select and drag the sides and angles of various shapes, and also select from a choice of congruency conditions. From each set of actions, feedback is provided from the system. This is likely to influence learners’ subsequent actions. Thus the system offers opportunities for students to learn proofs in a way that is different from traditional textbook-based learning. As such, we are interested in how the tasks in our system can be characterised in terms of the conception of congruency, and whether we might be able to identify similarities and differences between tasks in the textbook and our system.

From our analysis of a commonly-used Japanese Grade 8 textbook (Jones and Fujita, 2013), we know that the Japanese textbook includes many tasks which are related to congruent triangles. Some of the tasks entail identifying congruent figures, while others focus on proving properties of geometrical figures using congruency-based arguments. Because our web-based learning support system especially focuses on proof-related task, we chose the tasks shown in Table 1 as our sample for analysis. These tasks are similar to each other at a first glance. Our intention is to see if different intended conceptions might be observed in our system because of the technology that underpins it.

Following the approach of González and Herbst, we undertook an *a priori* analysis of the tasks in following way:

- we used the quadruplet (Problems; Operations; Representation system; Control structure) to characterise the sample tasks selected from the Grade 8 textbook and from our geometry proof system;
- we used the information from our analysis to characterise the approach to triangle congruency utilised in the sampled tasks.

## FINDINGS AND DISCUSSION

Table 2 summarises the result of our analysis of Lesson 10 (textbook) and task II-1 (proof system). In terms of the four conceptions of congruency, both tasks can be characterised as being the correspondence conception of congruency (CORR) as both tasks require learners to identify corresponding parts to deduce congruent triangles. Similar characteristics were identified for other tasks we analysed.



Despite both tasks in Table 2 being characterised as CORR, the table suggests striking differences between the way in which the same intended conception is realised in the tasks in the textbook and in our proof system. In particular, whereas both tasks provide similar problems (P), learners would face quite different learning experience in terms of operation (R), representations (L), and control structure ( $\Sigma$ ) thanks to the technology in our system.

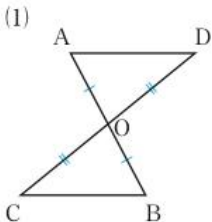
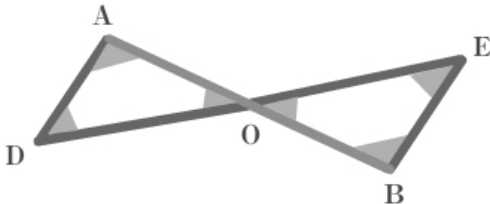
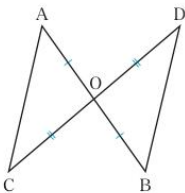
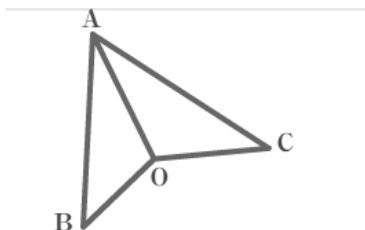
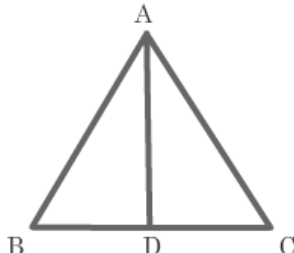
| Tasks from the Grade 8 textbook  | Tasks from the geometry proof system  |
|--|---|
| A task taken from lesson 10  | Lesson II-1   |
| <p>In the diagram below, identify a pair of congruent triangles and name them using the / sign. Also, name the congruence condition used. The sides and angles labelled with the same marks in each diagram may be considered equal.</p> | <p>In the diagram below, prove triangles ADO and BEO are congruent by assuming what is needed (AO=BO assumed)</p>   |
| <p>(I)</p>    |   |
| A task from lesson 13  | Lesson III-2  |
| <p>In the diagram below, if O is the mid-point of line segments AB and CD, then angles <math>\text{OAC}=\text{OBD}</math> (prove this).</p>  | <p>In the diagram below, prove that angles <math>\text{ABO}=\text{ACO}</math> by using triangle congruence and by assuming what is needed.</p>                                      |
|   |   |
| A task from Lesson 17  | Lesson V-1  |
| <p>In triangle ABC, we need to show if <math>\text{AB} = \text{AC}</math>, then angle B (ABD) = angle C (ACD).</p>   | <p>In the diagram below, if <math>\text{AB}=\text{AC}</math> and angle <math>\text{BAD} = \text{angle CAD}</math>, then angles <math>\text{ABD}=\text{ACD}</math> (prove this).</p> |
|   |   |

Table 1: tasks selected from the textbook and from the web-based learning support platform

In the textbook task, learners have to correspond figural elements to symbolic ones by themselves, but this can be quite hard for many learners who are just developing their CORR conceptions. The system supports this process by dragging and dropping figural elements to cells connected with the equal sign (=) or congruent sign ( $\cong$ ), and as a result learners can concentrate on formulating logical relationships in their proof. Also, the system does not have any measurement or superposition tools and these restrictions might help make learners aware that it is possible to study geometry theoretically as well as practically.

To complete a proof, learners have to exercise which condition should be applied, but our classroom observation suggest that often learners who have just learnt the conditions cannot use them effectively. The system supports learners as the known facts to be used are shown in the tabs. The system also gives various forms of feedback in accordance with learners' actions. For example, a learner might make mistakes when choosing an appropriate condition of the condition. Without our system, a learner might not know whether their proof is correct or not until the proof is shared with their peers or until a teacher points out their mistake; with our system the learner is supported within the system and this should help to activate their conceptual control structure.

|          | Tasks from G8 textbook  | Tasks from the proof system  |
|----------|---|--|
| P        | <p>10Pa: To identify two congruent triangles.</p> <p>10Pb: To identify the conditions of congruent triangles.</p> <p>10Pc: To use symbols correctly.</p>  | <p>II-1 Pa: To prove triangles ADO and BEO are congruent.</p>  |
| R        | <p>10Ra: To find pairs of congruent sides and angles.</p> <p>10Rb: To identify equal sides/angles including not symbolised ones.</p> <p>10Rc: To apply the conditions of congruent triangles.</p> <p>10Rd: To apply already known facts</p>   | <p>II-1 Ra: To identify what assumptions and conclusions are.</p> <p>II-1 Rb: To drag and drop sides, and angles.</p> <p>II-1 Rc: To choose statements (conditions of congruent)</p> <p>II-1 Rd: To check answers by clicking a button</p> <p>II-1 Re: To review already completed answers by clicking stars</p>   |
| L        | <p>10La: The diagram is the medium for the presentation of the problem.</p> <p>10Lb The symbols are the registers of equal sides and angles.</p> <p>10Lc: Already known facts such as vertically opposite angles or the conditions of congruent triangles mediate for the solution and reasoning.</p> | <p>II-1 La: The diagram on the computer screen is the medium for the presentation of the problem.</p> <p>II-1 Lb: Dragged sides/angles/triangles are the registers of equal sides/angles/triangles.</p> <p>II-1 Lc: The structure of the proof is visualised by the flow-chart format.</p> <p>II-1 Ld: Tabs are the medium of right statements to be chosen.</p> |
| $\Sigma$ | <p>10<math>\Sigma</math>a: If we can find three components of triangles (SSS, ASA, SAS).</p> <p>10<math>\Sigma</math>b: If one of the conditions of</p>   | <p>II-1 <math>\Sigma</math>a: If one of the conditions of congruent triangles is applied to two triangles.</p> <p>II-1 <math>\Sigma</math>b: If the system gives feedback 'your</p>  |

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|  |                    |
|--|--------------------|
| congruent triangles is applied to two triangles. | proof is correct’. |
|--|--------------------|

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Table 2: analysis of tasks selected from the textbook and from the web-based platform

## CONCLUDING COMMENT

Given the sparse research on the topic of congruency, as a starting point for our research we have analysed various congruency-related tasks in textbooks and in our web-based learning system through an analysis utilising the four congruency conceptions proposed by González and Herbst (2009). In our analysis of tasks in a Japanese textbook (see Jones & Fujita, 2013) we show that the textbook is based on a learning progression from PERC or MeaP to CORR, i.e. from a practical conception of congruency to a correspondence conception. Our analysis in this paper shows that our system can be used during the introductory stage of proof learning because the tasks provided in the web-based platform are similarly designed to help learners to bridge between PERC or MeaP and CORR. One reason for developing our web-based platform is that national survey data from Japan shows this progression might not be as straightforward as we might expect and that it might be necessary to support many more learners to develop CORR in their learning of proofs in geometry.

In addition to aiming to support the development of students’ CORR, with our web-based system we aims to support students’ learning in various other ways, including mediating figural and symbolic elements of geometrical proofs, scaffolding the students’ use of known facts, and supporting their control structure by providing relevant and timely feedback. We argue that such learning experience should be useful as students proceed to more complex and formal learning in geometry and proving, and that is why the learning with our system can be located in the introductory stage of proof learning.

Our next task is to characterise actual students’ conceptions when they interact with various congruent triangle problems. In this way we aim to examine more systematically how our web-based learning system would contribute to supporting the development of students’ correspondence conception of congruency.

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## DYNAMIC ALGEBRA IN EPSILONWRITER: A PEDAGOGICAL PERSPECTIVE

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\* Aristod Company, \*\* University of Siena

*Dynamic Algebra means ‘doing calculations with the mouse’. A Dynamic Algebra System has been implemented in Epsilonwriter software, allowing the user to perform mathematical calculations just by using the mouse. The system is based on the Theory of Movement in Formulas in which gestures have been associated with algebraic transformations. Moreover, it may provide, per each gesture performed, both a mathematical justification of the action and a description of the gesture. The Dynamic Algebra system has the potential for students to support learning and for teacher to write students’ worksheet. In this contribution we are going to analyse the potential of the ‘equivalent drag&drop’ gesture in manipulating an algebraic expression.*

### INTRODUCTION

This contribution aims at describing the potentials of the software Epsilonwriter<sup>1</sup> (Nicaud & Viudez, 2009; Nicaud & Mercat, 2012) in supporting students to manipulate properly an algebraic expression. The development of the software is rooted around the idea of Dynamic Algebra (DA). This refers to making algebraic transformations through directly moving specific parts of an expression by means of a mouse.

The pedagogical potentials of dynamicity in educational software has been broadly analysed as far as concerns Dynamic Geometry Environments, where the computational objects to be manipulated consist in ‘figures’ appearing on the screen (see for instance, Lopez-Real & Leung, 2006). Even if the idea of Dynamic Algebra (DA) has been used for the last twenty years, specific studies on the pedagogical advantages to move representations of algebraic expressions are still lacking.

Analysing some dynamic technological environment, Hegedus and Armella (2010) introduce the notion of ‘co-action’ in order to address the relationship between the user and the environment they interact with.

*‘Users are actively changing the status of the object of inquiry through the examination and executability of the environment’ (p. 26).*

In order to implement a Dynamic Algebra System in the Epsilonwriter software, the Theory of Movements in Formulas (TFM) has been developed so as to define a correspondence between mathematical properties of the mathematical object represented in the environment and gestures which are allowed on it. In this contribution, after a brief description of TFM, the ‘drag&drop’ gesture is analysed in how it works in solving an equation. Then, pedagogical implications for using DA, as it has been implemented in Epsilonwriter, are analysed.

### THE THEORY OF MOVEMENTS IN FORMULAS

The Theory of Movements in Formulas (TFM) (Nicaud, 2013) is based on the possibility to move sub-expressions in formulas respecting the equivalence between the starting formula and the transformed formula (in this paper we consider the words *formula* and *expression* as synonymous). The internal question to which TFM replies is the following:

*When can we move a sub-expression from here to there, getting an equivalent expression?*

As we are talking about algebraic expressions, a sub-expression is not just something which is ‘put there’ in a formula but it is something which is attached to the rest of the formula by means of an operator (+, −, ×, /). This gives to the sub-expression a *status* with regard to the closest sub-expression to which it belongs (its mother sub-expression). For instance, in the expression  $2x + 3 = \frac{a}{5}$ , 2 has a *multiplier* status in  $2x$  (*multiplier* of  $x$ ), 3 has an *adder* status in  $2x + 3$  (*adder* of  $2x$ ), 5 has a *divisor* status in  $\frac{a}{5}$  (*divisor* of  $a$ ).

TFM identifies three different statuses: *adder*, *multiplier* and *divisor* (no status is attached to the subtraction,  $a - b$  is seen as the sum of  $a$  and  $-b$ ), and considers the movements of a sub-expression  $u$  through operators (+ − × / and relations) which preserve equivalence. The internal question is the following:

*Is it possible to move  $u$  from here to there,  $u$  being unchanged or changed to  $-u$ , the rest of the formula being unchanged except possibly the orientation of an inequality?*

For instance, in the equation  $4 = 3x$ , the sub-expression 4 can be considered as an *adder* in the left-hand side (being viewed as  $4 + 0$ ); it can be moved to the right-hand side, keeping *adder* status and being changed to  $-4$ , coming to  $0 = 3x - 4$ . Moreover, it can also be considered as a *multiplier* (being viewed as  $4 \times 1$ ) and be moved to the right-hand side, remaining unchanged and getting the *divisor* status, obtaining  $1 = \frac{3x}{4}$ .

TFM is based on three classes of movements which can be described as follows: (1) *Moving at the same level*, which includes the application of the commutative law and movements in relations;

(2) *Entering* into an expression, e.g., 2 in  $2\frac{x}{3}$  can enter into the numerator of the fraction as a *multiplier*, providing  $\frac{2x}{3}$ ;

(3) *Exiting* an expression, e.g., 2 in  $\frac{2x}{3}$  can exit the numerator as a *multiplier*, providing  $2\frac{x}{3}$ .

Elementary movements can be combined, e.g., 2 in  $2x = \frac{5}{3}$  can move to get  $x = \frac{5}{2 \times 3}$ . This is a combination: (1) 2 exits of  $2x$  as *multiplier*, (2) 2 moves to the right as *divisor* of  $\frac{5}{3}$ ; (3) 2 enters into the denominator of  $\frac{5}{3}$  as *multiplier* of 3.

TMF also considers movements of minus sign taking into account that a minus sign can be viewed as a *multiplier*  $-1$ . For example, a minus sign can exit the numerator of  $\frac{-5}{3}$  getting  $-\frac{5}{3}$  because the multiplier  $-1$  can exit the numerator of  $\frac{(-1)5}{3}$  getting  $(-1)\frac{5}{3}$ .

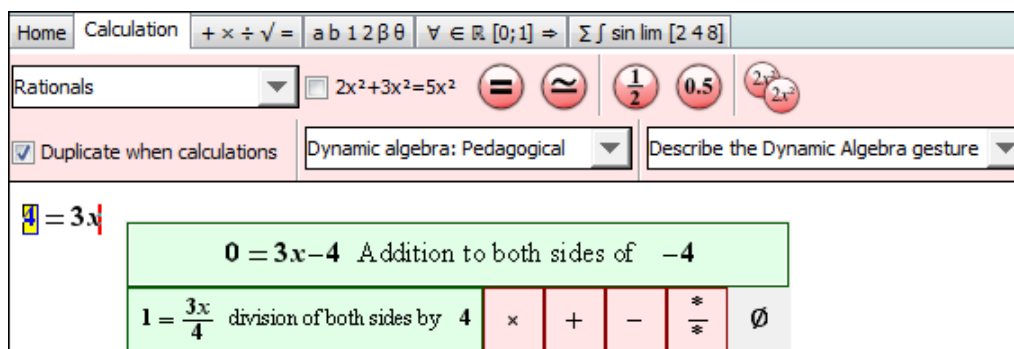
For the fundamental operators + − × / and relations, within the considered status and classes of movements, TMF identifies 74 cases and 56 of them produce correct transformations, which corresponds to 76% (Nicaud, 2013).

This percentage leads us to consider that the notion of movement of sub-expressions in formulas is a meaningful concept of algebraic manipulations.

## DYNAMIC ALGEBRA IN EPSILONWRITER

Dynamic Algebra in Epsilonwriter is a rich system with several modes of use. The *Pedagogical Dynamic Algebra* is one of them, the one which implements TMF. It is limited to movements in a formula (also called *equivalent drag&drop*, as it is a drag&drop which preserves the equivalence). The gesture can be carried out as follows: the user selects a sub-expression then drags this sub-

expression. During the dragging, drop proposals are shown in a pop-up menu. The user can choose to drop here choosing a proposal or to move to another place (Fig.1). Epsilonwriter has a parameter for choosing what will be displayed. It can be just the result, a mathematical explanation on the line of the result (Fig. 2), a table with a mathematical explanation and a description of the gesture (Fig. 3). The *Pedagogical Dynamic Algebra* also informs the user of incorrect moves (Fig. 4).



**Figure 1.** The manipulated expression is  $4 = 3x$  on the top left of the page. The sub-expression 4 is selected then dragged to the right. When the mouse is after  $x$ , the cursor is displayed after  $x$  and a menu proposes two equivalent drag&drops of the TMF, plus several structural drag&drops. If the user releases the mouse, the menu is still displayed so that the user can choose which drop to perform.

|                    |                        |
|--------------------|------------------------|
| $4 = 3x$           |                        |
| $1 = \frac{3x}{4}$ | division of both sides |
|                    | by 4                   |

**Figure 2.** Result of the drop shown in Fig 1 when “division of both sides” is chosen with the parameter *Explanation on the line of the result*

|  |   |
|--|---|
| $4 = 3x \rightsquigarrow 1 = \frac{3x}{4}$ | <i>Gesture: Multiplicative move to the other side, the factor becomes divisor</i> |
|  | <i>Explanation: division of both sides by 4</i>                                   |

**Figure 3.** Result of the same gesture and choice with the parameter *Describe the Dynamic Algebra gesture*. The user gets the formal transformation with the moved expression in red (the arrow indicates equivalence), the description of the movement with gesture words, and an explanation with mathematical words.

|            |                                    |   |   |   |
|------------|------------------------------------|---|---|---|
| $3x+4 = 5$ | No basic exit of a factor to a sum |   |   |   |
| x          | +                                  | - | * | / |

**Figure 4.** An incorrect attempt of movement in the Theory of Movements in Formulas. To go to the right of 5, the sub-expression 3 has to exit from  $3x$  which is possible with the *multiplier* status. Then it has to exit from  $3x + 4$  which is not possible with the *multiplier* status and which is written in the popup. The word *factor* has been used instead of *multiplier*; the ‘basic’ adjective refers to TMF.

The other modes of Dynamic Algebra in Epsilonwriter allow more complex calculations<sup>2</sup>. For example, 3 can exit from  $3x + 4$ , providing  $3\left(x + \frac{4}{3}\right)$ ; this is not a TMF movement as the rest of the expression is not unchanged. 3 as adder of the numerator in  $\frac{3+a}{2}$  can exit the numerator to provide  $\frac{3}{2} + \frac{a}{2}$ . There are also external drag&drops, like the drop of  $\begin{cases} x = 2a + 1 \\ y = a - 2 \end{cases}$  over  $\frac{x-y}{x+2y}$  which proposes to apply a substitution providing  $\frac{2a+1-(a-2)}{2a+1+2(a-2)}$  see demos.

Solving a typical linear equation like  $x + 2 = 3(2x - 5)$  in the TMF framework can be made by combining user calculations and equivalent drag&drop as follows. First, the user expands the right side, getting  $x + 2 = 6x - 15$ . Then she performs equivalent drag&drop of  $6x$  to the left and of  $2$  to the right getting  $x - 6x = -15 - 2$ . Then she adds like terms herself, getting  $-5x = -17$  then she makes a move of a minus sign to the other sides getting  $5x = 17$  and last a move of  $5$  getting  $x = \frac{17}{5}$ .

Using a more powerful mode, all the solving process of the above equation can be made with Dynamic Algebra: expansion of  $3(2x - 5)$  can be made by entering  $3$  in  $(2x - 5)$  and adding like terms can be performed by dropping one of these terms on the other. With a more powerful mode, a quadratic equation can be solved either by completing the square or by using the quadratic formulas; even cubic equations can be solved using the Cardan formulas (Nicaud & Viudez, 2013).

## PEDAGOGICAL IMPLICATIONS

The potentialities of the TFM as it is implemented in Epsilonwriter can be exploited for different educational goals concerning both the introduction and/or the remedial of algebraic manipulation. The key idea on which is rooted TFM is allowing manipulation of formulas by means of doing *gestures*. Performing a gesture produces a physical change on the object one is acting, as emphasized in a comprehensive perspective by Lakoff and Núñez (2000), which argues that bodily experience is necessary to develop mathematical understanding. Moreover, in the domain of algebraic manipulation, not only is a gesture performed to transform a formula, but what is crucial is that it brings along the mathematical meaning which justifies such a movement.

In fact, as far as algebraic calculation is concerned, many studies (see for instance, Kieran, 2007; Wittmann et al., 2013) highlight that students (in particular students doing correct calculations) use gestures: *'gestures and ambiguous speech of moving are the only algebra used at that moment'* (Wittmann et al., 2013, p.169)

Moreover, the authors analyse more deeply the potentials of gestures arguing they are not simply related to procedural aspects but they also embed a semantic value.

*'Making sense of students' algebraic manipulations and what happens in the complex moments of a mathematical operation enacted by hand is incomplete without considering the action itself as part of cognition'* (p. 173).

Together with the perspective of dynamic representations (Goldin & Kaput, 1996), we can analyse the pedagogical potentials of the *equivalent drag and drop* feature as it is implemented in the mode of use called Pedagogical Dynamic Algebra in Epsilonwriter. Selecting a sub-expression, moving it and finally positioning it at a desire place in the formula gives the possibility to experience directly the meaning of 'transforming a formula'. First of all, the transformation does not occur as if one works in paper and pencil environment, i.e. by writing a new formula, but it is exactly the given formula which can be modified. According to Duval (2006), a 'congruence' can be hypothesized between a formula together with the ways to transform it (seeing it as a mathematical object) and an



inscription on Epsilonwriter together with the available gestures representing such a mathematical object. From an epistemological point of view, the way to conceive a formula can change dramatically thanks to the dynamicity the DA software includes.

Secondly, the DA system implemented in Epsilonwriter may help the user to conceive a formula as a structure and not only as a procedure. This is particularly significant as far the teaching and learning of algebra is concerned, in which many difficulties are interpreted as a lack of structural view of the formulas by students (Kieran, 2006). Booth (1989) argued that

*‘our ability to manipulate mathematical symbols successfully requires that we first understand the structural properties of mathematical operations and relations which distinguish allowable transformations from those that are not’ (p.57).*

The modes to act on an expression in Epsilonwriter prevent from selecting sub-expressions in a formula which cannot be dragged to produce an equivalent formula (see Fig. 3).

Thirdly, the system may provide a particular type of feedback when the user performs a gesture. In fact, as explained in the previous section, each gesture may be accompanied by a description of the movement and an explanation of how such a movement modifies the formula the user is treating. As stressed by Hattie & Timperley (2007) in their research study, feedback are proved to be more effective the more are directed to the process to carry out a task, instead only to the task solution. This is consistent with the duality highlighted by Balzer et. al. (1989) when they classify feedbacks according to two types: the ones which inform only about the correctness task solution or an intermediate step which they called ‘outcome’ feedback and the ones which adds a piece of information about the reason for which something is wrong, which they called ‘cognitive feedback’. The feedback offered by Epsilonwriter (see Fig. 2), can be classified as a cognitive feedback since it adds the mathematical explanation which stand behind a specific gesture.

## CONCLUSIONS

As we tried to underline in the paper, the way to carry out algebraic manipulation can change dramatically when using Epsilonwriter in comparison with using paper and pencil. Obviously it cannot be taken for granted that as a consequence students will learn algebra much easier. The role of teacher, as in every teaching/learning process, remains crucial. In Epsilonwriter, the teacher can decide how to set the parameters of the system, for example if the mathematical explanation is included with a gesture or it is not. These choices will be consistent with the educational goal she/he is pursuing.

The dynamicity offered for carrying out algebraic manipulation, makes it possible to reflect about designing different tasks, compared to which given in paper and pencil. Investigating on these new opportunities could be crucial for teaching/learning algebraic manipulation in a different way, which, emphasizing the movements on which is based, should help students to either not experience or overcome the well-known difficulties usually they met.

This may require a different educational approach from the beginning, that is, from the introduction of algebraic manipulation. The teacher can decide to use Epsilonwriter through an interactive whiteboard during their lessons so as to make students by one side being introduced to new concepts, by the other side familiarizing with the modes of use of the tool. The issue of combining theory and practice seems to be particularly crucial in this case, since, as stated in the paper, the objects represented in Epsilonwriter are not mere formulas, but are formulas with enclosed a movement. As a consequence, acting on them by means of gestures, aims at making emerge a meaning of a formula as a whole between the inscription and the possible movements.

Epsilonwriter is not just a DA application. It is also a rich and flexible editor of text and math, able to import Latex, to paste from Wikipedia (getting editable formulas) to export web pages and to copy to Word (getting editable formulas). It has also communication facilities, sending emails with pretty formulas and allowing synchronous collaborative work. Besides the pedagogical interest, Dynamic Algebra can be seen as a help for writing documents for students and teachers: it is often easier to modify a formula to get an expected one with DA than with usual editing functions. For teachers, the production of solved exercises is very easy; the *Explanation on the line of the result* parameter (Fig. 2) provides a usual representation of explained calculation steps. Furthermore, as all occurs in an edited document, the teacher can modify some explanations and delete unnecessary explanations.

## NOTES

1. <http://www.epsilonwriter.com>
2. <http://www.epsilonwriter.com/en/top10demos>

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## **INTERACTIVE WORKSHEETS FOR CONNECTING SYMBOLIC AND VISUAL REPRESENTATIONS OF 3D VECTOR EQUATIONS**

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*Learning the close relation between symbolic and visual representations is a key to conceptual understanding of 3-dimensional (3D) vector equations. For learning such a relation, it is valuable that students manipulate and transform the graphic objects directly with observing the simultaneous change of related symbolic equations. The interactive change of graphic and symbolic objects provides the students with opportunities to recognize their relations experimentally. This paper describes how such interactivity is designed as electrical worksheets, implemented into our learning-support www-system, and what reflections they received from students and teachers.*

### **INTRODUCTION**

It has been and still is difficult to improve students' conceptual understanding in vector equations of lines or planes in 3D space although the concept is foundation to mathematics, science and engineering. The annual INCT (Institute of National Colleges of Technology, Japan) achievement tests repeatedly show the field as the weakest of our students (INCT, 2012). Some of the students could not even tell if an equation represented a plane or a line in 3D space. Those students tended to memorize a series of formulas without examining their features or understanding the relations of the formulas with the features of graphic objects. In another words, they cannot use visual reasoning and their analytic reasoning is very shallow.

Analytic reasoning is based on the use of symbolic representations and construction of logical inference chains. Instead, visual reasoning is based on visual interpretations of mathematical concepts. Visualization makes it possible to perceive abstract mathematical objects through senses. Visual representations can be considered more concrete than analytic ones, because they are based on external objects. Analytic reasoning is often exact and detailed, but visual reasoning is needed to reveal wider trends of the whole problem solving process and holistic features of the problem situation (Viholainen, 2008). But many students have difficulties in analysing visual representations, and therefore, they cannot utilize them in problem solving (Stylianou & Dubinsky, 1999). In the case of our students, we think that their shallow analytic reasoning and no visual reasoning have been causing their poor performance in 3D linear algebra at the INCT achievement tests.

So we made a lesson reform following the MODEM approach (Haapasalo, 2003), in which he stresses the necessity of the link between conceptual understanding and procedural knowledge, and proposes to include concept-building steps (concept orientation, definition, identification, production, and reinforcement) into learning activities. According to this approach, our lesson plan is directed from concrete applications to abstract mathematical ideas, from handling graphic objects and identifying their characteristics toward defining symbolic expressions and rewriting them (Nishizawa, Yamada, & Yoshioka, 2009). Interactively changing graphics are to provide students with the opportunity to make their own experiments in the concept-building steps by connecting visual and symbolic representations closely together.

The paper explains how we designed the interactive worksheets for learning the relation of planes in 3D space and their vector equations using interactive graphic interface and built-in programming

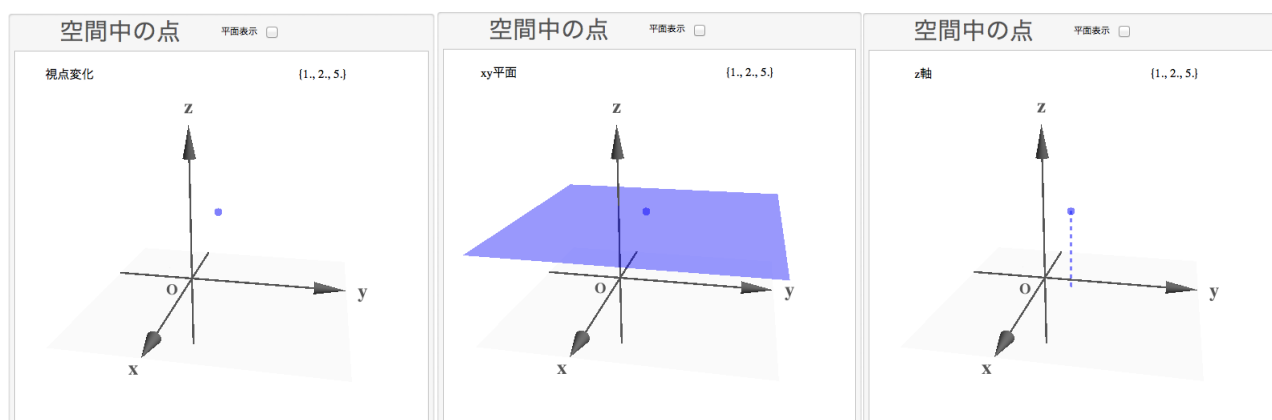
language of a CAS: Mathematica. Attractive features and limitations of those interactive worksheets are to be demonstrated and discussed at the presentation.

## RESEARCH METHOD

Our research question is if the students are given the opportunity to manipulate 3D graphic objects in virtual space by dragging specific points of the graphic objects and observe how the vector equations change their parameters in a lesson, do they recognize more easily the relations of the parameters and the graphic objects through their experiments? To ask this question, we have designed the following interactive worksheets and let our students use the worksheets in a lesson at the computer laboratory on February 2013. We made interviews to some students and gave all the students paper-tests on April and May 2013. The first test was taken two weeks after they started learning the 3D vector equations in traditional lessons. The test scores and their answer sheets were compared to the ones of former students, who had learnt without the worksheets.

## INTERACTIVE WORKSHEETS FOR LEARNING VECTOR EQUATIONS

Mathematica's built-in functions offer us a powerful interface, on which we could directly manipulate a graphic object and observe the simultaneous change of related symbolic expression. Although computer screens are 2D, we can locate the position in a 3D virtual space as the cross-point of a plane or a line in the virtual space and the view-line to the 3D space. Every point on the view-line looks like a single point on the screen and cannot be separated from each other. But it indicates a specific location as a cross-point with another graphic object in the space. Dragging the cross-point on the screen with a mouse is the foundation of handling 3D graphic objects in the virtual 3D space.



**Figure 1: Handling a point in a virtual 3D space**

Figure 1 shows how we move a target point in the 3D virtual space, which is shown as a blue dot on the screen, in our worksheets. The left picture is the default mode, where a student can change the viewpoint to the space by clicking any place other than the target point and dragging the whole picture. When she changes the viewpoint, the graphic objects including the target point and three axes on the screen rotate according to the movement of her mouse. The middle picture shows the *xy-plane mode*, which appears when she clicks once on the target point. In this mode, she can move the target point in the plane, which is parallel to the *xy-plane*, with her dragging motion. When she clicks again on the target point, the *z-axis mode* appears as shown in the right picture. In this mode, she can move the target point vertically, in parallel to the *z-axis*. If she clicks more times on the point, the mode changes between the *xy-plane mode* and *z-axis mode* in turn. She needs to click on the place other than the target point to return to the default mode.

Figure 2 shows the first interactive worksheet, which uses the handling technique shown in Figure 1. In this worksheet, a student is asked to find the straight line which crosses the two given points A and B in the 3D space. She can move the base and tip of the arrow as shown in the left picture. If she rotates the viewpoint, and observes the 3D space from the top (from a positively distant point on the z axis) or from the side (for example, from a positively distant point on the x axis) in turn, she can easily recognize the locations of the points, and adjust the position of the arrow's base to point A. When she adjusts the arrow's base exactly on point A, the point expands the diameter for her to confirm her success as shown in the middle picture. When she moves the arrow's base, she may recognize that the vector equation displayed on the top right position also changes its parameters along with the movement of the arrow, which represents the direction vector of the line. A careful student also recognizes that the direction vector in the equation does not change the parameters and only the position vector to the base change the parameters when she moves the arrow's base. The direction vector only changes its parameters when she moves the tip of the arrow. When she adjusts the line defined by the arrow to cross both points A and B, the displayed vector equation becomes the right answer to this worksheet. Thus she learns the relation of symbolic and visual representations experimentally.

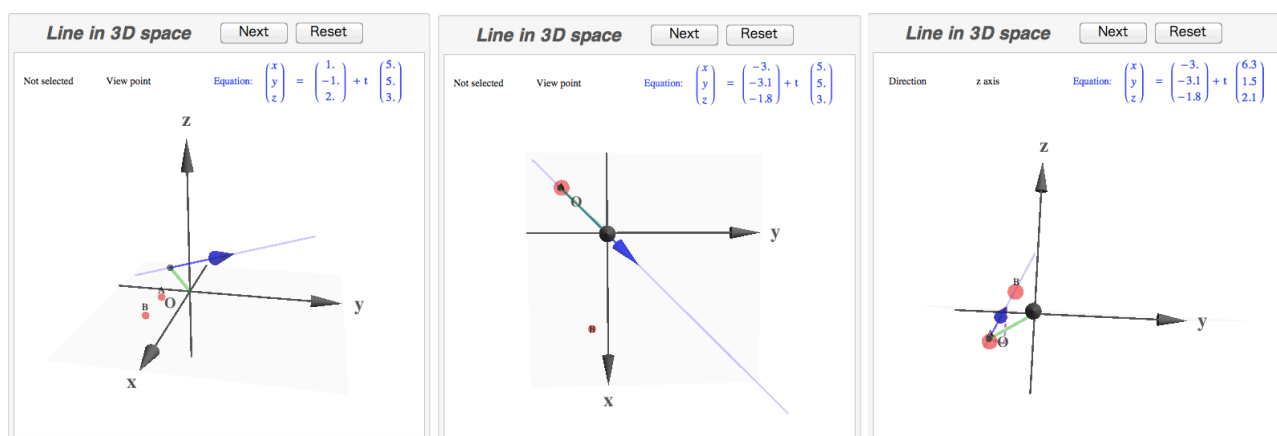


Figure 2: Fixing a straight line in the 3D space

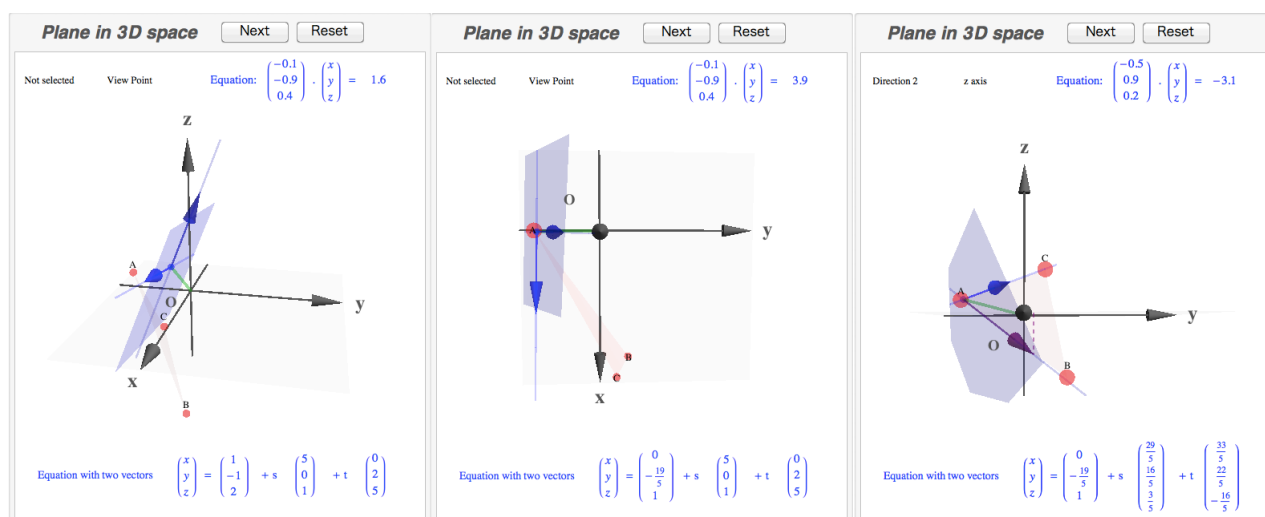
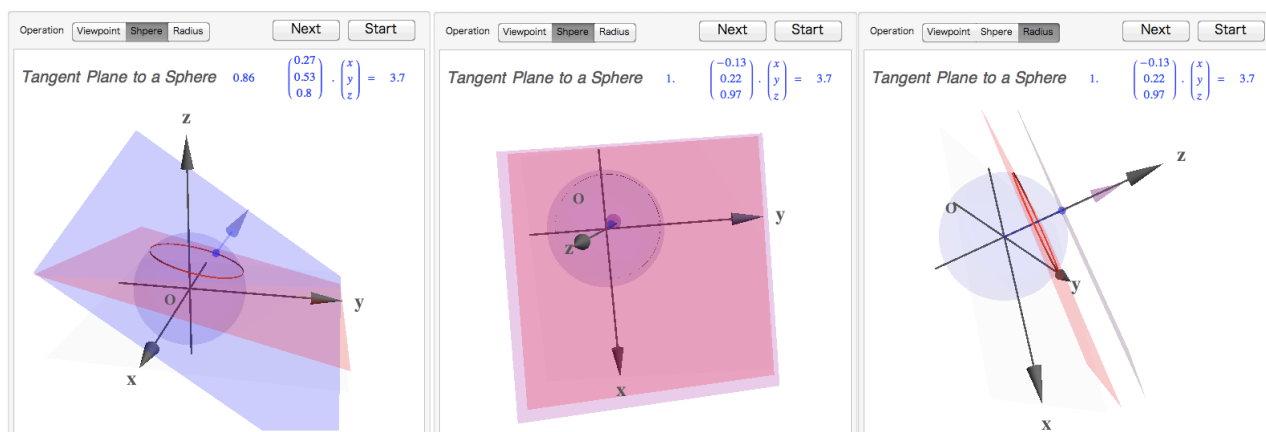


Figure 3: Fixing a plane in the 3D space

In the worksheet shown in Figure 3, a student is asked to find the plane which crosses the given three points A, B, and C in the 3D space by adjusting the base point of two arrows and their tip

points. The two arrows represent the direction vectors of the vector equation displayed at the bottom of the window. The equation is related to the arrows in the 3D space and changes its parameters simultaneously with the graphic objects. As in the worksheet of finding lines, the worksheet has two activities; matching the plane to the target graphically and finding the values of parameters. Although the worksheet also shows another vector equation at the top right corner of the window, which is expressed with the normal vector to the blue plane, the relation between this equation and the graphical representation is not clear in this worksheet.

The plane's vector equation that uses the inner product with the normal vector is easier to relate to the visual representation in the worksheet shown in Figure 4. In this worksheet, a student is asked to overlay the adjustable plane to the target plane by dragging the target point. The adjustable plane is the tangent plane to the sphere centered at the origin, and the target point is the touching point of the sphere and the adjustable plane. In the *sphere mode*, a student can drag the target point on the surface of the sphere, so the adjustable plane changes its direction but not the distance to the origin. If she switches the mode to the *radius mode*, the target point moves on the line starting from the origin to the target point and far beyond. In the *radius mode*, the adjustable plane changes its distance to the origin without change its direction. In this way, the target point controls the adjustable plane's direction and the distance to the origin.



**Figure 4: Finding the vector equation based on an inner product for a plane**

The cross points of the target plane and the sphere make a circle in the virtual space if the target plane cuts the sphere. This circle helps students to adjust the direction of the adjustable plane to the same direction of target plane as shown in the middle picture. Once the directions of two planes are matched, the resulting task is to adjust the distance to the origin as shown in the right picture. When a student changes the distance of the adjustable plane to the origin, she recognizes that the direction vector in the equation, which is shown at the upper right corner of the window, does not change its parameters and only the number on the right hand side of the equation changes its value.

In this worksheet, we had to choose menu buttons instead of directly clicking on the graphic objects for switching the operating mode because of the slower response of this worksheet than the other ones. The slow response irritated our students and made them switch to the unwilling mode. The menu buttons are located on the upper left corner of the window, and students can change the viewpoint or move the target point either on the sphere or along the radius after selecting a mode from the menu.

## TEMPORARY RESULTS

The interviews to some of the students showed that the connections between symbolic and visual representations were easy for lines (Figure 2), and manageable for planes expressed by two vectors (Figure 3). Many students found the direct connections of the position and direction of the arrows and the parameters of the vector equation during their experiments.

However the interviews also revealed two opposite reflections to the worksheet of Figure 4. A student liked this display and thought it deepened his conceptual understanding. He even admired the simplicity of the vector equation. Another student could not find the connection between symbolic and visual representations only with this worksheet. He felt he needed additional explanations, for example the relation of symbolic and visual representations of 2D vector equations, to convince himself of the connection.

The test scores also showed opposite results (Table 1). The 2013 students' average score is significantly higher than the 2011 and 2012 students' averages for the first test of deducing equations of lines, but it is lower for the second test of deducing equations of planes and calculating the planes' distances from the origin. 2013's average is significantly lower than 2011's, and is also lower than 2012's although not significantly. The lower score in the second test made sense with the negative reflection to the worksheet of Figure 4, and implied the limited effectiveness of the worksheets for expressing planes when it was used in a short lesson.

| Academic year  |   | 2011               | 2012           | 2013           |
|--|---|--------------------|----------------|----------------|
| Test   | Used interactive worksheets                   | No                 |                | Yes            |
|  | Deduce equations of lines in 3D space (April) | Number of students | 42             | 40             |
| Average / Full mark (standard deviation)   |   | 1.9 / 4 (1.2)      | 1.1 / 4 (1.1)  | 3.1 / 4 (1.3)  |
| Deduce equations of planes in 3D and calculate the distances from the origin (May) | Number of students                            | 41                 | 44             | 42             |
|  | Average / Full mark (standard deviation)      | 4.7 / 10 (3.1)     | 4.0 / 10 (2.9) | 3.1 / 10 (2.2) |

**Table 1: Test results of deducing equations of given lines or planes in 3D space**

\* The test composed of two or three problems, each requested the vector and algebraic equation of a line in 3D space. One of them was perpendicular to an axis, and the algebraic equation for a line became a system of two equations.

## DISCUSSIONS

One of the benefits of the 3D interactive worksheets is that they could show the advantage of vector equations over algebraic linear equations in connecting symbolic and visual representations. Novice students tend to ignore the advantage when they are explained in 2D. Algebraic equation is much more familiar for them, and they don't feel the need to select another way of expressing the same graphic object. Either they select to use a newly learned vector equation or the already familiar algebraic equation is irrelevant when they express a straight line in 2D. Vector equation stays to be only one of many selections. It might not be a good approach to start teaching vector equations in 2D for the sake of simplicity in drawing on chalkboards. Maybe we should show 3D graphic objects at first without explaining the detail, then use 2D objects as their special case and start explaining.

The advantage of vector equations is finally obvious in 3D when algebraic linear equations become complex. Algebraic equations change their forms in special cases, for example, where a plane or a line is parallel or perpendicular to one of the axes. They have different forms from the other cases. The changing forms of algebraic equations always annoy the students who tend to memorize



formulas without considering the mechanism. By using the 3D interactive worksheets, we could start our lessons from vector equations in 3D. Algebraic equations in 3D may be learnt after vector equations. It could change the learning of vector equations more meaningful to some students if we use the close connection between symbolic and visual representations from the start.

On the other hand, understanding the vector equations that use inner product of the normal vector to the plane may need more students' experience on concrete examples other than our interactive worksheets. A single lesson with the worksheet could not deepen the conceptual understanding. For deeper understanding, students may have to feel confident with the visual meaning of inner product as they already have with the visual meaning of arithmetic product. One of such our approaches may be the use of a virtual game, which uses inner product of 3D vectors to decide the winner of the game (Nishizawa, Shimada, & Yoshioka, 2013).

## CONCLUSION

Interactive worksheets presented in this paper connected visual representations of straight lines and planes in 3D virtual space with symbolic vector equations in students' mind. In some cases, such connections lead students to quick understanding of how the vector equations were constructed. In other cases, it was not tight enough to deepen the conceptual understanding. However, they have a potential to improve the learning of vector equations to more meaningful and enjoyable one if they are appropriately incorporated into the lessons and are combined with additional explanations.

## ACKNOWLEDGEMENT

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## COMBINING THE SPREADSHEET WITH PAPER AND PENCIL: A MIXED ENVIRONMENT FOR LEARNING ALGEBRAIC METHODS

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*We consider the combined environment of electronic spreadsheet with paper and pencil for the learning of the formal method of substitution for solving systems of linear equations. The data presented comes from one student, and were collected during a teaching experiment with 9<sup>th</sup> graders. We find that informal experiences with the spreadsheet, as the identification of variables, the translation of conditions and the identification of the numerical solution provided a suitable approach to the formal method of substitution performed with paper and pencil in algebraic language.*

### INTRODUCTION

Students are expected to learn how to use formal algebraic methods in several situations in the course of their study of Algebra. However, it is common to find many students in the use of formal algebraic methods, such as solving equations, operating with symbols in a mechanical way, without understanding the meaning of the operations they perform. In Portugal the national curriculum recommends engaging students with informal experiences before entering the formal algebraic manipulation and it also indicates problem solving as a privileged activity for such purpose. Additionally, the spreadsheet is suggested as an appropriate tool to establish functional relationships and to create bridges with the learning of algebraic language.

Starting with a contextual problem, students need to model it, and this may include informal notations and expressions as it occurs with using a spreadsheet to address the problem. Working on those models helps students to reinvent the intended more formal mathematics. Initially, the models relate to specific situations that are real experiences for students, thus allowing them to come up with informal strategies. Later, when students face similar problems, the models become more general and useful as a basis for mathematical reasoning and not solely as a way to represent a problem in its context. Therefore a model of informal mathematical activities develops into a model for mathematical reasoning. According to this approach, students move to higher levels of general understanding through contextual problem solving (Gravemeijer, 2005).

### ALGEBRAIC THINKING

Algebraically thinking involves not only knowing various forms of representation, including the symbolic, but also flexibility in changing between modes of representation and ability to operate with symbols in context and when appropriate (Schoenfeld, 2008). It also includes working with mathematical structures and using symbols in problem solving, and understanding the meaning of the symbol (Arcavi, 2006). Kieran (2007) states that, in a more advanced level, this kind of thinking manifests itself through the use of symbolic expressions and equations instead of numbers and operations. However, for students who have not yet learned the algebraic notation, generalized ways of thinking about numbers, operations and notations, such as the equal sign, can be considered algebraic. The use of algebraic symbolism should be taken as an indicator of algebraic thinking but the absence of algebraic notation should not be judged as the inability to think algebraically (Zazkis & Liljedahl, 2002). In this study we assume this broader perspective of algebraic thinking,

considering that it is manifested not only by the use of algebraic symbolism, but also through other representations involving words and general relations between numbers.

The substitution idea within the use of algebraic language isn't easy for all. Filloy, Rojano & Solares (2004) have found that some students are unable to solve problems with two unknowns using formal methods. In problem-solving situations involving two linear equations such as  $4x - 3 = y$  and  $6x + 7 = y$ , students have trouble to get  $4x - 3 = 6x + 7$ . This can be explained by the difficulty in applying the “transitivity of the equal sign” or because they consider the two  $y$  as being different. Godfrey & Thomas (2008) argue that if students look at the equal sign as the result of a procedure, and have not constructed the properties of an equivalence relation, they will not be able to interact fully with the mathematical equation.

## **THE SPREADSHEET**

Although the spreadsheet has not been conceived as an educational tool, it has proved to be a powerful educational resource for the construction of algebraic concepts, namely to establish functional relationships. It is a powerful tool in problem solving, namely in the development of algebraic thinking (Ainley et al. 2004). Its use in problem solving emphasizes the need to identify all relevant variables and stimulates the need to establish dependency relationships between variables. The breakdown of a dependency relationship in successive simpler relations is one of the aspects that can be emphasized by the use of this tool, which can have significant consequences in the problem solving process (Haspekian, 2005). We agree also with Friedlander (1998) who claims that “spreadsheets build an ideal bridge between arithmetic and algebra and allow the student free movement between the two worlds. Students look for patterns, construct algebraic expressions, generalize concepts, justify conjectures, and establish the equivalence of two models as intrinsic and meaningful needs rather than as arbitrary requirements posed by the teacher” (p. 383). However, we find a gap in the literature with regard to the contribution that the spreadsheet can give to the learning of formal methods, in particular of the method of substitution to solve systems of linear equations.

## **METHODOLOGY**

The aim of this study is to understand how a 9<sup>th</sup> grade student progressed in the learning of the substitution method for solving systems of equations in the context of a teaching experiment that relies on an environment combining the spreadsheet with pencil and paper and is strongly organized around problem solving tasks.

We followed a qualitative and interpretative methodology. This research uses a design experiment outline, in which the first author takes simultaneously the roles of teacher and researcher. We present and analyze the case of Ana, a 14-years old student who is generally a committed student and who has been using the spreadsheet to solve problems in her math classes as reported in Nobre, Amado & Carreira (2012). Ana has also used the spreadsheet regularly in the previous school year. The empirical material of the study was gathered by collecting students' productions, the recording of computer screens, the audio recording of their dialogues and taking field notes from observation.

## **RESULTS**

In the following we intend to highlight the key ideas in the learning of the formal method of substitution for systems of linear equations in Ana's learning path, giving visibility to the main aspects of algebraic thinking that were developed.

## Establishing indeterminate equations and looking for solutions

Sofia loves to pose challenges to her colleagues. On the first math class she gave the following one: To find my birthday just multiply the day of my birth by 12 and the month by 30 and add the two values obtained. The result is 582. What's the day and month of my birth?

**Figure 1: The problem “Guess the birthday”**

Ana started to solve the problem with the spreadsheet by selecting the independent variables: day and month. Then she decided to assign one column to the variation of day and used several constant columns to define month as a parameter (equal to 1, 2, ..., 12), as shown in Figure 2.

| Day of Birth      |       | Month of birth/January        |       | Total  |   | February  |        | Total |
|-------------------|-------|-------------------------------|-------|--------|---|-----------|--------|-------|
| C                 | D     | E                             | F     | G      | H | I         | J      | K     |
| dia do nascimento |       | mês do meu nascimento/Janeiro |       | Soma   |   | Fevereiro |        | Soma  |
| 1                 | =1*12 | 1                             | =1*30 | =D3+F3 | 2 | 60        | =60+D3 |       |
| 2                 | =2*12 | 1                             | =1*30 | =F4+D4 | 2 | 60        | =J4+D4 |       |
| 3                 | =3*12 | 1                             | =1*30 | =F5+D5 | 2 | 60        | =J5+D5 |       |
| 4                 | =4*12 | 1                             | =1*30 | =F6+D6 | 2 | 60        | =60+D6 |       |
| 5                 | =5*12 | 1                             | =1*30 | =F7+D7 | 2 | 60        | =60+D7 |       |

**Figure 2: Excerpt of Ana's production**

More than one solution to the problem appeared in the class. Students concluded that there were three possible dates, which meant that the problem had more than one solution. After discussing the problem, the teacher prompted the students to find an algebraic relationship using the variables  $d$ , the day, and  $m$ , the month.

Teacher: Could you write a condition relating 582 with the day and the month?

Carolina: So it's the formula! (Referring to the formula she had used)

Tatiana: So, it's the day times 12... and the month times 30.

Abigail: Added.

Tatiana: Added to the month times 30.

Professora: And after that?

Alunos: Equals 582.

Students wrote the algebraic relationship  $12d+30m=582$ , and checked the solutions with paper and pencil. This task was useful for having provided the opportunity to deal with a linear equation involving two unknowns. It allowed giving meaning to the unknowns in an indeterminate equation and seeing that such equations can have multiple solutions.

## The notion of system of linear equations

Mr. José has three daughters who love to eat sweets: Alice, Beta and Célia. As the summer approached, they started to worry about their figure. The three decided to go on a diet and regularly weighed themselves on a big scale that their father kept in his store. When they started the diet, they weighed themselves, in pairs. Alice and Beta together weighed 132 Kg. Beta and Célia together weighed 151 Kg. Célia and Alice together weighed 137 Kg. What was the weight of each of Mr. José's daughters?

**Figure 3: The problem “The weights of the three sisters”**

Ana starts by naming two columns as “Alice” and “Beta”, probably due to the first sentence in the statement that concerned those two sisters.

Teacher: Which values will the columns hold?

Ana: All numbers are possible provided that the sum is 132 because we have to combine with the weight of Beta. The total must be 132... The values will vary from 1, 2, 3, 4, 5...

Ana selects the weight of Alice as an independent variable and uses formulas to get the other weights: Beta's weight depends on Alice's and Célia's weight depends on Beta's. She uses the joint weight of Célia and Alice as a control to find the solution (figure 4).

| Alice | Beta | Célia | Célia e Alice |
|-------|------|-------|---------------|
| 1     | 131  | 20    | 21            |
| 2     | 130  | 21    | 23            |
| 3     | 129  | 22    | 25            |
| 57    | 75   | 76    | 133           |
| 58    | 74   | 77    | 135           |
| 59    | 73   | 78    | 137           |
| 60    | 72   | 79    | 139           |

| Alice | Beta     | Célia    | Célia e Alice |
|-------|----------|----------|---------------|
| 1     | =132-D9  | =151-E9  | =F9+D9        |
| 2     | =132-D10 | =151-E10 | =F10+D10      |
| 3     | =132-D11 | =151-E11 | =F11+D11      |
| 57    | =132-D65 | =151-E65 | =F65+D65      |
| 58    | =132-D66 | =151-E66 | =F66+D66      |
| 59    | =132-D67 | =151-E67 | =F67+D67      |
| 60    | =132-D68 | =151-E68 | =F68+D68      |

Figure 4: Excerpt of Ana's production

In the class discussion the symbolic writing of the relationships involved in the problem was encouraged, and the students conclusions were written on the board. This allowed the introduction of the algebraic term and symbolic writing of "system of equations".

### The idea of variable substitution using paper and pencil

In the problems solved with the spreadsheet the idea of substitution is already present, although implicitly, because the calculations are done automatically by the computer giving students a more active role in identifying variables and in defining relations. The following task was planned to work on the idea of substitution, this time with pencil and paper, and presented a set of situations as exemplified in figure 5.

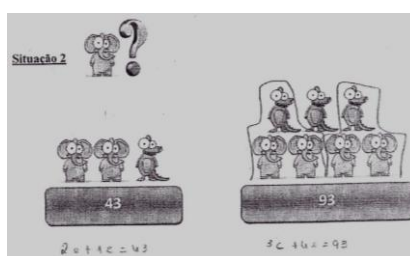


Figure 5: The problem "The values of the animals"

Ana translated the relations to equations but did not use them. She explained here reasoning to obtain the value of an elephant: "We know that 2 elephants plus a crocodile weighs 43, then we can split the figure in sets and we get 43 plus 43 equals 86, which is the weight of two groups. Then we do 93 minus 86 equals 7, thus 7 is the weight of a crocodile. Then the weight of 2 elephants is 43 minus 7 equals 36. We divide 36 by 2 to find the elephant weight: 18 kg". Ana's procedure corresponds to the typical steps of the formal substitution method for solving systems of two equations.

### From the spreadsheet to the formal method of substitution

In a farm there are chickens and rabbits. Altogether there are 212 heads and 700 paws. How many chickens and how many rabbits are there on the farm?  
Translate to algebraic language the conditions you created with the spreadsheet.

Figure 6: The problem "Chickens and rabbits"

Ana started by choosing the independent variable and established the relations between this and the other variables. The "Sum of the paws" was used for control (figure 7).

| Rabbits |          | Chickens         |         | Sum of the heads |                | Paws of rabbits |          | Paws of chickens |         | Sum of the paws |                |
|---------|----------|------------------|---------|------------------|----------------|-----------------|----------|------------------|---------|-----------------|----------------|
| Coelhos | Galinhas | Soma das Cabeças | Coelhos | Galinhas         | Soma das patas | Coelhos         | Galinhas | Soma das Cabeças | Coelhos | Galinhas        | Soma das patas |
| 1       | 211      | 212              | 4       | 422              | 426            | 1               | 211      | 212              | =F4*4   | =G4*2           | =L4+K4         |
| 2       | 210      | 212              | 8       | 420              | 428            | 2               | 210      | 212              | =F5*4   | =G5*2           | =L5+K5         |
| 3       | 209      | 212              | 12      | 418              | 430            | 3               | 209      | 212              | =F6*4   | =G6*2           | =L6+K6         |
| 4       | 208      | 212              | 16      | 416              | 432            | 4               | 208      | 212              | =F7*4   | =G7*2           | =L7+K7         |
| 137     | 75       | 212              | 548     | 150              | 698            | 137             | 75       | 212              | =F140*4 | =G140*2         | =L140+K140     |
| 138     | 74       | 212              | 552     | 148              | 700            | 138             | 74       | 212              | =F141*4 | =G141*2         | =L141+K141     |
| 139     | 73       | 212              | 556     | 146              | 702            | 139             | 73       | 212              | =F142*4 | =G142*2         | =L142+K142     |

Figure 7: Excerpt of Ana production

Ana presented her solution to the class and the algebraic translation was made from there in a dialogue between the teacher and the students. The purpose was to relate the work on the spreadsheet with the substitution method (see Figure 8).

| F       | G        | I                | J       | K        | L              | N |
|---------|----------|------------------|---------|----------|----------------|---|
| Coelhos | Galinhas | Soma das Cabeças | Coelhos | Galinhas | Soma das patas |   |
| 1       | 211      | 212              | =F4*4   | =G4*2    | =L4+K4         |   |
| 2       | 210      | 212              | =F5*4   | =G5*2    | =L5+K5         |   |
| 3       | 209      | 212              | =F6*4   | =G6*2    | =L6+K6         |   |
| 4       | 208      | 212              | =F7*4   | =G7*2    | =L7+K7         |   |
| 136     | 76       | 212              | 544     | 152      | 696            |   |
| 137     | 75       | 212              | 548     | 150      | 698            |   |
| 138     | 74       | 212              | 552     | 148      | 700            |   |
| 139     | 73       | 212              | 556     | 146      | 702            |   |
| 140     | 72       | 212              | 560     | 144      | 704            |   |

|                 |  |  |
|-----------------|--|--|
| $g = 212 - c$   | $\begin{cases} 4c + 2(212 - c) = 700 \\ c = 138 \end{cases}$ | $\begin{cases} g = 212 - 138 \\ c = 138 \end{cases}$ |
| $4c + 2g = 700$ |  |  |

Figure 8: Translation from Ana spreadsheet work to paper and pencil

The independent variable is the number of rabbits ( $c$ ), a variable-column in the spreadsheet. The number of chickens ( $g$ ) was obtained from a descending column for the difference between the total of animals (212) and the number of rabbits. In algebraic language it corresponds to  $g=212-c$ . Intermediate columns were used for the paws of the rabbits and for the paws of the chickens. These variables were translated symbolically by  $4c$  and  $2g$ . Knowing that the total of paws has to be 700, the equation  $4c+2g=700$  appears. It was emphasized that the substitutions performed to get the solution are similar to the operations performed with the spreadsheet (figure 8).

Students realized a correspondence between the usual procedures in the spreadsheet and the substitution method to solve systems of equations and this was the basis to start solving systems using this formal method. Ana, in particular, easily learned how to use formal methods to solve linear systems, particularly the substitution method.

## CONCLUSION

The proposed tasks allowed Ana to make a progressive transition in the learning of formal methods from informal computational experiments. Although, at first she did not use an algebraic formal language this did not prevent her to develop algebraic thinking (Kieran, 2007; Zazkis & Liljedahl, 2002). We observed, as Haspekian (2005) and Friedlander (1998) indicate, that the spreadsheet may support the route from Arithmetic to Algebra. Besides it proved to be a useful tool in problem solving, providing an environment without the constraints of using algebraic symbolism. Ana began by developing an understanding of the symbolic writing of relations, starting by using an essentially arithmetical language, in the hybrid environment of the spreadsheet, and later she was able to represent a system of equations in algebraic language. The idea of variables substitution, usually presenting difficulties to the students (Filloy, Rojano & Solares, 2004; Godfrey & Thomas, 2008), came up naturally as a bridge between the work done with the spreadsheet and with pencil and paper. We found that the formal method of substitution arose naturally and became meaningful.

We claim that the spreadsheet benefits the learning of the substitution method for solving linear systems of equations given the close proximity between the way of solving problems in the spreadsheet and the formal method with pencil and paper, thus allowing a better understanding of the sequence of steps involved in the method through reinvention (Gravemeijer, 2005).

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# **MATHEMATICAL LEARNING DERIVED FROM VIRTUAL COLLABORATION, EXPLORATION AND DISCUSSION OF FREE-FALL VIDEOS, AMONGST CONTINUING EDUCATION STUDENTS**

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*We discuss here the collaborative aspect of online explorations of videos of free-fall phenomena, by five adult students enrolled in a continuing education course on “Physics and its algebra”. These explorations are part of an ongoing research project on promoting mathematical learning via the process of building math models in a context of rich experimentation and virtual collaboration in an online environment. The project includes various independent, constructionist explorations connected to different mathematical and physical ideas. The specific tasks presented here, involved discussions on previous knowledge about free-fall phenomena, taking measurements from videos, plotting and building mathematical models. But the focus of this paper is to illustrate the role of the participants’ discussions thru the project’s social network (that included forums, blogs and chat), to strengthen their internal constructions of the mathematics associated with free-fall phenomena.*

## **INTRODUCTION, BACKGROUND AND THEORETICAL FRAMEWORK**

The data discussed here is part of a larger ongoing research project centred in a distance environment (a virtual laboratory) where community members are encouraged to explore, collaborate and reflect, in a social network type of setting, on various types of mathematical problems. The study then aims to analyse how the reflective, social interaction and collaboration processes can promote learning in the participants. Specifically, the focus is on mathematical explorations through distance collaboration of various problems linked to real-life phenomena. The research is inspired in a former European project in mathematics education called WebLabs involving schools and research institutions in six countries; in that project, which ended around 2005, a community of students, teachers and researchers worked collaboratively to explore mathematical ideas and scientific phenomena through computational and virtual infrastructures (e.g., see Simpson, Hoyles and Noss, 2005). From that project we have taken some technical and methodological aspects, including the principle that students explore and share their discoveries online, with the aim of constructing knowledge thru collaboration and collective reflection.

An important feature of our research is the use of social networks: we consider a group of distance students collaborating virtually in some mathematical explorations. A social network can be defined as a “social structure of nodes that represent individuals (or organizations) and the relationships between them within a certain domain” (Liccardi et al., 2007; p. 225). In the theoretical conception of the implementation of a social network in our research, we take into account the relationships between the elements present, between the *learners* (the users collaborating within the social network) and the structures thru which these learners can potentially benefit from, in that network, thru *webbing* (Noss & Hoyles, 1996) and *connectivism*. Webbing “is meant to convey the presence of a structure that learners can draw upon *and reconstruct* for support – in ways that they choose as appropriate for their struggle to construct meaning for some mathematics” (Noss & Hoyles, 1996; p. 108). Connectivism, described by Siemens (2005), also considers that “how people work and function is altered when new tools are utilized”, focusing on the social aspects, presenting “a model of learning that acknowledges the tectonic shifts in society where learning is no longer an internal,



individualistic activity” and “provides insight into learning skills and tasks needed for learners to flourish in a digital era” (ibid, p.6).

Thus, in our theoretical model, technology plays two roles. On the one hand, drawing on *constructionist* ideas (Papert & Harel, 1991), it provides tools to think with, carry out the explorations, express ideas, and build models, thus serving as an instrument of mediation; as Kaput (1999, p. 327) expressed it: “the computer heralded a new kind of culture – a virtual one – which differs crucially from preceding cultural forms. Not only is there a new representational infrastructure but also the externalization from the human brain”. The technology also offers a kind of interactivity less possible in the past, such as social networks which are a medium thru which learners can interact during the activities, exchange information, collaborate and discuss ideas with other students; an external process that we consider can strengthen –following the ideas presented by Wilensky (1991)– the internal webbing process (the internal constructions and connections) of a student during his/her experiences during the tasks.

In the next sections we provide an example of one of the explorations is our study – the free fall phenomena – where we attempt to show how the virtual collaboration and discussion helped the participants exchange knowledge, have feedback and learn from their mistakes and successes, in order to develop a better understanding of some of the mathematical concepts involved in the situation. Some tools used in the example episodes below, include software to measure directly over a video (a virtual rule), and a statistical package for analyzing and building the mathematical model.

## **THE FREE-FALL EXPLORATIONS: AN EXAMPLE OF VIRTUAL COLLABORATION**

### **Methodological aspects**

The free-fall exploration consists of several tasks:

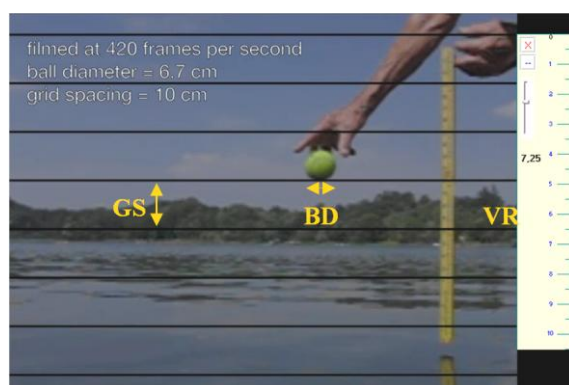
1. An online forum discussion on free-falls: Since this topic is taught in schools from elementary school level, and there is also intuitive knowledge on this, we provide a space of initial discussion so that participants can share their prior knowledge about free-falls.
2. Taking video measurements of height and time when a ball is thrown downward: students measure directly on a video using a virtual rule and then discuss if the data is correct.
3. Establishing scales to build the mathematical model: the video-measures need to be translated into real data using an appropriate scale.
4. Building graphs to analyse data: plotting the data (thru a quadratic function) in order to visualise it and analyse and discuss if it is correct.
5. Developing a mathematical model using quadratic regression: From the data analysis, students need to find a mathematical model that fits the data, i.e. the quadratic equation ( $h=ax^2+bx$ ), and build a mathematical model using the Modellus (see below) software.
6. Analysis of the gravity on the Moon: the same activities as before are now done using a video of an astronaut throwing a hammer on the Moon with the aim of finding out the gravity constant on the Moon.
7. Testing the theoretical mathematical models to see if they fit the real data.

This exploration offers the opportunity to engage with various mathematical ideas, such as functions of height over time and analysing the rate of change of distance over time; finding parameters for a quadratic function using quadratic regression; finding values (e.g. the gravity constants on Earth and the Moon) and mathematical equations that model the situation; and plotting the graphs of those models and understanding why the free-fall is represented by parabolae.

Though parts of the explorations are carried out individually by each student (since they are located

in remote areas), it is important for them to share their results and work collaboratively thru the virtual environment (a type of social network) that includes forums, blogs, chat and messaging.

The materials used in those explorations include: (i) Free-fall videos: We provide an educational video of a ball being dropped (Fig. 1), downloaded from the Internet (from [http://serc.carleton.edu/sp/library/direct\\_measurement\\_video/video\\_library.html](http://serc.carleton.edu/sp/library/direct_measurement_video/video_library.html)) – though we encourage students to also work on videos they produce themselves – as well as NASA videos (e.g. [http://www.youtube.com/watch?v=5C5\\_dOEyAfk](http://www.youtube.com/watch?v=5C5_dOEyAfk)) of astronauts throwing things (a hammer, a feather) on the Moon. (ii) Video software (e.g. Quicktime) that allows running the video frame by frame, in order to take measurements of time and height. (iii) A virtual ruler, such as JR Screen Ruler (<http://www.spadixbd.com/freetools/jruler.htm>) or similar, to measure directly over the video. (iv) The CurveExpert (<http://www.curveexpert.net>) or Excel software to find a mathematical equation that best fits the data. (v) The Modellus software (<http://modellus.fct.unl.pt>) which students can use to build their mathematical model and compare it to the real data (the videos).



**Figure 1. Video of a ball being dropped down**

The participants discussed in this paper, were five continuing education adult students, in an age range from 43 to 67 years old, located in different parts of the country and enrolled in an open university program in Mexico (<http://www.unadmexico.mx>) to study for a college (first university degree) in mathematics. The explorations reported were carried out at the end of a “Differential Calculus” distance education course. The first author of this paper acted as participant researcher: he was the guide and moderator of the online explorations thru the forum.

### **A couple of episodes to illustrate the levels of collaboration and mathematical thinking**

We now present two episodes that attempt to illustrate different levels of collaboration and mathematical thinking, during the free-fall explorations:

#### **Episode 1: Brainstorming on the concept of Scale**

As explained in the previous section, one of the online tasks was to measure on the video the height of a dropping ball at a certain time stamp. Students have two options to measure the video: The first one is to use the ruler on the video and grid space (marked as GS in Fig. 1) to make approximate measures by considering the separation between the lines, which represent 10 cm intervals. The second option is to use the virtual ruler (e.g. JR Screen Ruler) (marked as VR in Fig. 1), to measure directly in pixels or centimetres (though the virtual ruler is shown in Fig. 1 on the right hand side, it can actually be placed anywhere over the video). After each participant carried out his/her measurements and later shared them. In Fig. 2, we see one student’s measurements table that he posted on the virtual forum, with the values obtained using the virtual ruler.

An online discussion ensued (Fig. 3) where the students brainstormed about how to find the scale that related their measurements from the video to the real measurements of the ball and the setting:

on the video (Fig. 1) the real values of the ball's diameter (BD) and the grid spacing (GS) are given; but since all five students used the virtual ruler, their measurements were in pixels or centimetres of the video scale (not real-life). The discussion was a starting point of mathematical thinking because they had to deal proportions when comparing the virtual object with the real object.

tome las mediciones segundo a segundo, le da con la flecha hasta que empezaba el segundo y despues media con la regla en centimetros.

| t | h cms |
|---|-------|
| 1 | 6.06  |
| 2 | 6.22  |
| 3 | 6.91  |
| 4 | 8.55  |
| 5 | 10.69 |

**Figure 2. Sharing their measurements on the online forum**

From the discussion, two students who had measured in pixels, decided to re-measure in centimetres because they realised it was even more complicated to find the equivalence between pixels and the real-life measures. Others also considered possible distortions:

Chayito: the ball is measured in cms on the screen and then compared with the actual ball

Beatriz: but we have to consider distortion on the video, there may be some deformity

**Figure 3. An initial discussion about scales**

During the discussion, one student (Deyánira) proposed a first approximation to get the scale:

Deyánira: Scale = actual diameter of the ball / ball diameter in the video. Everything is reduced by this factor.

Another student (Felipe) then refined the idea,

Felipe: in the video, the ball measures 0.79cms, in real life the ball measures 6.7 cm, this means that the scale is  $6.7 / 0.79 = 8.48101265822785$

Thus, through the discussion, a first theoretical idea of the scale was proposed (by Deyánira), that led to a numerical value for the scale (given by Felipe). As has been explained, students had to find the gravity constant. For that it is important to get a convenient scale, as was done by the students, as shown above; as well as data accuracy in order to get closer to the actual gravity value. A student (Beatriz) recognized the importance of this, such as using more decimal values in the measurements and the scale, that would lead to a better approximation of the gravity constant.

Beatriz: the more decimal places are used, the closer to the constant of gravity:  $9.8 \text{ m/s}^2$

It is interesting that her statement is similar to a limit definition, with a focus on approximation and getting closer. In this way the forum served as an exchange and reflection space for students, and the discussion and virtual collaboration facilitated the development of the mathematical ideas: Although initially each student provided different ideas (comparison, equivalence, proportion) of the meaning of scale, thru the discussion they built a synthesized and useful concept of scale incorporating all the concepts mentioned above. We see here the value of the connectivism (Siemens, 2005) where learning can be facilitated thru the social interaction provided by the technology (the virtual forum).

## Episode 2: Learning from mistakes

In the initial discussion on free-fall, prior to the explorations, the students had discussed, from their previous knowledge, the formula  $h = 1/2gt^2$ , where  $h$  = height,  $t$  = time and  $g$  = the gravity constant. They thus knew that they could use this to find the constant of gravity ( $g$ ) from the data they had collected. In order to get an accurate value for  $g$ , and since the mathematical model involves a quadratic function  $y = ax^2 + bx + c$ , where in this case  $h = y$ ,  $t = x$ ,  $a = 1/2g$ , the moderator suggested they apply a quadratic regression, since they had previously studied this during the “Differential Calculus” course, and they had been trained to use the CurveExpert software with which one can run quadratic regressions easily: they could thus apply a CurveExpert quadratic regression on their measurements, which formed a set of data of time and height ( $x$ ,  $y$ ), and find the best quadratic equation to fit the data. One of the students (Chayito), applied the quadratic regression using her measurements from the video, and shared the computational output (Fig. 4) on the forum but without analyzing it, and she seemed unable, at this stage of the discussion, to interpret the values she had obtained by quadratic regression. But another student (Puentes) posted a translation of the computational output of quadratic regression (Fig. 5), thus helping her and the others to interpret the computational results. The collaboration here served as a bridge towards better understandings.

El modelo matemático con la regresión cuadrática en CurveExpert

Distributing the calculation over 2 cores...

Final Result [Linear Regressions/Polynomial Regression (degree=2)]:

Equation :  $a + b*x + c*x^2 + \dots$

$a = -8.618671428571403 E+00$

$b = 3.064928571428565E+00$

$c = 5.759699999999999$

Standard Error : 7.962884617308698E-01

Correlation Coefficient : 9.994167646726891E-01

**Figure 4. A student shares the computational output by CurveExpert of the quadratic regression.**

LA REGRESION CUADRATICA AJUSTA LOS VALORES A UNA FUNCION DE SEGUNDO

Equation :  $a + b*x + c*x^2 + \dots$

$a = -8.618671428571403 E+00$

$b = 3.064928571428565E+00$

$c = 5.759699999999999$

ALTURA= $5.75X^2 + 3.06X - 8.61$

**Figure 5. Another student translates the output into an equation for the height (ALTURA, in Spanish).**

Los datos están mal porque los están tomando como notación científica,

Results

Model Information

Name: Polynomial Regression (degree=2)

Type: Regression

$y = a + bx + cx^2 + \dots$

Parameters

| Name | Value                  |
|------|------------------------|
| a    | 6.663999999999995E+00  |
| b    | -9.452857142857094E-01 |
| c    | 3.507142857142848E-01  |

$a = 6.66$

$b = -0.945$

$c = 0.35$

Si multiplicas  $0.3507 \times 8.8 = 3.08$  (escala es 8.8)

3.08 y se compara con 4.9 se acerca pero todavia esta lejos de la constante

**Figure 6. The quadratic regression using CurveExpert, with correct values.**

But in the equation (Fig. 5) proposed by Puentes (as well as in Chayito’s data –Fig. 4—, from which it was derived) there is a mistake: the value of  $a = \frac{1}{2}g = 5.75$ , which would give a  $g = 11.5$ , far from the true constant of 9.8. Students didn’t notice this immediately, but there was a discussion that followed where other students wondered how accurate was the value of gravity they had obtained and whether it was near or far from the real value. It was then that Felipe discovered that there were

errors, and identified these as derived from the methodology because of misreadings of the scientific notation when inputting the values in the quadratic regression, as well as from misapplying the scale that had been found. So he proposed another model (Fig. 6) from which he claimed to find as constant of gravity 8.8 (closer to the real one). In the above interchange, we see some collaborative work in action, with students exploring together (albeit virtually), giving feedback, and learning from mistakes. Again, in this process of finding the gravity constant, we see the value of the support of the social structure for the construction of meaning which is an important aspect of our theoretical framework based on the ideas of webbing (Noss & Hoyles, 1996) and connectivism (Siemens 2005): starting from the raw data (the results of the quadratic equations), thru the collective discussion and analysis between peers (facilitated by the technology), new connections between concepts were made and valuable meanings were given to the information.

## FINAL REMARKS

The virtual collaboration between the participants was quite dynamic. It began initially as brainstorming, where each member expressed his/her opinion not minding if they were incorrect, but providing good ideas on how to tackle the problem. They tried to solve the different tasks using the resources provided, but also helping each other; and some of the mistakes provided an opportunity for rich exchange and learning, as well as for proposing new ideas. During the discussions, some students reflected deeply on the results and reasoning of their classmates.

The technology served on the one hand as an instrument of communication, thru the forums mainly; and also, through the different tools (e.g. virtual rule, videos and statistics software), as a vehicle for exploring the situation and the mathematical ideas. It was thru the tools that they could structure a way of checking which findings were correct and recalculate them if necessary (such as when a student applied the quadratic regression without first applying a scale).

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## PEER FEEDBACK IN THE CONTEXT OF A CONSTRUCTION TASK WITH GEOGEBRA

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*In this paper we present an episode of a school year teaching experience with 7th grade students where it was privileged the work with the computer in the teaching and learning of mathematics. We intend to discuss and analyse the nature of coactions between the computer and students and peer feedback, when solving a triangle classification task in pairs using GeoGebra.*

### INTRODUCTION

The pedagogical integration of the computer in the mathematics teaching and learning process seems to gather consensus among mathematics education researchers as it may be seen in several released articles, particularly in the vast set of papers that have been presented in ICTMT in the last years.

In Portugal the current mathematics curriculum for grades 1-9 recommends that students must use dynamic geometry software, mainly when doing exploratory and investigative tasks. Therefore the computer emerges as a new element that interacts with students giving them immediate visual feedback of their actions. This feedback allows students to reflect on the mathematical implications of their actions (Burril, 2011), making possible to try new ideas in a quick way, going forward and backwards, at the speed of a single click, using the software functions, in a way which is impossible with paper and pencil.

This permanent interaction between students and the computer provides a new context, where students can simultaneously share ideas with their peers and the computer. Those dialogues between peers arise while the images appear on the computer screen and from the students' necessity of justifying or clarifying their conjectures (Yu, Barrett & Presmeg, 2009). In this context the formative assessment concept, which comprises the students' contributions to their peers' learning, is found appropriate. In this perspective, the application of a triangle construction task is the basis to understand the role and relevance of peer feedback to students' learning in triangle classification when using GeoGebra.

### TECHNOLOGY AND PEER FEEDBACK

For decades that feedback deserves a particular attention in the area of formative assessment (Black & William, 1998). Particularly when a regulation of the learning process is intentional, feedback allows students to improve their learning (William, 1999). Mory (2003) suggests four reasons to the use of feedback as a learning support. First, feedback can be considered as an incentive to promote the answers; second, it can be faced as reinforcement; third, it can be considered as information that students can use to validate or change a previous answer and, finally, feedback can be considered as a support to help students to build and analyse their solving processes.

Arzarello & Robutti (2010) refer that technology can support the communicational and interpersonal interactions between students and between students and the teacher according to the tools affordances. The integration of technologies in the teaching and learning process brings out

new forms of feedback in the classroom, as it is stated in the studies presented by Bokhove & Drijvers (2011) and Irving & Crawford (2011) at ICTMT 10. In this study we particularly highlight peer feedback between students when solving tasks using the computer.

Gielen, Peteers, Dochy, Onghena & Struyven (2010) examine the importance of peer feedback for learning. These authors highlight the main difference between the teacher and the peer feedback: peers are not domain experts, as opposed to teachers. Nevertheless, peer feedback can be beneficial for learning, which might even be due to the difference from teacher feedback, since the absence of a clear “knowledge authority” (e.g., the teacher’s knowledge) alters the meaning and impact of feedback. The communication opportunities promoted by group work with the computer encourage the appearance of peer feedback and although peer feedback may be less accurate than teacher’s feedback, it can lead to fruitful learning. However, peer feedback must be validated by the teacher in order to avoid the propagation of mistakes and inaccuracies.

Appelbaum (2008) highlights the exchange of ideas between students in their peers learning, giving as example the questioning between them while solving a task. Gadsby (2012) summarizes these ideas as “activating students as owners of their own learning” and “activating students as learning sources for one another”. This last, when it functions properly, places students’ work as the focus of productive discussions.

According to several authors (Black *et al.*, 2003; Gielen, Peteers, Dochy, Onghena & Struyven, 2010), feedback practices are important given that:

1. Peers are able to criticise and freely give advice in a language that others understand better than the teacher’s.
2. Students react differently to the feedback when coming from their peers or from their teacher.
3. Students take on self regulation practices.
4. Students improve their self assessment abilities.
5. Feedback is interactive, face to face, more plentiful and continuous.
6. Feedback promotes autonomy, development and meaningful learning.

## METHODOLOGY

This study follows a qualitative methodology of interpretative character and it is inserted in a teaching experiment with 7th grade students (12-13 years old). The data was gathered in the classroom using students’ dialogues audio recording, students’ performance on the computer screen digital recording and collecting the files produced by students.

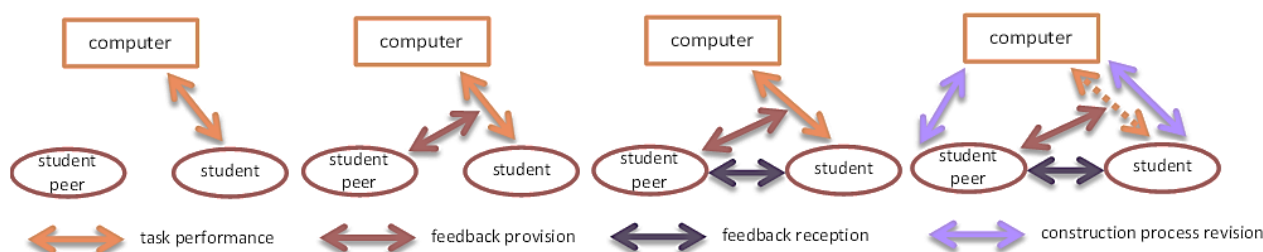
To analyse data, a descriptive model of the feedback phases adapted from Kollar & Fischer (2010) was used. These authors consider the following phases:

*Task performance* A student interacts with the computer performing an action in a given task. That action can be the result of a joint strategy, or not, with a partner.

*Feedback provision* - Through the student’s action, the visual feedback appears provided by the computer which is followed by possible oral feedback by a peer. This feedback focus on the process used by the student, it may reveal doubts, discordance or approvals, indications or simple findings about his actions, procedures or reasoning.

*Feedback reception* - The feedback reception by a student can guide her/him to a partner’s answer, leading to an interactive dialogue, whose aim is to clarify actions, procedures or reasoning possible to use as a lever to the co-regulation process.

*Construction process revision* - It starts a new pursuit and strategies establishment, open to the participation of each student, which leads to the improvement of a new action. This step is composed by reactive and proactive strategies. Reactive strategies are the ones where the student reacts to the visual feedback while he handles geometric pictures. Proactive strategies are the ones where a student determines which actions to take before taking them. While in reactive strategies the student is more dependent from the computer feedback, in proactive strategies the student starts from mathematics properties and uses the computer to implement a specific plan.



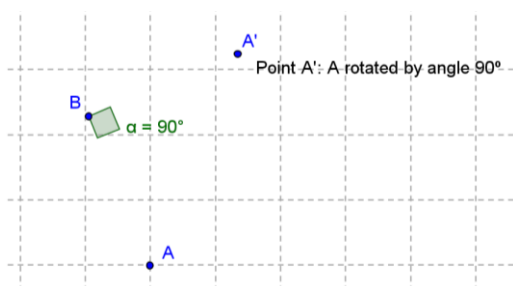
**Figure 1: Model of the feedback phases**

## THE ANALYSIS OF AN EPISODE

When solving tasks using GeoGebra, students worked in pairs having each pair a computer at their disposal. Here, we will describe and analyse a short episode related to the construction of different types of triangles. This episode focuses specially on a pair of students whose fictional names are Andre and Lukas. However, sometimes there is ideas exchange with other pairs. The choice of one episode was due to space limitations and although it refers to a single task, it offers relevant material to find evidences of the role of peer feedback interrelated with the coactions between the tool and the students on the construction process. Other episodes have shown to support the main conclusions derived. The task had several questions related to scalene, isosceles, equilateral, and right-angled triangles construction and classification. At start, Lukas assumed the computer work but later Andre suggested taking turns. Overall it was Andre who took over the computer most of the time. One of the tasks was to build a right-angled triangle that would not deform when its vertices were dragged. The question was illustrated with a scalene right-angled triangle at vertex A.

Andre takes the lead of building the triangle after Lukas having stated that it must have a 90 degrees angle. His first action is to mark on the plan a 90 degrees angle using the “angle with given size” tool. With this tool, they get a triangle ABA’ right-angled at B and isosceles (figure 2).

This is how the tool works and the resulting points constitute the visual feedback of the action performed by the student who takes advantage of this software affordance.



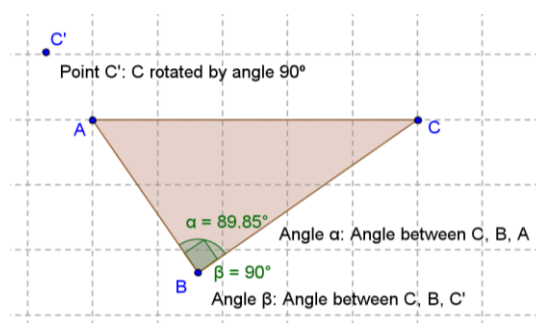
**Figure 2: Angle with given size**

Lukas immediately reacts noticing that the 90 degrees angle should have the vertex at A and not at B (he was referring to the figure shown in the worksheet). So he offers himself to perform another construction but Andre doesn't accept it and deletes the previous construction.



He starts it all over again by marking three points A, B and C, this time using the “polygon tool”. Moving the vertices he tries to make a 90 degrees angle at B. In order to understand how the angle was attained, Lukas asks Andre some questions. Lukas wants to be sure that the angle is unchangeable. They discuss about it and Lukas realizes that the angle actually changes when the vertices are dragged. After receiving the computer visual feedback, Andre accepts that he didn't solve the problem, agreeing with Lucas that it was supposed to be always 90 degrees.

Based on the initial strategy of using “the angle with given size” tool, Lukas suggests that Andre should repeat the initial procedure of marking a 90 degrees angle but this time using two of the vertices of the polygon already made (a general triangle). Afterwards he drags one of the previous polygon vertices so that side AB sets up on one of the sides of the right angle as figure 3 shows.



**Figure 3: Setting up with side AB**

Lukas keeps raising questions about the result and insists that he wants to be sure. Andre replies that it is OK and well done but Lukas undoes Andre's last action on the computer and repeats the procedure of dragging the vertex.

Lukas: Wait, let me see it.

Andre: No. It is OK.

Andre: Well, you made the triangle again.

Lukas: It's 89.85. It's not 90!

Andre: Why did you delete the other?!

Lukas: Come on, Andre!

Andre: Look, this is how you should do it. Use your head.

Andre carefully moves one vertex, measuring the angle and obtains 90.42. He tries to drag the point again and he manages to get 90 degrees. At this point both students understood that they didn't solve the problem because the angle kept changing.

Lukas: I have an idea. May I? [He takes the computer and chooses to show the axes].

Lukas: Andre, why haven't we thought of this? [He constructs a right-angled triangle plotting the vertices on points of the grid].

Lukas: The shape is the same (referring to the triangle given in the worksheet).

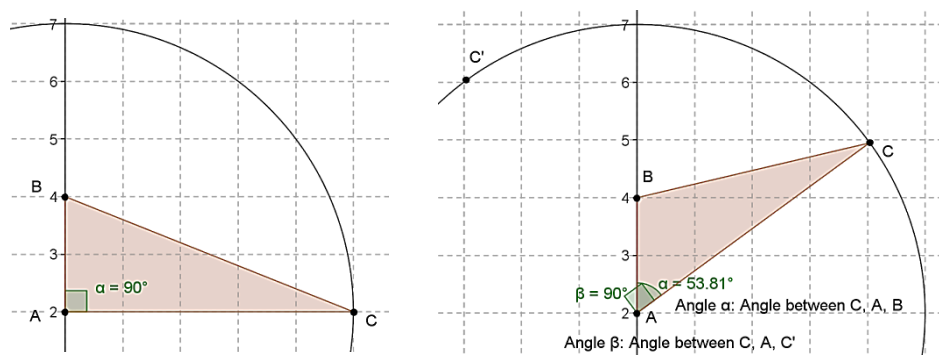
Andre: No, it isn't.

Lukas: Yes, it is.

In trying to convince his peer, Lukas decided to show that the triangle could rotate and still not changing its shape. He constructed a circle with centre on the vertex of the right angle (A) and through another vertex (C) as shown in figure 4. Next, Lukas moved point C and realized that his

conjecture was wrong, because the computer visual feedback showed that the angle size at point A changed. When this happened, Lukas and Andre felt unable to solve the question and decided to call the teacher for help. However, soon after Lukas states:

Lukas: Now I know! Andre, do you know what the challenge is? The challenge is to drag [the vertices] and keeping the angle with 90 degrees!



**Figure 4: An attempt to rotate the triangle**

The teacher approached the students and asked them how the straight lines that made the right angle were. This helped the students to think about perpendiculars and opened the way for a construction based on creating two perpendiculars to define two sides of any right-angled triangle.

According to the peer feedback model involving technology proposed above we can identify a sequence of steps where performance, feedback and revisions were important for understanding the purpose of the task. In each step the pair of students learned something about properties of a right-angled triangle and about Geogebra.

|                    |   |
|--------------------|---|
| Performance        | Andre constructs an angle given size: 90 degrees.   |
| Feedback provision | The computer shows an isosceles right-angled triangle; Lukas disagrees about the vertex of the right angle (they are trying to reproduce the figure given in the worksheet).  |
| Reception          | Andre accepts the feedback and deletes the construction.  |
| Performance        | Andre constructs a 3-sided polygon and drags vertices to display a right angle (by trial).  |
| Feedback           | The computer shows a figure similar to the given one. Lukas questions the robustness of the right angle.  |
| Reception          | Andre accepts.  |
| Revision           | Lukas suggests not to delete the figure and to use the angle with given size as before.   |
| Performance        | Andre creates a right angle using 2 vertices of the previous polygon and tries to adjust the triangle to the right angle by moving the vertices.                              |
| Feedback           | The computer shows two angles with the same vertex: one right and one approximate. Lukas doubts that the triangle is right-angled and proves he is right.                     |
| Reception          | Andre tries to ignore the remark but eventually agrees.   |
| Revision           | Lukas offers a new strategy: to use the grid.   |
| Performance        | Andre protests but makes a right-angled triangle with the aid of the grid.  |
| Feedback           | The computer shows a right-angled triangle but its position isn't the wanted one. Lukas understands that the position is different but the shape is correct and they discuss. |
| Revision           | Lukas presents a new strategy: a way of rotating the triangle by introducing a circle.  |
| Performance        | Lukas creates a circle with centre on the right angle vertex through another triangle vertex and drags this vertex.   |
| Feedback           | The computer shows a changed triangle which isn't right-angled.   |
| Revision           | The students conclude by understanding the aim of the problem presented.  |

## CONCLUDING REMARKS

As research has been emphasising the nature of the tasks to be solved with the use of computers is a key feature to develop learning opportunities. The task involved in this episode is an interesting challenge because it allows for different ways to be solved with Geogebra. Moreover the option of students working in pairs contributed to a fruitful context to explore both the geometry underling the problem and the potentialities of the software. The successive cycles of peer feedback indicate an obvious exploratory work of the two peers. As the cycles were happening the peer feedback has improved and became more consistent, promoting more thinking and more strategy search. The episode clearly reveals how co-regulation was a product of peer feedback and visual feedback from the computer. This process led students from an initial focus on reproducing a given figure to the full understanding of the aim of the construction problem even though it was still required to get a hint from the teacher to achieve the solution.

## ACKNOWLEDGEMENTS

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## THE IMPACT OF COMPUTER USE ON LEARNING OF QUADRATIC FUNCTIONS

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*Studies of the impact of various types of computer use on the results of learning and student motivation have indicated that the use of computers can increase learning motivation, and computers can have a positive or a negative effect, or no effect at all on learning outcomes. Some results indicate that it is not computer use itself that has a positive or negative effect on the achievement of students, but the way in which computers are used. This study explores the impact of computer use on learning quadratic functions in the ninth grade. The study involves five classes where computers are used alongside traditional methods and five classes with only traditional forms of learning. There are no significant differences in learning outcomes between the students who use computers and those who do not; the students who use computers have higher motivation for learning functions compared with those who do not use computers.*

### INTRODUCTION

Functions are among the most important and complex subject matters in school mathematics. When learning functions, students often struggle to see the connection between algebraic and graphic representations of a function. As learning functions seems difficult for students, it has a negative impact on students' overall assessments about mathematics classes. According to Pihlap (2009), learning functions was less appealing to students of 7<sup>th</sup> grade than learning mathematics in general, but the students who had used computers found more enjoyment in the lessons about functions than those students who had not used computers. No significant differences were found between the learning outcomes of the students who had or had not used computers. Baki and Güveli (2008) studied the teaching of functions in 9<sup>th</sup> grade. In the final test, the students who had used the aid of computers achieved better results than those who had not used computers, but the difference was statistically insignificant.

In both of the aforementioned studies, computers were used alongside traditional teaching methods. In the first study 6 of 30 classes were held in a computer room, and in the second study 10 classes were held in a computer room over a period of five weeks.

Weigand and Weller (2001) studied the use of computers for teaching quadratic functions in 11<sup>th</sup> grade and found that the students who had used computers did not have better knowledge, but their knowledge was different. Weigand and Weller believe that many students learning with the aid of computer programs suffer from the problem that "learners sitting in front of a computer seldom have the patience to read representations on the screen, and then to interpret and reflect on them. Computer-generated representations are often viewed only as pictures, unrelated to the mathematical content behind them." They recommend asking questions to guide students to reflect on and comprehend the mathematical content of the representations. It is important to have study arrangements where students can be active participants (Leinbach, Pountney, Etchells, 2002).

According to Papanastasiou *et al.* (2003), the use of computers in itself has no positive or negative impact on learning outcomes, but the manner in which computers are used is important. The use of computers should be studied across the entire study programme in order to identify effective and ineffective uses (McCoy, 1996).

The objective of the current study is to clarify the impact of computer use on learning results and motivation of the students who learn quadratic functions in the 9<sup>th</sup> grade. This article presents the preliminary results of a comprehensive study.

## **METHOD**

### **Participants**

The study was conducted in September and October of the 2012/13 school year in 9<sup>th</sup> grades of five Estonian schools. Five ninth grade groups were used as control classes where no computers were used for teaching, and five were experimental classes where teaching involved some computer use. The division of groups into experimental and control classes was based on the teachers' willingness and ability to give some lessons in a computer room. A total of 199 students participated in the study. 105 of them belonged to control classes and 94 to experimental classes. Nine teachers participated in the study, with one of them teaching two experimental classes.

### **Materials and procedure**

Before the study, the researcher met with all participating teachers, explaining the procedure of the study and giving teachers copies of all necessary materials. The worksheets for lessons in computer rooms were sent to the teachers by e-mail and were also available online.

Before starting with quadratic functions, all groups spent five lessons on rehearsing past study topics, which are required for learning quadratic functions. This rehearsal was based on harmonised study materials prepared by the researchers. Next, all groups took a preliminary test on the subjects rehearsed in previous five lessons. In all groups, the subject of quadratic functions was covered in 15 lessons. In experimental classes, three lessons (of 15) were supposed to be held in a computer room. For various reasons, two classes were able to have three lessons in a computer room; two classes had one lesson in a computer room and completed the work of the remaining two lessons independently as home assignments; in one class, two lessons were held in a computer room and one was completed in the form of home assignments. The idea of the study was to mimic the normal pattern of computer use in mathematics lessons in Estonian schools. Therefore, computers were used to supplement, not replace, traditional forms of learning. Only traditional lessons took place in the control classes, without any visits to a computer room. Furthermore, the students in these classes did not use computers during mathematics lessons and did not complete any tests in a computer. The use of a data projector was permitted in both experimental and control classes.

Worksheets prepared by the researcher were used in all experimental classes. Two worksheets guided the students to explore the properties of quadratic functions with the help of the GeoGebra dynamic geometry software. The third lesson in a computer room was used for compiling a pattern of dynamic lines using the GeoGebra. The students were given this worksheet, because students who have participated in Estonian student competitions have stated that compiling such a pattern helps them to understand the connection between the equation and graph of the function (Pihlap, Sild, Kreutzberg, 2011). All worksheets concluded with links to computerised tests, enabling students to test their new knowledge.

All participating teachers completed a study log, received from the researcher, recording student participation levels, the content of lessons and teaching methods used, home assignments, etc.

The lessons of quadratic functions ended with a final test for all classes. In addition, all students filled out two questionnaires, one at the start of the first lesson after the preliminary test and the other at the start of the first lesson after the final test. The questionnaires focussed on students' attitudes towards mathematics lessons. The students of experimental classes filled out an additional questionnaire about the process of learning functions with the help of computers.

## **RESULTS AND DISCUSSION**

A t-test was made to compare the mean scores of preliminary and final tests in experimental and control classes. Only the results of the students who submitted both the preliminary and the final test

were included in the data. In total, there were 90 such students in control classes and 83 in experimental classes. The maximum score was 50 in both tests. The mean scores in the preliminary tests were 33.21 and 31.24 for the control and experimental classes, respectively. In the final test, the mean scores were 34.41 for the control group and 32.6 for the experimental group (Table 1). Consequently, the control group had somewhat better results in both the preliminary and the final test, but the difference was not statistically significant ( $p < 0.05$ ).

|          | Type of group | N  | Mean  | Standard deviation | p    |
|----------|---------------|----|-------|--------------------|------|
| Pretest  | Control class | 90 | 33.21 | 10.03              | 0.23 |
|          | Exp. class    | 83 | 31.24 | 11.43              |      |
| Posttest | Control class | 90 | 34.41 | 9.18               | 0.24 |
|          | Exp. class    | 83 | 32.60 | 11.02              |      |

**Table 1. Learning results**

Before starting with quadratic functions, the students were asked about the necessity of learning mathematics. The students could express their opinion on a scale from one to five, where one meant that learning mathematics was unnecessary and five meant that it was necessary. After the lessons about quadratic functions, the students were asked the same question about the necessity of learning functions. A t-test was made to compare the mean results in experimental and control classes. The mean rating of the necessity of learning mathematics was 4.3 in control classes and 4.4 in experimental classes (Table 2).

The subsequent rating of the necessity of learning quadratic functions was 3.1 and 3.2, respectively. Both control and experimental classes believed that learning functions was not as necessary as learning mathematics in general. The explanations indicated that the students did not see any practical use for functions in their future life or career. Both control and experimental classes believed that learning functions was easier than learning mathematics in general. The mean rating of the difficulty of mathematics lessons was 2.8 in control classes and 3.0 in experimental classes. After learning quadratic functions, the respective ratings were 3.2 and 3.5.

The students were also asked whether learning mathematics was interesting for them or not. The mean rating in response to this question was 3.2 for control classes and 3.3 for experimental classes. After learning quadratic functions, the same question was asked about functions. The students gave a rating of 2.9 in control classes and 3.3 in experimental classes. The latter difference was also statistically significant ( $p = 0.02$ ).

| Question  | Before studying quadratic functions |    |      |      | After studying quadratic functions |    |      |      |
|---|-------------------------------------|----|------|------|------------------------------------|----|------|------|
|   | Group                               | N  | Mean | p    | Group                              | N  | Mean | p    |
| Studying maths is unnecessary (1)...<br>... necessary (5) | Contr.                              | 95 | 4.3  | 0.80 | Contr.                             | 88 | 3.1  | 0.53 |
|   | Exp.                                | 90 | 4.4  |      | Exp.                               | 77 | 3.2  |      |

|  |        |    |     |      |        |    |     |      |
|--|--------|----|-----|------|--------|----|-----|------|
| Studying mathematics (functions) is difficult(1)...easy(5)   | Contr. | 95 | 2.8 | 0.35 | Contr. | 88 | 3,3 | 0.05 |
|  | Exp.   | 90 | 3.0 |      | Exp.   | 77 | 3,5 |      |
| Studying mathematics (functions) is dull(1)...interesting(5) | Contr. | 94 | 3.2 | 0.68 | Contr. | 88 | 2.9 | 0.02 |
|  | Exp.   | 88 | 3.3 |      | Exp.   | 75 | 3.3 |      |
| Do you like studying maths (functions)?                      | Contr. | 95 | 3.2 | 0.49 | Contr. | 88 | 2.6 | 0.00 |
|  | Exp.   | 90 | 3.3 |      | Exp.   | 77 | 3.3 |      |

**Table 2. Attitudes**

The ratings given in response to the question, “Do you like studying maths?” were 3.2 and 3.3 in control and experimental classes, respectively. After the lessons about quadratic functions, the respective ratings were 2.6 and 3.3, with the difference being statistically significant ( $p=0.00$ ).

Consequently, it seems that the students who used computers found the study of functions more interesting and they enjoyed it more than the students who did not use computers. This is similar to the findings of previous studies (Pihlap, 2009; Baki and Güveli, 2009).

The students of experimental classes also filled out a questionnaire about computers as study aids, using a Lickert scale. A total of 71 students filled out the questionnaire. 35 students (50.7 % of the 69 students who answered this question) believed that computers significantly helped them to understand the contents of lessons and they would like to use computers as study aids in the future; 30 students (43.5 %) stated that there was no difference whether to study with or without computers. Four students (5.8%) found computers to be a distraction to understanding and they would not like to use computers in the future for learning mathematics.

The worksheets used in the lessons were seen as easy and comprehensible by 38 students (53.5 % of 71 who answered this question); 27 students (38 %) stated that some worksheets were easy and some difficult, and 7 students (9.9 %) believed that they were too difficult.

The students of experimental classes were also asked whether the lessons with computers had changed their attitude towards maths. This question was answered by 69 students, with 24 of them stating that their attitude had changed. In case of 23 students, the attitude had improved, with one student giving the opposite response. 45 students stated that their attitude towards mathematics had not changed. In this question, the students could also write an explanation of their answer. The students whose attitude had improved, explained it with the following comments (the text in italics presents some examples of student statements):

- Improved understanding of the contents of lessons. *I can understand much better and would like to learn more; I have started to think more about how something can be done; For example, if I can move axes on the computer and see how things work, it helps me to understand; I understood this part and it slightly changed my attitude, but this could be reversed if I encounter a more difficult topic.*
- Learning is easier, more interesting, fun, convenient, exciting. *Learning has become a little bit more interesting compared to simply learning and solving problems on paper; These worksheets have been exciting.*

One student reported a change in attitude, but the explanation (*because I can't be bothered with the computer*) was somewhat unclear. The same student stated in the preliminary questionnaire that he/she was comfortable with computers but used them less than once a week.



The students whose attitude had reportedly not changed, gave the following explanations:

- The attitude was already positive. *This has not changed my opinion, it was a good subject for me anyway; My attitude towards maths is still good.*
- It makes no difference whether computers are used or not. *The principles of studying maths are always the same; It does not matter too much whether I complete a test in a computer or on paper. It could take more time on paper, while it is significantly faster in the computer; No difference, both are difficult.*
- Do not like mathematics. *I don't like and will never like maths; If I can't understand something, it does not make it more interesting.*

Some students also stated that their *opinion did not changed but working on a computer was very helpful* or that there were not enough lessons with computers to facilitate a change of opinion.

In conclusion, this study indicates that using computers to teach quadratic function does not change the learning outcomes but it makes the learning process more interesting and appealing to students. 34 % of questioned students stated that their attitude towards maths had improved. A limitation of this study is the fact that it did not include any groups where computers were used in more than three lessons. Furthermore, even the number of groups where computers were used in all three envisaged lessons was lower than planned.

#### ACKNOWLEDGEMENT

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## DIFFICULTIES IN ALGEBRA: NEW EDUCATIONAL APPROACH BY ALNUSET

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*This paper discusses the difficulties students encounter in algebra, considering in particular those students affected by dyscalculia. It is generally ascertained that dyscalculic students have difficulties arithmetic but, according to a strictly didactic point of view, we will try to answer the following questions: How they can approach algebraic manipulation and how they can grasp the meaning of algebraic manipulation? The aim of this paper is to present a qualitative analysis of the potentiality of a new software of dynamic algebra named AlNuSet, which favours not only a axiomatic-deductive approach to algebra but also a dynamic-perceptive one. The devised perspective has been based on the research developed in the domain of mathematics education and it has been integrated with the resulting data of the fields of psychology and of the neuro-sciences. The analysis presented in this report is a part of a work in progress, which involves students at the Upper Secondary School.*

### INTRODUCTION

The need to deal with different cognitive demands and in particular to those of students having learning difficulties in mathematics, is discussed in mathematics education research, in cognitive psychology and in the research regarding learning difficulties. In Italy, students with learning disorders are estimated to be between 3% and 5%, and recent data indicates that 0.9% of the school population is positive in diagnostic testing (Italian Ministry of Education, 2011b), so the number certified students with learning disorders is increasing. A conscious use of specific teaching strategies suitable for students diagnosed with learning disorders, and in particular with developmental dyscalculia (DD) (Butterworth, 2005; Dehaene, 1997), is also important for those students who are not certified but with learning difficulty profiles very similar to those of dyscalculic students. In addition, national and international studies (Ianniti & Lucangeli, 2005, Di Martino, 2009) indicate that the process of learning in mathematics for many students (even without certification DD) appears hampered due to many factors, including lack of motivation, or increased anxiety generated by mathematics compared to other disciplines (Zan, 2007; Kogelman & Warren, 1978). Therefore, the development and the supply, to teachers, of an innovative teaching support to mathematics, looks like an ever more necessary goal for research in mathematics education and for teacher training. We will look at, in particular, the software named AlNuSet (Algebra of Numerical Sets), a new software which performs dynamic algebra. In this paper we will present a qualitative study carried out on a teaching experiment, still in progress, with students at secondary school level. Our aim is to analyse the potential of AlNuSet in designing activities that take into account students' difficulties in Algebra, as described in literature, and, at the same time, involve all students in the class, as much as possible (Baccaglini-Frank, Robotti, 2013). We have taken into consideration the algebraic domain, since it represents one of the main difficulties for high school students. Considering the algebraic domain we needed to refer to theoretical key aspects that are described in the following. These theoretical assumptions haven't yet assumed the appearance of an organic theoretical framework but which play an essential role for the characterization of the potential of AlNuSet.

## **DEVELOPMENTAL DYSCALCULIA**

According to Butterworth (2005), Developmental Dyscalculia (DD) is a *learning disability* that affects the acquisition of knowledge about numbers and arithmetic: “DD children have problems with both knowledge of facts and knowledge arithmetical procedures [...], although Temple (1991) has demonstrated, using case studies, that knowledge of facts and grasp of procedures and strategies are dissociable in developmental dyscalculia” (Butterworth, 2005, p.459).

Often the words "learning disabilities" and "learning disorders" are used interchangeably, but each of them refers to very different situations. The term “learning disabilities” denotes any difficulty that students encounter during their studies. These difficulties can be treated through repeated learning activities and revision exercises which lead to a positive development. Very different are the situations falling into the category of specific “learning disorders”. They refer to more serious problems which have uncertain evolution. In fact, they are not a result of a disability, or of external factors such as cultural differences, insufficient or inappropriate instruction, but are connected to dysfunctions of the central nervous system and, as such, cannot be completely solved. Therefore, "difficulty" and "disorder" are not synonymous and should be used in an appropriate way according to the situation to which they relate. As we described above, DD concerns difficulties in learning about number and arithmetic, but what about DD and algebra? Some research on dyscalculic learners showed that there is a dissociation between the recovery ability of arithmetic facts, which are compromised, and algebraic manipulations, which are intact (Hittmair-Delazer et al., 1995). These results have been interpreted as evidence for the existence of two independent processing levels of mathematics: a formal- algebraic level and an arithmetic-numeric level. These results have also been the basis for determining the existence of specific, independent neural circuits dedicated to algebraic and arithmetic knowledge (Dehaene, 1997). Moreover, we observe that certain neuroimaging research results, focusing on algebraic transformations, have highlighted how the visual-spatial areas of the brain are activated at the expense of the language. For example, it has been shown that when we solve equations, the expressions are manipulated mentally so fit visual rather than verbal elaboration (Landy and Goldstone, 2010).

## **DIFFERENT CHANNELS OF ACCESS TO INFORMATION AND LEARNING STYLE**

Research in cognitive psychology identifies four channels of access to information: visual-verbal, visual-non-verbal, auditory and kinesthetic, (Mariani 1996). Mariani showed that they condition individual learning styles. The visual-verbal channel favours reading and writing; who has access to information mainly through this channel learn by reading. The non-verbal visual channel, favours pictures, diagrams, graphs, maps; who has access the information in this way learns on the basis of a visual memory that uses images to remember and use " memo-techniques through images" for the storage of data in memory. The auditory channel favours listening; who has access to information mainly through this channel learns by listening. The Kinesthetic channel favours concrete activities, such as tackling a direct problem to understand its nature; who has access to information mainly through this channel learns by doing. Research shows that, for the access to information, the DD students prefer visual non-verbal, kinesthetic and auditory channels (Stella, Grandi 2012). Therefore, it is legitimate to assume that the use of these channels, as support to the teaching activities, could be the basis for effective teaching practice capable of mediating learning styles peculiar to students with learning difficulties in mathematics and DD students.

## **ALNUSET**

Alnuset is a system developed in the context of ReMath (IST - 4 - 26751) EC project for students of lower and upper secondary school (yrs 12-13 to 16/17). It is constituted by three integrated components: the Algebraic Line component, the Algebraic Manipulator component, and the

Functions component. Even if the educational relevance of this system emerges better through the integrated use of these three components, in this paper we only consider the Algebraic Manipulator component to show how it can be used to modify the approach to algebra in class and, in particular, with DD students and students having learning difficulties. To have a more complete idea about this system you can see [www.alnuset.com](http://www.alnuset.com).

### Algebraic Manipulator of AlNuSet

The Algebraic Manipulator component of Alnuset is a structured symbolic calculation environment for the manipulation of algebraic expressions and for the solution of equations and inequations. Its operative features are based on pattern matching and rewriting rules techniques. Pattern matching is based on a structured set of basic rules that correspond to the basic properties of operations, to the equality and inequality properties between algebraic expressions, to basic operations among propositions and sets. These rules appear as commands on the left part of interface and they made active only if they can be applied to the currently selected part of expression. The user can easily control the whole process of algebraic transformation exploiting feedback given by the system (Figure 1).

The screenshot shows the Algebraic Manipulator interface. On the left, there are several categories of rules: Addition, Multiplication, Powers, Compute, Logic and Set, and Algebraic Line. The 'Compute' category is currently selected, showing rules like 'A → (A)', 'Simplify 0', 'Simplify', 'Collect', and 'Substitute Variable'. The right panel displays a sequence of algebraic steps. The current step is highlighted in yellow and shows the expression  $x^*x + -1$ . The previous step is highlighted in green and shows  $(x-1)^*x + (x-1)^*1$ . The interface also includes buttons for 'User Rules', 'Show User Rules', 'Import', 'Export', and 'Clear'.

**Fig. 1: Example of algebraic manipulation in Manipulator component of AlNuSet: only the commands which can be applied to the currently selected expression are available.**

Note that mediation provided by AlNuSet is profoundly different from that proposed by the software traditionally used for the teaching of algebra like CAS where the basic rules (commutativity, associativity, etc.) are used internally in a sequence generally not controlled by the user, to produce a higher level result, like “factorize” or “combine”. As a consequence, the techniques of transformation involved in CAS can be obscure for a non expert user.

In Algebraic Manipulator it is possible to create a new rule once it has been proved. This feature supports the construction of the idea of structured theory. We can observe that all the features of Algebraic Manipulator component of AlNuSet support the development of skills regarding the algebraic transformation and they contribute to assign a meaning of proof to it. Finally, note that

several researches (Pedemonte, Robotti, 2013; Chiappini, Robotti, Trgalova, 2010) highlighted the educational potentialities of this software showing how the approaches described above are effective in order to understand some basic algebraic concepts (fractions, expressions, equations,...).

## **QUALITATIVE ANALYSIS OF THE USE OF ALNUSET**

In order to discuss a new educational approach to algebra, which allows the handling of learning difficulties in algebra and developmental dyscalculia, we present a qualitative analysis on the use of AlNuSet and, in particular, on its Algebraic Manipulator component. In other papers we discussed with our colleagues a new approach to the meaning of some algebraic notions (variable, unknown, parameter, equality, equation, identity..) that have been developed exploiting the dynamic and perceptive functionalities of Algebraic Line of AlNuSet (Robotti, 2013; Chiappini, Robotti, Trgalova, 2010). In this paper we put in evidence the fact that the Algebraic Manipulator of AlNuSet allows students to have a better comprehension of the algebraic manipulation and of its meaning due to a new visual rather than verbal approach to the manipulation itself. To this extent we need to identify the main difficulties encountered by students in algebraic manipulation. According to K. Weber (2001), for example, current researches can be broadly categorized in two main categories of students' difficulties: 1) students don't have an accurate conception of what is an algebraic proof 2) students may misunderstand a theorem or a concept and systematically misapply it (e.g. Harel and Sowder, 1998). In our qualitative analysis we are particularly interested in two of them: i) students do not see the meaning of algebraic manipulation as a process that transforms an expression in an another equivalent, ii) students do not understand symbolic language. In particular, in algebraic manipulation performed by dyscalculic students or students having difficulties in mathematics, we have identified two kinds of major difficulties that are linked to those previously described: the first is related to the process (difficulties in handling procedural sequence), the second is related to the meaning (algebraic manipulation as equivalence between expressions). At this point the questions are: Which kind of algebraic manipulation can be done with the Algebraic Manipulator of AlNuSet? How can we make this manipulation effective in order to overcome some of the difficulties described above? In this circumstance, we show the analysis of some particular features of Algebraic Manipulator which are: selecting a part of an expression, making available only the rules applicable to the selected part of the expression, viewing directly the result of the rule to be applied, being able to go back and, if necessary, cancel the application of the rule. In selecting part of an expression we can observe that the kinaesthetic channel is involved. In fact, to select the part, the user has to drag the mouse along the part. The selected part will be highlighted in yellow and, in this case the non-verbal channel is exploited to access the information. The information is: identify the part of the expression, which has to be manipulated. At this point, only the rules applicable on the selected part of the expression are made available by the system. In order to choose the most effective rule to manipulate the expression, the students have to recognize an analogy between the algebraic structure of the rule and the algebraic structure of that part of the expression to which the rule is going to be applied. It must be noted that the selected part of the expression is highlighted in yellow so that the non-verbal channel become essential in order to access the information. The identification of this analogy is facilitated by the fact that the rule is visible on the interface and the student does not need to recall it from memory. In other words, the comparison between the two structures (that of the selected part of the expression and that of the rule) is a predominantly visual task that does not involve linguistic aspects. But, what is the difference between a set of rules written on a sheet of paper and the Algebraic Manipulator? On paper the rules in the table are all visible and applicable, on the contrary the interface of Algebraic Manipulator highlights in yellow only the rules effectively of use for that particular manipulation. This fact supports the neuro-images research results, according to which during the algebraic

manipulation, mainly the visual-spatial areas of the brain are activated. From an educational point of view, this kind of approach seems to develop the initial stage of a compensatory strategy. Thus, this approach seems to support both students with difficulties in algebra and dyscalculic students in developing the meaning of algebraic manipulation. Moreover, we can observe that Algebraic Manipulator is a memory support for student since it makes available a set of rules and axioms. For this reason, dyscalculic students do not need to recall rules from their memory to manipulate expression, because they are available on the interface. In fact, dyscalculic students and students with difficulties in mathematics often have difficulty in recalling the formulas because their long-term memory does not support them in this process. Thus, we can say that the Manipulator plays the role of compensatory instrument. The application of an algebraic rule in a manipulation without the support of Alnuset, requires computation which involve arithmetical aspects (procedural aspects linked to the arithmetical computation, retrieval from the long-term memory of arithmetical facts,...). For a dyscalculic subject this could represent a difficulty that could hinder the algebraic manipulation. Moreover, we can observe that, once the rule is applied by the system, the subject must activate a dual system of control: a control on the computation and a check on the semantic aspects related to the transformation itself. In the latter case, he must evaluate whether the application of the chosen rule is useful to achieve the assigned learning objective (for example, prove the equivalence between  $(a+b)^2$  and  $a^2+b^2+2ab$ ). In this monitoring process, an error could be due to one or both of two causes: incorrect application of the rule (miscalculation) or wrong choice of the rule to be applied. The algebraic manipulation on the Manipulator of AlNuSet allows dyscalculic students and students with learning difficulties to overcome this difficulty because they can be sure of the correctness of the application of the rule. In fact, the calculation is entirely dependent by the software. The subjects are able to completely focus on the educational aim, which, in the mentioned example, is represented by the demonstration of equivalence between two expressions. The meaning of the algebraic transformation becomes the main objective of the manipulation, leaving apart the mere application of a technique.

## CONCLUSION

analysis shows that Algebraic Manipulator of AlNuSet can assume different mediation roles in learning algebra to support DD students and students with learning difficulties in Math. First of all, it allows students to develop compensatory strategies, not only in arithmetical calculus, but also in algebraic ones. In fact, the Algebraic Manipulator of AlNuSet supports students in the memorization of rules necessary to manipulate expressions or propositions and in calculus, allows students to focus on the educational aim of task concerning the equivalence between algebraic propositions. Moreover, in Algebraic Manipulator is possible to add new rules, which have been previously proved in Manipulator. These new rules become available in the Manipulator interface and students can use them in the next manipulation. This fact supports the construction of the idea of a structured theory that students can build themselves little by little. Thus, this feature allows students to assign, to the set of algebraic rules available on the interface, the meaning of tools useful to transform expressions algebraically. In other words, in Algebraic Manipulator it is possible to understand the operative, axiomatic and deductive nature of the algebraic manipulation, which, often, is interpreted as a mere application of techniques. Thus, it seems that, grasping meaning of algebraic manipulation and having memory support and calculus support, can help students and, in particular, DD students and students having learning difficulties, to face difficulties encountered in that manipulation.

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## GEOGEBRA ELEARNING LAB

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*Since its foundation in 2010, the GeoGebra Institute of Turin has proposed training projects and experimental teaching methodologies to Italian teachers. In the same year the GeoGebra Institute of Bari was founded and immediately embarked on a close working relationship with its partner in Turin. The reasons for this collaboration can be found not only in the need to have an equal standard in the certificates that institutes provide, but above all because of their shared research background and their experience in teacher training. The keywords that characterize the experience are: project-community-practice. In the light of the experience gained by both institutes, we present the 'Geogebra eLearning Lab' using a Moodle platform which we will develop, where a different type of learning experience called "a Learning Event" takes place. These activities are designed to stimulate the participants to think and to interact with each others, as well as to produce further activities and resources.*

### THE GEOGEBRA INSTITUTE OF TURIN

Since its foundation in 2010, the Geogebra Institute of Turin has proposed training projects and experimental teaching methodologies to Italian teachers. In particular, we have proposed some training projects to teachers from the Piedmont Region. This was made possible thanks to a contribution from the Province of Turin, which has supported the organization through the Teachers' House, a social promotion association, which is a partner of the Mathematics Department of the University of Turin in the Geogebra Institute.

The keywords that characterize the experience are: project-community-practice (Robutti 2013).

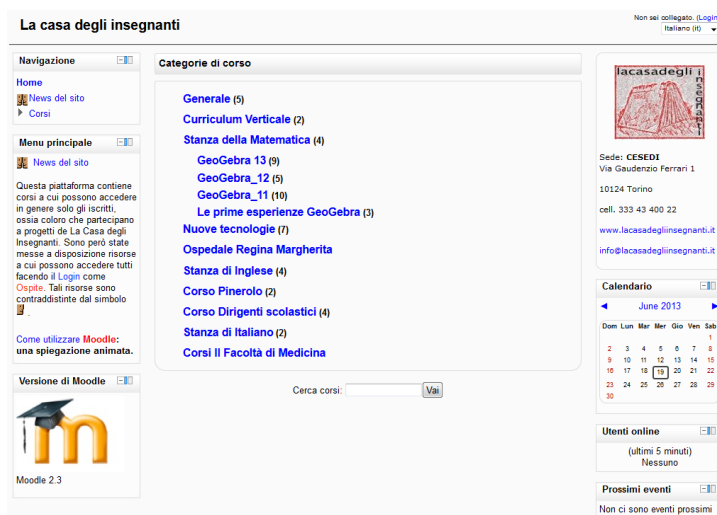
**Project:** as we do not wish only to train in the use of Geogebra software, but also to support and stimulate thinking and discussion on teaching methodologies that its use requires. Moreover the beginning of the experience coincided with changes in school curricula and therefore the need for reflection was requested on the part of teachers. Precisely for this reason the courses were complemented by seminars (Robutti 2013); (Gallo & Cantoni 2010-2013).

**Community:** as we wanted to create working groups, especially pertaining to schools or centers of schools. Moodle platforms helped us to establish a connection that went beyond the face-to-face workshops for teachers. Indeed, the GeoGebra Institute can count on two Moodle platforms: the University DI.FI.MA. (Teaching physics and mathematics <http://teachingdm.unito.it/porteaperte>) and La Casa degli Insegnanti (Teachers' House <http://lacasadegliinsegnanti.wizshelf.org/>).

**Practice:** as we did not want training to remain a moment disconnected from teaching. Instead, Geogebra became an essential teaching tool. Teachers tested it in the classroom with the mentorship of experts and recognized the need for an effective integration of technologies in classroom activities.

The teachers proved to be very enthusiastic. In Piedmont alone, about 250 teachers have been trained in three years. Some of these, after receiving the certification of GeoGebra user, have become experts, in a path that has led them to become trainers themselves. A "cascade" process has thus been generated. (Sargenti & al 2011). A first publication with materials from courses and experimental school activities will be available during GeoGebra day, on October the 4th 2013 in Turin.





**Figure 1. Teachers' House website** <http://lacasadegliinsegnanti.wizshelf.org/>

As well as the projects leading to certification, the blended training activity also involved the National project of “Lauree Scientifiche” (Scientific degrees): teachers were trained in maths activities using Geogebra with the aim of experimenting with these activities in their classroom.

Both the projects described are planned considering a theoretical model of teacher education called “Meta-Didactical Transposition” (Arzarello & al., in press), which describes teacher training as a dynamic process involving dialectical interaction between the community of teachers and that of researchers. One of the main results of this interaction is the teachers’ development of both new awareness (on a cultural level) and new competences (on a methodological-didactical level), which lead them to activate, in their classes, a didactical transposition – different from the traditional one, based on lectures – in tune with the recent educational trends (that means not only the use of GeoGebra, but also problem solving activities, mathematics laboratory, working groups, discussions). The change in teaching practices that we have observed in these teachers is contagious and causes a “cascade” process as previously mentioned, when teachers work in their schools.

## THE GEOGEBRA INSTITUTE OF BARI

The mission of the GeoGebra Institute of Bari is to empower teachers at all levels to use GeoGebra in student-centered learning, highlighting the pedagogical aspects of the software rather than technical ones as far as we firmly believe that teachers should learn how to use technological tools (such as GeoGebra) as a methodological resource (Faggiano 2009).

Our team has experience in teacher and pre-service teacher face-to-face and online training courses. The teachers and researchers from the local teacher community and the Department of Mathematics at the University of Bari are engaged in the design of free teaching and professional development materials, in research projects concerning GeoGebra and IGI, in publications in journals and in presentations at national and international conferences.

In March and November 2010 we carried out two eTwinning Lab events called “eTwinning ideas for maths”. They were online courses for Maths teachers which lasted 15 days each, on the use of Geogebra in on line twinning projects between European schools involved in the Lifelong Learning Program.

During the eTwinning Lab event we proposed some tasks concerning loci, the golden section and polygons using Geogebra to give teachers the opportunity to be subject of a “mise en situation” (as Chevallard’s well-known approach). In this way teachers can experience by themselves the difficulties students can encounter and have to overcome, the cognitive processes they can put into

action and the results they can achieve. They also have the opportunity to reflect on the changes occurring when using technologies and become more skilful and self-confident when deciding to exploit the potential of new technologies in mathematics education (Faggiano & Ronchi 2011). Around 120 European teachers took an active part in the courses, doing all the proposed tasks concerning the use of Geogebra in school activities.

In May 2011 we organized a seminar with the participation of M. Dawes (Cambridge University) and O. Robutti (Mathematics department – University and GI of Turin) as speakers focused on: Geogebra – the school mathematician’s tool and research and didactic experimentation using a dynamic geometry software.

### **THE GEOGEBRA ELEARNING LAB – AN ONGOING PROJECT**

The Italian GeoGebra Institutes of Turin and Bari, were founded in the same year and immediately embarked on a close working relationship. The reasons for this collaboration can be found not only in the need to have an equal standard in the certificates that institutes provide, but above all because of their shared research background and their experience in teacher training.

Over the past three years many materials and teacher training projects have been promoted on how to use GeoGebra’s various features and tools, but what specific teaching methods, strategies or models can be used to organize effective GeoGebra training?

Above all, is it enough to learn how to use Geogebra, without reflecting on the methodologies, contents and objectives of Mathematics teaching? Does using Geogebra automatically improve teaching and students’ learning? Research has shown that technological tools are not useful if they are not accompanied by a serious study of the mental processes which they generate, the limitations which these tools have, and the added value which they bring. (Marconato 2009).

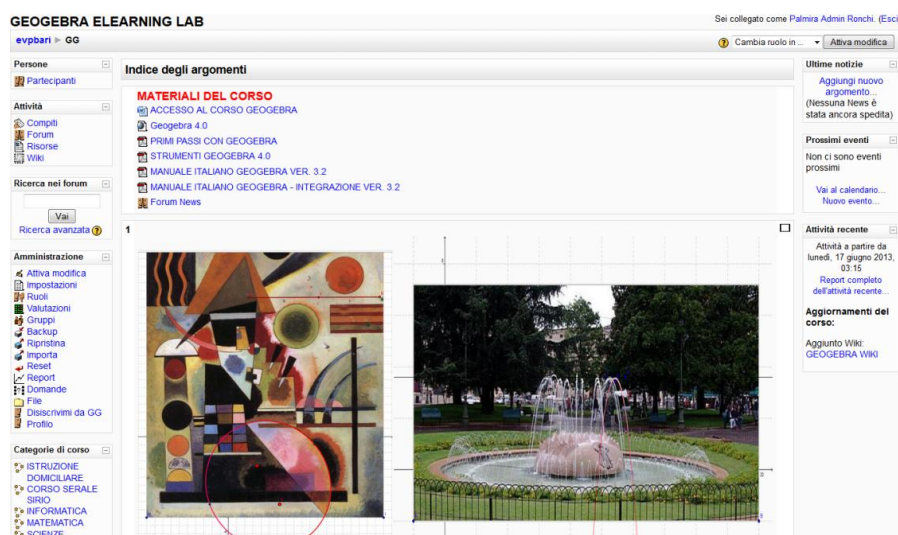
Considering the above questions, the aims of "Geogebra elearning lab" will be to structure and implement a Moodle Learning Management System, including Course Management Systems, and GeoGebra applications in elementary, middle and secondary classrooms that are based on active learning, inquiry and creative problem solving.

A learning experience for teacher trainers called “a Learning Event” will take place in the lab with the aim of providing a Geogebra certification, however the lab activities will be designed to stimulate the participants to think and to interact with each other, as well as to produce further activities and resources using Geogebra in a more meaningful way in the school activities.

The implementation of Geogebra lab was achieved by incorporating the necessary Moodle server software into a dedicated website, Moodle (an acronym of Modular Object-Oriented Dynamic Learning Environment) is a free and open-source e-learning software platform which enables teachers:

- to develop learning materials supported by several tools and plug-ins (including a Geogebra plug-in),
- to keep track of teacher trainees and their achievements,
- to log and evaluate activities.

Some typical features of Moodle such as: assignment submission, discussion forum, files download, grading, online news and announcement, pool and WIKI, will be used during the event.



**Figure 3. The Moodle “GEOGEBRA ELEARNING LAB” pilot platform**

The pilot "Geogebra elearning lab" event is based on five scheduled tasks. Four of them are linked to the National standard curriculum topics and international and national student assessments, in particular the proposed maths topics came from the Italian mathematics association (Unione Matematica Italiana 2001-2003-2004).

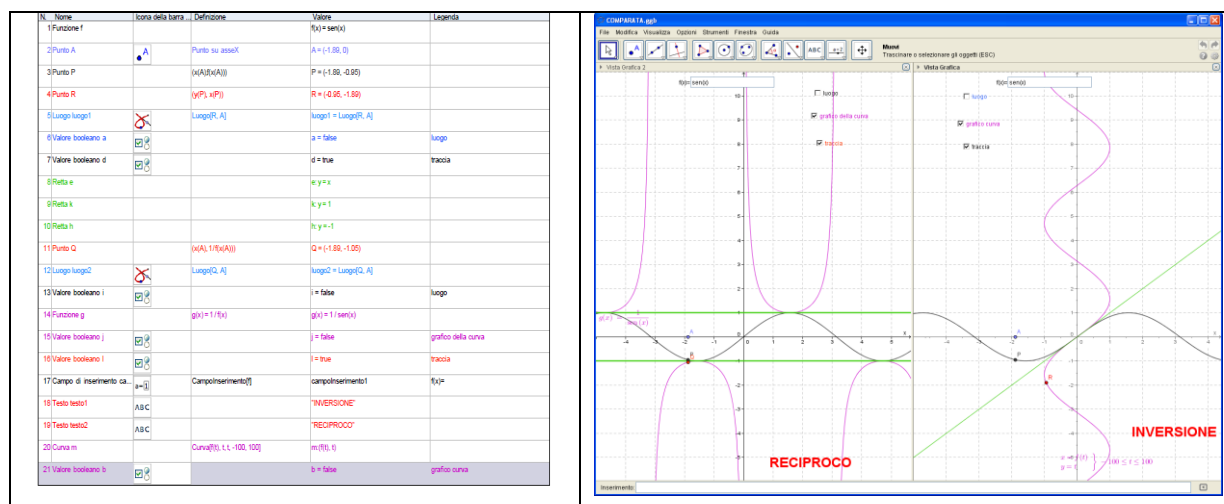
| <b>The pilot "Geogebra elearning lab" event – SCHEDULE</b> |  |  |
|--|--|--|
| <b>Tasks</b>   | <b>Topics from National standard curriculum and international and national student assessments</b> | <b>Some operations of Geogebra from Geogebra certification Guidelines</b>  |
| 1. two days  | Geometry   | <ul style="list-style-type: none"> <li>- Installation and introduction of Geogebra. - Getting started with Geogebra</li> <li>- First construction using Geogebra</li> <li>- Customizing and simplifying the user interface - User defined tools</li> </ul>   |
| 2. two days  | Numbers  | <ul style="list-style-type: none"> <li>- Basic algebraic input, commands and functions</li> <li>- Inserting static text and pictures into the graphics window</li> <li>- Inserting dynamic text into the graphics window</li> <li>- Creating Dynamic work-session</li> <li>- Export of Pictures to the clipboard or as image file</li> <li>- Free and Dependent Objects</li> </ul> |
| 3. two days  | Function   | <ul style="list-style-type: none"> <li>- Loci and conic sections</li> <li>- Slider</li> <li>- Graphics' Transformations</li> </ul>   |

|   |   |  |
|---|---|--|
|   |   | <ul style="list-style-type: none"> <li>- Slope</li> <li>- Derivative and Integral Commands</li> </ul>  |
| 4. one day  | Uncertainty & data  | <ul style="list-style-type: none"> <li>- Spreadsheet View</li> <li>- Auxiliary Objects</li> <li>- Trace to Spreadsheet</li> <li>- List Tools and Analysis Tools</li> </ul> |
| 5. one day  | Virtual conference session with an expert and creation of a discussion forum. | <ul style="list-style-type: none"> <li>- Geogebra news and updates</li> <li>- GeoGebra: CAS, 3D, Chrome, ...</li> </ul>  |
| <b>Final Questionnaire and end of the Geogebra lab event.</b> |   |  |

The first four tasks are structured in two parts: a step by step reconstruction of a Geogebra work-session and a challenge concerning the work session topic. In the figure below there is an example of the third task, for secondary school teachers, concerning the inverse and reciprocal function. The related challenges are: “Observing the function graphic, which suggestions can be obtained in order to draw its reciprocal function graphic?”. “Observing the function graphic, which suggestions can be obtained in order to draw its inverse function graphic?”. Is the results still a graphic of a function?

In each task trainees do practice in Geogebra construction, save the generated ggb file, answer questions and then send the work session file to the task lab-repository.

After the two online training weeks teachers have a maximum of one month to plan a lesson using Geogebra and to try it out in the classroom under the monitoring of the online tutor who provides support both in the planning of the lesson and if there are problems regarding Geogebra. The teacher may decide to experiment with his own idea or with a suggestion coming from the course activities.



**Figure 4. An example of “GEOGEBRA ELEARNING LAB” task.**

There is no special schedule or exact timeline for this online event, meaning that teachers can work on their sessions and tasks whenever they like. The online tutor will put the next day’s sessions and tasks in the last evening of the previous session, so that teachers have time to work them for the whole two days.

To receive the final certificate trainees have to insert their work-files in the dedicated task repository and use the Moodle WIKI until the end of the course – where they have to write a short reflective paper about their experience using Geogebra in their school activities.

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# TOUCHCOUNTS: AN EMBODIED, DIGITAL APPROACH TO LEARNING NUMBER

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*This paper describes the design of a digital technology focussed on early number sense (especially counting and adding). This “TouchCounts” application (designed for the iPad) takes advantage of the easily shareable, multimodal touchscreen interface that provides direct mediation through fingers and gestures. After describing the affordances of the application and its relation to current literature on the role of fingers in the development of number sense. Using a new materialist theoretical lens, I analyse the way in which a group of four 3-4 year old preschool children become fluent with cardinal aspects of number.*

## INTRODUCTION

Current mathematics education software has been developed for the desktop/laptop paradigm of technology use where the mouse and keyboard are essential interfaces. Even software for interactive whiteboards (IWBs) does not take full advantage of touch-screen capacities because the mouse/keyboard interface is the default interaction mode. In addition, IWBs, while providing a social space for interaction, do not allow individual students, or small groups of students, to each interact directly with the software. In contrast, iPad devices permit both whole-class, small group and individual interactions. Also, depending on the application’s design, iPads enable collaborative interaction between two or three students on a single device (as it recognizes multiple inputs from different individuals simultaneously), something that computers, with a single mouse, have not been able to offer. As well, their small size overcomes obstacles faced by teachers using school computer labs (e.g., awkwardness of obtrusive monitors in rows of desks).

The touchscreen iPads also enable direct mediation, allowing children to produce and transform objects with fingers and gestures, instead of through a keyboard or mouse. Recent research has shown that there is a neurofunctional link between fingers and number processing and that finger-based counting may facilitate the establishment of number practices (Andres, Seron, and Olivier, 2007; Sato et al., 2007; Thompson et al., 2004). Research has already shown that consistent use of fingers positively affects the formation of number sense and thus also the development of calculation skills (Gracia-Baffaluy & Noël 2008). This suggests that using the fingers to create numbers in a correctly ordered way, with both visual and auditory feedback, can support the development of number sense and provide the foundation for arithmetic achievement.

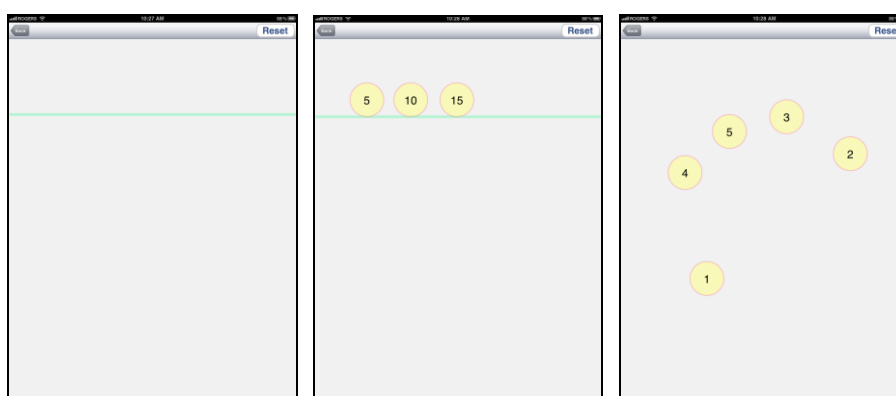
While many number-related applications can be found for the iPad, the large majority of them are designed for game-like interactions in which learners practice arithmetic operations. Our interest was in developing an expressive technology that supports the development of meanings related to numbers and operations. A similar project has been undertaken by Ladel & Kortenkamp (2011), who have developed a multi-touch-table environment in which children can place a certain number of virtual tokens on the table using their fingers.

## DESIGN OF TOUCHCOUNTS

The design of the application is modular, which means that the main app will contain several different sub-applications that are meant to offer an evolving sense of number for the learner. Currently, there are two sub-applications, one for Counting (1, 2, 3, ... ) and the other for Adding (1+2+3+...). In this paper, we focus exclusively on the Counting world, in which learners tap their

fingers on the screen to create small numbered circles that are also represented through both symbol (written numeral) and sound (spoken word) as fingers are placed onscreen. In the default mode, gravity makes these circles fall off the screen, unless they are placed on the horizontal bar (see Figure 1a, b). Being able to select specific numbers to “pull out” is evidence of having objectified number, that is, of being able to think about a number  $n$  not only as an element in the process of counting, but also as a number that has a particular relationship to  $n-1$ . More specifically, in order to place a given number on the bar, one must know what the previous number will be. This objectification of number enables the move from ordinality to cardinality, the latter being necessary for answering the “how many?” question.

Fingers can be placed onscreen all at once to create a group of numbers. So, for example, placing five fingers on the screen creates five numbered circles but produces only the word ‘five’ orally. If the user repeatedly touches the screen with two fingers, she will see pairs of numbered circles appear but will only hear “two, four, six”, etc. Every finger touch produces a number; this means that it is not possible to move existing objects on the screen. In the no-gravity mode, numbers that are placed on the screen remain there (see Figure 1c).



**Figure 1: (a) Default Counting world; (b) Numbers on the bar; (c) No gravity Counting world**

The goal of this simple application is to assist young children (ages 3-6) in developing an understanding of the one-to-one relationship between their fingers and numbers. Children at this age, do not necessarily associate the words for each number with the objects being pointed to. They tend to recite the numbers as if it were a song (this is called “rote counting”) and point at the same time, but not always coordinating the two actions (Fuson, 1988). For small groups of objects, children can also use subitising to find the number of objects in a group, but research has shown that by the age of seven, the ability to subitise typically increases to 4 to 7 objects (Rouselle & Noël, 2008).

The *TouchCounts* Counting world directly supports two of the five aspects of counting identified by Gelman & Meck (1983): (1) when counting, every object gets counted once and only once (one-to-one correspondence principle); (2) the number words should be provided in a constant order when counting [1]. Further, when the gravity option is turned off, it becomes evident that the last number that is counted is the number of items on the screen, which is a third principle of counting. This paper focusses most directly on the third principle, especially as it relates to finger subitising.

## THEORETICAL FRAMING

Broadly speaking, we take a participationist, non-dualistic perspective on thinking and learning. More specifically, we adopt a new materialist approach in which the tool (in this case, the iPad application) is seen as participating in an agential relationship with the user so that the tool and the user mutually constitute each other through interaction (see de Freitas & Sinclair, 2013). Indeed,

insofar as the tool ‘speaks’ and moves, in interaction with the user, it takes on an animate role in the interaction, enabling and also preventing activity. Given the age of the learners, we attend specially to the broad and varied ways of communicating involved in mathematical activity—including gestures, bodily movements, tone of voice, gaze, etc.. This is in accord with principles of embodied cognition that posit that cognitive functions are “directly and indirectly related to a large range of sensorimotor functions expressed through the organism’s movement, tactility, sound reception and production, perception, etc.” (Radford, 2012). The new materialist approach extends also to mathematical concepts, not just to the concrete tools and bodies in the environment. This means that I will be focussing on how the assemblage of children-tool-number changes over time; how new materialities become part of the activity and affect the progression of activity.

## **METHODOLOGY**

The study takes place in a daycare that is located close to a North American University. It is a play-based environment in which children are free to choose from a range of activities, each of which offers different sets of materials. In order to fit into the environment, I placed the iPad on the carpet in the corner of the room and children were free to come or leave as they wished. At the beginning, many children crowded around the iPad, jostling to get a chance to play, but after about twenty minutes, a small group of four children formed and stay for the remaining twenty minutes. The analysis will begin at the point the group forms, when it was possible to record the interactions and actions of the children. The four children in the group were all between 3 and 4 years old. There were three girls and one boy. The focus of the five-minute interaction reported in this paper involves a task where children are asked to use many fingers to make numbers (instead of counting up to a given number one finger at a time). I have chosen to focus on this 5-minute interval because it was the beginning of the group’s work together, and there is a clear change in the way the way they use the iPad to create numbers.

## **USING MORE THAN ONE FINGER**

The children had spent some time tapping on the iPad, creating lots of numbers. They tended to use their whole palms (and sometimes both palms) or just one finger at a time. Based on prior pilot work, Sinclair and SedaghatJou (2012) conjecture that one-finger tapping (as opposed to two or more) would be more common amongst younger children, whose sense of number was dominated by rote counting and that increased experiences with number would engender the use of more fingers.

I asked a pair of children (Owen and Ramona) to try to “get four together”. They each tapped the screen with one finger once, making the iPad say “one”, “two” then almost at the same time, so that the iPad said “four” (see Figure 2a). They stopped tapping. Romana screamed and Owen said excitedly “I did it”. I then moved the iPad to face the two other girls, Katherine and Christine, and invited them to do make four together. They each tapped with one finger, without listening to the iPad, or following each other’s tapping, stopping when the iPad said “eight”. Christine sat back, Romana screamed and Katherine kept tapping. Owen pressed Reset. They tried again, this time going up to sixteen. Thinking that perhaps the girls were having difficulty coordinating their work, I asked Katherine to “get to four by yourself”. She placed all five fingers on the iPad, which said “five” (see Figure 2b). Romana screamed, Katherine and Christine laughed and Owen pressed Reset. Prompted by her use of more than one finger, I asked her to “use lots of fingers to get to four”. She placed her whole palm on the iPad. She tried again and crashed the application. When it was her turn, Christine did the same thing. I then moved the iPad in front of Owen.

N:                    You try to use lots of fingers to get to four.



- Owen: *Moves his whole hand, stretched out, toward the iPad, then pulls it back, looking at his fingers, then tucking his thumb in and touching the screen with four fingers (see Figure 2c).*
- iPad: Four
- Ramona: Ah. (*Very high pitch*) He did it.
- N: He did it.



**Figure 2(a) Ramona and Owen working together to get 4; (b) Katherine placing her five fingers on the screen; (c) Owen placing four fingers on the screen.**

Katherine reached over to grab the iPad but I pulled it away and offered it to Ramona. She raised her hand in the air and lifted her fingers one by one, then placed four of them on the screen. The iPad said “six”. Noticing that she had inadvertently tapped other parts of her hand on the screen, I rolled up her sleeve and let her try again. This time she tapped successively four times on the screen so that the iPad said “one, two, three, four”. When asked to do it with lots of fingers, Ramona placed her whole palm on the screen, getting “twelve”. She screamed, rolled over and, when I ask if that’s what she wanted, she exclaimed “no!”. Christine was next to try with “lots of fingers” and slapped the iPad with her whole right hand. The children all laughed; Owen pressed Reset. Christine tried again, as did Katherine, who imitated Christine’s hand slap.

I then picked up the iPad and said that we would try something different. I gave the iPad to Owen and asked him to “use lots of fingers to get to two”. He immediately put out his hand with his index and middle fingers outstretched and placed them on the screen. When it was Christine’s turn, she also extended two fingers, but got “three”, having inadvertently touched the screen with another part of her hand. She tried three more times, always holding out her two fingers, but each time making too many numbers. Katherine decided to press Reset and to tap successively twice. Then Christine placed two fingers on the screen and got “two”. I moved the iPad to Ramona, who lifted her left hand deliberately, extending one finger at a time and placing two fingers on the screen to get “two”. I then asked Owen to “do three with lots of fingers”, which he did successfully, as did Christine. Katherine then successively placed three fingers on the iPad, as did Ramona. I congratulated the children for all doing “three with lots of fingers” and asked them to “do four”. Owen succeeded quickly, as did Christine and Katherine. Ramona stretched out four fingers, but placed her palm on the screen so that the iPad said “five”. This happened twice, and then she decided to tap successively four times.

## **ORDINAL AND CARDINAL TOUCHING**

By the end of this five-minute time span, all the children could use their four fingers all at once on the screen, to make the iPad say “four”. Owen was able to do this early on, but not the three girls. This might be seen as evidence of subitising 4, or treating ‘four’ cardinally rather than ordinally. Such an interpretation, however, fails to account for the important role played by the gestures, the voice of the iPad, the developing expertise around tapping and the very concept of four. Also, it

should be noted that producing four fingers and using them to touch the screen differs from the usual subitising tasks of determining the number of given objects.

At the very beginning, Ramona and Owen used their fingers to create numbered circles and count up to four while Christine and Katherine used them simply to create numbered circles, without attending much to the number of circles on the screen or the number spoken aloud by the iPad. Indeed, Ramona and Owen managed to get to 4 by using the feedback from the iPad to stop tapping once they heard “four”, but Katherine and Christine tapped, without anticipating or listening for four. Perhaps the excitement of tapping outweighed the interest in performing well on the task. However, even on her own, Katherine did not use her fingers to get to four. Despite having tapped with one finger previously, both Katherine and Christine tried to get to 4 by slapping the iPad with their hands, all the while giggling. For them, four was some big number, unrelated to the small numbers they were familiar with, and unrelated to what they could do with their fingers.

Owen’s deliberate hand gesture introduced a new element to the assemblage; all the children saw his hand, which became joined up to the vocalised four of the iPad. The children heard that the gesture made four all-at-once, without passing through other numbers. When it was her turn, Ramona seemed to take up Owen’s gesture, but used her fingers differently, counting up to four with them instead of stretching them all out at once. She had difficulty getting the iPad to say “four” though, and decided to get to four by sequential tapping. Now the verbal sequence “one, two, three, four” had joined the assemblage. The intra-action between Ramona’s fingers and the iPad thus became ordinal in nature, so that four was what came after 1, 2 and 3. Despite seeing/hearing the Ramona-fingers-iPad intra-action, Katherine and Christine both stretched out their fingers, like Owen, but tapped with their whole hands on the screen. It was as if they were mimicking Owen’s gesture, without paying attention to the number of outstretched fingers or to the way in which those fingers tapped the screen. Ramona and Owen responded by screaming and resetting, respectively, obviously aware that the action was incorrect and that the girls needed to try again.

When I initiated the new round of tasks, in which the children were asked to use many fingers to make 2, then 3 and then 4, Christine and Katherine began to use their fingers very differently. Whereas a few minutes ago, when asked to “do four”, Christine and Katherine had slapped the screen almost haphazardly, this time they both held up and placed four fingers on the screen. The speed at which Christine first held out her two fingers suggests some kind of subitising, and she was confident enough about using these two fingers that she was willing to try several times to get the iPad to do as she wished. Katherine’s impatience, and choice to proceed with sequential taps, may have stemmed from a strong ordinal meaning for 2, or from having watched Ramona count up to 4. However, the speed and dexterity with which each child made 3 shows the gaining momentum of the subitising gesture. The making of three had a certain rhythm to it as each child lifted their fingers, tapped the screen, and ceded the iPad to the next child. This rhythm persisted for making 4, with only Ramona extending her fingers one at a time. Her shift from using all three fingers at once to using her fingers sequentially suggests that she was not mimicking the other children’s gesture; she knew, however, that counting up to four on her fingers would produce four numbered circles on the screen, as well as the sound “four”.

## **DISCUSSION**

Knowing how to put a certain number of fingers on the screen requires not just extending the correct number of fingers, but also making sure that only these fingers (and not other parts of the hand), touch the screen. When Ramona failed to achieve what she expected when she placed her fingers on the screen, she opted for the more careful, and also slower, strategy of tapping sequentially. Within this situation, Ramona’s concept of number tended to the ordinal, as her fingers lifted up one by one and she found it easier to tap sequentially. For Owen, Katherine and Christine, the concept of

number did not involve counting or sequence, but instead a visible, motor gesturing that could tap the screen.

The short episode shows learning occurring in that three of the children were able to do something they could not do at the beginning. I have argued that this learning cannot be separated from the materialities and interactions of the situation. *TouchCounts* was thus centrally involved in the learning. However, what particular role did it play in supporting this learning? Based on the above analysis, three features seem relevant: (1) the children could create numbers one by one or all-at-once, without having to be familiar with the numbers they were creating; (2) the oral numbers could be connected to the tapping, providing feedback that encouraged self-correction, without external prompting. The emotional engagement of the children—the screams, giggles, smiles, as well as the concentration, confusion and cooperation—cannot be overlooked. Further analysis of the affective flow in this episode would provide even greater insight into the nature and movement of the assemblages.

## NOTES

1. The other two principles are: (4) it doesn't matter in which order objects are counted; and (5) it doesn't matter whether the items in the set are identical.

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## **COMPUTER SIMULATION IN SOLVING PROBLEMS FROM PROBABILITY**

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*Probability is part of school mathematics which students are often afraid of it. Not infrequently it is taught by using formulas and without great clarity. Teachers do not usually use computers for simulation of random chances in the theory of probability in the Czech Republic. Students rarely come across some example which should be solved by computer simulation. Although there are exceptions. We would like to show that computer simulation can play an important role in understanding of secondary school students - for example while solving problems from random walk. In this case, students can use stochastic graph. Then it is possible to build a formula for the solution of the problem. In our lessons, we have seen that for a lot of students it is not illustrative enough to be it. They can't interpret it because they don't understand the graph. At that time computer simulation has the irreplaceable role. It gives an illustrative and more comprehensible solution (or a way to solve it) of this problem.*

### **INTRODUCTION**

There are numerous reasons why students think stochastic is demanding. One aspect is they can not often imagine the situation from some example. Teachers rarely initiate students to sketch diagram to model situations. Not many teachers use aids for lessons such as cubes, matches, or some lottery equipment. Computer simulations are involved much less.

Computer simulations do not have to be too complicated (create, apply, and understand). Problems which will be demonstrated in this article, show it. Even more it is possible to use software which teachers quite widely uses in teaching – program GeoGebra. No less important argument is that students can simulate some examples themselves.

We focused on three examples that can be solved with program GeoGebra. Our paper describes the designing of materials in GeoGebra. In addition, it summarizes the feedback of students who have passed our trial hour.

### **SOLVING PROBLEMS FORM PROBABILITY**

In our lessons, we have seen that stochastic is a quite difficult part of school math and students often have a problem with solving problems from probability.

Students look for some formula according to a similar example but unfortunately it often fails. It is necessary that students have to find some regularities first and then they have a bigger chance to find an appropriate formula.

They can use more ways for solving - outlining the situation, writing some points step by step, describing a situation or construct a stochastic graph (it is an advanced level of outlining).

Another way is doing experiments and simulations – for example throwing a dice. We can bring few dices in the classroom and try it with students. Then we can make a table with results. For a better and accurate idea it is good to complete the experiment with a computer simulation – many of these are created in programming languages and freely available in the internet.

## RANDOM WALK

Random walk belongs to an indispensable part of probability and is included in school curriculum too. It is a random process, a path that consists of a succession of random steps. This kind of problem is unusual for student and they do not know how to solve it without stochastic graphs. Computer simulation can help them to find the way how to construct and calculate these graphs.

As we have already mentioned we would like to show three examples to random walk. We used program GeoGebra in this examples. All of them are intended for high school students.

First example was simulated students from two Czech grammar schools during Week of science [1]. It is one of the basic examples of probability. It is usually the introductory example, which should help to reveal patterns in examples of a random walk.

Second example is from a series of classic textbook examples of a random walk. It is one of the more difficult kinds examples from probability, but solvable. This example makes easier to deepen the understanding of random walks. It is also an example which is not difficult to simulate and students can simulate it in math lessons alone.

Third example is much more complex. However, it is intended for high school students. It was presented at the first round of Mathematical Olympiad [2]. Accordingly it is assumed that only the best students know how to solve it. But if we use computer simulation in a math lesson it can increase the number of students who know how to deal with this problem.

### 1<sup>st</sup> example

Entering is: *We walk 5 steps. One step means to move about one place forward or back. What place has the biggest probability? What can you read from the result of the simulation?* Students usually need graph or simulation to find a solution because they lack the clarity and sufficient experience with a similar type of examples. In Figure 1 we can see the simulation of the 1<sup>st</sup> example. There are 3 windows – Algebra, Graphics and Spreadsheet. There are 6 free object – 1 line ( $a$ ) and 5 sliders (algebra window). If we want to make sliders for this problem we choose Object Prosperities → Slider. We have to insert limits from -1 to 1 there which mean a step backward and forward, and length of interval – 2. We cannot use 1 because the step cannot be zero (it would mean standing). There are 12 points in dependent objects. Point  $A$  has a definition  $Point[a] + i + j + k + l + m$ .  $B - R$  are points where  $A$  can occur after one to five throws coins.

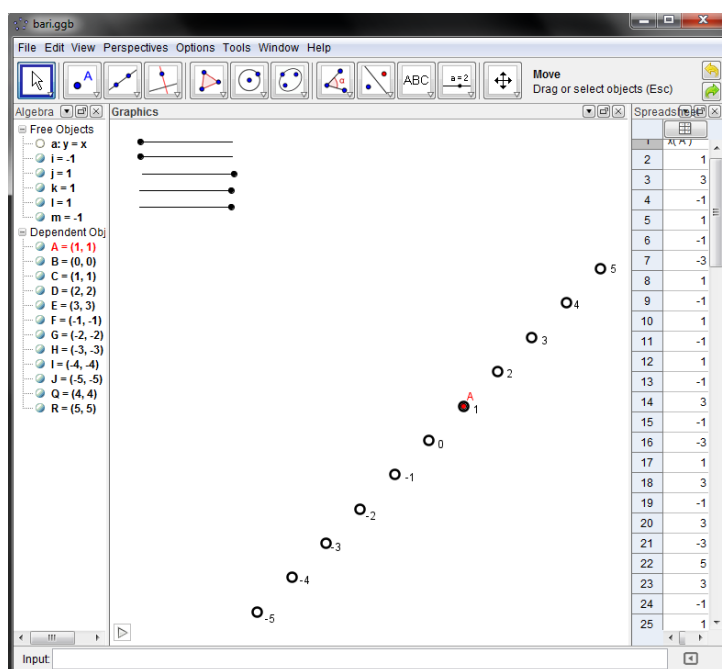
You can see sliders and points in the graphics window. Point  $A$  moves in the points as when five coin toss (thanks sliders) if you press *play* in graphics window. We can record to spreadsheet the location of the point  $A$ . Students can use results from spreadsheet, put it in Excel or other similar spreadsheet and find frequency of every point. If we have enough steps (few thousands) we divide frequency of each point the total number of steps.

After the simulation, students created a visual chart from which derived the Gaussian distribution. Moreover, they derive the following conclusions:

A point has always ended in an odd position. This is caused by the number of steps, which is also odd.

If we take an even number of steps, the point is in even position (in the beginning too). If we want to have a chance occupy even position after odd steps, we have to add the opportunity to remain in the place.

The greatest probability is symmetric in the points which are closest to the starting point.

Figure 1 – simulation for the 1<sup>st</sup> example

## 2<sup>nd</sup> example

The second example is similar to the first one. *We walk 12 steps - forward or back. What has bigger probability - to get at position 1 or 2 and at -1 or -2? Which probability does position 12 have?*

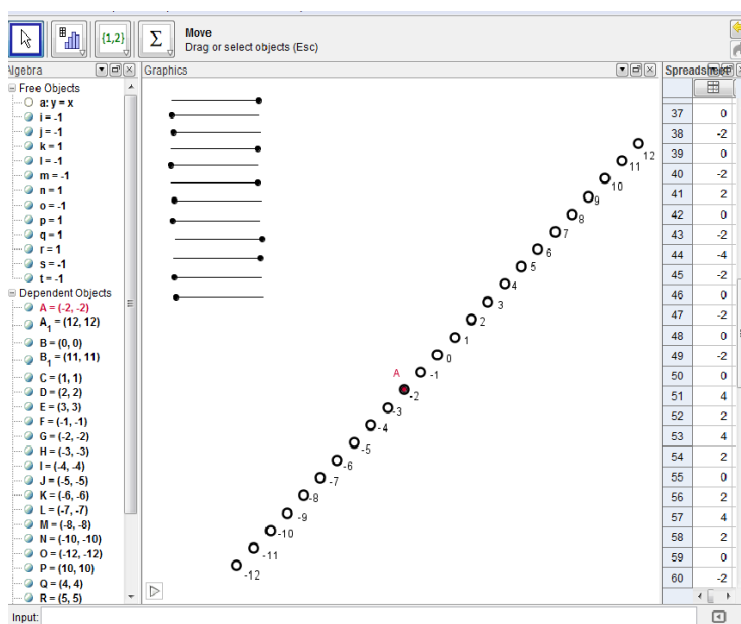


Figure 2 - Simulation of the 2nd example

We got similar simulation (Fig. 2) from which students made a table in Excel. (Table 1)

| probability | - | +  | probability |
|-------------|---|----|-------------|
|             |   | 0  | 0,22        |
| 0,19        |   | 2  | 0,19        |
| 0,12        |   | 4  | 0,12        |
| 0,06        |   | 6  | 0,058       |
| 0,021       |   | 8  | 0,019       |
| 0,003       |   | 10 | 0,004       |
| 0,0004      |   | 12 | 0,0002      |

Table 1: Probability of points from the 2<sup>nd</sup> example

Students saw symmetry according to the starting point again and that probability for all points is very close to real probability. They created a simple schema representing stochastic graph based on the table and calculated probability of every point.

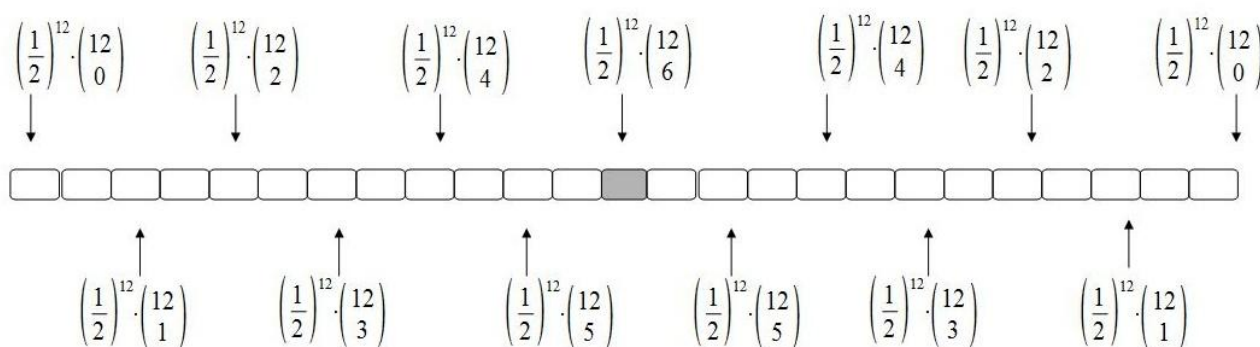
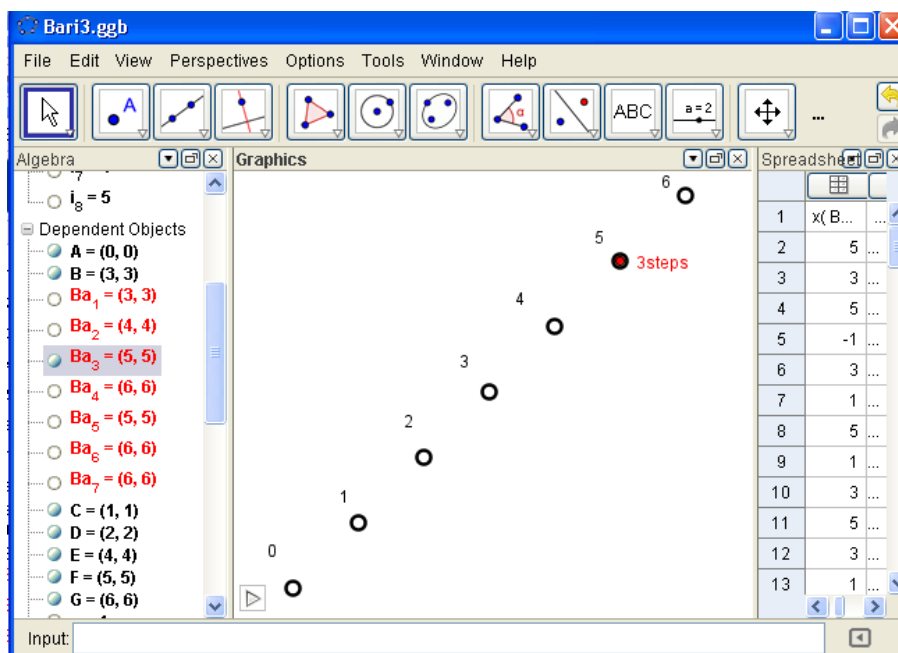


Figure 3 – Scheme of 12 steps random walk

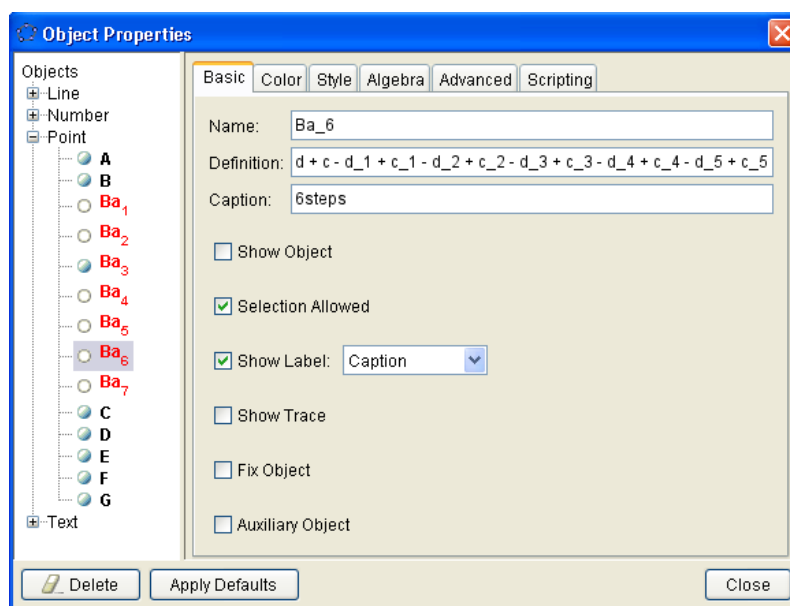
Moreover it is quite easy to make this simulation. If students know GeoGebra it takes only few minutes with help of the teacher.

### 3<sup>rd</sup> example

Third example has this assignment. *Two players (A and B) play this game: They have 6 positions (1 to 6). Game starts at position 2. A is the winner if he is the first at position 1. B is the winner if he is the first at position 6. They throw the dice - numbers which can be delighted by 3 mean step back, others mean step forward. Do they have the same probability to win?* This assignment is not so different from the first one but it is general. It means that the task will be more difficult. And not only the calculation but the creation a simulation too. Therefore most of high school students can not simulate this example alone. It will be used mainly for better phenomenon of this example. It is impossible to simulate infinity steps. But we can create it step by step for a few steps. It can help students to understand how the point behaves first after one step - at 50% it ends at position 1, but how the number of steps increases, the point 6 has bigger probability.

Figure 4 – Simulation of the 3<sup>rd</sup> example

We can see the simulation of the third example (Fig. 4) Situation is quite similar to previous two examples. But points  $Ba_1$  to  $Ba_7$  are the most important. These points illustrate the shift after 1-7 throws the dice. If we want to make this point, we have to use sliders and other points. We can see the definition in the picture number 5.

Figure 5 – Definition of points  $B_a$



This example is really difficult, but simulation can help some students to make this calculation

$$p_2 = \frac{1}{3} + \frac{2}{3p_3} \quad p_3 = \frac{1}{3p_2} + \frac{2}{3p_4} \quad p_4 = \frac{1}{3p_3} + \frac{2}{3p_4} \quad p_5 = \frac{1}{3p_4} \cdots \rightarrow p_2 = \frac{15}{31}$$

for A and similarly for B.

$$p_2 = \frac{16}{31}$$

It is nearly impossible in order that all students know how to solve this example after the simulation. But all of them can understand that the probability for one is with more step smaller and therefore probability for the number one and six could be either similar or that six has bigger chance.

## CONCLUSION

Above examples show that we can use software which was designed primarily for geometry for examples from stochastic (in our case from random walk) too.

The other important thing is that some simulation are not so difficult and students are able to create it themselves or with insignificant help of the teacher.

In conclusion, the simulation facilitates understanding and helps to find a way how to create graphs and subsequently calculation.

If we would be more interested in this topic, we can find other suggestions in [5], [6], [7].

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## **THE IMPACT OF TRAINING COURSES ON MATHEMATICS TEACHERS' USE OF ICT IN THEIR CLASSROOM PRACTICE**

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*Are teachers more likely to incorporate ICT in their teaching if they have done (a) a training course that is specifically devoted to exploring ICT packages to use in the classroom or (b) if they have done a training course that incorporates ideas for using ICT as part of a course focused on developing specific subject knowledge and pedagogy? As a significant provider of professional development for post 16 mathematics teachers in England, the Further Mathematics Support Programme reflects on the impact of their training courses on teachers' use of ICT in the classroom.*

### **INTRODUCTION AND CONTEXT**

The Further Mathematics Support Programme (FMSP), managed by Mathematics in Education and Industry (MEI) has been working to ensure universal access to a pre-university qualification in Mathematics in England since 2005. An Advanced Level General Certificate of Education (GCE), commonly referred to as an A level, is a pre-university qualification offered by schools and colleges. A levels are typically required for a student to gain entry to a university. Mathematics is unique in that there are two A levels available for students; A level Mathematics and A level Further Mathematics. However, many schools were unable to provide access to the latter. One reason for this was the shortage of teachers with appropriate skills at this level. Since the inception of the FMSP in 2009, the number of students studying A level Further Mathematics has risen by 125%, which has further increased the demand for teachers to acquire the necessary skills and experience to teach mathematics at this level.

The FMSP runs an extensive programme of professional development courses aimed at both improving teachers' subject knowledge and developing pedagogical skills. Each year approximately 2000 teachers take part in FMSP training courses.

The FMSP's training courses broadly fall into one of three types: a one day course, a series of live online sessions lasting 10 hours and a 14 month programme which includes four one-day courses and a series of online sessions. The amount of ICT that course tutors use in these courses varies greatly. In particular, in the courses where ICT is used, the FMSP is interested to find out which type of training course has the greatest impact on a teacher's use of ICT in the classroom. In 2012, approximately 90% of teachers who enrolled in the FMSP's programme of professional development took a one day course.

This paper looks at the impact of incorporating ICT into our one day courses which we have divided into two types: training courses that are specifically devoted to exploring ICT packages and training courses that focus on developing specific subject knowledge, incorporating ideas for using ICT.

### **OBJECTIVES OF THE RESEARCH**

This research is focusing on one day stand-alone courses. Approximately one tenth of these one day courses each year are devoted specifically to training teachers to use maths specific ICT software and hardware, for example graphing calculators, Geogebra and Autograph. In addition, approximately one fifth of the one day courses have significant elements of ICT embedded within

the subject knowledge focused content. Whilst the content of these days is driven by the need to improve subject knowledge and associated skills of the delegates, ICT is sometimes incorporated where it may illuminate the mathematical concepts, with the aim of improving teachers ICT skills and confidence so that they may be more willing to incorporate it in class.

In cases such as this, there is emerging anecdotal evidence that our courses not only support subject knowledge and pedagogy but also build confidence for teachers to engage with the use of maths specific ICT without explicitly aiming to do this. Thus, the research question for this work was – Are teachers more likely to incorporate ICT in their teaching if they have done (a) a training course that is specifically devoted to exploring ICT packages to use in the classroom or (b) if they have done a training course that incorporates ideas for using ICT as part of a course focused on developing specific subject knowledge and pedagogy?

## **METHOD**

Twenty one-day courses, scheduled in 2012 were identified. One half of these courses, attended by a total of 108 delegates, were specifically focused on ICT. The other half, attended by 115 delegates, were courses where ICT was incorporated but not the main focus.

Of the ten courses that specifically focused on ICT, seven focused on Geogebra, one on Autograph, one on graphing calculators and one looked at a variety of ICT applications in a teaching context.

Of the ten courses where ICT was not the focus, nine of the courses were supporting Pure Mathematics topics, and one course focused on Statistics.

Practising teachers who had attended these courses were contacted by email and invited to complete an online survey about their use of ICT with post 16 mathematics classes. For the purpose of this study, the teachers were instructed to consider their use of maths-specific software or hardware such as graphing calculators, CAS calculators, spreadsheets, dynamic geometry software, graphing software but not to include the use of an interactive whiteboard, Powerpoint, scientific calculators or Maths websites.

Key questions in our survey asked teachers to describe their practice before and after their course. In particular, we asked them to indicate the frequency with which they used ICT in their lesson preparation, used ICT to demonstrate in class and gave their students opportunities to use ICT in class. We also asked teachers a direct question about whether as a result of attending, the course had changed the amount they used ICT in their teaching.

## **SUMMARY OF FINDINGS**

We received responses from 39 of the 108 who attended courses specifically for ICT and 32 of the 115 who attended courses where ICT was incorporated. A summary of the responses to the key questions is shown in the tables below. The first two tables indicate teachers' responses when asked to describe their practice in three areas, prior to and after the course.

|                      | I use ICT in my lesson preparation |                  | I use ICT to demonstrate in class |                  | My students use ICT |                  |
|----------------------|------------------------------------|------------------|-----------------------------------|------------------|---------------------|------------------|
|                      | Prior to the course                | After the course | Prior to the course               | After the course | Prior to the course | After the course |
| every lesson         | 25.6%                              | 25.6%            | 15.4%                             | 15.4%            | 5.1%                | 5.1%             |
| at least once a week | 35.9%                              | 43.6%            | 35.9%                             | 51.3%            | 7.7%                | 17.9%            |
| once a month         | 15.4%                              | 17.9%            | 30.8%                             | 25.6%            | 30.8%               | 43.6%            |
| rarely               | 23.1%                              | 12.8%            | 17.9%                             | 5.1%             | 51.3%               | 30.8%            |
| never                | 0.0%                               | 0.0%             | 0.0%                              | 2.6%             | 5.1%                | 2.6%             |

**Table 1: Comparing responses from 39 teachers on ICT specific courses about their use of ICT before and after their training course**

|                      | I use ICT in my lesson preparation |                  | I use ICT to demonstrate in class |                  | My students use ICT |                  |
|----------------------|------------------------------------|------------------|-----------------------------------|------------------|---------------------|------------------|
|                      | Prior to the course                | After the course | Prior to the course               | After the course | Prior to the course | After the course |
| every lesson         | 40.6%                              | 43.8%            | 18.8%                             | 18.8%            | 0.0%                | 0.0%             |
| at least once a week | 28.1%                              | 28.1%            | 31.3%                             | 46.9%            | 18.8%               | 28.1%            |
| once a month         | 15.6%                              | 15.6%            | 37.5%                             | 25.0%            | 15.6%               | 21.9%            |
| rarely               | 15.6%                              | 12.5%            | 12.5%                             | 9.4%             | 56.3%               | 43.8%            |
| never                | 0.0%                               | 0.0%             | 0.0%                              | 0.0%             | 9.4%                | 6.3%             |

**Table 2: Comparing responses from 32 teachers on ICT incorporated courses about their use of ICT before and after the training course.**

Table 3 indicates teachers' responses to the direct question asking them whether as a result of attending, the course had changed the amount they used ICT in their teaching.

| As a result of this CPD course I have used ICT in my teaching... | much more than before | slightly more than before | about the same as before | slightly less than before | much less than before |
|--|-----------------------|---------------------------|--------------------------|---------------------------|-----------------------|
| Courses that are specifically ICT based (39 responses)           | 7.7%                  | 51.3%                     | 38.5%                    | 0.0%                      | 2.6%                  |
| Courses incorporating ICT (32 responses)                         | 0.0%                  | 34.4%                     | 65.6%                    | 0.0%                      | 0.0%                  |

**Table 3: Comparing responses from the two groups**

## DISCUSSION AND CONCLUSION

### Comparing the results in tables 1 & 2

Looking at the responses for lesson preparation it can be seen that some of the teachers on the ICT specific course indicated that they were more likely to use ICT in their lesson preparation after attending the training. However, from the results in Table 2, it can be seen that teachers who attended an ICT incorporated course indicated that there was little or no impact.

In both groups there was a slight increase in the number of teachers using ICT to demonstrate in class. The ICT specific group seem to have a marginally greater increase.

Some teachers on the ICT specific course indicated that they were more likely to involve students in the use of ICT through student-centred tasks after attending the training. However, from the results it can be seen that teachers who attended an ICT incorporated course again indicated that there was little or no impact.

The results in Table 1 and Table 2, although comparing the overall differences for each group, only provide a summary of each of the two groups as a whole. Table 4 breaks down the responses in order to ascertain to what extent each teacher's practice had increased, stayed the same or decreased after the course.

By considering the changes in practice of individual respondents shown in Table 4, it can be seen that that there was no change for the majority of teachers in all categories in the amount they used ICT in the areas surveyed. However we do note that for both types of course there has been a shift towards more student use of ICT in class and that this is slightly more marked among the group who attended the ICT specific courses.

From this it appears that the impact of a one day course, albeit ICT focused or incorporated, is limited. Crisan said "teachers' own learning experiences with ICT is of paramount importance in their uptake and implementation of ICT, more powerful than simply adopting the resources and ideas presented to them by others or acquired through attending professional development programmes, for example" (Crisan, C.: 2008)

However, when we consider the results in Table 3 reporting the teachers' perception of the impact of their training it appears to contradict the evidence. More than half of the teachers that attended the ICT specific course and approximately one third of teachers who attended an ICT incorporated course perceived that there had been an impact on their practice as a result of the training but this is

not really supported by the analysis in Table 4. Overall teachers seem to overestimate the value of a one day course on impacting on their use of ICT in the classroom.

| After attending the course... | I use ICT in my lesson preparation                     |  | I use ICT to demonstrate in class                      |  | My students use ICT                                    |  |
|-------------------------------|--|--|--|--|--|--|
|                               | Increase   | 17.9%  | 9.4%   | 25.6%  | 15.6%  | 33.3%  |
| Stayed the same               | 82.1%  | 90.6%  | 71.8%  | 84.4%  | 64.1%  | 71.9%  |
| Decrease                      | 0.0%   | 0.0%   | 2.6%   | 0.0%   | 2.6%   | 0.0%   |
|                               | Courses that are specifically ICT based (39 responses) | Courses with ICT incorporated (32 responses) | Courses that are specifically ICT based (39 responses) | Courses with ICT incorporated (32 responses) | Courses that are specifically ICT based (39 responses) | Courses with ICT incorporated (32 responses) |

**Table 4: Comparing the three questions about how individual responses have changed after the course, for the two types of course**

It appears that the courses impact slightly more on teachers who have chosen the ICT specific courses however the sample may well be self-selecting as these teachers have chosen to do an ICT course so we would expect them to be more receptive to the ideas. Teachers on ICT incorporated courses may well be less receptive as their primary purpose in attending the course will usually be to improve their subject knowledge in a specific area of mathematics rather than developing their ICT skills. They may also be less inclined to use ICT when they return to the classroom if they are just becoming confident with the mathematical concepts of the units they are teaching.

A longer study would be needed to ascertain whether any increase was sustained and whether it had impact beyond the A level classroom.

On the basis of this research and our extensive work with teachers, we suspect that the type and length of course that we offer may have an influence on the extent to which a teacher's engagement with math specific ICT has an impact on their practice. The FMSP's Teaching Further Mathematics course, which runs for a full year, has maths specific ICT embedded throughout. Teachers have access to Autograph, Geogebra and Graphical calculator emulators and are given opportunities to use the technology alongside learning the mathematics.

In 2011 the JMC produced a report which stated "The development of electronic professional development resources for teachers, possibly offered as e-CPD' is an approach that has been explored (but not exploited)..." (Digital technologies and mathematics education, Joint Mathematical Council (2011), Pg 26)

The Live Online Professional Development (LOPD) programme (de Pomerai, Tripconey 2009 and 2011) offers e-CPD courses which are both ICT specific and have ICT incorporated. The online sessions are recorded giving teachers the chance to revisit and refresh their knowledge allowing them more opportunity to embed the skills within their teaching.

It is our intention to continue this research to try to measure the impact of ICT training delivered in the FMSP's extended and online courses.

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## TRANSITIONS BETWEEN MICRO-CONTEXTS OF MATHEMATICAL PRACTICES: THE CASE OF ARC LENGTH

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*The study presented here is concerned with transitions between micro-contexts of mathematical practices. These micro-contexts are determined by the use of different software and material. The main focus is on a task in which students had to participate in investigating activities related to the arc-length. The task completion demonstrates a productive interaction of tool use which combines "instrumental approaches" to achieve the given purpose. At the same time it provides a framework for the observation of knowledge and skills transfer during transitions from one micro-context to another.*

### INTRODUCTION

Research in the area of the “transition” of knowledge seeks to explain the reason in some cases the transition from one context to another is possible, whereas some others it isn’t. The discussion debate around the issue of “transfer” has revealed the following two types: the Similarity transfer as well as the Dynamic transfer. Similarity transfer happens when people are able to identify that they have well- formed ideas beforehand and be able to use them advantageously in order to describe another situation in a new context: “*this is like that*”. In contrast, in the Dynamic transfer, the context helps people link elements (knowledge, skills, or abilities), so that they can create new “coordination classes” (diSessa & Wagner, 2005): “*this goes with that.*”

With the aim to study issues of knowledge transfer we designed and developed a research project with field research, different communities of 8<sup>th</sup> grade students and in different contexts. What we present here relates to the activities that were implemented in one grade 8 class of students. Our aim was to engage the students in the use of tools and encourage them to investigate mathematical concepts related to the notion “*arc-length*”. The role of the tools and their transformation, when used for the carrying out of specific activities, comprise the major issues currently of concern to researchers in the area of ICT. The tools used in the present research were a) tactile material (paper roll, paper tapes, protractors) b) Geometer’s Sketchpad, (a dynamic geometry environment) c) Microworld pro, (a Logo-based environment /Turtle Geometry) software) as well as c) robotics material (NXT-Mind storms-lego). The teaching contexts in which the materials were converted from artefacts to instruments by the students were considered to be micro-contexts. We opted for those four tools-contexts as they present Different: potentialities, restrictions, access, epistemological foundation of the representations. Our intention was that students with the productive use of tolls and the coordination of knowledge, skills, or abilities, manage to reach the formulation of the analytical relation, related to the arc’s length, without any explicit statements and implicit references. While in the overall research our goal was to interpret the function of the transfer within the theoretical framework of “instrumental approaches” [1], due to space considerations, in this paper we will focus on tuning of those activities that formed the lens to study the transitions between micro-contexts of mathematical practices.

### METHODOLOGY- THE SETTING OF THE RESEARCH

As our interest lies, on the one hand, on studying learning in specific contexts and, on the other, on exploring how students involved in learning activities think, we believe that anthropological research orientation is a suitable kind of approach. So we utilised ethnographic research techniques for data collection. The ethnographic research was supplemented by the implementation of teaching



experiment techniques. The researcher intervened at various stages using pre-designed activities which intended to trigger events to be studied. The role of the researcher was principally that of a participant observer, who focused either on observation or on participation accordingly through the assignment of relevant activities or the learning of some software. The ethnographic equipments that we used, included tape recordings, field notes, and the material handed in by the students both in final form and during the tackling of an activity.

The activities presented and analyzed here were implemented in one grade 8 class (20 students), during eight hours of instruction in a range of two months. The four meetings took place in the typical classroom; the rest in the computer lab. The students worked jointly at five.

### **Design and implementation of the research**

The central question that that students were invited to investigate was: “how much is it needed to turn the paper roll (toilet paper) as to get just half a meter of it?”

Our first concern was to concoct a boundary object [2]. The paper roll played in our research this role. It gave rise to develop a communication channel between the meanings that we thought concerned the subject area of arc-length and the world of the cultural meanings of the students associated with this specific object. This communication determined two different paths of development: the community of the students and the community of the researchers. These two communities seem to intertwine influencing each other. The development of mathematical meanings from the students’ side affected the design development activities while the introduction of new activities by the teachers determined the students’ actions. Our actions, finally, strike a balance between teaching and research. We consider that knowledge is the result of interactions between the students and the special environment that the researcher organized under a “didactical situation” (Balacheff, N, 1993). The arc length as a concept is embedded in a conceptual field (Vergnaud, 1983) determined by the way it is used, its various representations and its relatives concepts. The goal of Didactics is to accomplish conceptual coordination of all the contracting notions. The standard way of teaching consists in using the notion of proportional amounts in order to compose the detailed correlation of the quantities, arc length ( $s$ ) and its corresponding central angle ( $\varphi$ ). Below we present the implementation elucidating some characteristic points.

### **Episodes/ Discussion of the Episodes**

The class was divided in four groups. The tools were distributed (paper rolls, tape measures and protractors) and the central question – already been mentioned, was posed. The students were invited to perceive the mathematical concepts involved in the solution of the problem and develop effective strategies for solving it. The analysis of the dialogues shows that the students’ strategies were directly related to the handling of the paper roll. The following strategies emerged.

Student A: " .....*Since he tells us half meter, we shall rotate half circle....*."

In student A’s action schema we notice Similarity Transfer. The notification of his point of view, motivates his team and with the use of “trial and error” towards the direction of rejection. During the discussion process the students swivel the paper roll about its axis using a pencil.

Student B: " *Pull the paper to see by looking at it how many circles do.....*"

In student B’s action schema we notice Analogical Transfer [3]. By using the corresponding rotation formula about the axis, they measure half meter with the number of turns.

Student C: " *I found it!! I mark the starting point of the circle... I have counted on the desk 50 centimeters. Then unroll the paper. Till here it has rotated once. Then mark again, it needs a bit more to swivel...*"

Student C's action schema, although he uses a similar logic with turns to Student B's, it differentiates as to how he coordinates the different elements of the environment as well as how to handle the paper. He doesn't rotate the toilet paper about its axis; instead he rolls it on the surface.

Student D: *"I noticed something...A spin of the cycle is almost three pieces of paper...I will count how many pieces of paper are 50 cm and multiply it by 3, no I will divide it..." Oh, it does not come exactly"*

Researcher: *"What precisely do you mean with it does not come exactly?"*

Student D: *"The pieces are four and it needs 2 more cm"*

Researcher: *"Let us suppose that you don't need 2 more cm. How would you respond to the question?"*

Student D: *"Not even four by three comes....."*

Student D's action schema is close to that of Students B's, but here, additionally, is pinpointed the process of finding a common measure. The measure counts both the rotation and the length of 50 cm. So it seems as if he wants to express the given length with the number of turns. Student's reasoning seems to be largely self-explanatory rather than based on the interaction with the environment. The non-integer outcome is an obstacle and stops the deductive reasoning.

Student E: *"...Let's weigh the length we want...."*

Researcher: *"How can we weigh it?"*

Student: *"Hm....as we weighed the squares in Pythagorean Theorem."*

Researcher: *"In Pythagorean Theorem we weighed the squares of the vertical sides with the square of the hypotenuse. Here what shall we weigh?"*

Student E: *"...The 50 cm with the entire circle."*

Researcher: *"And what shall we achieve by this?"*

Student E: *"The scales will tilt to the circle. Nothing...nothing...forget it..."*

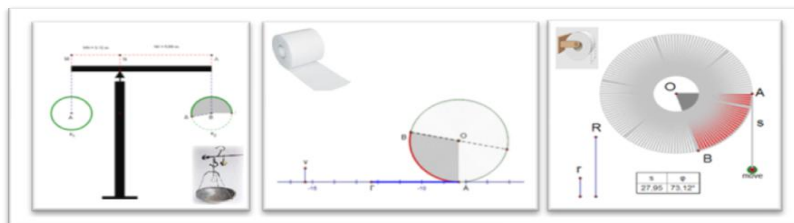
Students E'S action schema reveals a superficial Similarity Transfer. He carried elements from the previous situation, but alas he couldn't support it in the new one.

Student Z: *"Why did he hand out the protractor?"*

Student H: *"..Didn't you get the bluff? What he had us measure is more than one protractor."*

From this short dialogue between the two students it is apparent how difficult it is to identify the engaged parameters. The students remained anchored to the previous schemata; therefore not being able to move on to abstraction/generalization.

In brief, during the discussions of the First phase emerged different action schemata, and strategies as well as many cultural references in relation to the paper roll. The aim of phase Two from on the one hand was to use computing technology and create microworlds that simulate ideas of the children that came to light in phase one; on the other we wanted to direct students attention from the specific length relation (50 cm) and the respective angle, to the 1-1 length-angle relation. That is the students realize that the continuous change of the length corresponds to the related continuous change of the angle. With this in mind, we modeled the unrolling of the paper (figure 2), the rotation as well as student's E idea, which seemed to be at the moment unreal. The models were discussed in plenary of the class.



**Fig.1: Scaling    Fig.2: Unrolling    Fig.3: Rotating**

The students' engagement with the above models led us focus at greater depth in relation to the transfer of knowledge of their skills and abilities on the new surroundings. The new question been emerged was how it is possible to know the corresponding angle for each length. Illustrative is the following dialogue:

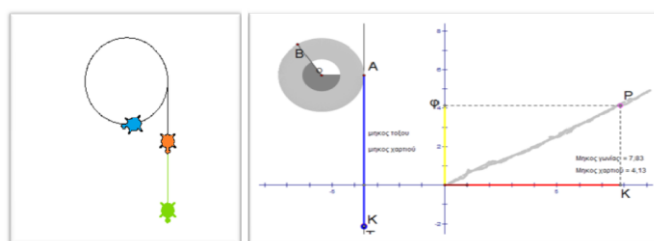
Student H: "How does the computer know the relation between the paper and the angles?"

Student E: "The bigger the paper, the bigger the angle. The quantities are proportional."

Researcher: "How do you know so?"

Student E: "Since the angle grows, so does the paper, sir "

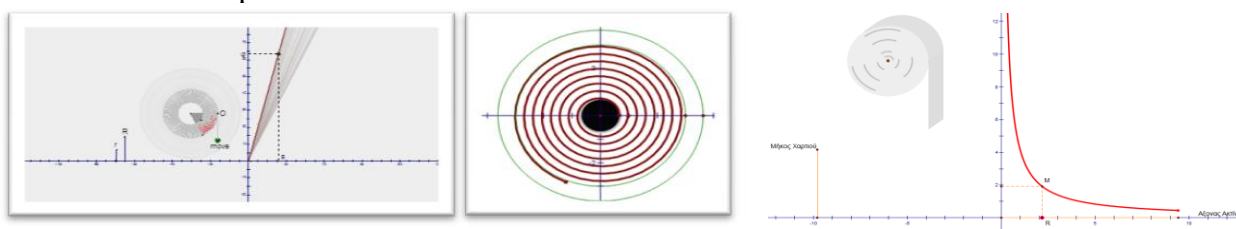
In our effort to tackle with such questions (superficial transfer) we altered the microworlds by 'dismantling' them; i.e. the unrolled paper and the angle are not in agreement. Consequently, the students would have to devise strategies related to the interdependence of the variables.



**Fig.4: Bugged microworld 1    Fig.5: Bugged microworld 2**

The first figure is indicative of the microworld created in Microworld pro. The students through programming had to debug the position of the green turtle so that it always coincides with the red turtle no matter how we turn around the paper roll (blue turtle). The second figure is illustrative of the microworld created in the environment of the dynamic geometry incorporating the rationale of the previous microworld, while the values of the lengths and angles are input as coordinates  $(k, \varphi)$  with the movement of point P. Point's P trace approximates the linear relationship  $(k, \varphi)$ .

The possibility of changing the radius of the paper roll in the microworlds, as well as questions of the type: 'The toilet roll became smaller. Now we must rotate more....' has led us to the construction of more parametric and realistic simulations.



**Fig 6: Parametric linear relation    Fig.7: Spiral    Fig.8: Parametric hyperbolic relation**

The first figure on the left (Fig. 6) depicts the microworld that is designed to study the linear relationship between length and angle with parameter the radius of the paper roll, that resulted in the construction of the algebraic relation  $\varphi = s \cdot 180 / \pi \cdot R$ . The second is based on the logic of parametric spiral so that the students manage the situation of the continuous change of the radius

while the paper is unrolled. That finally led to the design of a microworld (figure 8) with the aim that the students would construct meanings related to the co-variation of the measures angle and radius of the paper roll when the length of the roll, that we desire to cut, is fixed.

The difficulty in constructing a pragmatic model for the managing of the change of the algebraic relations with the unfolding of the paper roll drove us to introduce robotics technology which contributed substantially to the transfer of knowledge through instrumental genesis [1].



**Fig.9: Programming Fig.10:Transmission Fig.11: Rolling up and Cutting Fig.12: Communication**

The above figures cast light to the constructions and the contrivances. The first concerns a simple construction, which the students however had the possibility to program it acquiring gradual access to the software. The restatement of the problem ‘fixed length’ brought to light new solving strategies. *We par excellence* mention the brilliant idea of student E to change the position of the transmission of the motion from a central pivot of the reel to an external, which rotates two wheels transmitting the movement to the paper (fig. 10). The follow-up, much to our surprise, was utterly unpredictable. New questions-problems from the students were posed. How can we cut the paper? How can we roll it up? How can the machine let us know of the paper left over? In this way the initial object, the toilet paper roll, as a boundary object, attained important properties both for the instruction of the concept and the research. As related to the instruction an ample environment of interesting actions, whereas for the approximation of the research a lens to study the transitions.

## CONCLUSION

While the lens of our research in the beginning focused on the emerging transfer of similarity from one instrumental context to the other, *instrumental jump* (Tsitsos, et al, 2011) later on we moved to the thorough study of the transfer, as an evolution of knowledge and the utilization schemes, through instrumental genesis [1]. The various types of the chosen tools gave the students the chance for mental jumps from one instrumental environment to another, forming personal as well as communal actions. These mental passages led to the creation of mixed instrumental approaches. The knowledge and the utilization schemes transferred from one micro-context to the other were informed both by the inherent features of the former environment and the characteristics of the later environment. This combination resulted in the creation of hybrid tools. Knowledge transfer was achieved in some cases while in others it was hindered, as shown by the brief presentation of the episodes. In some cases we sought for the transition whereas in others arose spontaneously. We saw innovative ideas be born and we contributed to the development of a *dynamic type of transfer*. Decisive role played the communication route gradually being developed between the two communities. The community of practice with the tools used, produces, creates, and alters products, develops ‘*its own identity, its own atmosphere, its own crucial facts, its own thankings, its own judgments... It has its own history, created by, and shared between, and remembered by the people and the group*’ (Bishop, 1985: 26).

## NOTES

1. As mentioned in the literature (Artigue, et.al. 2010) researchers have become more and more sensitive to mathematical education, that is the processes of *instrumentalisation* and *instrumentation* that drive the transformation of a given artefact into an instrument of mathematical work. This perspective combines both the Piagetian and the Vygotskian theoretical frameworks. The instrumental approach is based on the distinction between an artefact and an instrument. The term artefact describes a human-made object, either material or symbolic. The term instrument describes a mixed entity with artefact-type components as well as *utilization schemes* (Trouche, 2005), which indicate the functional value of the instrument for the individual. These schemes concern the strategies developed by the individual in order to carry out a task. Utilization schemes are formed gradually through the use of the artefact. As a result the instrument is a mental construction of the individual and has psychological qualities. The process of the transformation of an artefact into an instrument is called *instrumental genesis*.
2. Boundary objects, as described by Etienne Wenger, are entities that can link different communities as to allow to their groups cooperate on a common task.
3. The notion ‘Analogical transfer’ is encapsulated in the notion ‘Similarity transfer’ (Star, 1989).

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## APPLICATIONS MATHEMATICA INTO TEACHING OF LINEAR ALGEBRA: THE CASE OF LEAST-SQUARES

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*In recent years, it can be seen from the available literature, not only various studies on the teaching of calculus appear but also a considerable number of studies on the teaching of linear algebra are presented by researchers. According to results, linear algebra has always been a challenge for learners and teachers. At this point, researchers agree on the use of educational technologies in the teaching of linear algebra. In this work, a teaching design of a special topic “least-squares” by the aid of Mathematica<sup>®</sup> software is developed. The tasks are formulated according to teaching of linear algebra frameworks. The steps of the design (use of matrices, plotting data, least-squares lines and curves) are explained in detail. Further research proposals are also posed.*

### THEORETICAL BACKGROUND

#### Teaching and Learning of Linear Algebra

Studies on the teaching of linear algebra first developed with a meeting to review the linear algebra curriculum by 16 educators from mathematics departments. These people, the Linear Algebra Curriculum Study Group (LACSG), as Harel (2000, p. 177) stated, generally focused on some special questions including “how students learn”; “how mathematics should be taught”, and “what pedagogical and epistemological considerations are involved in the learning and teaching of linear algebra”. Besides, LACSG also considered individual experiences and the role of applications of linear algebra in science. Finally, LACSG reported five recommendations for teaching and learning of linear algebra (Harel, 1997): i) The first course in linear algebra should take into consideration the needs of the different client disciplines (such as engineering, computer science, economics and statistics to name a few; ii) The first course in linear algebra should be matrix-oriented; iii) The first course in linear algebra should consider students’ need and interests as learners; iv) Technology should be used as part of the course; and v) Every mathematics curriculum should have at least a second course in matrix or linear algebra as a priority (cited as in Zamora, 2010, p. 15). In light of these suggestions, Harel (2000) developed a theoretical framework for teaching and learning of linear algebra. This framework consists of three major principles: the *concreteness* principle, the *necessity* principle and the *generalizability* principle. The concreteness principle is related to students’ pedagogical needs and according to Harel (2000), the premise of the concreteness principle is that students build their understanding of a concept in a context that is concrete to them (p. 180). Besides this principle states that “for students to abstract a mathematical structure from a given model of that structure the elements of that model must be conceptual entities in the student’s eyes; that is to say, the students have mental procedures that can take these objects as inputs” (Harel, 2000, p. 180). The necessity principle states that “for students to learn, they must see a need for what they are intended to be taught; by ‘need’ is meant an intellectual need, as opposed to a social or economic need” (Harel, 2000, p. 185). In Harel and Sowder (1998), three intellectual needs were defined: the need for *computation*, the need for *formalization* and the need for *elegance*. The third principle stated is that “when instruction is concerned with a ‘concrete’ model, that is, a model that satisfies the concreteness principle, the instructional activities within this model should allow and encourage the generalizability of concepts” (Harel, 2000, p. 187). To be included in the teaching and learning processes of linear algebra students, instructional designs (tasks, activities etc.) of the

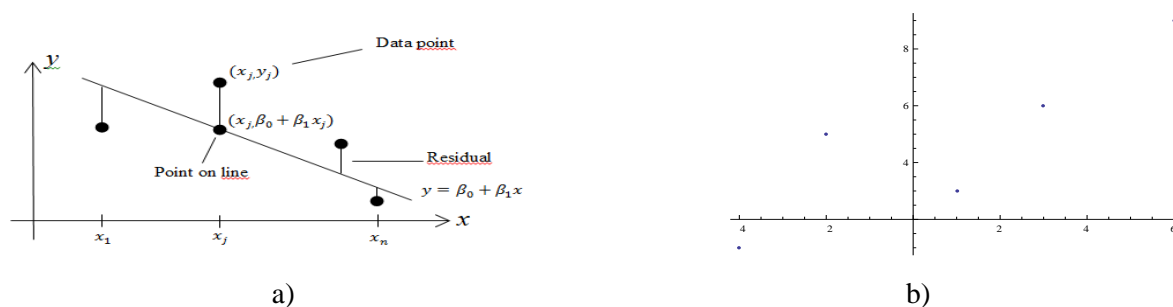
lectures should be developed according to this framework. Use of technology in this process is one of the research recommendations conducted recently (Klasa, 2010; Turgut, 2010).

### Use of Technology in the Teaching of Linear Algebra

It can be said that integration of technology with the design of the educational planning process has a twofold importance. The first is *visualization*; the second is *involvement* of students throughout the lecture. Calculators and software enable student to be free to concentrate on what computations mean, and when and why to perform them (Diković, 2007, p. 112). Pecuch-Herrero (2000) developed strategies and computer projects for teaching of linear algebra. The following strategies were developed and followed: i) exploration of new concepts through computer exercises; ii) teaching linear transformations as early as possible; iii) emphasis on geometry; iv) teaching to write mathematics through development of a portfolio and v) using computer projects for motivation and applications. After the treatment, it was reported that improvement in student learning was remarkable. Dogan (2001) investigated the effects of use of *Mathematica*<sup>®</sup> in learning basic linear algebra concepts. The study was implemented by means of comparing two first year linear algebra classes. According to results, the experimental group performed significantly better than the traditional group in tasks involving only conceptual knowledge. According to Diković (2007, p. 111) software packages such as Maple, MATLAB or *Mathematica*<sup>®</sup> have numerous functions of the following: i) instantaneous numerical and symbolic calculations; ii) data collecting, analysis, exploration, and visualization; iii) modelling, simulation, and prototyping; iv) presentation graphics and animation in 2D and 3D; and v) application development. In the light of these statements and results, we understand that use of software in the teaching of linear algebra is important and using it carefully corresponds to principles of Harel's (2000) framework. Although there are lesson plans developed by MATLAB and *Mathematica*<sup>®</sup> (ATLAST, 1997), we could not find explicit *Mathematica*<sup>®</sup> tasks focusing on least-squares line and least-squares parabola in the available literature. We follow same solution procedure as in Lay (2006).

### LECTURE DESIGN IN CONTEXT OF LINEAR ALGEBRA

In this work, in order to make notations more comprehensive, expressions of least-squares were used by the terms of statistics and engineering:  $X\beta = y$ , in which  $X$  is called the design matrix,  $\beta$  is the parameter vector and  $y$  is the observation vector. There are a lot of ways to construct least-squares models. First, least-squares lines should be introduced. Given two real numbers  $\beta_0$  and  $\beta_1$ , this can be represented a line by these values as  $y = \beta_0 + \beta_1 x$ . For each given point  $(x_j, y_j)$ , there are corresponding points  $(x_j, \beta_0 + \beta_1 x_j)$  on the line which have the same  $x$  value.  $y_j$  are called as the observed value of  $y$ . A residual is the difference between an observed  $y$ -value and a predicted  $y$ -value shown by  $\beta_0 + \beta_1 x_j$  for  $1 \leq j \leq n$ . Geometric interpretation of the least-squares line is rendered in Figure 1a (Lay, 2006, p. 435).



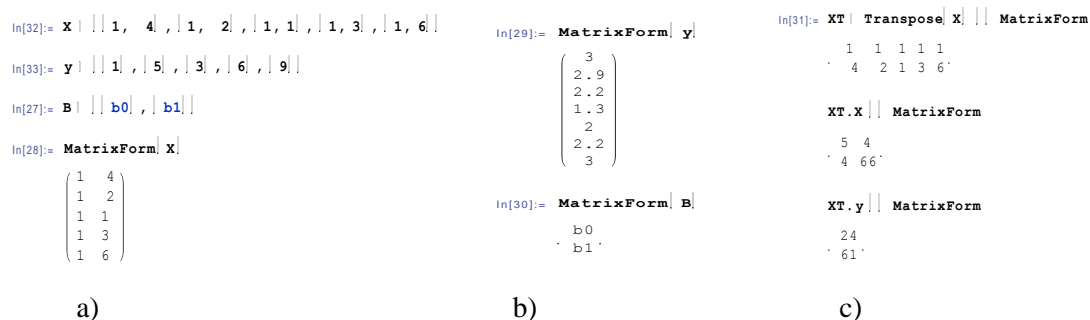
**Figure 1:** a) Geometric interpretation of Least-Squares Line; b) Given the set of Data 1

Using data points on the line, it can be expressed as  $y_j = \beta_0 + \beta_1 x_j$  for  $1 \leq j \leq n$ . In terms of matrix notation, this system of equations may be denoted as  $X\boldsymbol{\beta} = \mathbf{y}$  by

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}. \quad (1)$$

The square of the distance between the vectors  $X\boldsymbol{\beta}$  and  $\mathbf{y}$  is precisely the sum of the squares of the residuals (Lay, 2006, p. 436). In order to minimize the distance between  $X\boldsymbol{\beta}$  and  $\mathbf{y}$ , Lagrange Multipliers method is used (see Edwards & Penny, 2001, p. 884).

First, suffice it to say that this lecture should be interpreted in a computer laboratory. Before this course begins, some of basic commands of *Mathematica*<sup>®</sup> such as “Transpose”, “Inverse”, “MatrixForm” and “ListPlot” are needed to be presented to students. Focusing the first “concreteness” principle of the teaching of linear algebra (Harel, 2000), begin the lecture with the following statements: “We have a data set as in Figure 1b. How do we find the closest line lies throughout the data?” Next, take students’ opinions and discuss possible ways that students propose. Thereafter, introduce to students the following example: “Find the equation  $y = \beta_0 + \beta_1 x$  of the least-squares line that best fits the data points  $(-4,1), (-2,5), (1,3), (3,6)$  and  $(6,9)$ .” Considering “necessity” principle of Harel (2000), let the students form matrices which are going to be used in computations. Give a direction to students to form matrices in the equation (1) with *Mathematica*<sup>®</sup> commands. Remind them of the command “MatrixForm”. Help the students form Figure 2a and 2b:



**Figure 2:** a), b) *Mathematica*<sup>®</sup> output of the matrices in (1); c) *Mathematica*<sup>®</sup> output of the matrices in (2)

Ask students the next step. Remind them that the key process is the determination of  $\beta_0$  and  $\beta_1$ . At this point, ask students how you obtain the solution, since this is a simple linear algebra problem. Discuss the possible solutions. Because  $X$  is not a square matrix, first, matrix of  $X^T X$  should be formed. Let the students form  $X^T X \boldsymbol{\beta} = X^T \mathbf{y}$ . The *Mathematica*<sup>®</sup> output of the matrices  $X^T X$  and  $X^T \mathbf{y}$  will be as in Figure 2c. Ask students the next step. Help them to find the  $\beta_0$  and  $\beta_1$ . They will be able to solve the following system of linear equations

$$\begin{bmatrix} 5 & 4 \\ 4 & 66 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 24 \\ 61 \end{bmatrix}. \quad (2)$$

In order to solve the system (2), give directions to use the command “inverse”. Encourage the students to find the inverse of  $X$ . Take students’ comments for further computations. The command will be as in Figure 3a.

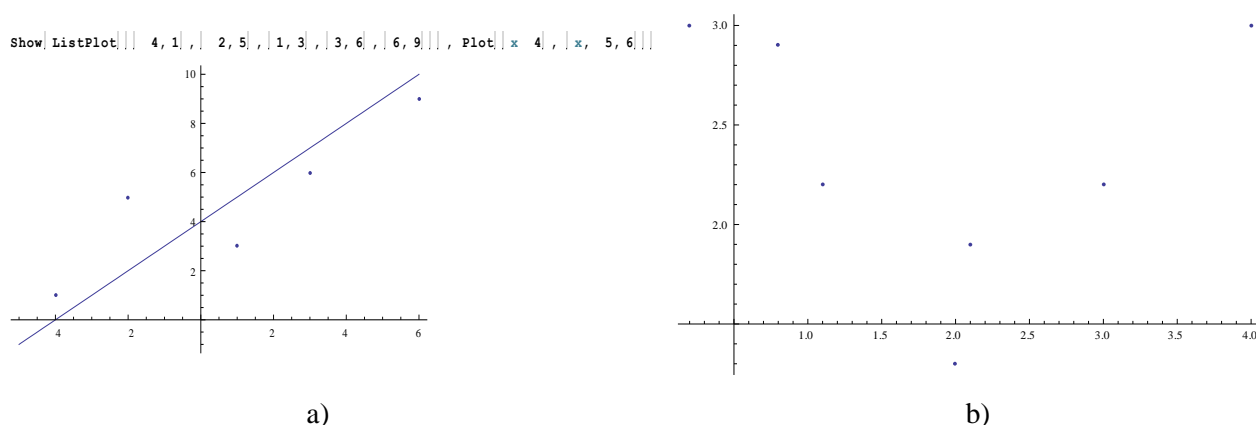


$$\begin{array}{cc} \text{Inverse} \left[ \text{XT} \cdot \text{X} \right] \left| \left| \right. \text{MatrixForm} & \text{Inverse} \left[ \text{XT} \cdot \text{X} \right] \cdot \text{XT} \cdot \text{y} \left| \left| \right. \text{MatrixForm} \\ \left( \begin{array}{cc} \frac{33}{157} & \frac{2}{157} \\ \frac{2}{157} & \frac{5}{314} \end{array} \right) & \left( \begin{array}{c} \frac{670}{157} \\ \frac{209}{314} \end{array} \right) \end{array}$$

a) b)

**Figure 3:** a) Command to find the inverse of  $X$ ; b) Command to find the matrix of  $\beta$

Now the matrix in the Fig. 3a will give a key to finding the least-squares line. Multiplying both sides of (2), students will be able to obtain Figure 3b. So the least-squares line is  $y = \frac{670}{157} + \frac{209}{314}x$ . The important part of this lecture is the geometric interpretation of the line within the given data set. Therefore, students are also able to use “show” and “plotlist” commands together to demonstrate the obtained results. The command and desired interpretation is rendered in Figure 4a.



**Figure 4:** a) Least-squares line of the given data set; b) Given the set of Data 2

After this step, enable students think the distribution of all data may not occur in the first (example) problem. According to Harel’s (2000) generalizability principle of the teaching of linear algebra, this is an also important part of the lecture providing students think generalization or other possible facts geometrically. At this point, ask students how to consider an arbitrary data? What can be possible situations? Take their comments and evaluate their questions. Next, ask students “about graphs of one year air temperatures of a place?” Thereafter, go on showing with the following Figure 4b. Then ask about the closest line which lies throughout the data given in Fig. 4b. Take their comments. The students are able to get a solution for this problem by finding a curve which lies closest throughout the data. Give explanations if students find a “parabola”. Inductively, this data represents a parabola instead of a line. And this parabola may be expressed as  $y = \beta_0 + \beta_1x + \beta_2x^2$ , where  $\beta_0, \beta_1$  and  $\beta_2$  are real numbers. The coordinates of the  $j$ th data point  $(x_j, y_j)$  satisfy the equation  $y = \beta_0 + \beta_1x + \beta_2x^2$  and it can be also expressed by  $y_j = \beta_0 + \beta_1x_j + \beta_2x_j^2 + \epsilon_j$  for  $1 \leq j \leq n$ . Here  $\epsilon_j$  is residual error between the observed values  $y_j$  and the predicted values  $y_j = \beta_0 + \beta_1x_j + \beta_2x_j^2$  (Lay, 2006, p. 438). The matrix form, which is similar to the least-squares line matrix, is as the following

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

or in a short way by  $X\beta = y$ . After these explanations, give the next example: “Find the equation  $y = \beta_0 + \beta_1x + \beta_2x^2$  least-squares curve (parabola) that best fits the data points

$(0.2, 3), (0.8, 2.9), (1.1, 2.2), (2, 1.3), (2.1, 2), (3, 2.2)$  and  $(4, 3)$ ." Next, provide students form  $X$  and  $y$  matrices with *Mathematica*<sup>®</sup>. The matrices will appear in the *Mathematica*<sup>®</sup> interface as in Figure 5.

```
x | | 1, 0.2, | 0.2 ^2, | 1, 0.8, | 0.8 ^2, | 1, 1.1, | 1.1 ^2, y | | 3, | 2.9, | 2.2, | 1.3, | 2, | 2.2, | 3 |
| 1, 2, 4, | 1, 2.1, | 2.1 ^2, | 1, 3, 9, | 1, 4, 16 |
MatrixForm X |
( 1 0.2 0.04
 1 0.8 0.64
 1 1.1 1.21
 1 2 4
 1 2.1 4.41
 1 3 9
 1 4 16 )
MatrixForm y |
( 3
 2.9
 2.2
 1.3
 2
 2.2
 3 )
```

**Figure 5:** Matrix forms of the given Data 2

Due to type of matrix  $X$ , let students form a transpose of  $X$ . And ask the next step. Help students find a square matrix in terms of  $X^T$  and  $X$ . Students will be able to find Figure 6a.

```
Transpose X | | MatrixForm
( 1 1 1 1 1 1 1
 0.2 0.8 1.1 2 2.1 3 4
 0.04 0.64 1.21 4 4.41 9 16 )
Inverse Transpose X | . X | | MatrixForm
( 1.05803 1.02418 0.201498
 1.02418 1.33514 0.296166
 0.201498 0.296166 0.0707906 )
Transpose X | . X | | MatrixForm
( 7. 13.2 35.3
 13.2 35.3 110.112
 35.3 110.112 374.323 )
Inverse Transpose X | . X | . Transpose X | . y | | MatrixForm
( 3.50132
 1.56494
 0.361131 )
Transpose X | . y | | MatrixForm
( 16.6
 30.74
 86.458 )
```

a)

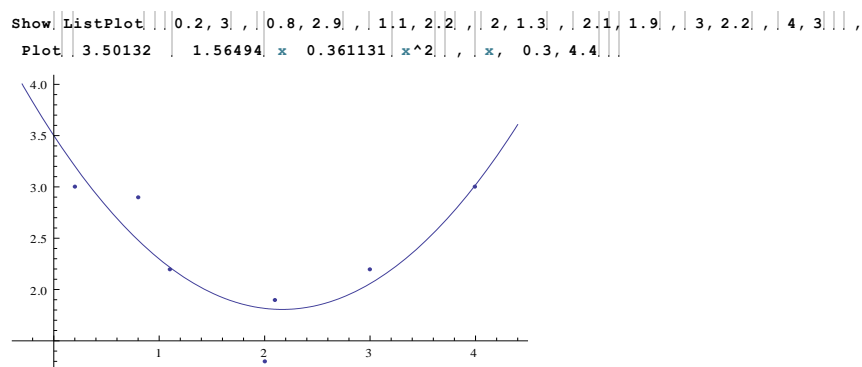
b)

**Figure 6:** a) Computations of  $X^T X$  and  $X^T y$ ; b) Computations of  $\beta_0, \beta_1$  and  $\beta_2$

Then the students have the following system of linear equations

$$\begin{bmatrix} 7 & 13.2 & 35.3 \\ 13.2 & 35.3 & 110.112 \\ 35.3 & 110.112 & 374.323 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 16.6 \\ 30.74 \\ 86.458 \end{bmatrix}.$$

Finally, using the inverse of  $X^T X$ , students can compute Figure 6b. And finally, they will be able to form:



**Figure 7:** Least-squares parabola of given Data 2

## CONCLUSION AND FURTHER REMARKS

The present paper pointed out a lecture design with *Mathematica*® in order to teach the least-squares method for linear algebra students. Due to page limitation, discussion with arguments and literature, as well as intellectual dimension of Harel and Sowder (1998) and generalizations of least-squares planes and least-squares surfaces, their use of technology and engineering could not be presented. As a remark, I suggest that linear algebra educators take into consideration these further examples. This fact is important with respect to Harel's (2000) generalizability principle of the teaching and learning of linear algebra. By extending the examples and tasks in this proceeding, a website could be developed which any student could personally use at home. By this way, students are able to evaluate and rethink classroom assignments. Furthermore, online learning modules of the least-squares method and other linear algebra topics may be implemented. This idea may be of interest and researchers may treat it as such in the future.

## NOTES

1. *Mathematica*® is a computational software developed by Stephen Wolfram, which is used in a lot of disciplines of science and engineering.

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## SPATIAL ABILITY TRAINING WITH 3D MODELLING SOFTWARE

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*In this study, activities supported by SketchUp, a 3D modelling software which is used to improve the spatial ability of primary school students have been developed. These activities have been developed considering the exemplary applications in the 6<sup>th</sup>-8<sup>th</sup> grade mathematics curriculum in Turkey and also in technical literature related to spatial ability training. The activities include “unit cubes” for the 6<sup>th</sup> grade students and “surface closure” for the 8<sup>th</sup> grade students. The unit cube activity aims to improve students’ ability to visualise structures given in different positions; the surface closure activity aims to improve students’ ability to visualise the surface development of basic geometric objects. Keywords: Spatial ability training, primary school, 3D modelling software.*

### THEORETICAL BACKGROUND

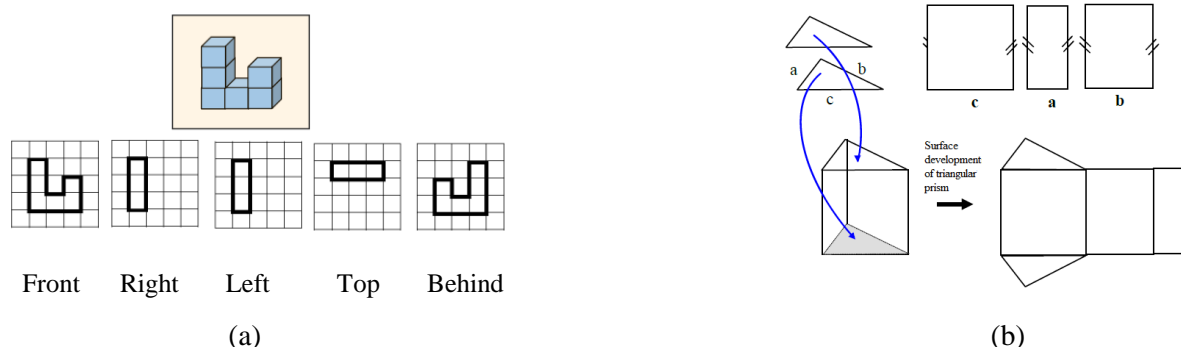
#### Spatial Ability

Spatial ability is one of the important skills regarded in various disciplines such as engineering, art and geometry. For this reason, many researchers have performed studies related to definition, components and development of spatial ability. According to many studies, spatial ability can be defined as a kind of skill related to visualizing views and transformations of two and three dimensional (3D) objects in the mind (Linn & Peterson, 1985; Lohman, 1988; Sutton & Williams, 2007). McGee (1979) expressed two main components of spatial ability in terms of spatial visualization and spatial orientation. Spatial visualization was defined as the ability to represent and manipulate visual objects mentally. Secondly, spatial orientation is related to representing views of an object from different perspectives in the mind. So spatial orientation ability, differently, includes imagination of body movement. On the other hand, some researchers defined mental rotation as another component of spatial ability (Linn & Peterson, 1985; Lohman, 1988; Maier, 1998). Mental rotation is relevant to rotating 2D and 3D objects mentally and includes less complicated activities in comparison to spatial visualization. Many studies express that spatial skills are used effectively in many areas such as mathematics and geometry. Thus national mathematics education programs at different levels attach importance to the training of these skills (MEB, 2009a; MEB, 2009B; NCTM, 2000).

#### Spatial Ability in Mathematics Education

Because of that spatial thinking is crucial for understanding and interpreting geometry, NCTM (2000) stated that students’ spatial thinking skills need to be developed by essential instructional tool such as concrete materials and geometry software. Also, Turkish national mathematics education programs carried out at primary, secondary and high school levels include educational objectives to improve spatial ability (MEB, 2009a; MEB, 2009b). Primary school mathematics education curriculum (1<sup>st</sup> – 5<sup>th</sup> grade) includes an objective that “students will be able to develop and use their own skills related to spatial relations”. Secondly, one of the purposes of secondary school mathematics education curriculum (6<sup>th</sup> – 8<sup>th</sup> grade) was explained this way: “students will be able to develop their own spatial ability, using objects formed by multiple cubes”. Therefore the secondary mathematics education curriculum includes various activities relevant to visualizing and building objects by using unit cubes (Fig. 1a). Because of that, these kinds of activities include visualization of body movement while drawing of views, and the skill that is trained displays properties of spatial orientation ability.

Students will draw views of the below object from front, right, left, top and behind sides on plotting or dotted paper.



**Figure 1:** (a) An activity related to views of 3D objects (6<sup>th</sup> grade), (b) an activity related to surface developments (8<sup>th</sup> grade)

Additionally, some activities in the educational program are related to visualization of surface developments of 3D objects and building the 3D objects by closing surface developments (Fig. 1b). Because these activities include imagination and manipulation of objects in the mind, the skill that is trained during the activities involves properties of spatial visualization ability.

In addition to activities of education curriculums which aim to improve spatial skills of students, many researchers have studied alternative methods and tools such as concrete and virtual 3D models which can be used effectively in spatial ability training. At this point, 3D modelling via computer technologies has a special place in the context.

### Spatial Ability Training via 3D Computer Applications

In the NCTM (2000) report, it is stressed that the visual technological tools are basic instrumentalities in teaching mathematics and that these tools give students the chance to see and analyse visualised depictions of abstract concepts. Therefore, it is stressed that since the beginning of the primary school years, the students are capable of acquisitions such as visualising geometrical concepts, analysing them and understanding their features by using concrete materials and software. In this sense, there has been a variety of studies about the effective use of software including 3D modelling. Cohen and Hegarty (2008) and Güven and Kösa (2008) have aimed to improve university students' spatial abilities in the activities they developed using different 3D software. At the end of these studies, it was seen that the students who participated in the activities supported with software proved to be more successful at spatial ability tests than the students who used traditional education tools. Moreover, Boytchev et al. (2007) have developed dynamic 3D computer program with the aim of developing spatial ability of the students ranging from primary school to secondary school ages, through a project called DALEST (Developing an Active Learning Environment for the Learning of Stereometry). These applications included activities which required spatial reasoning such as guessing the surface development of geometrical objects through dynamic 3D models and building 3D structures through unit cubes. In another study, Cohen and Hegarty (2008) designed activities to create cross sections through 3D computer modelling with the aim of improving the spatial ability of the students who had low spatial ability. The participants in these activities were seen to be more effective in their spatial abilities in a meaningful way. Within the scope of this study, it was intended that the activity which would improve the spatial ability of primary school students would be through the use of 3D modelling software SketchUp. These activities were developed considering the applications about spatial thinking in the primary school mathematics teaching in Turkey.

## CONTEXT AND TASK DESIGNS

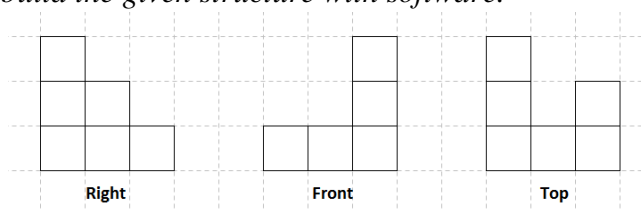
The activities included in this study involved modelling a shape whose appearance is given in different positions and which is formed by unit cubes through software, creating geometrical cross sections of 3D objects and building a geometrical object on software by beginning with its surface development. SketchUp's having geometrical cross sections, rotating with protractor and tools to measure length were used in the actualisation of these studies. The first activity done is the unit cube activity. In this activity which was designed for the 6<sup>th</sup> grade teaching, a structure whose three sides were given in different positions would be constructed on a software basis through unit cubes.

### Activity 1: Unit Cubes

Through this activity;

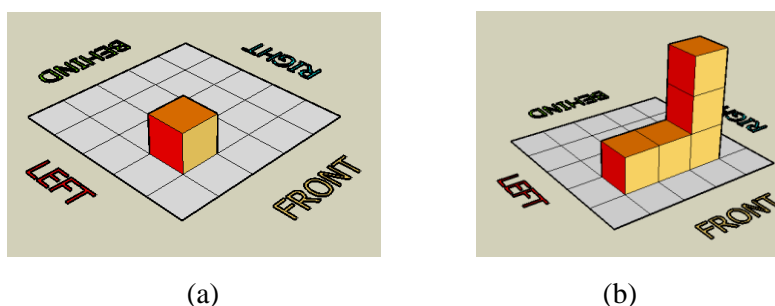
1. In their minds, the students will be able to visualize the shape of a geometrical structure from different dimensions.
2. In their minds, the students will be able to visualise geometrical structures whose aspects are given from different positions.

**Task:** In Figure 2, there are the appearances of a 3D structure from right, front and top. The students would be asked to build the given structure with software.



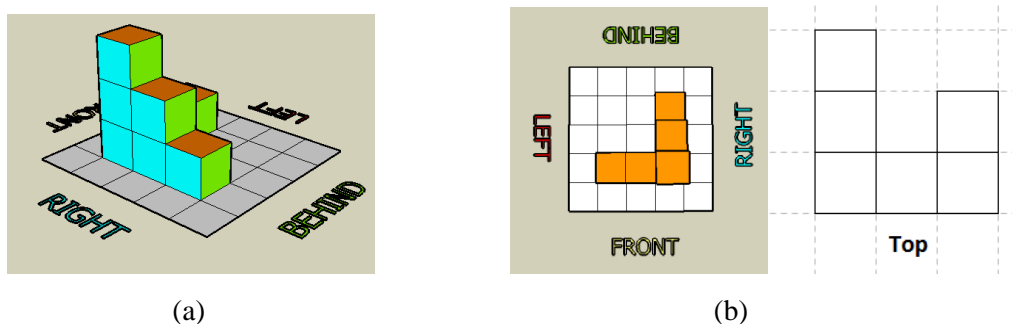
**Figure 2:** A geometrical structure given in different positions.

At the beginning of the formation process, the students are made to build a specific part of the structure using unit cubes (Fig. 3a and Fig. 3b).



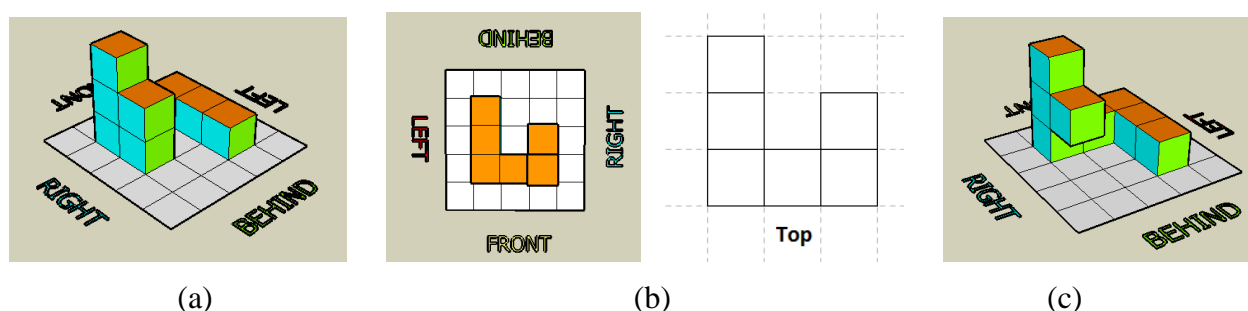
**Figure 3:** (a) Unit Cube Model in the Software, (b) building a part of the structure through copying and pasting of the unit cube.

Thereafter, the students are made to continue the building process by using the data given in the problem and in the 3D dynamic features of the software (Fig. 4a and Fig. 4b).



**Figure 4:** (a) A possible mistake that the student can make, (b) the comparison of the structure after the mistake and the appearance of the structure given in the problem.

At this point, it can be considered that the students can make mistakes as in Figure 4a. In such situations, the students are made to compare the structure given in the problem and the structure that he or she built in the software and to move the spatial reasoning process one step forward. In Figure 4b, the mistake is visible when the structure is observed from the top. The student is made to re-think the structure.



**Figure 5:** (a) The step the student takes to the conclusion, (b) the evaluation of the completed structure from the top, and (c) producing different solutions.

At this point, the students can reach the correct conclusion by following the step in Figure 5a. The student is made to compare the structure from the top and the structure given in the problem by asking him/her why this particular answer is the correct one (see Figure 5b). After the solution, these questions are asked of the students so as to create other solutions as in Figure 5c and improve their reasoning:

*Is there only one solution to the problem? Can we reach the conclusion by using fewer unit cubes?*

The second activity is surface development. With this activity targeted for 8<sup>th</sup> grade students, visualisation of surface development of 3D geometric objects is encouraged.

### **Activity 2: Closure of the Surfaces**

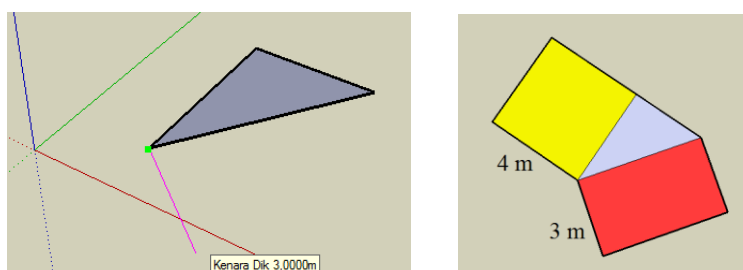
With this activity;

1. In their minds, the students will be able to visualise the surface development of basic geometrical objects.
2. The students will be able to understand the relationship between sides and surfaces on the surface development.

*Task: Draw the surface development of a right triangle prism on software and build the structure by developing the surfaces.*

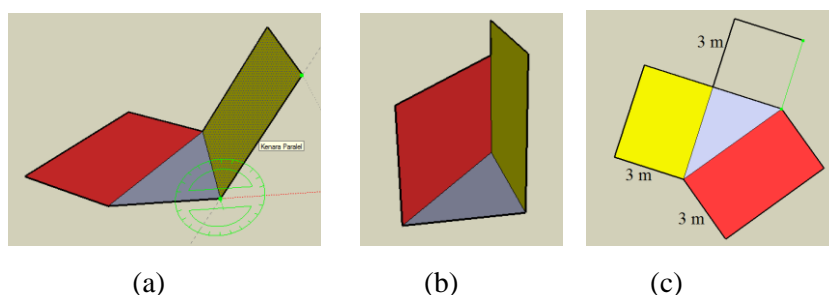
At the beginning of the process, the students are made to draw a right triangle on software and to think that this triangle is the bottom surface (Fig. 6a). Then, the students are asked how they can

draw the side surfaces. At this point, the students can find out the side lengths by overlooking the fact that side surfaces are overlapping (Fig. 6b).



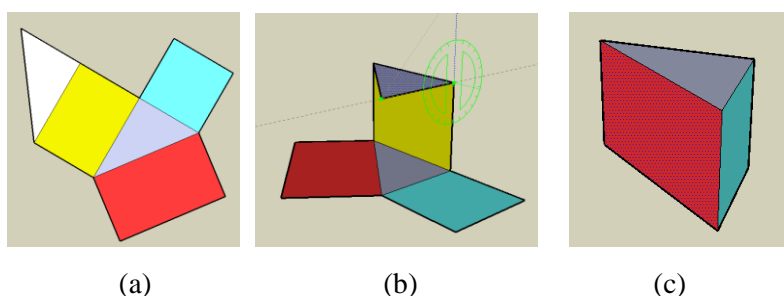
**Figure 6:** (a) Drawing of the bottom surface, (b) a mistake that students can make related to the sides.

At this point, the students are asked to evaluate their drawings by closing the sides (Fig. 7a). The students are made to realise that the sides of the triangle are drawn in different lengths (Fig. 7b). Afterward, the surface development is re-conducted within the scope of discovered features (Fig. 7c).



**Figure 7:** (a) Rotating the sides, (b) evaluating the drawings of side surfaces, (c) re-drawings of side surfaces.

In the next step, the students are asked about the relationship between the top side and bottom side of the prism and asked to create the top surface (Fig. 8a). During this, the students are made to create the symmetry of the bottom surface. In the last step, all the surfaces are 90 degrees rotated (Fig. 8b) and the right triangle prism is completed (Fig. 8c).



**Figure 8:** (a) Drawing of the symmetry of the bottom surface during the drawing of the top surface, (b) rotating the surfaces in a right angle, (c) completing the right triangle prism.

## CONCLUSION AND FURTHER REMARKS

In this proceeding, activities and educational examples which aim to improve students' spatial abilities through SketchUp software have been presented. Activities have been explained in detail and possible student mistakes have also been included. The developed activities have been limited to unit cubes and surface development. The activities in this proceeding are initial version of project. With the same software, activities involving rotating in the mid, spatial perception and spatial orientation can be carried out. In the technical literature, it is known that these sub-skills are



also related to math and geometry success (Tarte, 1990; Turgut & Yılmaz, 2012). On the other side, by expanding these activities, experimental studies can be developed. It is known that a geometry course designed with SketchUp software is able to improve math teacher candidates' spatial abilities (Kurtuluş & Uygan, 2010). Similar studies can be carried out at the primary and secondary school levels. Moreover, a web site involving all such activities appropriate to the curriculums can be created. In this way, students can revise, evaluate and offer feedback to the activities which take place within the class in the house environment.

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## **FLASHING BACK AND LOOKING AHEAD – DIDACTICAL IMPLICATIONS FOR THE USE OF DIGITAL TECHNOLOGIES IN THE NEXT DECADE**

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*Advantages and disadvantages of the use of digital technologies (DT) and especially of computer algebra systems in mathematics lessons are discussed controversially worldwide. What will be the meaning of DT in the next years or even the next decade? The basis of the following considerations is a long-term empirical project  $M^3$  (Model-project New Media in Mathematics lessons) which was started ten years ago in 2003 to test the use of symbolic calculators (SC) in Bavarian “Gymnasien” (grammar schools) in Germany. In 2013 there exists a widespread experience in the use of SC in the grades 10 to 12 in classroom activities, student and teacher documents as well as test and examination results of students. The implications of this project are going to be focused in 10 theses or hypotheses of possible, gainful developments in the future. These theses will be explained with examples from the project  $M^3$ .*

### **VISIONS**

The NCTM standards of 1989 (and in the revised version of 2000) have been visionary – concerning the field of mathematics education – by representing a vision for the future of mathematics education. This is especially true for the use of new technologies in mathematics classrooms, expressed in the ‘Technology Principle’:

“Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.” (p. 24)

and:

“Calculators and computers are reshaping the mathematical landscape ... Students can learn more mathematics more deeply with the appropriate and responsible use of technology.” (p. 25)

The first ICMI study in 1986 “The Influence of Computers and Informatics on Mathematics and its Teaching” (Churchhouse) was also affected by a great enthusiasm concerning the perspectives of mathematics education in view of the availability of new technologies. Many mathematics educators, for instance Jim Kaput, forecasted that new technologies would change all fields of mathematics education quite quickly.

“Technology in mathematics education might work as a newly active volcano – the mathematical mountain is changing before our eyes” (1992, p. 515).

### **DISILLUSIONS**

In the current ICMI Study 17 “Mathematics Education and Technology – Rethinking the terrain” (Hoyle & Lagrange 2010) disappointment is quite often expressed about the fact that – despite of the countless ideas, classroom suggestions, lesson plans and research reports – the use of DT has not succeeded, as many had expected at the beginning of the 1990s. In her closing address concerning “The Future of Teaching and Learning Mathematics with DT” Michèle Artigue summarizes in the ICMI Study:

“The situation is not so brilliant and no one would claim that the expectations expressed at the time of the first study (20 years ago) have been fulfilled.” (p. 464)

This study gives a good overview of the numerous activities in the last years concerning the use of new or digital technologies in mathematics education (also see Weigand 2010). But the book is not a vision; it rather poses questions, which are, however, quite similar or very similar to those 20 years before. One may interpret that as – partial – resignation, but one can also see it as an indicator of how hard these questions are to be answered. Finally, one can also understand it as a request and as a challenge to develop new ideas – visions – in order to make progress with the integration of DT in mathematics education.

### THE M<sup>3</sup>-PROJECT

In 2003 the M<sup>3</sup>-project started with the use of symbolic calculators (SC) in grade 10. The project was initiated by the Bavarian Ministry of Education, funded and supported by “Texas Instruments” and “Casio”. Students used the TI-Voyage 200, later the TI-Nspire as well as the Casio ClassPad and they were allowed to use SC in mathematics lessons, for homework and in examinations. The results of these project classes have been compared in a pre-post-test design with control classes, in which students had had traditional lectures without any use of graphing calculators or SC. During school year of 2006/07, the project was implemented in grade 11 – the content taught was calculus. Since 2012 SC have also been allowed in the state-wide final examinations at the end of grade 12. During the last ten years nearly 5000 students have been taken part in the project. For some more detailed information concerning the project, especially research questions and intermediate results see Weigand (2008) and Weigand & Bichler (2010b). At the moment the final report of the project is being developed (and might already be available during the ICTMT conference in Bari). The results of this long-term empirical project are the basis of the following implications for future developments.

In the following we see SC as prototypes of DT which integrate computer algebra systems, dynamic geometry software and spreadsheets. Ten *theses* are set up concerning DT which are rather pragmatic views than visions to strengthen the influence of DT on mathematics education.

### TEN THESES CONCERNING FUTURE DEVELOPMENTS

There are some general results of the M<sup>3</sup>-project which are also confirmed in some other *long-standing quantitative empirical investigations*, e.g. working with SC brings a greater variety of problem solving strategies; students work more individually and with partners; SC are well-accepted by students and teachers; successful working requires the orchestration of learning situations. But there are also some special results of the M<sup>3</sup>-project concerning *qualitative empirical investigations*, which are based on student and teacher interviews, on special individual answers in student questionnaires and on the interpretation of individual student solutions of test and examination problems. Both the quantitative and the qualitative results give rise to the following theses and hypotheses about the use of SC and DT in future mathematics classrooms.

We are going to explain these theses by means of authentic student solutions obtained in the frame of the M<sup>3</sup>-project. We are especially going to refer to final baccalaureate examinations. In Bavaria, the baccalaureate is a state-wide (final) examination with the same problems for all students set by the Ministry of Education. In May 2012 the first final baccalaureate examination was set for students at the end of school in grade 12. The next one will be in May 2013. It is intended to present examples from these examinations at the ICTMT 11. These considerations are also a further stage of Weigand (2011a).

## 1. Convincing Others

Despite of the countless ideas, classroom suggestions, lesson plans and research reports, nowadays the use of DT has not succeeded, as many had expected in the last decades. In the ICMI Study 17 “Mathematics Education and Technology – Rethinking the terrain” (Hoyles & Lagrange 2010) the disappointment that “Technology still plays a marginal role in mathematics classrooms” (p. 312) is quite often expressed.

*Thesis 1:* We underestimated the difficulties of DT-use – in a technical sense and in relation to the contents – and we have not been able to convince teachers, lectures at university and parents of the benefit of DT in the classrooms.

## 2. Examination Problems

Examinations influence the way contents are taught in the classroom. Only if DT are allowed in examinations they will also be used for classroom work. Moreover, examination problems set standards in the way of teaching. DT show advantages especially if you work on open problems and if modeling of realistic problems or discovery learning are the focus of the classroom work. But these problems are little suitable in – traditional – examinations because they require time and patience to try different solution strategies or to allow following dead-end streets in the problem solving process. If we stay on traditional ways of examination forms, new kinds of problems are necessary. Moreover, the question of the relation between expected knowledge, abilities and skills in examinations has to be discussed under new aspects.

*Thesis 2:* The construction of – good or meaningful – test and examination problems – if we think about a traditional oral or written exam – is even more challenging if DT are allowed to use.

But new ways of examination forms are also thinkable, e.g. (digital) portfolios, project work and oral presentations.

## 3. DT and their Tradition

The integration of DT into the classroom is an evolution and not a revolution. The use of DT in the classroom and in examinations has to be integrated into traditional learning and teaching and also into traditional – paper and pencil – examinations. The meaning of examinations in school lesson is very crucial to teachers and students and should not be underestimated. Quite often – especially in written tests – the solution of a problem has to be documented on paper. Solving a problem with the DT the documentation of the solution on paper has to combine the work with the tool (the input commands), the screen display (the output) and the calculations and notes of the student on paper. Even after one – or more – year(s) of SC-use students expressed their uncertainty about the correct way to document solutions on paper and they asked for a clear instruction for the documentation (see Schmidt-Thieme & Weigand 2012).

*Thesis 3:* Criteria for the adequate documentation of solutions in written examinations have to be developed.

We think that there are no algorithmic rules or norms how to document solutions on paper. Criteria for (non-)correct or (in-)adequate documentation forms of solutions might be, e.g.:

- The solution has to be understandable ‘for others‘ and it has to point out when and where DT were used.
- The solution describes the mathematical activities; it is not only a description in a special „calculator language“.

- The meaning of “keywords” (operators) in the problem definition has to be well-known to student, e.g. “show”, “explain”, “determine”, “prove”, ...

#### 4. Mental Representations

The main reason for working with DT in the classroom is the goal of a better understanding of mathematical concepts. Understanding means developing *mental structures or representations*. They are built by the knowledge, the abilities and the skills of using and interpreting real – symbolic, graphic and numeric – representations.

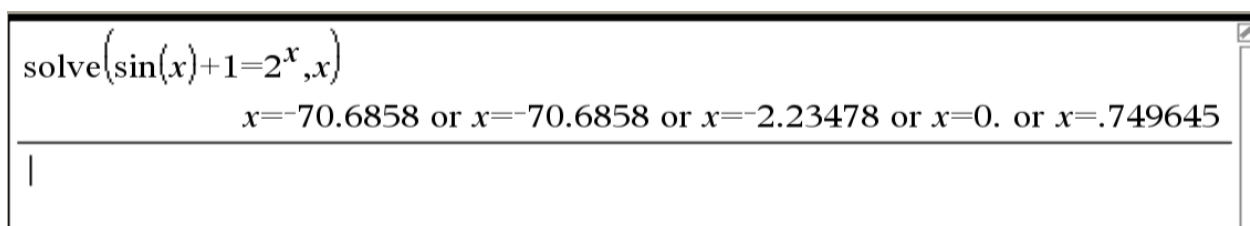
*Thesis 4:* In spite of the existence of interactive, dynamic and multiple digital representations the main challenge is the development of mental representations.

#### 5. Control Strategies

The solutions of problems are quite often not represented or displayed on a computer screen the way students are familiar with. This is especially true for symbolic representations. For the interpretation and evaluation of these results a more global view of the problem and the solutions as well as the ability to work with different representations necessary.

*Example:* Given are  $f$  and  $g$  with  $f(x) = \sin(x) + 1$  and  $g(x) = 2^x$ . How many intersection points do the graphs have for  $x \in \mathbb{R}$ ? Give reasons!

Solving this equation on the symbolic level with SC delivers the result (Fig. 1):



**Fig. 1.** A warning sign appears on the display: “Some more solutions may exist”.

It is quite difficult – for students (nearly) impossible – to interpret this display’s numeric solution. Changing to the graphic screen and zooming into interesting sections might be a good alternative strategy.

*Thesis 5:* Users (students) of DT need to have strategies to control, verify and revise solutions obtained with DT.

#### 6. Tool Competencies

Working with DT has different technical aspects, e.g. working with static, dynamic or multiple representations, and these aspects have to be seen in relation to contents or to the level of understanding of the contents you want to achieve. Moreover, you might see these two aspects also in relation to the cognitive activation of the considered problem. The relationship between these technical, content and cognitive activation aspects forms the “*Tool Competence*” of the user of DT. A competence model for tool competencies is a quite complex construct of technical, content-related and cognitive activation aspects (see Weigand & Bichler 2010a, Weigand 2011b).

*Thesis 6:* The construction of such a model may be helpful for diagnostic reasons and for creating *strategies* for developing tool competencies.

## 7. Limits

Computers are quite limited tools compared to the properties of mathematical concepts. Numeric calculations can only be done to a fixed accuracy; graphic representations on a screen show only a small part of the whole object representation in a discrete pixel world. Students have to be aware of these limitations of DT (examples at the ICTMT 11).

*Thesis 7:* Working with DT requires the knowledge of the limits of and the restrictions of DT concerning the mathematical contents.

## 8. Empirical Investigations

We need more long-standing empirical investigations which evaluate the use of SC in authentic situations of the real classroom work. But we also know that changes in the classroom do not happen only while using a new tool. Strategies of a meaningful use in relation to special mathematical concepts are necessary. And we also have to be aware that empirical investigation will only give answers to specified – relatively – small questions: what is the benefit of using dynamic representations for the understanding of families of quadrat functions? Will a DGS be helpful in discovering proving strategies in the field of chord-quadrilaterals? Empirical investigations cannot give answers to the big questions like “Shall we use DT in mathematics lessons? Yes or no?”

*Thesis 8:* We need long-standing empirical investigations to develop strategies for learning and teaching in the mathematics curriculum.

## 9. Connectivity

An integrated global concept of the use of new technologies has to follow different aspects; it concerns the interaction of different digital components such as laptops, netbooks, the Internet and pocket computers under technical aspects; it concerns the use of classroom materials, and it should support the cooperation between the teachers of a school, the teachers of different schools and the cooperation between school and school administration as well as school and university.

*Thesis 9:* Connectivity and interconnectedness will be key words in the future. The *acceptance* of new technologies and their gainful use require a global concept of teaching and learning.

## 10. Visions

Visions will be important in the future, in all fields of scientific and public life. Without visions there are no further developments. We need *visions* which are based on *empirical results and theoretical considerations*, but we also need visions which are based “only” on new and creative ideas, and we have to have the courage to also discuss visions which – nowadays – look like illusions.

*Thesis 10:* We need visions of the integration of DT into the (mathematics) classroom.

## FINAL REMARKS

The M<sup>3</sup>-project is going to be finished in July 2013. In Bavaria now all grammar schools (Gymnasien) are allowed to choose whether they want to use SC in mathematics classes and in final examinations. From the results of the M<sup>3</sup>-project schools and teachers expect specific answers to their questions concerning *why* and *how* they shall use SC or DT in their classes. These questions are at the heart of mathematics education: you are – as a mathematics educator – in the situation of an advisor or a consultant, who can “only” give some advice: “If you do this .... you have to care for this ..., you have to be carefully with ... and you can expect this ... “. The results of the empirical

research and the future-oriented considerations provide a basis for this kind of advice, no more, but also no less.

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# **Workshops**



## **DESIGNING RESOURCES FOR TEACHER EDUCATION WITH TECHNOLOGIES: EDUMATICS PROJECT**

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*Using ICT such as spread sheets, geometry packages or computer algebra when learning and teaching mathematics is recommended or compulsory in the curricula of many European countries by good reasons. Technology offers the potentiality to enhance the learning of mathematics (Zbiek, 2007). But to develop this potentiality it is important to support teachers to come along with this challenge. This was the aim of the EU project EdUmatix (European development for the use of mathematics technology in classrooms). In the frame of this project researchers and teachers of ten countries worked together and developed systematically and theory-based an internet platform for teachers of mathematics in secondary schools and their teacher trainers with material for classrooms and professional development.*

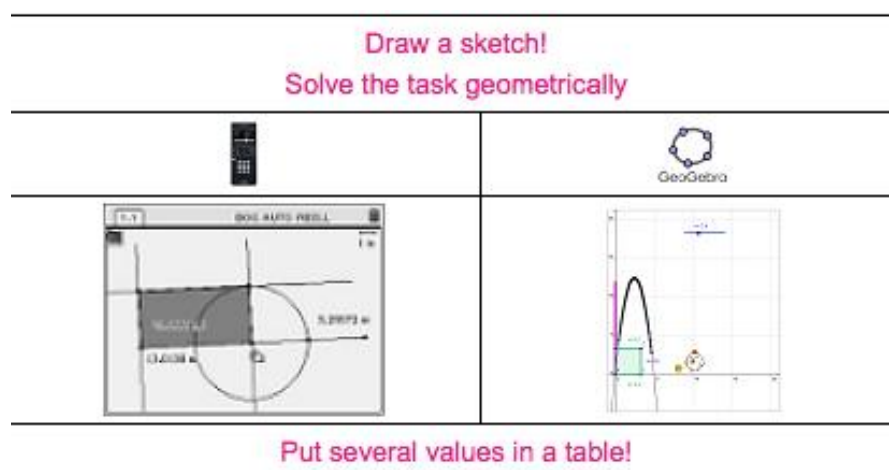
### **THE PROJECT EDUMATICS: DESIGNING RESOURCES**

Edumatics is a Comenius Project aimed at developing resources for mathematics teacher education in the field of integrating technology into mathematics teaching. It comprises of 5 modules and each one is aimed at presenting some particular issues of professional development: an introduction to the use of technologies, the use of representations in static and dynamic way, videos for teacher training, functions as models of phenomena and mathematical configurations. The resources for professional development, although aimed at teachers, include a range of tasks for school students to enable them to use technology within modelling and problem-solving activities. The themes chosen are different routes to the same common end-points, and offer a choice of different starting points depending on the motivation and the preliminary knowledge and skills. Moreover, they make connections with some of the important research ideas that have underpinned the design of the resources. This paper, and the accompanying conference presentation will focus on three of the EdUmatix modules and articulate these connections.

#### **Module 1: Starting to work with ICT**

This module aims to introduce how to work with a certain technology. This introduction is from the beginning along mathematical activities which can be used in the classroom with students, but can be initially experienced by teachers in a teacher training course to give an idea how to use the technology in secondary school mathematics classrooms. These activities are elaborated for two different types of technological tool (TI-Nspire and GeoGebra). The aim with the activities is to represent application areas and some characteristic aspects, where the use of technology can support mathematics teaching and enrich its efficiency. Simple optimization problems such as the isoperimetric problem (Figure 1) to find the rectangle with a maximum area by given circumference serve very well this intention as different representations and approaches can be integrated easily. There are quite often solutions using basic geometry and there are approaches using functions, which integrate numerical ways as well as graphical and algebraic approaches. Therefore these activities help to introduce the different applications of ICT and they help to promote an individual access to mathematics to enrich the efficiency of learning. According to the different necessities and capabilities when using technology we offer four different types of technical guide to learn the

handling of the technology - short and long versions and both offered in an electronic and in a paper version.

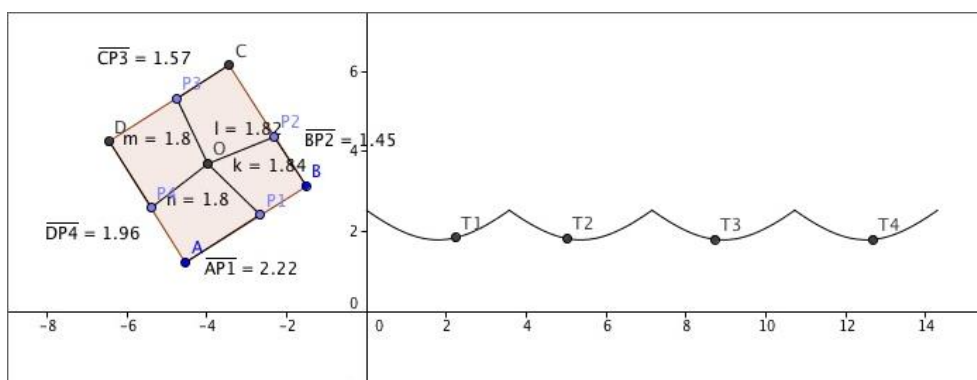


**Figure 1: Module 1 problem of optimisation.**

## Module 2: From static to dynamic representations

The focus of this module is on the use of dynamic representations in the mathematics classroom and the teaching approaches that support these representations to enhance students' mathematical construction of meanings. The activities in the module (that are directed towards students, but can be initially experienced by the teachers) help the participating teachers to understand the role of multiple representations within the learning of mathematics and the opportunities that the use of ICT offers to represent mathematical objects in a dynamic way. It is a fact that the act of doing mathematics requires us to work with mathematical objects that are visible through their various representations. For example, it is possible to work with different representations of a circle:  $x^2+y^2=R^2$ , or: "the circle of centre O and radius R", or an image of a circle, and so on. They are semiotic representations of a circle as a mathematical object, and according to the goal, one or the other representation may be more useful in a particular context. In order to understand what a circle is, one must know how to represent it in a given register, how to treat this representation in the register, and how to convert the representations from one to another register (Duval, 1993). These three actions on representations are at the basis of knowledge construction, and technologies may help in acting dynamically on them.

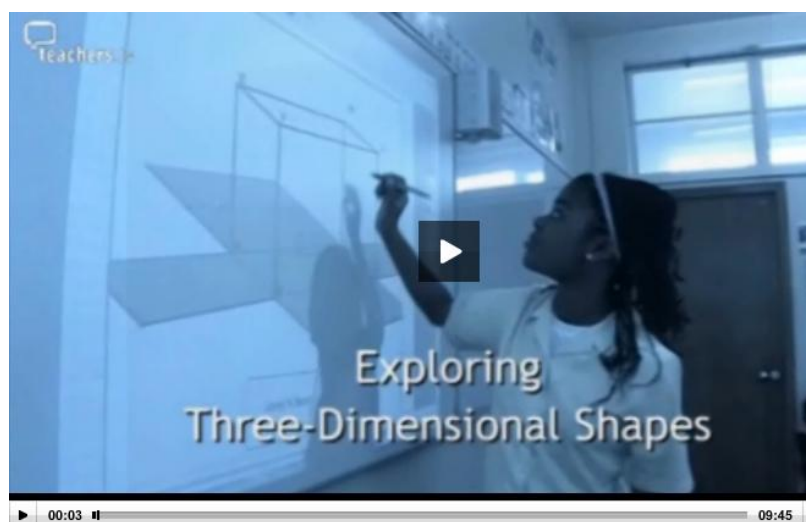
The main aim of the module is to explore dynamicity in different ways: cognitively (treating representations within dynamic situations, converting between registers of representation facilitated by technology); pedagogically (discovering and exploring different teaching methods computer-aided, facilitating discussions and debates in the mathematics classroom) and technologically (analysing the possibilities and constraints of technology, using different software within the same problem). The module places importance on the representations of mathematical objects, with an emphasis on the dynamic representations that are only possible as a result of the technology (Figure 2).



**Figure 2: Dynamic representations in Module 2**

#### **Module 4: Using ICT in the classroom**

This particular EdUmatics module uses the context of a set of video clips of mathematics classrooms (Figure 3) that show approaches to the use of technology for different: countries; teachers; ages and abilities of students; mathematical topics; classroom orchestrations and types of technology. The video clips are accompanied by published articles, which serve to support teachers to read more deeply about the classroom practices and research findings that have some relevance to the context. The bank of clips can be used in many ways, however the designers of Module 4 envisage that teachers would be working in small collaborative groups, alongside a professional development ‘leader’, who would support the teachers to choose appropriate video clips to use in cycles of reflective activity. The Module includes a series of question prompts and tools to support this professional reflection to enable the teachers to learn about new classroom orchestrations.



**Figure 3: Videoclip of a mathematics classroom.**

#### **THE DESIGN OF RESOURCES: CONNECTING FRAMEWORKS**

Designing professional development materials necessarily involves the use of theoretical frameworks by the research groups working at the project. In EdUmatics, the different teams used their theoretical approaches in the design phase, in the sense that they prepared the activities for

teachers that took account of recent research studies. The result, even if different sometimes in the production of the materials (not all the modules in the website have the same structure), can be considered positive, in that there is a richness of approaches, grounded on various theoretical frames. In this section we present some of them, highlighting the role they had in the design of activities, and how these were connected within the overall project.

### **Multirepresentations and multimodality: two sides of the same coin?**

Today we can represent a mathematical object with paper and pen, or using a technology. The potentiality offered by ICT is not only the precision in drawing, or the rapidity in calculating, but, more important, the possibility of representing an object in different registers (graphical, numerical, symbolic), integrating them and simultaneously using them with the same tool (the screen of the calculator, or the computer), or different tools (the screens of the various calculators present in the classroom). We call this affordance multirepresentation, intended both as using different registers in the same tool or using the same register with different tools (Arzarello & Robutti, 2010).

Multirepresentation of mathematical objects in technological tools goes on with another affordance, typical of ICT: the dynamicity, which let possible moving a graph, a draw, a list of numbers, a slider, and so on, in different ways. Dynamic representations are excellent supporters of exploration, discovering, formulation of conjectures, argumentation, in all the mathematical activities and particularly in problem solving. In this way, designing activities for teacher professional development that involve dynamic and multiple representations can help them in teaching practice. Moreover, researchers may investigate on the role of multiple and dynamic representations when used by the students, and if they are helpful in supporting cognitive production while students are engaged in doing mathematics.

Multimodality in linguistics is the use of two or more forms of communication: words, gestures, signs on paper or on tools, gazes, and so on. This term is used in neurology (the sensory-motor system of the human brain is multimodal), linguistics, and recently also in mathematics education (Arzarello and Edwards 2005; Arzarello 2008). If human activity is multimodal in the brain, in communication, and in the classroom, we can analyse students' productions in different modalities to understand their cognitive processes during the mathematical activities, in that they produce a variety of signs such as words, gestures, actions on the tools, inscriptions of whatever nature. Not only students, but also the teacher can interact with them and produce various kinds of signs. This approach is based on semiotic studies on class interactions and gives insight on the process meaning construction (Arzarello & Robutti, 2008). This multimodal production may be particularly rich when subjects use technology with multiple and dynamic representations (Robutti, 2010), and technologies may influence and support it (Arzarello & Robutti, 2010). For this reason, according to research results, we can suggest that multirepresentations and multimodality can be considered as the two sides of the same coin: on one side the cognitive aspects, on the other the technological ones. Taking this framework into account, the design of the material has been particularly devoted to the implementation of activities that make use of multiple (static and dynamic) representations and, in this way, can support multimodal production.

### **Resources and documents: how to use them in teaching?**

The great number of resources available for teachers and students, particularly on the internet has led to extend the instrumental approach to the teachers and students' documentation. The documentational approach (Gueudet & Trouche, 2010) models the interactions of subjects with the resources at their disposal. Teachers integrate resources throughout the development process of his teaching, and built its own documents related to a class of situations in which Gueudet and Trouche (2010, p.58) called documentary work. Like the instrumental genesis describes the transformation of

an artifact into an instrument as the result of a double movement, subject to the artifact and artifact to the subject, the process of transformation of "resources" in "document", called documentational genesis, is the result of instrumentalization, from subject to resources and instrumentation, from resources to subjects. The instrumentalization appears as shaping, forming a set of resources for specific uses of the subject, the instrumentation amending the behavior of the subject. At any given time, the document is then the combination of these resources and a scheme of use in a given context. The process is not complete, since the documents thus formed can be considered as new resources.

Documentational genesis is a process that concerns both students and teachers. The research showed that an important component of teachers' professional learning is the awareness of a possible discrepancy between the teacher's intentions and the actual students' learning; teachers' conception and construction of lesson leans on resources that are both available for teachers and students; the processes of appropriation and transformation of the resources can provoke "didactical incidents" (Aldon, 2011) when they are "parallel" or "divergent". A didactical incident can be seen as a clue of a misunderstanding between the teacher's intentions and the actual behaviour of students.

We take into account this framework by giving examples of resources and their implementation into some classes. Readers are invited to consider the resulting document and to build their own document fitting with their teaching conditions.

### **The use of CAS in mathematical activity: why not?**

Although a lot of studies show the potential of integrating Computeralgebrasystems (CAS) in teaching and learning mathematics (Heid & Blume 2008; Zbiek et al. 2007), a lot of countries still do not involve CAS in their examinations and mathematics classrooms. Therefore it is important to make the benefits and chances explicit to teachers and to show concrete ways of integration. This challenge is also served by the EdUmathics material. Integrating CAS into mathematics classrooms does not only support the capacity of multirepresentations and dynamic visualization in the field of functions, but offers also new advantages coming along with the feature of manipulation algebraic expressions within the technology. In a meta-study the conditions that ensure a successful use of CAS in mathematics classrooms were investigated (Barzel 2012). A main issue pointed out by this study is that the advantage of CAS depends on the design of mathematical activities. In this sense CAS can have an effect as catalyst towards a student-centred and understanding-oriented teaching. The acquisition of conceptual knowledge especially in the field of algebra can be fostered via CAS, for example by enabling discovery learning using the technology when finding patterns and structures in algebraic expressions and objects. This focus on the development of activities and tasks to enhance conceptual understanding is an important goal within any teacher training material.

### **Describing classroom practices with technology**

Whilst researchers in mathematics education might carry out studies to try to establish the best ways of embedding technology within classroom practices, it is widely acknowledged that this research is slow to reach classroom practitioners (if at all). Other studies have taken an alternative stance and sought to articulate and explain the emergent practices of teachers as they develop their classroom approaches that may (or may not) have been influenced by the research. These more naturalistic studies have sought to categorise what teachers think and do as they implement the use of technology within the classroom. As Module 4 is most closely associated with teachers' existing practices, it uses a number of references from this research domain from peer-reviewed academic papers (Paul Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010; Ruthven & Hennessey, 2002; Trouche, 2003). However, as the designers of EdUmathics were aware that many classroom

practitioners were not used to reading and discussing such papers, a number of papers from conferences presentations and professional journals (Clark-Wilson & Oldknow, 2009; Drijvers, 2011; Ruthven, Hennessy, & Deaney, 2005) were also included to act as a bridge to the more research-oriented publications.

### **The connection of frameworks: developing teacher professionalism**

The EdUmatic project gave us the opportunity to work together and to learn from different partners, researchers and teachers. It could be surprising to emphasize the collaboration between teachers and researchers in education, but it appears that very often the class is considered only as a field of experiment. However, the links between research and teaching are not new and the literature gives us examples of action researches where teachers are seen as reflexive practitioners or experts developing new theories about their practices. Kieran & al. (2013) spoke of co-producers of professional and/or scientific knowledge. The collaboration during the EdUmatic project is a good example of such a co-production in which researchers and teachers brought in the design of resources their expertise and competencies. For example, in module 2, the three activities have been constructed after experiments in classes, observations, discussions and analysis: in each of these steps, the dialogue between teachers and researchers allowed the activity to evolve, taking into account both the feasibility, the knowledge construction and the added value of technology. The theoretical frameworks appeared as tools allowing conceptualization of events for which teachers can build adapted pedagogical answers that led to theoretical refinements. The intersection of the two worlds is a neutral place of investment of theory in the practice and of confrontation of contingency with theory.

Bikner-Ahsbas and Prediger (2010) introduce the notion of *connecting strategies* when considering different theories highlighting a same research object. The authors distinguished different perspectives of connecting strategies among which strategies comparing and contrasting different approaches or strategies coordinating and combining perspectives. In the EdUmatic project, the necessity of confrontation with contingency and the will to build an on line in-training site led us to confront the analysis and the mutual contributions coming from the different theoretical frameworks. The aims of the project and the common work implied at least to really understand the other theories and to make explicit our own approach. On another hand, the partners working on the same module, need to put the different approaches in a shared resource and the necessity of consistency of the frameworks is directly linked to the global analysis of the proposed activities. Thus, in module 2, for example, the adapted strategy combined and coordinated the theoretical frameworks in order to propose coherent analysis of the different activities.

Both the collaboration between researchers, between researchers and teachers but also collaboration between teachers working in different national and institutional contexts participated to the richness of the activities and as a domino effect, to the richness of professional development.

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## USING DYNAMIC GEOMETRY FOR PROBLEM-SOLVING AND INQUIRY FROM 16-18

David Harris and Kate Mackrell

*St. Clares, Oxford and Institute of Education, University of London*

*This series of three workshops will focus on the theme of exploration, a central theme of the new International Baccalaureate (IB) mathematics courses that is also relevant to other mathematics curricula. Using dynamic geometry software (Cabri II Plus, Cabri 3D and Geometer's Sketchpad), participants may explore a set of recently developed resources designed to facilitate the learning of sequences and series, vectors, functions, and calculus through problem-solving and inquiry. Ways to use these resources in the classroom will be discussed, and participants will be shown how the resources may be modified and how selected resources were developed.*

### INTRODUCTION

A new International Baccalaureate (IB) curriculum in mathematics was introduced in September 2012, which, at both Standard and Higher level, encourages teaching and learning to be held in a spirit of inquiry, involving the development of curiosity, creativity, research and critical thinking skills, independent learning and the ability to take risks and defend beliefs.

In particular, students are now required to submit a report of an independent exploration of an area of mathematics as part of the summative assessment. This is a report of 6-12 pages in which the student explores an area of mathematics for his/herself.

The specific purposes of the exploration are to:

- develop students' personal insight into the nature of mathematics and to develop their ability to ask their own questions about mathematics
- provide opportunities for students to complete a piece of mathematical work over an extended period of time
- enable students to experience the satisfaction of applying mathematical processes independently
- provide students with the opportunity to experience for themselves the beauty, power and usefulness of mathematics
- encourage students, where appropriate, to discover, use and appreciate the power of technology as a mathematical tool
- enable students to develop the qualities of patience and persistence, and to reflect on the significance of their work
- provide opportunities for students to show, with confidence, how they have developed mathematically. (IBO, 2012)

In order to more effectively implement this important aspect of the curriculum, the authors above have been collaborating in the development of learning materials using the dynamic geometry softwares Cabri 3D (Bainville & Laborde, 2004), Cabri II Plus (Laborde, 2003), and Geometer's Sketchpad (Jackiw, 2009).

## THE LEARNING MATERIALS

These consist of prepared dynamic geometry files, together with lesson plans, which are given at two levels: detailed, with support for using the technology; and overviews, with indications of how the technology might be used. It is expected that at both levels lesson plans would be adapted by the teacher to their own teaching style and situation.

Development of the learning materials has been influenced by the following:

- The requirements of the IB, both in terms of specific content and in terms of suggested teaching approaches and the need to prepare students to be able to engage in an independent exploration;
- Research evidence concerning student learning issues with particular areas of study (e.g. Alcock & Simpson (2004), Tall (2008), Zazkis, Liljedahl & Gadowsky (2003), Biza, Dikoumopoulos, & Souyoul (2007), Poynter & Tall (2005));
- Approaches developed by other researchers (e.g. Henning & Hoffkamp, 2013);
- The experience of the developers;
- The history of mathematics;
- Exploring connections between different topics;
- Feedback from colleagues and students (materials have been trialled in IB classrooms);
- The affordances and constraints of the softwares used.

The particular rationale for each activity will be discussed and the details of the development of selected software files will be given, with the aim of giving some working knowledge of the software and also some of the specific techniques used in order to facilitate teachers in adapting existing resources or developing their own.

Development of the materials is ongoing: at this conference we will present a selection of materials drawn from the following topics:

### Sequences and Series

The focus is on building student understanding of convergence and divergence through the exploration of dynamic models, both in general and as specific to geometric sequences and series by contrasting these with arithmetic series and the harmonic series. In particular, we seek to add to students' experience of convergence and divergence through the exploration of spirals constructed from convergent or divergent series. In this way we aim to stimulate students' intuition through perceiving series in a new way, and motivate the need for rigorous proof in the classification of the behaviours the students perceive.

### Functions

The focus is on exploring different representations of functions (including that of Descartes) and in particular looking at why function transformations have certain graphical results. We seek to help students' achieve ownership and understanding of the reasons behind graphical transformations admitted by  $y = p f(q(x-a)) + b$  by use of the Cartesian representation and hence to also give this topic a historical dimension. Exponential and trigonometric functions will also be explored.

### Vectors

The focus is on the student learning in an environment that allows exploration by dragging to alter the parameters of the vector equation of a line. The aim is to help students experience that there are infinitely many representations of a line in vector form. In addition, a resource based on a "space rendezvous" provides an environment in which students can plan and test approaches to solving problems with vectors. By changing initial conditions, students can pose "what if?" questions,

predict the solution, and test their prediction. In this way we seek to develop confidence in students to use technology to explore new ideas and solve problems by constructing mathematical arguments through the use of precise statements. Scalar and vector products will also be represented and explored.

## Calculus

Here we present a resource that gives students the opportunity to experience, describe and test conjectures regarding secants and tangents to curves and the features of the graph of a function and how they relate to the graphs of its first and second derivatives. We aim to help students become more knowledgeable about the derivation and proof of the product and chain rules of differentiation through guided exploration tasks. Differential equations and implicit differentiation will also be explored.

## THE SESSIONS

In each of the sessions, the participant may choose to focus exclusively on exploring the learning materials. However, we feel that, although the materials do not demand particular expertise with Cabri 3D, an introduction to the stand-alone use of this software in 16-19 would be useful. We also felt that there might be some interest in some of the advanced techniques used to develop the files. The sessions would hence be structured as follows:

### Session 1

Introduction to the softwares used and the learning materials, giving the research rationale and the classroom results of each. Opportunity for the participants to engage in hand-on exploration of the materials, and to discuss the use of the materials in the classroom and begin to adapt materials to their own purposes.

### Session 2

Continuing exploration of the learning materials, with the optional reconstruction of one of the Cabri 3D files, introducing the construction of lines, planes, vectors, and function graphs and also the creation of a hide/show slider.

### Session 3

Continuing exploration of the learning materials, with the optional reconstruction of one of the Cabri II Plus/Geometer's Sketchpad files, focusing on some of the advanced techniques used to create this file, such as various types of action sliders and feedback.

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## THE GEOMETER'S SKETCHPAD WORKSHOP: EXPLORING NON-EUCLIDEAN GEOMETRY WITH THE POINCARÉ DISK

Nicholas Jackiw

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*This paper briefly introduces a hands-on computer workshop offered at ICTMT11, and orients participants to reusable resources.*

### OVERVIEW

The study of hyperbolic geometry—and non-euclidean geometries in general—dates to the 19<sup>th</sup> century's failed attempts to prove that Euclid's fifth postulate could be derived from the other four postulates. Lobachevsky, Bolyai, and Gauss all independently conceived a geometry in which the 5<sup>th</sup> postulate is "broken" by allowing *many* lines—rather than just one—to be defined as parallel to a given line through a point not on that line. The resulting *hyperbolic geometry* can be made particularly vivid by Henri Poincaré's remarkable disk model, which allows that geometry to be visualized—and, using geometry software, manipulated—within the Euclidean plane. In this workshop, we'll use *The Geometer's Sketchpad* to examine some of the implications of breaking the 5<sup>th</sup> postulate by constructing and exploring hyperbolic geometry, using Poincaré's disk model of the hyperbolic plane.

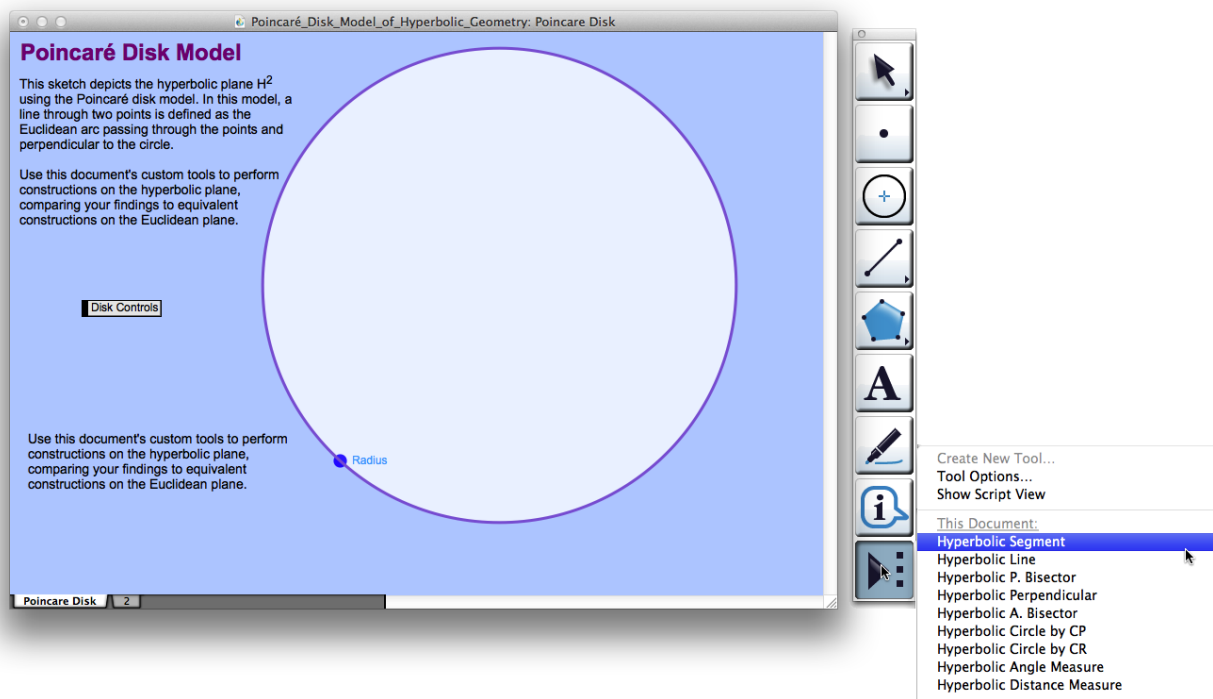
### SKETCHPAD & NON-EUCLIDEAN "DRAWING WORLDS"

The Geometer's Sketchpad (Jackiw, 1991, 2009) is available for Macintosh and Windows computers. A free preview edition can be accessed here: <http://www.keycurriculum.com/download-free-preview-of-sketchpad>

While offering a Dynamic Geometry environment focused on an essentially Euclidean geometry marked up by various transformational and analytic capabilities, Sketchpad allows users to augment both the software's virtual plane and its mathematical toolkit with custom contents and operations, which in turn allows other geometries to be fabricated in standalone and disseminable "drawing worlds" (Jackiw, 1997). These act as self-contained microworlds offering their own mathematical "terms of engagement."

In the workshop, we will use a Sketchpad "drawing world" implementing the Poincaré disk model of hyperbolic geometry as well as some basic operations in that world. Its ingredients are illustrated in Figure 1. On the left, the basic Sketchpad document template includes a Euclidean circle whose interior represents Poincaré's hyperbolic plane. On the right, attached to the Custom Tools icon in Sketchpad's basic toolbox, are the series of new tools we can use in this disk model. Much of our workshop will focus on comparing the operation of these non-Euclidean tools, within this disk model, to those of their Euclidean counterparts (the top-level tools in Sketchpad's toolkit) within the Euclidean plane (a basic blank Sketchpad sketch).

The basic Poincaré Disk Model drawing world can be accessed from within Sketchpad by choosing Help | Sample Sketches and Tools, and then navigating to the Advanced Topics Folder.



**Figure 10. The Poincaré Disk Model as a Sketchpad Drawing World**

Additional Sketchpad drawing worlds, such as for the half-plane model of hyperbolic geometry or for other geometries (elliptical geometry, Minkowskian geometry, etc.) can be freely downloaded from the Advanced Sketch Gallery

([www.dynamicgeometry.com/General\\_Resources/Advanced\\_Sketch\\_Gallery.html](http://www.dynamicgeometry.com/General_Resources/Advanced_Sketch_Gallery.html))

or the Sketchpad Sketch Exchange (<http://sketchexchange.keypress.com>).

## WORKSHOP QUESTIONS

Given participant time and interest, we will attempt to construct the following understandings or demonstrations in hyperbolic geometry:

- the suspension of the fifth postulate
- the variability of the triangle angle sum
- the equilateral triangle (Euclid's first proposition)
- hyperbolic tessellations by equilateral triangles
- hyperbolic conic sections

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## EPSILONWRITER: EDITING TEXT AND FORMULAS, DYNAMIC ALGEBRA, QUESTIONNAIRES AND COMMUNICATION WITH MATH

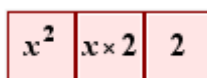
Jean-François Nicaud  
Aristod, Palaiseau, France

*Epsilonwriter is Java software running on Windows, MacOS and Linux, for editing text and formulas. Beyond allowing the production of documents and web pages with a very flexible formula editor, Epsilonwriter implements Dynamic Algebra, an innovative and rich mechanism for step by step calculations. It includes a module for questionnaires (Multiple choice questions and questions with open mathematical answers). Epsilonwriter also has a chat, and allows working on Live documents (several persons share a document in real time). In all cases, math formulas are easy to type and are received as objects that can be edited.*

### EDITING TEXT AND FORMULAS

Epsilonwriter contains a very flexible editor of formulas (Nicaud & Viudez 2009). You can type  $3x^2-4x/x-1$  and get  $\frac{3x^2 - 4x}{x-1}$ .

During typing, a popup allows choosing the interpretation of the last entry, e.g., when 2 is typed, the popup proposes:



With Epsilonwriter you can paste images and import Latex files.

You can paste some Wikipedia content, getting editable formulas, e.g. the following part of a webpage:

**If the discriminant is negative, then there are no real roots. Rather, there are two distinct (non-real) complex roots**

$$\frac{-b}{2a} + i \frac{\sqrt{-\Delta}}{2a} \quad \text{and} \quad \frac{-b}{2a} - i \frac{\sqrt{-\Delta}}{2a},$$

where the formula is a big image is pasted as:

**If the discriminant is negative, then there are no real roots. Rather, there are two distinct (non-real)**

**complex roots**  $\frac{-b}{2a} + i \frac{\sqrt{-\Delta}}{2a}$  and  $\frac{-b}{2a} - i \frac{\sqrt{-\Delta}}{2a}$

where the formula can be modified.

Epsilonwriter documents can be:

- Copied then pasted in Word, providing editable formulas for Word 2007 and later;
- Sent by email (directly or via copy and paste in a mailer); formulas are transformed into images to be correctly displayed;
- Exported in HTML webpages; formulas are transformed into images to be correctly displayed;
- Exported in XHTML webpages; formulas are written in MathML to get the pretty display of formulas included in XHTML.



Epsilonwriter documents can be directly published on the <http://www.epsilon-publi.net/> website where index allowing formulas are automatically generated.

## QUESTIONNAIRES

In the author mode, the questionnaire module allows to insert multiple choice questions and questions with open mathematical answers. For the last ones, one or several expected answers are typed by the author and the mechanism for comparing the student and the expected answer is chosen (identical, same modulo associativity and commutativity, same modulo numerical calculations). For both, explanations can be inserted. Explanations are displayed in the evaluation phase.

In the student mode, students choose items of multiple choice questions or type answers to questions with open mathematical answers. They can have the evaluation phase after each question or at the end, with a score in this case. Completed questionnaires can be sent to a tutor who can add annotations.

## DYNAMIC ALGEBRA

Dynamic Algebra is “doing algebraic calculations on a computer by direct manipulation”. Dynamic Algebra has been implemented in the Epsilonwriter (Nicaud & Maffei 2013). For that purpose, a “Theory of Movements in Formulas” (TMF) has been elaborated. It describes, in a precise context, how we can move a sub-expression in a formula with preservation of equivalence. It has been implemented in Epsilonwriter for the Pedagogical Dynamic Algebra mode. Dynamic Algebra in Epsilonwriter has other modes with more complex actions of different classes: equivalent drag & drop internal to a formula, external drag & drop, on click calculations and rewriting rule application. This makes Epsilonwriter a very powerful and flexible calculation tool for step by step calculations; and a tool able to provide explanations. Epsilonwriter contains different options to control the possible movements and the feedback on them. It has been developed to help students learn formula structure and transformations, and to help every user produce step by step explained calculations.

## CHAT AND LIVE DOCUMENTS

The chat of Epsilonwriter allows to type text and formulas, and also to paste images. This is a very pleasant tool to discuss with formulas.

Live Documents are documents having several readers who see the modifications of the documents a few second after they occurred. Epsilonwriter has two sorts of Live Documents. The “one writer at a time” sort is well adapted when the reader wants to follow the work of the writer, which is the case in distance tutoring. The writer can select an expression to highlight it for the reader. The “several writers at a time” sort allows several persons writing on different paragraphs of the document.

## NOTES

1. [jeanfrancois.nicaud@laposte.net](mailto:jeanfrancois.nicaud@laposte.net)
2. <http://epsilonwriter.com/>

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# Posters



## **EDUMATICS PROJECT: TEACHERS EDUCATION WITH TECHNOLOGIES**

Gilles Aldon, Bärbel Barzel, Alison Clark-Wilson, Ornella Robutti

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*The EdUmatICS project aims to provide teachers of secondary mathematics with support to learn to use and integrate technology within their classrooms. The resources for professional development, whilst aimed at teachers, include a range of tasks for students to enable them to use technology within modeling and problem-solving activities. The resources include links to free and trial software, applications and animations in addition to task sheets and help sheets that can be adapted for different scenarios.*

### **THE EDUMATICS PROJECT**

Edumatics is a Comenius Project containing mathematics teacher education resources in the field of integrating technology into mathematics teaching. It comprises of 5 modules and each one is aimed at presenting some particular issues of professional development: an introduction to the use of technologies, the use of representations in static and dynamic way, videos for teacher training, functions as models of phenomena and mathematical configurations. The themes chosen are different routes to the same common end-points, and offer a choice of different starting points depending on the motivation and the preliminary knowledge and skills. Moreover, they make connections with some of the important research ideas that have underpinned the design of the resources. This poster, along with the workshop presentation, gives some insight into the project.

#### **Module 1: Starting to work with ICT**

In this module participants learn about some typical activities that use technology in secondary school mathematics. These activities are elaborated for different types of technological tool, including TI-Nspire. These activities aim to represent some characteristic aspects and application areas, where the use of technology can support mathematics teaching and enrich its efficiency.

#### **Module 2: From static to dynamic representations**

In this module participants learn about the use of dynamic representations in the mathematics classroom and the teaching approaches that support them to be used to enhance students' mathematical understanding. This will support an understanding of the role of multiple representations within the learning of mathematics and opportunities that the use of ICT offers to represent mathematical objects in a dynamic way.

#### **Module 3: Constructing functions and models**

In this module participants learn how to use digital technologies (spreadsheet, dynamic geometry, computer algebra) for introducing students with functions as modeling tools in a diversity of contexts, and for exploring their properties through the dynamic interplay between representations that technology makes possible.

#### **Module 4: Using technology in the classroom: Teaching approaches**

In this module participants are encouraged reflect on their own classroom practice when using technology in their teaching, and to further develop these practices. Through the study of other

colleagues' practices, participants will get ideas on how to approach their own classes as they integrate technology in mathematics courses.

### **Module 5: Interrelationships between software**

In this module participants learn how to use the dynamic linkage between mathematical representations, which has been made available by ICT, as a beneficial tool in the classroom. They also consolidate knowledge on the use of dynamic geometry systems, spreadsheet software and computer algebra systems.

### **REFERENCES**

[www.edumatics.eu](http://www.edumatics.eu)

## REASONING WITH MULTIPLE AND DYNAMIC REPRESENTATIONS

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University of Wuerzburg

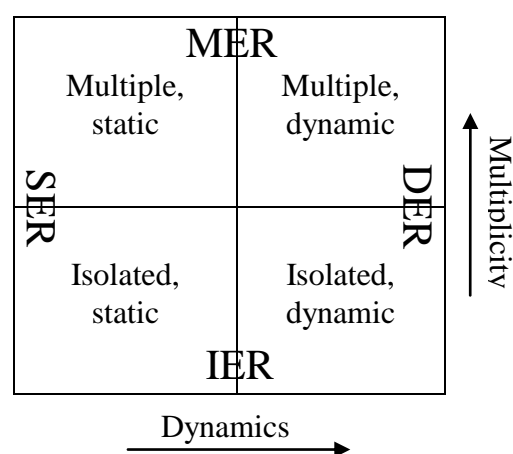
*This paper presents a research in progress which aims to answer the question if learners working with digital multiple and dynamic representations really refer to the dynamics and the multiplicity in their arguments. In an empirical investigation learners were given mathematical reasoning tasks along with the appropriate representations, which were partially multiple, dynamical, or both. The hand-written documents were analysed regarding the appearance of multiple or dynamic representations in the learners arguments. Results indicate that the given representations have a great influence most of the time, but not always.*

### MULTIPLE AND DYNAMIC REPRESENTATIONS

Representations play an important role in the learning and the use of mathematics. The emergence of new technologies such as computers and handheld devices, the quick availability of even complex representations, along with the possibility of creating multiple and dynamic representations of mathematical objects with the press of a button, brought their use in mathematics classrooms into research focus.

Using multiple representations yields potential benefits for learners. As Ainsworth (1999) pointed out in her taxonomy of the roles of multiple representations, they can not only provide learners with additional information and allow them to use more appropriate strategies for their task, but they can also help to decrease ambiguity and construct a deeper understanding of representations. We speak of a multiple representation (MER) if it integrates and displays more than one representation of the same mathematical object. Otherwise we call it isolated (IER). An example of a multiple representation is the simultaneous depiction of an equation, a graph and a table of a function.

Dynamic representations (DER), too, provide learners with additional information by enabling them to vary a given representation quickly to construct variations of a mathematical problem which may lead to conjectures about a solution (Arzarello, Ferrara, & Robutti, 2012). We will call a representation dynamic if it changes over time (automatically or through user input). Otherwise, we call it static (SER). As an example of a dynamic representation one may think of using sliders to vary parameters of an equation of a function resulting in a change of the shape of the function's graph.



**Fig. 1: Taxonomy of representations**

### RESEARCH QUESTION

The presented research was designed to answer the following question:

Does the representation category (s. figure 1) that is used when presenting the problem to learners have the most significant impact on the representations used by learners in their justifications, or is it their individual preference (or capability)?

## METHOD

The empirical investigation was taken in four German classes, grade 11 (students aged 16-17 years); a total of 89 probands with mixed levels in school mathematics. Their tasks were presented on a computer screen, with the students being in control of the input devices to enable them to vary the given representations if possible. The students were asked to solve the tasks with paper and pen, so they would not be limited by having to give mathematical input to a computer. The probands were divided into two groups (sized 44 and 45 students) to ensure that while all combinations of representational category switching were present, the working time did not exceed 45 minutes.

The tasks have been around the mathematical topic “functions”. The questions were purely inner-mathematical, and students were always asked to give justifications for their claims. Each student had to solve four mathematical problems, which consisted of two pairs a and b with analogue tasks, but different representations, in an “abab” order (see below in the results). This design was used to determine if the students were set on one category of representation as in figure 1, regardless of the one presented, or if they switched e.g. from using isolated to multiple representations, according to the one presented in the problem.

## RESULTS

Not all probands completed all tasks. This is why the following numbers do not add up to 89. The p values were calculated using the binomial test where the null hypothesis is that the considered types of representations in students’ arguments were distributed equally.

|             |            | A3 (IDR)   |           |             |            | A4 (MDR)   |            |
|-------------|------------|------------|-----------|-------------|------------|------------|------------|
| A1<br>(ISR) |            | Stat. Arg. | Dyn. Arg. | A2<br>(IDR) |            | Isol. Arg. | Mult. Arg. |
|             | Stat. Arg. | 1          | <b>9</b>  |             | Isol. Arg. | 2          | <b>14</b>  |
|             | Dyn. Arg.  | 2          | 28        |             | Mult. Arg. | 2          | 10         |

As the table above shows, 9 students who used SER in their arguments switched to DER when the representation depicted in the task switched in the same way from task 1 to 3. Although the majority of 28 students used DER in both cases, this is a significant result ( $p < .05$ ). In tasks 2 and 4, when multiplicity was added, but the representation remained static, the largest group of students (14) switched from isolated to multiple representations, which is a highly significant result ( $p < .01$ ).

|             |            | B3 (MSR)   |            |             |            | B4 (MDR)   |           |
|-------------|------------|------------|------------|-------------|------------|------------|-----------|
| B1<br>(ISR) |            | Isol. Arg. | Mult. Arg. | B2<br>(MSR) |            | Stat. Arg. | Dyn. Arg. |
|             | Isol. Arg. | 23         | 16         |             | Stat. Arg. | 14         | 5         |
|             | Mult. Arg. | 2          | 1          |             | Dyn. Arg.  | 11         | 9         |

While 23 students stuck to isolated representations in task 1 and 3, a large group of 16 switched to MER in their arguments, also a highly significant result ( $p < .001$ ). The switch from multiple static to multiple dynamic representations, however, did not show a clear picture, resulting in  $p > .10$ . The reason for this is not clear and suggests further investigation.



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## WIMS: A WWW INTERACTIVE MULTIPURPOSE SERVER

Marina Cazzola, Bernadette Perrin-Riou, Eric Reysat

*Università di Milano-Bicocca, Université Paris-Sud, Université de Caen*

*In this poster we wish to introduce WIMS, showing its potentials as a tool for the teaching of mathematics (and beyond).*

### WIMS

WIMS, “WWW Interactive Multipurpose Server”, is a flexible system that allows for the creation of a large variety of learning objects with automatic marking. It was conceived by Xiao Gang and made public in 1998 (Xiao, 2001). Installed on a WWW server, it provides standard LMS-like facilities (virtual classes and students’ management, discussion forum, repository of interactive documents, quizzes, ...; for a complete description see Guerimand 2004). WIMS’s main characteristic is the capability of interacting with softwares of various kinds: Maxima, Octave, Pari-GP, Gnuplot, Povray, Jsxgraph, GAP, GeoGebra, Graphviz, just to name a few. This capability, together with the possibility to use random parameters, allows for the design of exercises of very different kinds. A well designed use of random parameters allows WIMS to provide a virtually infinite set of copies of each single activity. Also WIMS has a builtin drawing engine able to efficiently create geometrical drawings on the fly. Examples of ready to use activities (i.e. “modules” in WIMS language) are available on any WIMS server. WIMS is distributed under GPL, it is available in many languages, and any installation of WIMS has access to the whole collection of public modules (about 1500).<sup>1</sup>

There are a number of descriptions of teaching experiences with WIMS which may constitute a first basis for an evaluation of the effectiveness of WIMS as a tool for teaching mathematics (e.g. see Cazes et al., 2006; Cazzola, 2011; Ducrocq, 2010; Kuzman, 2007; Ramage & Perrin-Riou, 2004; Reysat, 2013). Also WIMS meetings are held periodically to exchange experiences in the community of users; abstracts of the conferences given and posters presented are publicly available.<sup>2</sup> The underlining idea in all of these experiences (at different school levels, in different class situations, in different subjects), which might be the basis for future research, is that it seems that making available to students a wide variety of exercises with automatic correction of the response can motivate students and keep alive their attention. If the activities are well constructed, and activate different skills (in this sense, the multimodality of the exercises that are actively using images seems to be a key example), if such exercises provide the students with a good variety of different examples, it seems they can help the students develop mathematical abstraction.

Once again, the strengths of WIMS, that seem to have contributed to its spread, appear to be:

- variety of activities that activate many skills available “out of the box” and friendly user interface allowing teachers to create new learning objects and modify existing ones;
- longevity of the exercises: the first exercises created in 1998 are still operating perfectly;
- a modular structure that allows for the extension of functionality;
- an active community, coordinated by the association WIMS EDU, which updates and distributes the system, proposes new modules and gives active support to the users of WIMS around the world (through the WIMS EDU web site <http://www.wimsedu.info/>).

We realize that many questions remain open and any contributions from mathematics education researchers can contribute to the development of WIMS in a direction that improves its effectiveness in the teaching of mathematics (and beyond).

- Which are the most effective interactive activities, and can the strengths of WIMS help teachers offer such activities to their students?
- Can WIMS be used to evaluate students' achievements?
- Can WIMS be used to build personalized learning paths?
- How to develop the technological knowledge of teachers in order to enable them to exploit the full potential of WIMS?
- Can the capabilities of WIMS be embedded into other LMS?

Such questions have been partly covered by the reports listed in the references, but we hope they can be further investigated.

## NOTES

1. To experience WIMS one can start from <http://wims.auto.u-psud.fr/wims/> or <http://wims.matapp.unimib.it/wims/>.
2. The latest “colloques” were held in Nice, Bordeaux and Rennes (see Guérimand et al., 2008, [http://www.math.u-bordeaux1.fr/WIMS\\_2010/presentations.html](http://www.math.u-bordeaux1.fr/WIMS_2010/presentations.html) and [http://www.irem.univ-rennes1.fr/viedelirem/activites\\_1112/colloque-wims-2012/conferences.html](http://www.irem.univ-rennes1.fr/viedelirem/activites_1112/colloque-wims-2012/conferences.html)).

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## INTERACTION: A KEY COMPONENT TO SUCCESSFUL ONLINE LEARNING

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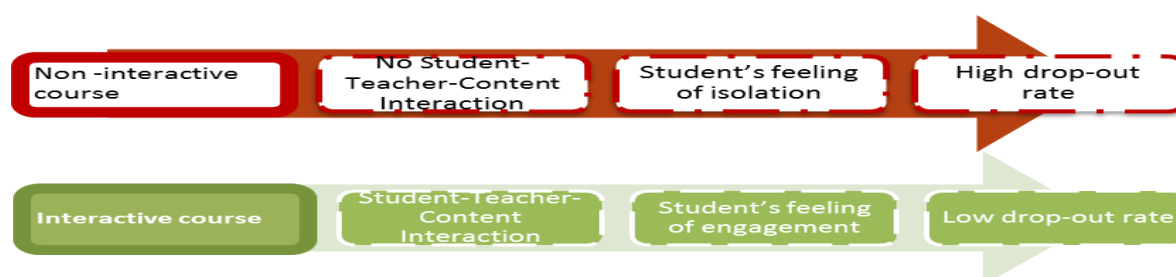
*E-learning, a process to create and provide access to learning when the source of information and the learners are separated by time and distance, has special kinds of barriers which need to be known and considered, for instance the high drop-out rate of e-learning and the suitability of e-learning to cover different subjects. Interactivity in e-learning is considered to be more than just clicking a mouse. This thesis argues that an in-depth understanding of interactivity in e-learning will reinforce and enhance the capabilities of this learning mode, and consequently have a measurable positive impact on the aforementioned limitations.*

### METHODS

Quantitative method is conducted in this study. The selected research instruments are a semi-structured questionnaire as well as an experiment. Respondents were students and teachers, studying or teaching different courses such as IT, business, and foreign languages in educational institutes or academic centres. Students were asked to study a short online course with a simple assessment at the end. After finishing the course, in which all interactive elements were functionalized, students who studied it with interactive elements were the experiment group, those who did not were the control group. Then, they were asked to answer the questionnaire after finishing the course.

### RESULTS

The concluded results and findings obtained from the experiment and questionnaires giving evidence to its related research question is illustrated in the following constructed diagram:

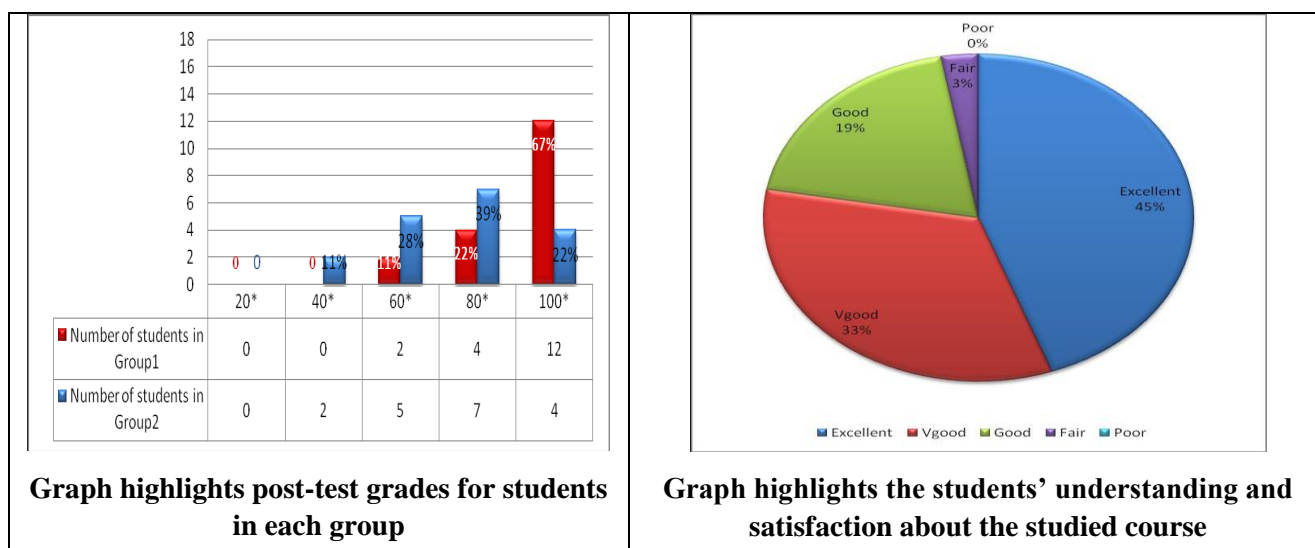


**The relation between interactive elements and drop-out rate in OL**

### DATA

Group1 = Experiment Group (Interactive Course) =18

Group2 =Control Group (non-Interactive Course) =20-2(missing) =18



## CONCLUSION

A good understanding of interactive e-learning is suggested to engage learners and increase their understanding and satisfaction of an OL course, and the potential to reduce the drop-out rate of OL is great. With regards to expanding the possibility of e-learning to cover more online subjects, it is sensible to accept that the potential to achieve this is limited because of a number of factors, such as:

- Lack of research about the web2 application in online courses
- Lack of staff training regarding online courses
- Teachers' limited access to technology
- Limited bandwidth in some geographical areas

However, interactive online study is not expected to cover all subjects without first overcoming these obstacles.

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## THE GAINS AND THE PIFALLS OF DESIGNING EDUCATIONAL MATH SOFTWARE BY PRINCIPLES OF MATEMATICS

Reinhard Oldenburg

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*This poster briefly describes two software prototypes developed by the author that address different aspects of the algebra taught in secondary schools. Both programs are designed to reflect specific aspects of algebra as a mathematical discipline.*

### SHOOL ALGEBRA, LOGIC AND EDUCATIONAL SOFTWARE

Algebra is an important and highly complicated subject of schools mathematics that can be structured according to various dimensions of analysis, e.g. it comprises different kinds of activities (Kieran 2004) and structures that obey different linguistic systems (Hodgen et al. 2013). Among the syntactical, semantical and pragmatic aspects of the algebraic language that students need to learn, the first two can be addressed by two programs developed by the author, FeliX1D and SCAS. They represent systems that try to deliver computer based mathematics for school students that is close to the framework of modern university mathematics. The paper's aim is to investigate what strengths and weaknesses follow from this mathematical foundation.

#### FELIX1D

Semantical aspects of algebra are those connected with the notion of reference. In mathematical logic formulas like  $x+y=y+x$  acquire meaning by choosing an *interpretation* i.e. a choice of a domain and choice of one object for all free variables of the domain.

The software FeliX1D (Oldenburg 2009) is a one-dimensional off-spring of FeliX (Oldenburg 2007). It is realized in two ways, as a stand-alone application and as a browser based version that is considered here. Basically it consists of a number line on which markers for variables can be moved with the mouse. There is a variable-table which shows the current value of each variable in tabular. This variable table corresponds to the current interpretation in the sense of logic applied to the formulae. These formulae may be entered and changed in a second table and are given in the form of equations and/or inequalities. The FeliX1D system tries to satisfy these equations. There is no distinction between dependent and independent variables. Every marker can be moved with the mouse if the equations allow it to move. From this description it should be clear that FeliX1D is a straightforward computerization of the algebra-logical conceptualization of arithmetic.

Intended tasks for students of the middle grades are e.g. to restrict three point such that one point will always be the center of the other two or to express that one number is always more than 1 to the right of another one. The didactical hope is that such activities integrate the various aspects of variables (unknowns, known quantities, changing numbers).

#### SCAS

The second tool that is presented here is a simple transparent computer algebra system that shows how it works. SCAS aims to support the learning of syntactical aspects of algebra as well to provide a simple mental model of what a computer algebra system (CAS) is and how it works.

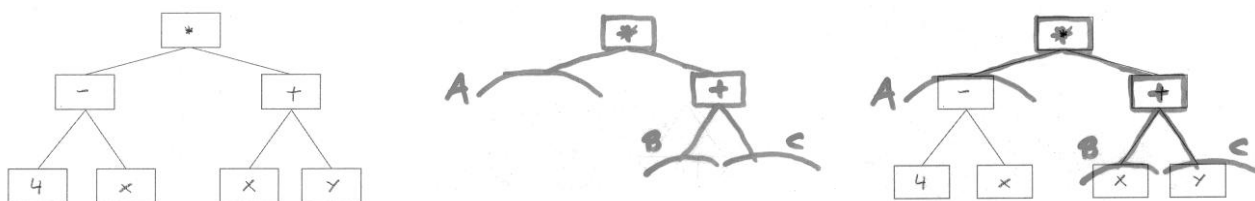
The basic use of SCAS is similar to other CAS: You enter an expression and the system answers with an output. E.g. if you enter  $2*x+y+x+3$  then the system answers  $3*x+y+3$ .

The following simple rules govern the behaviour of the system:

- All expressions are represented by binary trees (e.g.  $a+b+c$  is interpreted as  $a+(b+c)$ )
- There is a list of rewrite rules that are recursively applied. A rule consists of a pattern, possibly a condition, and a replacement. A handy notation is *pattern*→*replacement*.

The application step is the most complex one. It consists of the following sub-steps: First checking if a rule's pattern matches an expression, e.g. the pattern  $(A+B)^2$  matches the expression  $(x+\sin(45))^2$  with the assignment  $A=x$ ,  $B=\sin(45)$ , then applying the rule.

Despite the complex description, rule application can be visualized rather easily. Expression trees are written on a paper template sheet, rules on a similar template on a transparency slide. Putting the latter on top of the former shows how they match.



**Figure 1: Rule application: The expression  $(4-x)*(x+y)$  on the left, the template  $A*(B+C)$  in the middle and the template slide “applied” to the expression, showing  $A=4-x$ ,  $B=x$ ,  $C=y$**

SCAS can therefore be used in two ways: It can give a deeper understanding of formal manipulation systems and it can trigger reflections on the symbolic nature of mathematics. Moreover, it is powerful enough to handle much of high school algebra and calculus.

## COMPARISON AND SYNTHESIS

Although both programs obey the same design principles, they address very different aspects of mathematics as shown in the following table. Yet they have in common that they adhere closely to mathematical theories. For FeliX1D this is Tarski's semantics of logic, for SCAS it is the theory of term rewriting (Bader&Novikov 1999).

| linguistic aspect vs. mode of operation | syntactical    | Semantical  |
|---|----------------|---|
| relational                              | Formal provers | FeliX1D   |
| functional                              | SCAS           | functional programming languages; spreadsheets; dynamic geometry, AlNuSet |

The whole picture shows that it is possible to have rather simple software that faithfully represents modern mathematical theory beyond what is currently in the mainstream of computer tools.

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## **RECENT ADVANCES IN THE APPLICATION OF 3D GEOMETRIC MODELING SOFTWARE WITH FOCUS ON LINEAR PERSPECTIVE**

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*A problem with difficulty of studying and understanding 3D geometry at secondary schools and colleges is addressed in this article. One possibility of improvement the understanding geometry is suggested - modeling with modern geometric and algebraic systems. The integration of computer modeling into the studying and teaching geometry is seems to be as efficient aid how to innovate the schooling of geometry and achieve better results. The application of geometric modeling software will be demonstrated on example of linear perspective - the central projection.*

### **INTRODUCTION AND MOTIVATION**

Descriptive geometry (Pottmann et al., 2007), the branch which deals with principals of geometric projections, is the main field of our interest. The techniques of descriptive geometry are very important for architectures, builders, technicians, designers or civil engineers.

I have my own experiences of teaching classical, descriptive, and computational geometry at Charles University in Prague – Faculty of Mathematics and Physics. The study of geometry is very difficult for some students. Our conjecture is that this is the reason why descriptive geometry and geometry in general does not belong among popular subjects. One unfortunate tendency in recent years in the Czech Republic is decreasing interest in studying this interesting part of mathematics. For improvement the understanding of the geometry of 3D space and for increasing the interest in geometry we suggest using modern geometric computer systems. Geometric modeling software in the education process can motivate students. The study of geometry becomes modern disciplines due to using computers and descriptive geometry regains the importance in technical practice.

In above mentioned areas one is often concerned with representing three-dimensional objects on a two-dimensional display planar surface. Descriptive geometry deals with those representations which are one-to-one correspondent and the 3D objects can be easily and clearly derived from 2D view. Let us summarize, there are two points of view. Firstly we construct the geometric projections of 3D objects onto the two-dimensional plane and secondly we have to be able to reconstruct 3D object from two-dimensional result of projection. In both directions we can use suitable computer software for illustrating the situation in 3D space, for planar constructions, for proving some geometric problems or for automatic creation of projections.

Let us provide important note. We don't propose the absolute replacement of drawings by hand with the outputs of modern computer software. We still have to know the basic principles and rules in geometry even though we use those software.

### **THE APPLICATION OF 3D GEOMETRIC MODELING SOFTWARE IN LINEAR PERSPECTIVE**

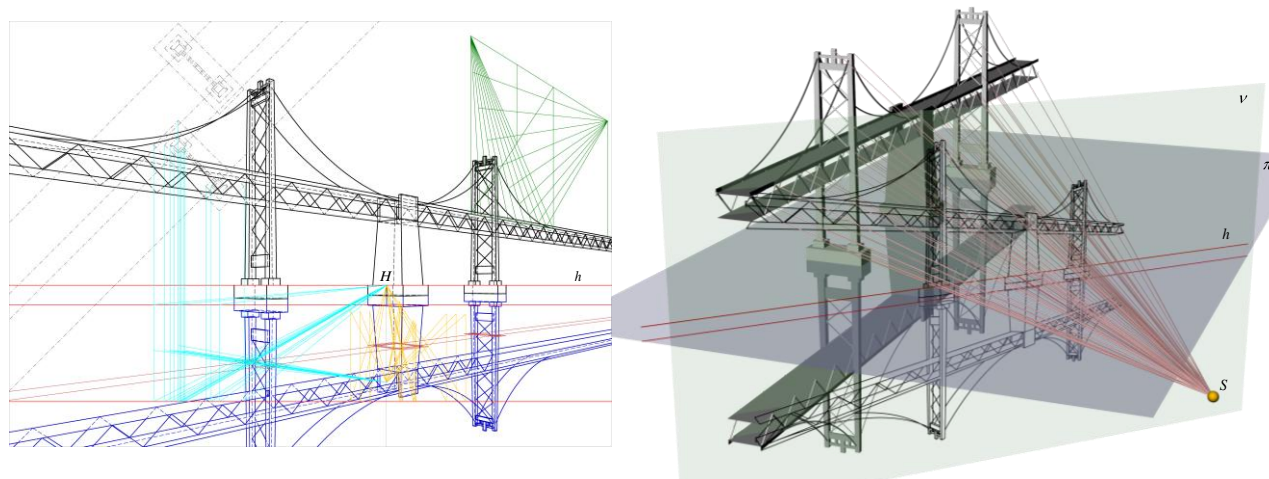
The main principles of well known and widespread central projection – linear perspective can be found in (Auvil, 1996). Perspective projection is use for realistic representation of 3D objects in the two dimensions. The results of this special type of central projections are very close to images as



they are seen by the eye. These kinds of projections are widely used in graphic arts, for presentation drawings, for advertising, for visualizations of architectural design or technical devices.

Typical geometric task is to construct perspective projection of some 3D object or geometric situation in the plane by hand. Complicated 3D structure can be very difficult for sketching because in this phase we work only with the planar situation. We can do the construction in the plane also using the computer. In modern modeling software, we can also work with rotations and other transformations; we can change the view of a designed object. Understanding the basics of linear perspective also improves our ability to find appropriate perspective views.

Let us show construction of perspective view of 3D real object - suspension Bay Bridge. The task is to construct perspective view of bridge and its reflection in water. To generate perspective image of bridge by hand is quite complicated. We did this construction in the plane using computer for more precise result, see figure 1 (left). We use known principles for creation of perspective view and display only a few lines and curves which are used for constructions. Geometric modeling software we use also for illustration the spatial situation, for demonstration the principals of perspective projection, see figure 1 (right). We made figures in software Rhinoceros.



**Figure 1: The result of perspective projection of Bay Bridge (left) and spatial situation and principle of perspective projection (right).**

## CONCLUSION

In this contribution we suggested the use of computer modeling software in the studying geometry. Suitable software can help us for better understanding geometric situation in 3D space. Another examples of 3D modeling can be found in (Surynková, 2011; Surynková 2012).

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