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DOI:10.1016/j.cam.2016.07.011

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Abstract

The paper presents some properties of Generalized T-splines (GT-splines), which are crucial to their actual application. In particular, we construct a dual basis for a noteworthy class of GT-splines, which allows to show that, under suitable conditions, they form a partition of unity. Moreover, we study the approximation properties of the GT-spline space by constructing a class of quasi-interpolants which belong to it and are defined by giving a dual basis.

Keywords: T-spline, generalized B-spline, partition of unity, spline approximation, isogeometric analysis.

1 Introduction

The introduction of the concept of T-mesh in the recent years was a significant advancements for the use of multivariate spline functions. The basic idea behind this approach was first introduced in [19], where the authors propose to construct spline surfaces using control points not necessarily lying, topologically, on a regular rectangular grid whose edges always interesct at “cross junctions”, allowing to have instead partial rows of control point lying on a grid whose vertices can intersect at “T-junctions”. This scheme gives several advantages, such as the possibility to locally refine the surfaces, a considerable reduction of the quantity of control points needed, the ability to easily avoid gaps between surfaces to be joined (see, e.g., [19], [20] and [21]).

While these ideas have been applied mainly to polynomial splines, recently they were generalized [6] and [7] to the noteworthy non-polynomial case of the generalized B-splines (GB-splines). The GB-splines are a particularly relevant class of non-polynomial splines, thanks to their adaptability and their applications like the isogeometric analysis (see, e.g., [12], [15] and [11]): in short, the GB-splines are basis of spaces of piecewise functions, locally spanned both by polynomials and by two other suitable functions. In [6], the T-spline approach was applied to the trigonometric GB-splines, and the basic properties and the linear independence of the obtained blending functions were studied. In [7] the results given in [6] were further generalized to any kind of GB-spline to define the Generalized T-splines, and the class of VMCR T-meshes, guaranteeing the linear independence of the associated T-splines and GT-splines, was introduced.

The goal of this paper is to study some additional properties of the GT-splines, which are crucial for their future practical use, especially considering that the polynomial T-splines are already employed in applications (see, e.g., [1]). In particular, we will show that, under certain conditions, it is possible to prove that the GT-splines form a partition of unity. This results, similarly to what was done in [3], will be achieved by constructing a suitable set of dual functionals. Moreover, this set of functionals will be used to construct a class of quasi-interpolant operators, which, in turn, will allow us to get some results about the approximation power of the space spanned by the GT-splines. In particular, in order to guarantee the full approximation order of the GT-spline space, it is crucial that the norms of the elements of the dual basis do not diverge as the T-mesh is refined. We will show in detail that, at least in some noteworthy cases, such bound can be obtained.

Section 2 contains the definition and main properties of univariate GB-splines, including an overview about the construction of their dual basis and a noteworthy example where the norms of the elements of the dual basis are bounded. In Section 3 we recall the notations and some results about GT-splines, needed in the following. In Section 4 we present the main results of the paper, that is, the construction of the dual basis for GT-splines and the approximation power of the GT-spline space by studying a quasi-interpolation projection operator. We will discuss some detailed examples where the norms of the elements of the dual basis can be bounded independently of the refinement of the T-mesh and, as a consequence, we can bound the constant involved in the study of the approximation order of the GT-spline space.

2 Univariate generalized B-splines

2.1 Definition and main properties

We recall the main definition and properties of the GB-splines, which can be also found in [7], [15], and [12]. Let $n, p \in \mathbb{N}$, $p \geq 2$, and let $\mathbf{\Sigma} = \{s_1 \leq ... \leq s_{n+p}\}$ be a non-decreasing knot sequence (*knot vector*); we associate to $\mathbf{\Sigma}$ two vectors of functions $\mathbf{\Omega_u} = \{u_1(s), ..., u_{n+p-1}(s)\}$ and $\mathbf{\Omega_v} = \{v_1(s), ..., v_{n+p-1}(s)\}$, where, for $i = 1, ..., n+p-1$, u_i, v_i belong to $C^{p-2}[s_i, s_{i+1}]$ and are such that the space W spanned by the derivatives

$$U_i(s) = \frac{d^{p-2}u_i(s)}{ds^{p-2}}, \quad V_i(s) = \frac{d^{p-2}v_i(s)}{ds^{p-2}}$$

is a Chebyshev space, that is, the two following conditions are verified (see also [10]):

$$\begin{aligned} &\forall \psi \in W, \text{ if } \psi^{(p-2)}(s_1) = \psi^{(p-2)}(s_2) = 0, \quad s_1, s_2 \in [a,b],\ s_1 \neq s_2 \\ &\qquad \text{then } \psi^{(p-2)}(s) = 0, \quad s \in [a,b]; \end{aligned} \tag{1}$$

$$\begin{aligned} &\forall \psi \in W, \text{ if } \psi^{(p-2)}(s_1) = \psi^{(p-1)}(s_1) = 0, \quad s_1 \in (a,b), \\ &\qquad \text{then } \psi^{(p-2)}(s) = 0, \quad s \in [a,b]. \end{aligned} \tag{2}$$

We remark that the condition (1) is essential to construct the GB-spline functions defined below, while the condition (2) is a key point to prove some approximation properties of the spaces spanned by the GT-splines in Section 4.
We consider the *generalized spline space* of the functions which, restricted to each interval $[s_i, s_{i+1}]$, belong to the space spanned by $\{u_i(s), v_i(s), 1, s, ..., s^{p-3}\}$ for $p \geq 3$ and by $\{u_i(s), v_i(s)\}$ for $p = 2$. For the generalized spline space it is possible to define a basis of compactly-supported splines called *Generalized B-splines*.
Having required that the space spanned $W = \langle U_i, V_i \rangle$ is a Chebyshev space, it is not restrictive to choose, as generating functions of W, $U_i(s)$ and $V_i(s)$ such that

$$U_i(s_i) > 0, \quad U_i(s_{i+1}) = 0, \qquad V_i(s_i) = 0, \quad V_i(s_{i+1}) > 0. \tag{3}$$

We will call the selected functions $U_i(s)$ and $V_i(s)$ *generating functions associated to* $[s_i, s_{i+1}]$. Following [12] and [15], we can define a basis of compactly-supported spline functions for the generalized spline space in the following way: for $p = 2$

$$N_i^{(2)}[\mathbf{\Sigma}, \mathbf{\Omega_u}, \mathbf{\Omega_v}](s) = \begin{cases} \dfrac{V_i(s)}{V_i(s_{i+1})}, & \text{if } s_i \leq s < s_{i+1}, \\ \dfrac{U_{i+1}(s)}{U_{i+1}(s_{i+1})}, & \text{if } s_{i+1} \leq s < s_{i+2}, \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

while, for $p \geq 3$,

$$N_i^{(p)}[\mathbf{\Sigma}, \mathbf{\Omega_u}, \mathbf{\Omega_v}](s) = \int_{-\infty}^{s} \left(\delta_i^{(p-1)} N_i^{(p-1)}(r) - \delta_{i+1}^{(p-1)} N_{i+1}^{(p-1)}(r)\right) dr, \qquad i = 1, ..., n, \tag{5}$$

where

$$\delta_i^{(p)} = \left[\int_{-\infty}^{\infty} N_i^{(p)}(r) dr\right]^{-1}, \qquad i = 1, ..., n. \tag{6}$$

Moreover, if $N_i^{(p)}(s) = 0$, we set

$$\int_{-\infty}^{s} \delta_i^{(p)} N_i^{(p)}(r) dr = \begin{cases} 1, & s \geq s_{i+p}, \\ 0, & s < s_{i+p}, \end{cases}$$

(see Figure 1 for some examples). For notational brevity, in the sequel we will drop $[\mathbf{\Sigma}, \mathbf{\Omega_u}, \mathbf{\Omega_v}]$ in $N_i^{(p)}[\mathbf{\Sigma}, \mathbf{\Omega_u}, \mathbf{\Omega_v}](s)$ if there is no risk of confusion.

Property 2.1. *The GB-splines satisfy the following properties.*

1. **Continuity:** *each* $N_i^{(p)}$ *is* $(p - m_j - 1)$ *times continuously differentiable at the knot* s_j*, where* m_j*,* $1 \leq m_j \leq p$*, is the multiplicity of* s_j *in the knot vector* $\{s_i, ..., s_{i+p}\}$*, that is, the cardinality of the set*

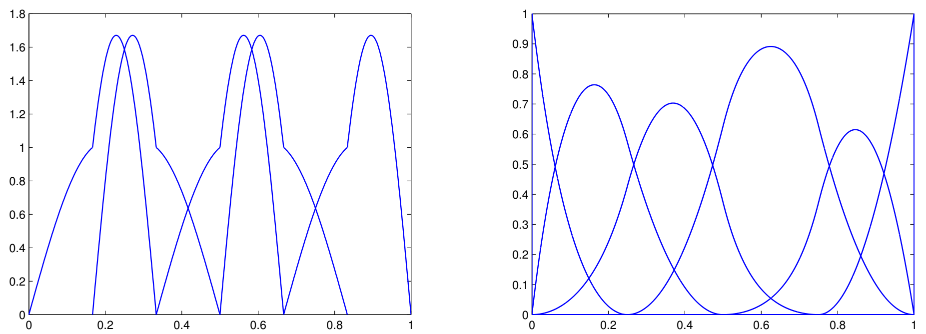
$$\{k : i \leq k \leq i + p, s_k = s_j\}.$$

2. **Positivity:** $N_i^{(p)}(s) \geq 0$ *for* $s \in \mathbb{R}$*,* $i = 1, ..., n$ *and* $p \in \mathbb{N}$*,* $p \geq 2$*.*

3. **Local support:** *if* $s \notin [s_i, s_{i+p}]$ $N_i^{(p)}(s) = 0$*,* $i = 1, ..., n$ *and* $p \in \mathbb{N}$*,* $p \geq 2$*.*

4. **Partition of unity:** *for* $p \geq 3$ *and* $s \in [s_p, s_{n+1}]$*,* $\sum_{i=1}^{n} N_i^{(p)}(s) = 1$*.*

5. **Linear independence:** *for any* $p \geq 2$ $N_1^{(p)}, ..., N_n^{(p)}$ *are linearly independent.*



(a) (b)

Figure 1: Two examples of GB-splines: (a) the GB-splines of order $p = 2$ defined on the knot vector $\mathbf{\Sigma} = \{0, 1/6, 1/3, 1/2, 2/3, 5/6, 1\}$ and locally spanned by the functions $\{u_i(s), v_i(s)\}$, where $u_i(s) = cos(\omega_i s)$ and $v_i(s) = sin(\omega_i s)$ with $\{\omega_i\}_{i=1}^{6} = \{8, 15, 8, 15, 8, 15\}$; (b) the GB-splines of order $p = 3$ defined on the knot vector $\mathbf{\Sigma} = \{0, 0, 0, 1/4, 1/2, 3/4, 1, 1, 1\}$ and locally spanned by the functions $\{1, u_i(s), v_i(s)\}$, where $u_i(s) = cosh(\omega_i s)$ and $v_i(s) = sinh(\omega_i s)$ with $\{\omega_i\}_{i=1}^{8} = \{1, 1, 8.5, 1.3, 12.3, 0.5, 1, 1\}$. In (a), since $p = 2$, the GB-splines do not form a partition of unity.

We also have a knot insertion formula stated below (see [6]- [7] for a detailed proof).

Theorem 2.2. *Let* $\mathbf{\Sigma} = \{s_1, ..., s_{n+p}\}$ *be a knot vector,* $\bar{\mathbf{\Sigma}} = \{\bar{s}_1, ..., \bar{s}_{n+p+1}\}$ *the knot vector obtained by inserting a new knot* $\bar{s}$*,* $s_i \leq \bar{s} < s_{i+1}$*. Let* $\mathbf{\Omega_u} = \{u_1(s), ..., u_{n+p-1}(s)\}$*,* $\mathbf{\Omega_v} = \{v_1(s), ..., v_{n+p-1}(s)\}$ *and* $\bar{\mathbf{\Omega}}_\mathbf{u} = \{\bar{u}_1(s), ..., \bar{u}_{n+p}(s)\}$*,* $\bar{\mathbf{\Omega}}_\mathbf{v} = \{\bar{v}_1(s), ..., \bar{v}_{n+p}(s)\}$ *be the corresponding vectors of functions, where*

$$\begin{aligned} \bar{u}_j(s) = u_j(s) \quad &and \quad \bar{v}_j(s) = v_j(s) \quad if \quad j \leq i \\ \bar{u}_j(s) = u_{j-1}(s) \quad &and \quad \bar{v}_j(s) = v_{j-1}(s) \quad if \quad j > i. \end{aligned} \tag{7}$$

If we denote by $N_i^{(p)}(s)$ *and* $\bar{N}_i^{(p)}(s)$ *the GB-splines of order* p*, respectively before and after the knot insertion, and by* $r+1$ *the multiplicity of* $\bar{s}$ *in* $\bar{\boldsymbol{\Sigma}}$*, then we obtain*

$$N_j^{(p)}(s) = \alpha_{j,p}\bar{N}_j^{(p)}(s) + \beta_{j+1,p}\bar{N}_{j+1}^{(p)}(s), \tag{8}$$

with, for $p > 2$*,*

$$\alpha_{j,p} = \begin{cases} 1, & j \le i-p, \\ \dfrac{\delta_j^{(p-1)}}{\bar{\delta}_j^{(p-1)}}\alpha_{j,p-1}, & i-p < j < i-r+1, \\ 0, & j \ge i-r+1, \end{cases}$$

$$\beta_{j,p} = \begin{cases} 0, & j \le i-p+1, \\ \dfrac{\delta_j^{(p-1)}}{\bar{\delta}_{j+1}^{(p-1)}}\beta_{j+1,p-1}, & i-p+1 < j < i-r+2, \\ 1, & j \ge i-r+2, \end{cases}$$

and, for $p = 2$*,*

$$\alpha_{j,2} = \begin{cases} 1, & j < i, \\ \dfrac{V_i(\bar{s})}{V_i(s_{i+1})}, & j = i, \\ 0, & j \ge i+1, \end{cases}$$

$$\beta_{j,2} = \begin{cases} 0, & j < i, \\ \dfrac{U_i(\bar{s})}{U_i(s_i)}, & j = i, \\ 1, & j \ge i+1, \end{cases}$$

where $\delta_j^{(p-1)}$ *and* $\bar{\delta}_j^{(p-1)}$ *are the constants defined by* (6) *for* $\boldsymbol{\Sigma}$ *and* $\bar{\boldsymbol{\Sigma}}$ *respectively, and* $U_i(s)$ *and* $V_i(s)$*,* $\bar{U}_i(s)$ *and* $\bar{V}_i(s)$*,* $\bar{U}_{i+1}(s)$ *and* $\bar{V}_{i+1}(s)$ *are the generating functions associated to* $[s_i, s_{i+1}]$*,* $[s_i, \bar{s}]$*,* $[\bar{s}, s_{i+1}]$*, respectively, and such that* $\bar{V}_i(s) = V_i(s), \bar{U}_{i+1}(s) = U_i(s)$*.*

2.2 Dual basis

It is possible to construct a dual basis for the GB-splines. We know, that, for given $n, p \in \mathbb{N}$, $p \ge 2$, the GB-splines $N_j^{(p)}$, $j = 1, ..., n$, are linearly independent, which, by basic linear algebra arguments, implies that there exists a unique dual basis, that is, n linearly independent linear functionals $\lambda_i^{(p)}[\boldsymbol{\Sigma}, \boldsymbol{\Omega}_{\mathbf{u}}, \boldsymbol{\Omega}_{\mathbf{v}}] : \langle N_j^{(p)} \rangle_{j=1}^n \to \mathbb{R}$, $i = 1, ..., n$ such that

$$\lambda_i^{(p)}[\boldsymbol{\Sigma}, \boldsymbol{\Omega}_{\mathbf{u}}, \boldsymbol{\Omega}_{\mathbf{v}}](N_j^{(p)}) = \delta_{i,j}, \qquad i, j = 1, ..., n, \tag{9}$$

where $\delta_{i,j}$ represents the Kronecker symbol. For notational brevity, we will drop $[\boldsymbol{\Sigma}, \boldsymbol{\Omega}_{\mathbf{u}}, \boldsymbol{\Omega}_{\mathbf{v}}]$ in $\lambda_i^{(p)}[\boldsymbol{\Sigma}, \boldsymbol{\Omega}_{\mathbf{u}}, \boldsymbol{\Omega}_{\mathbf{v}}]$ if there is no risk of confusion. While several methods can be used to construct the dual basis, we will mainly focus on the following one.

The elements of the dual basis can be expressed as linear combinations of function evaluations, following the scheme for the construction of linear functionals for generalized spline quasi-interpolants given in [9, 13]. Let us consider n domains γ_i, $i = 1, ..., n$, such that $\gamma_i \cap (s_i, s_{i+p}) \neq \emptyset$, and define the index sets $J_i = \{j : \gamma_i \cap (s_j, s_{j+p}) \neq \emptyset\}$. Moreover, for each i we choose a set of points $\{x_j^{(i)}\}_{j \in J_i}$ belonging to $\bar{\gamma}_i$ (we denote by $\bar{\gamma}_i$ the closure of γ_i), such that the collocation matrix $\mathbf{M}_i = \left(N_h^{(p)}(x_k^{(i)})\right)_{h,k \in J_i}$ is nonsingular. Then, we can express the dual basis in the form

$$\lambda_i^{(p)}(f) = \sum_{j \in J_i} \alpha_{i,j} f(x_j^{(i)}), \qquad i = 1, ..., n, \quad f \in \langle N_j^{(p)} \rangle_{j=1}^n. \tag{10}$$

Each vector of coefficients $\alpha_i = [\alpha_{i,j}]_{j \in J_i}$ is determined by the conditions

$$\lambda_i^{(p)}(N_j^{(p)}) = \delta_{i,j}, \qquad j \in J_i,$$

which are equivalent to the linear system

$$\mathbf{M}_i \alpha_i = \mathbf{\Delta}_i,$$

where $\mathbf{\Delta_i}$ is the vector of length $|J_i|$ whose components are all zeros except for the one corresponding to the element i, which is 1. Since the points $x_j^{(i)}$ belong to $\bar{\gamma}_i$, we have $\lambda_i^{(p)}(N_j^{(p)}) = 0$, also for $j \notin J_i$. Note that this construction allows, once all the coefficients $\alpha_{i,j}$ have been obtained, to compute $\lambda_i^{(p)}(f)$, for $i = 1, ..., n$ and $f \in C([s_i, s_{i+p}])$, only by using evaluations of f.
For example, we can set, for $i = 1, ..., n$

$$\gamma_i = (s_{r_i}, s_{r_i+1}), \qquad r_i = \min\{r : 1 \leq r \leq n-1 \wedge (s_i, s_{i+p}) \cap (s_r, s_{r+1}) \neq \emptyset\},$$

and choose the $x_j^{(i)}$ as $|J_i| = p$ distinct points in γ_i itself. In fact, by [12, Theorem 6.5] the corresponding matrix $\mathbf{M}_i$ is nonsingular. Note that this choice allows to consider local linear systems of dimension $p \times p$, which is the minimum size achievable, since γ_i must intersect (s_i, s_{i+p}) and therefore there must be at least p GB-splines whose support intersects γ_i.

Example 1. In the following sections of the paper, in order to prove the approximation properties of a quasi-interpolant belonging to the GT-spline space (see (25)), it will be crucial to have dual functionals whose norm stays bounded as the generalized spline space is refined (that is, as more knots are inserted in the knot vector $\mathbf{\Sigma}$ the norm does not diverge). Let us give us an example where this additional feature is guaranteed.
Let $n \in \mathbb{N}$, $p = 3$, and let $\mathbf{\Sigma} = \{s_1 < ... < s_{n+3}\}$, $\mathbf{\Omega_u} = \{\cos(\omega s), ..., \cos(\omega s)\}$ and $\mathbf{\Omega_v} = \{\sin(\omega s), ..., \sin(\omega s)\}$ with $\omega < \min_i \pi/(s_i - s_{i-1})$. In other words, we are considering the generalized spline space locally spanned by the functions $\{1, \cos(\omega s), \sin(\omega s)\}$. For $i = 1, ..., n$ we set

$$\gamma_i = (s_{i+1}, s_{i+2}) \Rightarrow J_i = \{i-1, i, i+1\}, \quad x_{i-1}^{(i)} = s_{i+1}, \quad x_i^{(i)} = \frac{s_{i+1} + s_{i+2}}{2}, \quad x_{i+1}^{(i)} = s_{i+2}.$$

Moreover, we assume that to have a local quasi-uniform distribution of the knots, that is, there exist two constants $m_1, m_2 > 0$ independent of i and such that

$$m_1 h_{i-1} \leq h_i \leq m_2 h_{i-1}, \qquad h_i = s_i - s_{i-1}, \qquad i = 2, ..., n+3. \tag{11}$$

Then, the dual functionals obtained by the construction (10) are

$$\lambda_i^{(3)}(f) = \sum_{j=i-1}^{i+1} \alpha_{i,j} f(x_j^{(i)}), \qquad i = 1, ..., n,$$

where the coefficients $\alpha_i = [\alpha_{i,i-1}, \alpha_{i,i}, \alpha_{i,i+1}]$, $i = 1, ..., n$, are obtained by solving $\mathbf{M}_i \alpha_i = \mathbf{\Delta}_i$, where

$$\mathbf{M}_i = \begin{bmatrix} N_{i-1}^{(3)}(s_{i+1}) & N_{i-1}^{(3)}(\frac{s_{i+1}+s_{i+2}}{2}) & 0 \\ N_i^{(3)}(s_{i+1}) & N_i^{(3)}(\frac{s_{i+1}+s_{i+2}}{2}) & N_i^{(3)}(s_{i+2}) \\ 0 & N_{i+1}^{(3)}(\frac{s_{i+1}+s_{i+2}}{2}) & N_{i+1}^{(3)}(s_{i+2}) \end{bmatrix}.$$

We will now show that, as we locally refine the generalized spline space, that is, as some of the steps h_i approach 0, the norms $\|\lambda_i^{(3)}\|$, $i = 1, ..., n$, do not diverge. Since the knots satisfy (11), it is sufficient to prove that

$$\lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} \|\lambda_i^{(3)}\| < \infty. \tag{12}$$

First, we observe that $\|\lambda_i^{(3)}\| \leq \max_{j=i-1,i,i+1} |\alpha_{i,j}|$ and that the coefficients $\alpha_{i,j}$, $j = i-1, i, i+1$ are rational functions of the elements of $\mathbf{M}_i$:

$$\alpha_{i,i-1} = \frac{-N_{i-1}^{(3)}(\frac{s_{i+1}+s_{i+2}}{2}) N_{i+1}^{(3)}(s_{i+2})}{\det(\mathbf{M}_i)},$$

$$\alpha_{i,i} = \frac{N_{i-1}^{(3)}(s_{i+1}) N_{i+1}^{(3)}(s_{i+2})}{\det(\mathbf{M}_i)},$$

$$\alpha_{i,i+1} = \frac{-N_{i-1}^{(3)}(s_{i+1}) N_{i+1}^{(3)}(\frac{s_{i+1}+s_{i+2}}{2})}{\det(\mathbf{M}_i)}.$$

It is possible to directly compute the elements of $\mathbf{M_i}$. For example, by using the recursive definition of the GB-splines given by (4)-(5) we get

$$N_{i-1}^{(3)}(s_{i+1}) = 1 - \frac{(1-\cos(\alpha h_{i+1}))\sin(\alpha h_{i+2})}{\sin(\alpha h_{i+2})(1-\cos(\alpha h_{i+1})) + \sin(\alpha h_{i+1})(1-\cos(\alpha h_{i+2}))}.$$

Then, by using the fact that $\lim_{u\to 0} \frac{\sin(\alpha u)}{\alpha u} = 1$ and $\lim_{u\to 0} \frac{\cos(\alpha u)}{1-\alpha^2 u^2/2} = 1$, we obtain that

$$\lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} N_{i-1}^{(3)}(s_{i+1}) = \lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} \frac{h_{i+2}}{h_{i+1}+h_{i+2}}.$$

Analogously, it is possible to obtain

$$\lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} N_{i-1}^{(3)}(\frac{s_{i+1}+s_{i+2}}{2}) = \lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} \frac{h_{i+2}}{4(h_{i+1}+h_{i+2})},$$
$$\lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} N_{i}^{(3)}(s_{i+1}) = \lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} 1-\frac{h_{i+2}}{h_{i+1}+h_{i+2}},$$
$$\lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} N_{i}^{(3)}(\frac{s_{i+1}+s_{i+2}}{2}) = \lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} 1-\frac{h_{i+2}}{4(h_{i+1}+h_{i+2})}-\frac{h_{i+2}}{4(h_{i+2}+h_{i+3})},$$
$$\lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} N_{i}^{(3)}(s_{i+2}) = \lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} 1-\frac{h_{i+2}}{h_{i+2}+h_{i+3}},$$
$$\lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} N_{i+1}^{(3)}(\frac{s_{i+1}+s_{i+2}}{2}) = \lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} \frac{h_{i+2}}{4(h_{i+2}+h_{i+3})},$$
$$\lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} N_{i+1}^{(3)}(s_{i+2}) = \lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} \frac{h_{i+2}}{(h_{i+2}+h_{i+3})}.$$

As a consequence, we have

$$\lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} \alpha_{i,i-1} = \lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} \frac{-N_{i-1}^{(3)}(\frac{s_{i+1}+s_{i+2}}{2})N_{i+1}^{(3)}(s_{i+2})}{\det(\mathbf{M}_i)}$$
$$= \lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} \frac{-h_{i+2}^2}{4(h_{i+1}+h_{i+2})(h_{i+2}+h_{i+3})}\frac{2(h_{i+1}+h_{i+2})(h_{i+2}+h_{i+3})}{h_{i+2}^2} = -\frac{1}{2},$$
$$\lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} \alpha_{i,i} = \lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} \frac{N_{i-1}^{(3)}(s_{i+1})N_{i+1}^{(3)}(s_{i+2})}{\det(\mathbf{M}_i)}$$
$$= \lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} \frac{-h_{i+2}^2}{(h_{i+1}+h_{i+2})(h_{i+2}+h_{i+3})}\frac{2(h_{i+1}+h_{i+2})(h_{i+2}+h_{i+3})}{h_{i+2}^2} = 2,$$
$$\lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} \alpha_{i,i+1} = \lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} \frac{-N_{i-1}^{(3)}(s_{i+1})N_{i+1}^{(3)}(\frac{s_{i+1}+s_{i+2}}{2})}{\det(\mathbf{M}_i)}$$
$$= \lim_{(h_{i+1},h_{i+2},h_{i+3})\to(0,0,0)} \frac{-h_{i+2}^2}{4(h_{i+1}+h_{i+2})(h_{i+2}+h_{i+3})}\frac{2(h_{i+1}+h_{i+2})(h_{i+2}+h_{i+3})}{h_{i+2}^2} - \frac{1}{2},$$

which proves (12). In other words, this means that, as we refine the generalized spline space, the norms of the dual functionals stay bounded.

3 Generalized T-splines

In this section, following [6] and [7], we will recall some notations, definitions and results about T-meshes and GT-splines.

3.1 T-meshes

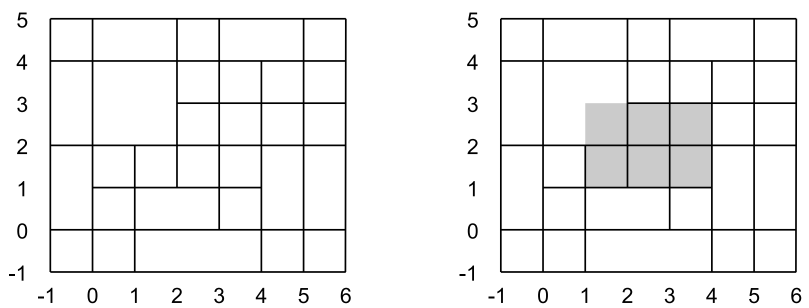
Let $\mathbf{\Sigma}^s = \{s_{-\lfloor p/2\rfloor+1}, ..., s_{\mu+\lfloor p/2\rfloor}\}$ and $\mathbf{\Sigma}^t = \{t_{-\lfloor q/2\rfloor+1}, ..., t_{\nu+\lfloor q/2\rfloor}\}$ be two knot vectors, where $\mu, \nu \in \mathbb{N}$, $p, q \in \mathbb{N}$ are equal to or greater than 2 and, for any real number k, $\lfloor k\rfloor$

stands for the largest integer smaller than or equal to k. Analogously, $\mathbf{\Omega}^s_u$, $\mathbf{\Omega}^s_v$, $\mathbf{\Omega}^t_u$ and $\mathbf{\Omega}^t_v$ are the associated vectors of functions.
An *index T-mesh* M on the index domain $[-\lfloor p/2\rfloor+1, \mu+\lfloor p/2\rfloor]\times[-\lfloor q/2\rfloor+1, \nu+\lfloor q/2\rfloor]$ is a collection of rectangles (called *cells*) which intersect only on edges and whose vertices have integer coordinates, and whose union is the whole domain (see Figure 2(a)). T-junctions are allowed but L-junctions or I-junctions are not, since the cells are rectangles. We call *edge* any segment, either horizontal or vertical, linking two vertices of the mesh. We denote the set of vertices by $\mathcal{V}$ and by hE, vE and E the sets containing only horizontal, only vertical and all the edges respectively. The valence of a vertex P is the number of edges $e \in E$ such that $P \in \partial e$. Finally, we denote by S the union of all the edges and vertices.
We define the *active region* $AR_{p,q}$ and *frame region* $FR_{p,q}$ (see Figure 2(b)) as

$$AR_{p,q} = [1, \mu] \times [1, \nu],$$

and

$$FR_{p,q} = \Big(\big[-\lfloor p/2\rfloor + 1, 1\big] \cup \big[\mu, \mu + \lfloor p/2\rfloor\big] \Big) \times \big[-\lfloor q/2\rfloor + 1, \nu + \lfloor q/2\rfloor\big]$$
$$\cup \big[-\lfloor p/2\rfloor + 1, \mu + \lfloor p/2\rfloor\big] \times \Big(\big[-\lfloor q/2\rfloor + 1, 1\big] \cup \big[\nu, \nu + \lfloor q/2\rfloor\big] \Big).$$



(a) (b)

Figure 2: A T-mesh (a) in the case $p = q = 4$, $\mu = 4$ and $\nu = 3$, and (b) the corresponding active region highlighted in gray, with the remaining part representing the frame region.

Definition 3.1. *A T-mesh* M *is admissible for the bi-order* (p, q) *if* $S \cap FR_{p,q}$ *includes the segments*

$$\{l\} \times \big[-\lfloor q/2\rfloor + 1, \nu + \lfloor q/2\rfloor\big] \qquad \text{for } l = -\lfloor p/2\rfloor + 1, ..., 1, \text{ and } l = \mu, ..., \mu + \lfloor p/2\rfloor,$$
$$\big[-\lfloor p/2\rfloor + 1, \mu + \lfloor p/2\rfloor\big] \times \{l\} \qquad \text{for } l = -\lfloor q/2\rfloor + 1, ..., 1, \text{ and } l = \nu, ..., \nu + \lfloor q/2\rfloor,$$

and all vertices belonging to $(-\lfloor p/2\rfloor+1, \mu+\lfloor p/2\rfloor)\times(-\lfloor q/2\rfloor+1, \nu+\lfloor q/2\rfloor)\cap FR_{p,q}$ *have valence 4.* $AD_{p,q}$ *will denote the set of admissible T-meshes for the bi-order* (p,q).

Definition 3.2. *A T-mesh* $M \in AD_{p,q}$ *belongs to* $AD^+_{p,q}$ *if, for any couple of vertices* $P_1 = (i_1, j_1), P_2 = (i_2, j_2) \in \mathcal{V}$ *both belonging to the boundary of a cell and such that* $i_1 = i_2$ *(*$j_1 = j_2$*, resp.), the segment* $\{i_1\}\times(j_1, j_2)$ *(*$\{j_1\}\times(i_1, i_2)$*, resp.) belongs to* S.

Definition 3.3. *Given T-mesh* $M \in AD_{p,q}$*, the set of anchors* $\mathcal{A}_{p,q}(M)$ *is defined in the following way:*

- *if both* p *and* q *are even,* $\mathcal{A}_{p,q}(M) = \{A \in \mathcal{V} : A \subset AR_{p,q}\}$;
- *if* p *is odd and* q *is even,* $\mathcal{A}_{p,q}(M) = \{A \in hE : A \subset AR_{p,q}\}$;
- *if* p *is even and* q *is odd,* $\mathcal{A}_{p,q}(M) = \{A \in vE : A \subset AR_{p,q}\}$;
- *if both* p *and* q *are odd,* $\mathcal{A}_{p,q}(M) = \{A \in M : A \subset AR_{p,q}\}$.

We associate to each anchor $A \in \mathcal{A}_{p,q}(M)$ a *local horizontal (vertical) index vector* $\mathbf{I}^s_l(A)$ ($\mathbf{I}^t_l(A)$) of length $p+1$ ($q+1$). The construction of these vectors is basic in the theory of T-splines and GT-splines, since they represent the dependence of each GT-spline function on the local topology of the T-mesh, as it will be clear also from the Definition (13) of GT-splines (a formal and extended definition of these vectors can be found for example in [4]).

The *T-mesh in the parameter space* is naturally obtained by considering in the domain $[s_{-\lfloor p/2\rfloor+1}, s_{\mu+\lfloor p/2\rfloor}]\times[t_{-\lfloor q/2\rfloor+1}, t_{\nu+\lfloor q/2\rfloor}]$ the cells

$$(s_{i_1}, s_{i_2})\times(t_{j_1}, t_{j_2}) \neq \emptyset,$$

where $(i_1, i_2)\times(j_1, j_2) \in M$. Let us introduce the notation

$$\begin{aligned}
\mathbf{\Sigma}^s(\mathbf{I}^s) &= \{s_i \in \mathbf{\Sigma}^s : i \in \mathbf{I}^s\},\\
\mathbf{\Sigma}^t(\mathbf{I}^t) &= \{t_j \in \mathbf{\Sigma}^t : j \in \mathbf{I}^t\},\\
\mathbf{\Omega}^s_u(\mathbf{I}^s) &= \{u^s_i \in \mathbf{\Omega}^s_u : i \in \mathbf{I}^s\backslash\{\max \mathbf{I}^s\}\},\\
\mathbf{\Omega}^t_u(\mathbf{I}^t) &= \{u^t_j \in \mathbf{\Omega}^t_u : j \in \mathbf{I}^t\backslash\{\max \mathbf{I}^t\}\},\\
\mathbf{\Omega}^s_v(\mathbf{I}^s) &= \{v^s_i \in \mathbf{\Omega}^s_v : i \in \mathbf{I}^s\backslash\{\max \mathbf{I}^s\}\},\\
\mathbf{\Omega}^t_v(\mathbf{I}^t) &= \{v^t_j \in \mathbf{\Omega}^t_v : j \in \mathbf{I}^t\backslash\{\max \mathbf{I}^t\}\}.
\end{aligned}$$

for any index vectors $\mathbf{I}^s \subseteq \{-\lfloor p/2\rfloor+1, ..., \mu+\lfloor p/2\rfloor\}$, $\mathbf{I}^t \subseteq \{-\lfloor q/2\rfloor+1, ..., \nu+\lfloor q/2\rfloor\}$. In this way, local knot and functions vectors naturally correspond to the local index vectors associated to each anchor (see, e.g., [6] and [7] for details).

3.2 GT-splines

Then, we define, for each anchor A, the bivariate *Generalized T-spline*:

$$\begin{aligned}
N_A(s,t) =& N^{(p)}_1\left[\mathbf{\Sigma}^s(\mathbf{I}^s_l(A)), \mathbf{\Omega}^s_u(\mathbf{I}^s_l(A)), \mathbf{\Omega}^s_v(\mathbf{I}^s_l(A))\right](s)\\
&\times N^{(q)}_1\left[\mathbf{\Sigma}^t(\mathbf{I}^t_l(A)), \mathbf{\Omega}^t_u(\mathbf{I}^t_l(A)), \mathbf{\Omega}^t_v(\mathbf{I}^t_l(A))\right](t).
\end{aligned} \tag{13}$$

Property 3.4. *The GT-splines enjoy the following properties, as direct consequence of their definition.*

1. **Continuity:** *each blending function $N_A(s,t)$, for any $A \in \mathcal{A}_{p,q}(M)$, is $(p - m_i^s - 1)$ times continuously differentiable with respect to s and $(q - m_j^t - 1)$ times continuously differentiable with respect to t at the point (s_i, t_j), where m_i^s and m_j^t are the multiplicities of s_i and t_j in the knot vectors $\mathbf{\Sigma}^s(\mathbf{I}_l^s(A))$ and $\mathbf{\Sigma}^t(\mathbf{I}_l^t(A))$, respectively.*

2. **Positivity:** *$N_A(s,t) \geq 0$ for $(s,t) \in \mathbb{R}^2$, $A \in \mathcal{A}_{p,q}(M)$ and $p, q \in \mathbb{N}$, $p, q \geq 2$.*

3. **Local support:**

$$\text{if} \quad (s,t) \notin [\min \mathbf{\Sigma}^s(\mathbf{I}_l^s(A)), \max \mathbf{\Sigma}^s(\mathbf{I}_l^s(A))] \times [\min \mathbf{\Sigma}^t(\mathbf{I}_l^t(A)), \max \mathbf{\Sigma}^t(\mathbf{I}_l^t(A))],$$

then $N_A(s,t) = 0$, for $A \in \mathcal{A}_{p,q}(M)$ and $p, q \in \mathbb{N}$, $p, q \geq 2$.

4. **Linear independence for tensor-product case:** *If M is a tensor-product mesh, then the corresponding blending functions are linearly independent.*

In [6] and [7], it was proved that there is a strong connection between the linear independence of the GT-splines and of the classical polynomial T-splines. In particular, it was shown that there exists a class of T-meshes for which both the associated GT-spline blending functions of bi-order (p,q) and the T-spline blending functions of bi-degree $(p-1, q-1)$ are linearly independent. This class, called the class of *VMCR (Void Matrix after Column Reduction) T-meshes*, is characterized in terms of the results obtained by applying the column reduction procedure to the matrix which expresses the relation between the GT-spline space and the coarser tensor-product spline space including it (see Corollary 4.9 in [6], Corollary 4.8 in [7], and related results).

Moreover, in [7] we also show that the class of VMCR T-meshes includes at least a couple of noteworthy classes of T-meshes which can be more easily characterized: the *weakly dual-compatible* T-meshes and the *analysis-suitable* T-meshes. The latter is the most known class of T-meshes guaranteeing the linear independence of the associated T-spline blending functions, and featuring several useful properties. This class of T-meshes was first defined and studied in [14], and later an equivalent class of T-meshes, called *dual-compatible*, was introduced in [3-4]. Since in the next section we will use it, let us recall the definition of dual-compatible T-meshes.

Let $M \in AD_{p,q}^+$ and let A_1 and A_2 be two anchors with local horizontal index vectors $\mathbf{I}_l^s(A_1) = \{i_1^s(A_1), ..., i_{p+1}^s(A_1)\}$ and $\mathbf{I}_l^s(A_2) = \{i_1^s(A_2), ..., i_{p+1}^s(A_2)\}$. We say that the local horizontal index vectors $\mathbf{I}_l^s(A_1)$ and $\mathbf{I}_l^s(A_2)$ *overlap* if

$$\forall k \in \mathbf{I}_l^s(A_1),\ i_1^s(A_2) \leq k \leq i_{p+1}^s(A_2) \Rightarrow k \in \mathbf{I}_l^s(A_2), \tag{14}$$

$$\forall k \in \mathbf{I}_l^s(A_2),\ i_1^s(A_1) \leq k \leq i_{p+1}^s(A_1) \Rightarrow k \in \mathbf{I}_l^s(A_1). \tag{15}$$

Analogously, if $\mathbf{I}_l^t(A_1) = \{i_1^t(A_1), ..., i_{q+1}^t(A_1)\}$ and $\mathbf{I}_l^t(A_2) = \{i_1^t(A_2), ..., i_{q+1}^t(A_2)\}$ are the vertical index vectors of A_1 and A_2, we say that they overlap if

$$\forall h \in \mathbf{I}_l^t(A_1),\ i_1^t(A_2) \leq h \leq i_{p+1}^t(A_2) \Rightarrow h \in \mathbf{I}_l^t(A_2), \tag{16}$$

$$\forall h \in \mathbf{I}_l^t(A_2),\ i_1^t(A_1) \leq h \leq i_{p+1}^t(A_1) \Rightarrow h \in \mathbf{I}_l^t(A_1). \tag{17}$$

Moreover, the anchors A_1 and A_2 are said to *partially overlap* if either their horizontal or vertical local index vectors overlap.

Definition 3.5. *A T-mesh $M \in AD^{+}_{p,q}$ is dual-compatible (DC) with respect to the bi-order (p, q) if any two anchors $A_1, A_2 \in \mathcal{A}_{p,q}(M)$ partially overlap.*

Theorem 3.6. *Any DC T-mesh $M \in AD^{+}_{p,q}$ is a VMCR T-mesh.*

Proof. See [7].

4 Approximation by GT-splines

4.1 Dual functionals

Studying the approximation properties for GT-splines associated with VMCR T-meshes is an open and interesting issue, but an analysis on the whole class is beyond the scope of this paper. Here we will construct the dual basis and then employ it to prove the partition of unity and to study the approximation properties for GT-splines associated with DC T-meshes, a special case of VMCR T-meshes (see Section 3.2). We start with the construction of a class of dual functionals for the GT-splines. For any $A \in \mathcal{A}_{p,q}(M)$, we define

$$\begin{aligned}\tilde{\mathbf{I}}^s(A) &= \mathbf{I}^s_l(A) \cup \{-\lfloor p/2 \rfloor + 1 \le i \le \mu + \lfloor p/2 \rfloor \,:\, i < \min \mathbf{I}^s_l(A) \vee i > \max \mathbf{I}^{\mathbf{s}}_{\mathbf{l}}(A)\} \\ \tilde{\mathbf{I}}^t(A) &= \mathbf{I}^t_l(A) \cup \{-\lfloor q/2 \rfloor + 1 \le j \le \mu + \lfloor q/2 \rfloor \,:\, j < \min \mathbf{I}^t_l(A) \vee j > \max \mathbf{I}^{\mathbf{t}}_{\mathbf{l}}(A)\}. \quad (18)\end{aligned}$$

Note that $\tilde{\mathbf{I}}^s(A)$ and $\tilde{\mathbf{I}}^t(A)$ contain the local vectors $\mathbf{I}^s_l(A)$ and $\mathbf{I}^t_l(A)$, respectively, but in general they do not coincide with them (see the example in Figure 3).

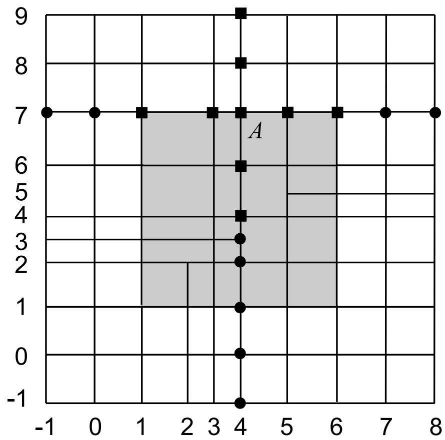


Figure 3: A DC T-mesh with $p = q = 4$: considering the anchor $A = (4, 7)$, the squared dots represent the local index vectors $\mathbf{I}^s_l(A) = \{1, 3, 4, 5, 6\}$ and $\mathbf{I}^t_l(A) = \{4, 6, 7, 8, 9\}$, while the circular and the squared dots together represent $\tilde{\mathbf{I}}^s(A) = \{-1, 0, 1, 3, 4, 5, 6, 7, 8\}$ and $\tilde{\mathbf{I}}^s(A) = \{-1, 0, 1, 2, 3, 4, 6, 7, 8, 9\}$.

For each $A \in \mathcal{A}_{p,q}(M)$, let $n^s(A) = |\tilde{\mathbf{I}}^{\mathbf{s}}(A)| - p$ and $n^t(A) = |\tilde{\mathbf{I}}^{\mathbf{t}}(A)| - q$; moreover, let $1 \leq i(A) \leq n^s(A)$ and $1 \leq j(A) \leq n^t(A)$ be the indices such that

$$N_A(s,t) = N^{(p)}_{i(A)}[\boldsymbol{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_u(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))] N^{(q)}_{j(A)}[\boldsymbol{\Sigma}^t(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_u(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_v(\tilde{\mathbf{I}}^t(A))].$$

Note that such indices always exist, thanks to the definition (18) of $\tilde{\mathbf{I}}^s(A)$. Then, we associate with each anchor $A \in \mathcal{A}_{p,q}(M)$ the functional λ_A defined by

$$\lambda_A = \lambda^{(p)}_{i(A)}[\boldsymbol{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_u(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))] \otimes \lambda^{(q)}_{j(A)}[\boldsymbol{\Sigma}^t(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_u(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_v(\tilde{\mathbf{I}}^t(A))]. \tag{19}$$

Note that if both $\lambda^{(p)}_{i(A)}[\boldsymbol{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_u(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))]$ and $\lambda^{(q)}_{j(A)}[\boldsymbol{\Sigma}^t(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_u(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_v(\tilde{\mathbf{I}}^t(A))]$ in (19) are constructed by using (10), that is

$$\begin{aligned} \lambda^{(p)}_{i(A)}[\boldsymbol{\Sigma}^s(\tilde{\mathbf{I}}^s_l(A)), \boldsymbol{\Omega}^s_u(\tilde{\mathbf{I}}^s_l(A)), \boldsymbol{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))](f) &= \sum_{h \in J^s_i(A)} \alpha^s_h(A) f\big(x^{(i(A))}_h\big) \\ \lambda^{(q)}_{j(A)}[\boldsymbol{\Sigma}^t(\tilde{\mathbf{I}}^t_l(A)), \boldsymbol{\Omega}^s_u(\tilde{\mathbf{I}}^t_l(A)), \boldsymbol{\Omega}^s_v(\tilde{\mathbf{I}}^t(A))](f) &= \sum_{k \in J^t_j(A)} \alpha^t_k(A) f\big(y^{(j(A))}_k\big), \end{aligned} \tag{20}$$

where

$$\begin{aligned} J^s_i(A) = \{1 \leq h \leq n^s(A) : &\operatorname{supp}(N^{(p)}_h[\boldsymbol{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_u(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))]) \cap \\ &\operatorname{supp}(N^{(p)}_{i(A)}[\boldsymbol{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_u(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))]) \neq \emptyset\}, \\ J^t_j(A) = \{1 \leq k \leq n^t(A) : &\operatorname{supp}(N^{(q)}_k[\boldsymbol{\Sigma}^t(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_u(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_v(\tilde{\mathbf{I}}^t(A))]) \cap \\ &\operatorname{supp}(N^{(q)}_{j(A)}[\boldsymbol{\Sigma}^t(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_u(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_v(\tilde{\mathbf{I}}^t(A))]) \neq \emptyset\}, \end{aligned}$$

then each dual functional (19) can be expressed in the form

$$\lambda_A(f) = \sum_{h \in J^s_i(A)} \sum_{k \in J^t_j(A)} \alpha^s_h(A) \alpha^t_k(A) f\big(x^{(i(A))}_h, y^{(j(A))}_k\big), \qquad f \in C\big([0,1]^2\big). \tag{21}$$

Proposition 4.1. *If the T-mesh is dual-compatible, the linear functionals* (19) *are dual to the GT-splines, that is,*

$$\lambda_A(N_B) = \delta_{A,B}, \qquad \forall A, B \in \mathcal{A}_{p,q}(M). \tag{22}$$

Proof. Let us assume that the GT-spline associated to B has the form

$$N_B(s,t) = N^{(p)}_1[\boldsymbol{\Sigma}^s(\mathbf{I}^s_l(B)), \boldsymbol{\Omega}^s_u(\mathbf{I}^s_l(B)), \boldsymbol{\Omega}^s_v(\mathbf{I}^s_l(B))](s) N^{(q)}_1[\boldsymbol{\Sigma}^t(\mathbf{I}^t_l(B)), \boldsymbol{\Omega}^t_u(\mathbf{I}^t_l(B)), \boldsymbol{\Omega}^t_v(\mathbf{I}^s_l(B))](t).$$

By construction, we have

$$\begin{aligned} \lambda_A(N_B) = \lambda^{(p)}_{i(A)}[\boldsymbol{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_u(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))]\Big(N^{(p)}_1[\boldsymbol{\Sigma}^s(\mathbf{I}^s_l(B)), \boldsymbol{\Omega}^s_u(\mathbf{I}^s_l(B)), \boldsymbol{\Omega}^s_v(\mathbf{I}^s_l(B))]\Big) \\ \times \lambda^{(q)}_{j(A)}[\boldsymbol{\Sigma}^t(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_u(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_v(\tilde{\mathbf{I}}^t(A))]\Big(N^{(q)}_1[\boldsymbol{\Sigma}^t(\mathbf{I}^t_l(B)), \boldsymbol{\Omega}^t_u(\mathbf{I}^t_l(B)), \boldsymbol{\Omega}^t_v(\mathbf{I}^s_l(B))]\Big). \end{aligned} \tag{23}$$

Since the T-mesh is dual-compatible, the two anchors A and B must partially overlap. Without loss of generality, we assume that the anchors overlap in the horizontal direction. As a consequence, we have that $\mathbf{I}^s_l(B) \subset \tilde{\mathbf{I}}^s(A)$ and therefore

$$N_1^{(p)}[\mathbf{\Sigma}^s(\mathbf{I}^s_l(B)), \mathbf{\Omega}^s_u(\mathbf{I}^s_l(B)), \mathbf{\Omega}^s_v(\mathbf{I}^s_l(B))] = \sum_{h=1}^{n^s(A)} \epsilon_h N_h^{(p)}[\mathbf{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \mathbf{\Omega}^s_u(\tilde{\mathbf{I}}^s(A)), \mathbf{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))],$$

for suitable coefficients $\epsilon_h \in \mathbb{R}$. This, by using (9) and the definition (18), implies

$$\lambda^{(p)}_{i(A)}[\mathbf{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \mathbf{\Omega}^s_u(\tilde{\mathbf{I}}^s(A)), \mathbf{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))]\Big(N_1^{(p)}[\mathbf{\Sigma}^s(\mathbf{I}^s_l(B)), \mathbf{\Omega}^s_u(\mathbf{I}^s_l(B)), \mathbf{\Omega}^s_v(\mathbf{I}^s_l(B))]\Big)$$
$$= \delta_{\mathbf{I}^s_l(A), \mathbf{I}^s_l(B)}.$$

Therefore, by the definition of λ_A, we have

$$\lambda_A(N_B) = \delta_{\mathbf{I}^s_l(A), \mathbf{I}^s_l(B)} \lambda^{(q)}_{j(A)}(N_1^{(q)}[\mathbf{\Sigma}^t(\mathbf{I}^t_l(B)), \mathbf{\Omega}^t_u(\mathbf{I}^t_l(B)), \mathbf{\Omega}^t_v(\mathbf{I}^t_l(B))]).$$

Then (22) is proved if $\mathbf{I}^s_l(A) \neq \mathbf{I}^s_l(B)$. If instead $\mathbf{I}^s_l(A) = \mathbf{I}^s_l(B)$, the anchors A and B are vertically aligned and so they must overlap vertically too. Since we can prove, analogously to the horizontal case, that

$$\lambda^{(q)}_{j(A)}[\mathbf{\Sigma}^t(\tilde{\mathbf{I}}^t(A)), \mathbf{\Omega}^t_u(\tilde{\mathbf{I}}^t(A)), \mathbf{\Omega}^t_v(\tilde{\mathbf{I}}^t(A))]\Big(N_1^{(q)}[\mathbf{\Sigma}^t(\mathbf{I}^t_l(B)), \mathbf{\Omega}^t_u(\mathbf{I}^t_l(B)), \mathbf{\Omega}^t_v(\mathbf{I}^t_l(B))]\Big) = \delta_{\mathbf{I}^t_l(A), \mathbf{I}^t_l(B)},$$

we have proved (22). □

Proposition 4.2. *If the T-mesh M is dual-compatible and the GT-splines of order $p > 2$ span a space containing the constant functions, then they are also a partition of unity.*

Proof. First, we remark that $\lambda_A(1) = 1$ for each $A \in \mathcal{A}_{p,q}(M)$. In fact, we have

$$\begin{aligned}\lambda_A(1) = &\lambda^{(p)}_{i(A)}[\mathbf{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \mathbf{\Omega}^s_u(\tilde{\mathbf{I}}^s(A)), \mathbf{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))](1) \\ &\times \lambda^{(q)}_{j(A)}[\mathbf{\Sigma}^t(\tilde{\mathbf{I}}^t(A)), \mathbf{\Omega}^t_u(\tilde{\mathbf{I}}^t(A)), \mathbf{\Omega}^t_v(\tilde{\mathbf{I}}^t(A))](1) = 1\end{aligned}$$

since

$$\lambda^{(p)}_{i(A)}[\mathbf{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \mathbf{\Omega}^s_u(\tilde{\mathbf{I}}^s(A)), \mathbf{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))](1) =$$
$$\lambda^{(p)}_{i(A)}[\mathbf{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \mathbf{\Omega}^s_u(\tilde{\mathbf{I}}^s(A)), \mathbf{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))]\Big(\sum_{h=1}^{n^s(A)} N_h^{(p)}[\mathbf{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \mathbf{\Omega}^s_u(\tilde{\mathbf{I}}^s(A)), \mathbf{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))]\Big) = 1,$$
$$\lambda^{(q)}_{j(A)}[\mathbf{\Sigma}^t(\tilde{\mathbf{I}}^t(A)), \mathbf{\Omega}^t_u(\tilde{\mathbf{I}}^t(A)), \mathbf{\Omega}^t_v(\tilde{\mathbf{I}}^t(A))](1) =$$
$$\lambda^{(q)}_{j(A)}[\mathbf{\Sigma}^t(\tilde{\mathbf{I}}^t(A)), \mathbf{\Omega}^t_u(\tilde{\mathbf{I}}^t(A)), \mathbf{\Omega}^t_v(\tilde{\mathbf{I}}^t(A))]\Big(\sum_{k=1}^{n^t(A)} N_k^{(q)}[\mathbf{\Sigma}^t(\tilde{\mathbf{I}}^t(A)), \mathbf{\Omega}^t_u(\tilde{\mathbf{I}}^t(A)), \mathbf{\Omega}^t_v(\tilde{\mathbf{I}}^t(A))]\Big) = 1,$$

and the univariate GB-splines of order $p > 2$ are a partition of unity. Since we assumed that the space spanned by the GT-splines includes constant functions, there must be a vector of real coefficients $\{a_B\}_{B \in \mathcal{A}_{p,q}(M)}$ such that

$$1 = \sum_{B \in \mathcal{A}_{p,q}(M)} a_B N_B(s,t).$$

Then, to prove the Proposition it is sufficient to note that

$$1 = \lambda_A(1) = \lambda_A\Big(\sum_{B \in \mathcal{A}_{p,q}(M)} a_B N_B(s,t)\Big) = a_A, \qquad \forall A \in \mathcal{A}_{p,q}(M). \qquad \square$$

4.2 Approximation power

Being able to obtain sets of dual functionals for the GT-splines associated to analysis-suitable T-meshes allows us to get some interesting approximation properties for the spline space spanned by the GT-splines themselves. In particular, analogously to what was done in [3] to study the approximation properties of spline spaces spanned by polynomial T-splines, we will use the dual functionals to construct a class of quasi-interpolants and show their approximation order.

Let us introduce the following notations. Given an anchor $A \in \mathcal{A}_{p,q}(M)$, let Q_A denote the support of the GT-spline $N_A(s,t)$; for any cell Q of the T-mesh, we define

$$\hat{Q} = \cup_{A \in S_Q(M)} Q_A, \qquad S_Q(M) = \{A \in \mathcal{A}_{p,q}(M) : \, Q_A \cap Q \neq \emptyset\}. \tag{24}$$

Proposition 4.3. *Let M be an analysis-suitable T-mesh, and assume that the space spanned by the GT-splines associated to M contains the constant functions. Then the operator $\Pi : C\big([0,1]^2\big) \to \mathcal{S} = \langle N_A \rangle_{A \in \mathcal{A}_{p,q}(M)}$ defined by*

$$\Pi(f)(s,t) = \sum_{A \in \mathcal{A}_{p,q}(M)} \lambda_A(f) N_A(s,t), \qquad f \in C\big([0,1]^2\big), \tag{25}$$

is a projector on S and satisfies, for any $f \in C\big((0,1)^2\big)$ and for any cell Q,

$$\|\Pi(f)\|_{L^\infty(Q)} \leq C_Q(M) \|f\|_{L^\infty(\hat{Q})} \tag{26}$$

where

$$C_Q(M) = \max_{A \in S_Q(M)} \left\{ \left\| \lambda^{(p)}_{i(A)}[\boldsymbol{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_u(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))] \right\| \right.$$
$$\left. \times \left\| \lambda^{(q)}_{j(A)}[\boldsymbol{\Sigma}^t(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_u(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_v(\tilde{\mathbf{I}}^t(A))] \right\| \right\}.$$

Proof. The operator Π is a projector on $\mathcal{S}$ by construction. By Proposition 4.2, for any $(s,t) \in Q$ we have

$$|\Pi(f)(s,t)| = \Big| \sum_{A \in S_Q(M)} \lambda_A(f) N_A(s,t) \Big| \leq \max_{A \in S_Q(M)} |\lambda_A(f)| \leq \max_{A \in S_Q(M)} \|\lambda_A\| \|f\|_{L^\infty(\hat{Q})},$$

where

$$\lambda_A = \lambda^{(p)}_{i(A)}[\boldsymbol{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_u(\tilde{\mathbf{I}}^s(A)), \boldsymbol{\Omega}^s_v(\tilde{\mathbf{I}}^s(A))] \otimes \lambda^{(q)}_{j(A)}[\boldsymbol{\Sigma}^t(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_u(\tilde{\mathbf{I}}^t(A)), \boldsymbol{\Omega}^t_v(\tilde{\mathbf{I}}^t(A))].$$

The Proposition is then proved. □

Corollary 4.4. *If the dual functionals used in the definition of Π are constructed according to (21), then the bound (26) holds with $C_Q(M) = \max_{A \in S_Q(M)} \sum_{h \in J^s_i(A)} \sum_{k \in J^t_j(A)} |\alpha^s_h(A) \alpha^t_k(A)|$.*

As a direct consequence of the previous Proposition, we have, for any $g \in \mathcal{S}$

$$\begin{aligned} &\|f - \Pi(f)\|_{L^\infty(Q)} = \|f - g + \Pi(g - f)\|_{L^\infty(Q)} \\ &\leq \|g - f\|_{L^\infty(Q)} + \|\Pi(g - f)\|_{L^\infty(Q)} \leq (1 + C_Q(M))\|g - f\|_{L^\infty(\hat{Q})}. \end{aligned} \tag{27}$$

Since this holds for any $g \in \mathcal{S}$, we can state that

$$\|f - \Pi(f)\|_{L^\infty(Q)} \leq (1 + C_Q(M))dist(f, \mathcal{S}), \tag{28}$$

where

$$dist(f, \mathcal{S}) = \inf_{g \in \mathcal{S}} \|g - f\|_{L^\infty}.$$

Moreover, since (28) holds for any cell Q, we have

$$\|f - \Pi(f)\|_{L^\infty} \leq (1 + C(M))dist(f, \mathcal{S}), \tag{29}$$

with

$$\begin{aligned} C(M) = \max_{Q \text{ cell of } M} C_Q(M) = \max_{A \in \mathcal{A}_{p,q}(M)} &\Big\{ \Big\| \lambda_{i(A)}^{(p)}[\mathbf{\Sigma}^s(\tilde{\mathbf{I}}^s(A)), \mathbf{\Omega}_u^s(\tilde{\mathbf{I}}^s(A)), \mathbf{\Omega}_v^s(\tilde{\mathbf{I}}^s(A))] \Big\| \\ &\times \Big\| \lambda_{j(A)}^{(q)}[\mathbf{\Sigma}^t(\tilde{\mathbf{I}}^t(A)), \mathbf{\Omega}_u^t(\tilde{\mathbf{I}}^t(A)), \mathbf{\Omega}_v^t(\tilde{\mathbf{I}}^t(A))] \Big\| \Big\}. \end{aligned}$$

In other words, this means that, whenever the constant $C(M)$ can be bounded independently of the refinement of M, Π provides the full approximation order of the spline space $\mathcal{S}$.

Finally, let us focus on the particular case where the functions vectors contain identical functions, that is,

$$\begin{aligned} u_1^s = u_2^s = ... = u_{m+p-1}^s = u^s, \quad v_1^s = v_2^s = ... = v_{m+p-1}^s = v^s, \\ u_1^t = u_2^t = ... = u_{n+q-1}^t = u^t, \quad v_1^t = v_2^t = ... = v_{n+q-1}^t = v^t. \end{aligned}$$

Let us use the notation:

$$\mathcal{P} := \text{span}\langle 1, s, ..., s^{p-3}, u^s(s), v^s(s) \rangle \otimes \text{span}\langle 1, t, ..., t^{q-3}, u^t(t), v^t(t) \rangle, \qquad (s,t) \in [0,1] \times [0,1].$$

Given a function $f \in \mathcal{C}^{(p,q)}([0,1]^2)$ and $(s_0, t_0) \in int(Q)$ (where $int(Q)$ denotes the interior of the cell Q), we define the interpolant $L(f; s_0, t_0)(s,t)$ as the function satisfying the two following conditions:

1. it belongs to $\mathcal{P}$;

2. its polynomial expansion of coordinate bi-degree $(p-1, q-1)$ coincides with the polynomial expansion of f of the same bi-degree, that is, it is a Hermite interpolant of coordinate bi-degree $(p-1, q-1)$.

Since $L(f; s_0, t_0)$ is a Hermite interpolant, the Taylor expansion of the difference $f - L(f; s_0, t_0)$ does not contain any term of degree smaller than or equal to k, where $k = \min\{p-1, q-1\}$, and then

$$\|f - L(f; s_0, t_0)\|_{L^\infty} \leq \tilde{C} h^{k+1} \sum_{|\mathbf{r}|=k+1} \|D^{\mathbf{r}} f\|_{L^\infty}$$

where $h = \|(s - s_0, t - t_0)\|$ and $\tilde{C}$ is a constant independent of f and h.
In order to show that $L(f; s_0, t_0)(s, t)$ exists and is unique for any $f \in \mathcal{C}^{(p,q)}(\Omega)$ and $(s_0, t_0) \in Q$, it is sufficient to write the explicit expressions of a generic element of $\mathcal{P}$, of its derivatives, and to require them to be equal to the derivatives of f up to the coordinate bi-degree $(p-1, q-1)$. This leads to a linear system, which can be proved to be non-singular by using the assumptions (1) and (2) (see [10]).

Proposition 4.5. *If the space $\mathcal{P}$ is included in $\mathcal{S}$ and $f \in C^{(p,q)}([0,1]^2)$, then*

$$\|f - \Pi(f)\|_{L^\infty(Q)} \leq (1 + C_Q(M))\, diam(\hat{Q})^{k+1} \tilde{C} \sum_{|\mathbf{r}|=k+1} \|D^{\mathbf{r}} f\|_{L^\infty(\hat{Q})}, \tag{30}$$

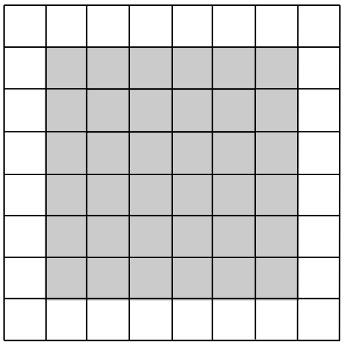
where $diam(\hat{Q})$ is the diameter of the set $\hat{Q}$, that is $diam(\hat{Q}) := \sup_{x,y \in \hat{Q}} \|x - y\|$, and $\tilde{C}$ is a constant independent of f and $diam(\hat{Q})$.

Proof. The Proposition is proved by applying (27) with $g = L(f; s_0, t_0) \in \mathcal{S}$. □

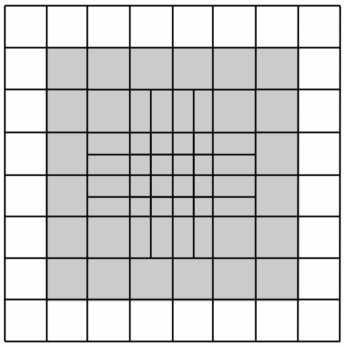
Proposition 4.5 is crucial for the study of the approximation power of the space spanned by the GT-splines. In fact, it implies that, if the constant $C_Q(M)$ does not diverge as we refine the T-mesh, the operator Π has approximation order $k + 1$.

Remark 1. For polynomial T-splines, results analogous to Propositions 4.3 and 4.5 have been proved (see, e.g., [3]). However, unlike the case of GT-splines presented here, for T-splines the assumption of having a dual-compatible T-mesh is enough to guarantee that the correponding constant $C_Q(M)$ is bounded (see [3]). It is then worth presenting some examples which show this difference, that is, examples of T-meshes where for the polynomial T-splines the constant $C_Q(M)$ is bounded independently of the refinement of the T-mesh M while for the GT-splines the same property cannot be guaranteed. Several examples when this assumption on $C_Q(M)$ is satisfied for GT-splines will be given too, that is, examples of sequences $\{M_\iota\}_{\iota=1}^{\infty}$ of refined DC T-meshes obtained by successive refinements where the corresponding constant $C_Q(M_\iota)$ does not diverge. In all the following examples, we will assume $p = q = 3$.

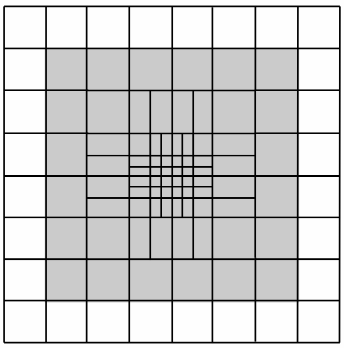
Example 2. First, let us consider the sequence of DC T-meshes $\{M_\iota\}_{\iota=1}^{\infty}$ whose first 4 elements are shown in Figure 4, obtained with the following rule. Starting from the tensor-product mesh of Figure 4(a) with $\mathbf{\Sigma}^s = \{s_0 < ... < s_8\}$, $\mathbf{\Sigma}^t = \{t_0 < ... < t_8\}$, $\mathbf{\Omega}^s_u = \{\cos(\omega s), ..., \cos(\omega s)\}$, $\mathbf{\Omega}^s_v = \{\sin(\omega s), ..., \sin(\omega s)\}$ and $\mathbf{\Omega}^t_u = \{\cos(\omega t), ..., \cos(\omega t)\}$, $\mathbf{\Omega}^t_v = \{\sin(\omega t), ..., \sin(\omega t)\}$ ($\omega < \min_i \pi/(s_i - s_{i-1})$). At each refinement step we subdivide into 4 equivalent parts the 4 central cells, and its 8 adjacent cells are either horizontally or vertically split (see Figures 4(b)-(d)). This refinement rule produces a sequence of T-meshes $\{M_\iota\}_{\iota=1}^{\infty}$ where for each anchor $A \in \mathcal{A}_{p,q}(M_\iota)$ both $\mathbf{\Sigma}^s(\tilde{\mathbf{I}}^s_l(A))$ and $\mathbf{\Sigma}^t(\tilde{\mathbf{I}}^t_l(A))$



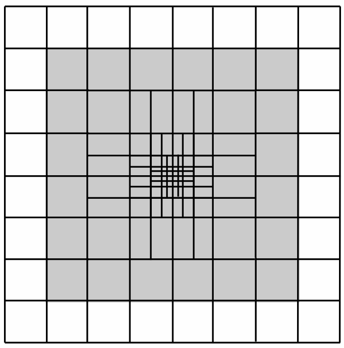
(a) M_1



(b) M_2

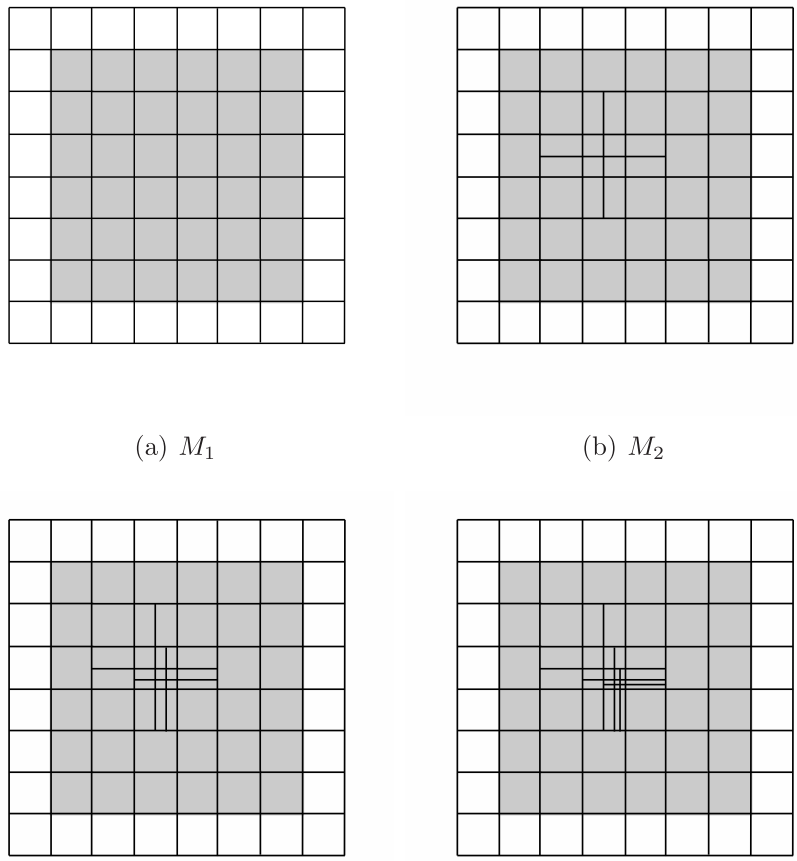


(c) M_3



(d) M_4

Figure 4: Sequence of T-meshes M_ι which guarantees the boundedness of $C_Q(M_\iota)$ both for polynomial T-splines and for GT-splines.



(a) M_1 (b) M_2

(c) M_3 (d) M_4

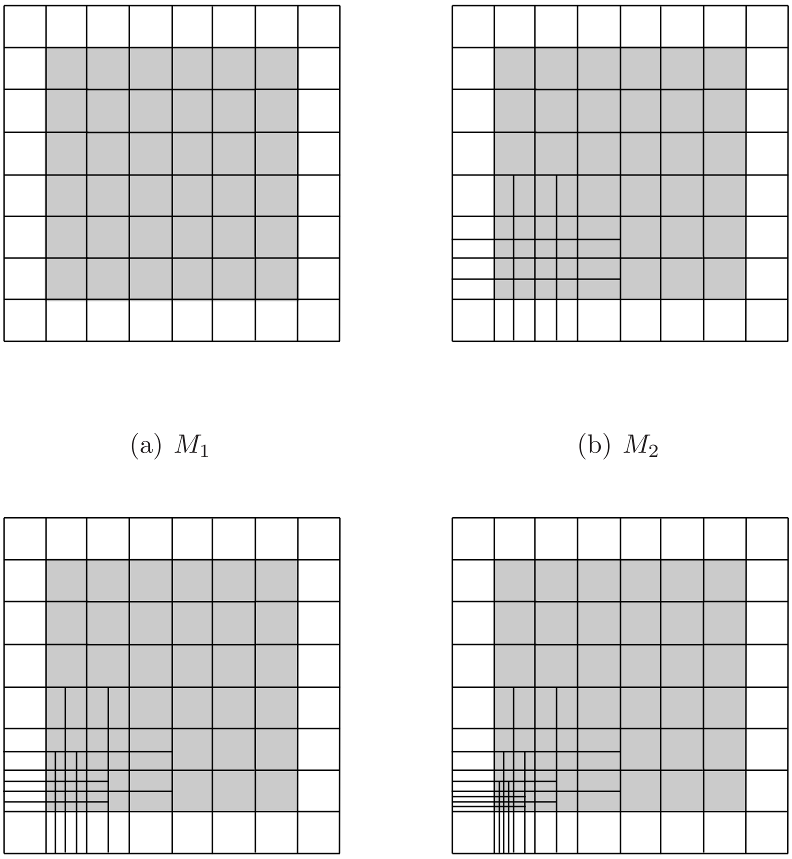
Figure 5: Sequence of T-meshes M_ι which guarantees the boundedness of $C_Q(M_\iota)$ for polynomial T-splines, but not for GT-splines.

satisfy the local quasi-uniformity assumption (11) with $m_1 = 1$, $m_2 = 2$. By employing the results obtained in Example 1 for the univariate case, we get that

$$\lim_{\iota\to\infty} C(M_\iota) = \lim_{\iota\to\infty} \max_{Q \text{ cell of } M_\iota} C_Q(M_\iota) < \infty.$$

which means that the constant $C_Q(M)$ in (30) in this case does not diverge as we refine the T-mesh.

Example 3. Now we consider instead the sequence of DC T-meshes $\{M_\iota\}_{\iota=1}^{\infty}$ whose first 4 elements are shown in Figure 5, obtained with the following rule. Starting from the same tensor-product mesh of Example 2 (see Figure 5(a)), at each refinement step we subdivide into 4 equivalent parts the cell having the central vertex of the T-mesh as its right-bottom corner, and its 4 adjacent cells are either horizontally or vertically split (see Figures 5(b)-(d)). Note that each T-mesh M_ι is dual-compatible and therefore, by the results in [3] (in particular Proposition 5.4), the corresponding $C_Q(M_\iota)$ is certainly bounded. On the other hand, in this case we cannot guarantee that for the GT-splines



(a) M_1

(b) M_2

(c) M_3

(d) M_4

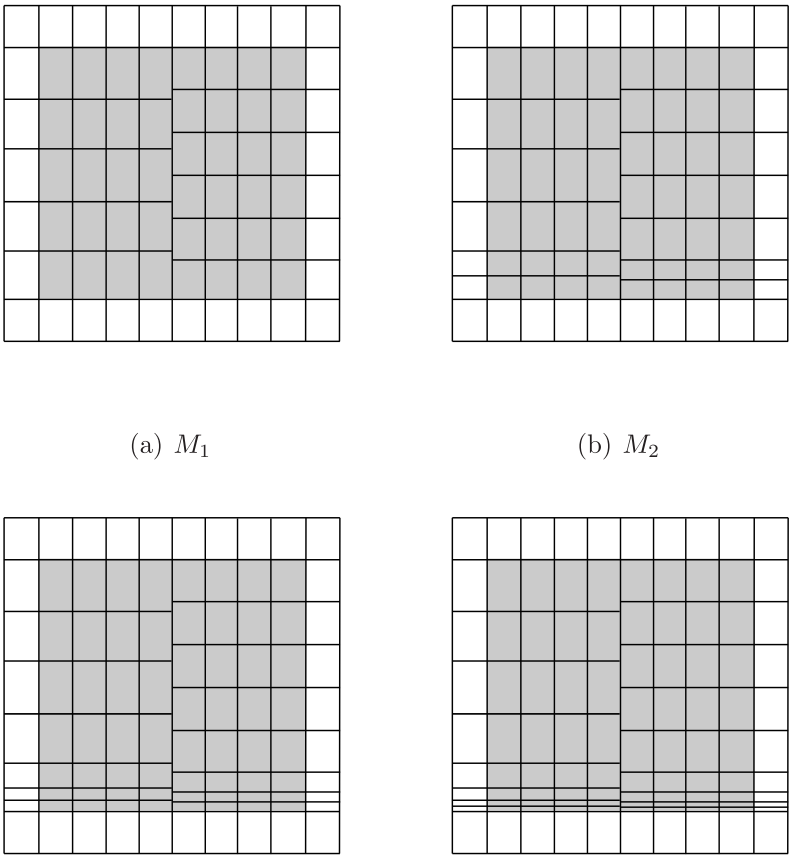
Figure 6: Sequence of DC T-meshes.

the corresponding constant $C_Q(M_\iota)$ is bounded, since there exist no constants m_1, m_2 by which every anchor $A \in \mathcal{A}_{p,q}(M_\iota)$ has $\mathbf{\Sigma}^s(\tilde{\mathbf{I}}^s_l(A))$ and $\mathbf{\Sigma}^t(\tilde{\mathbf{I}}^t_l(A))$ satisfying the local quasi-uniformity assumption (11).

Example 4. Let us consider the sequence of DC T-meshes $\{M_\iota\}_{\iota=1}^{\infty}$ whose first 4 elements are shown in Figure 6, obtained with the following rule. We start from the same tensor-product mesh of the two previous examples (see Figure 6(a)). This time, at each refinement step in the active region the cells of the leftmost-bottom 2×3 rectangle are vertically split, and the cells of the leftmost-bottom 3×2 rectangle are horizontally split (see Figures 6(b)-(d)).

Note that the refinement rule produces a sequence of T-meshes $\{M_\iota\}_{\iota=1}^{\infty}$ such that for each anchor $A \in \mathcal{A}_{p,q}(M_\iota)$ both $\mathbf{\Sigma}^s(\tilde{\mathbf{I}}^s_l(A))$ and $\mathbf{\Sigma}^t(\tilde{\mathbf{I}}^t_l(A))$ satisfy the local quasi-uniformity assumption (11) with $m_1 = 1$, $m_2 = 2$. Then, by using same arguments of Example 2, we can conclude that the constant $C_Q(M)$ in (30) does not diverge as we refine the T-mesh.

Example 5. Similar arguments hold also when considering the sequence of T-meshes whose first 4 elements are shown in Figure 7. To obtain this T-meshes we start from



(a) M_1 (b) M_2

(c) M_3 (d) M_4

Figure 7: Refinements of anisotropic DC T-meshes.

the T-mesh in Figure 7(a) with $\mathbf{\Sigma}^s = \{s_0 < ... < s_{10}\}$, $\mathbf{\Sigma}^t = \{t_0 < ... < t_{12}\}$, $\mathbf{\Omega}_u^s = \{\cos(\omega s), ..., \cos(\omega s)\}$, $\mathbf{\Omega}_v^s = \{\sin(\omega s), ..., \sin(\omega s)\}$ and $\mathbf{\Omega}_u^t = \{\cos(\omega t), ..., \cos(\omega t)\}$, $\mathbf{\Omega}_v^t = \{\sin(\omega t), ..., \sin(\omega t)\}$ ($\omega < \min_i \pi/(s_i - s_{i-1})$). Then, at each refinement step in the active region the bottom row of cells is horizontally split (see Figures 7(b)-(d)).

We remark that (1) the sequence of DC T-meshes shown in Figure 6 could be used in corner singularity problem on helical-shaped domains and (2) the sequence of DC T-meshes shown in Figure 7 could be used in boundary layer problems on helical-shaped domains obtained by gluing two standard tensor product patches (see [2, 16, 18]). The study of these applications is one of our future research topics.

Remark 2. It is worth noting that the procedure presented to construct the dual basis could be easily extended to the multivariate case, as recently done for polynomial T-splines (see, e.g., [5] and [17]). In [5], the concepts of anchors and of local index vectors associated to them have been generalized: given a d-variate mesh M and a vector of orders $\mathbf{p} = (p_1, ..., p_d)$, d local index vectors $\mathbf{I}_l^1(A), ..., \mathbf{I}_l^d(A)$ are associated to each anchor A. Then, denoting by $\mathbf{\Sigma}^1, ..., \mathbf{\Sigma}^d$, $\mathbf{\Omega}_u^1, ..., \mathbf{\Omega}_u^d$ and $\mathbf{\Omega}_v^1, ..., \mathbf{\Omega}_v^d$ the knot and function vectors, the *d-variate Generalized T-spline* associated to an anchor A can be defined as

$$\begin{aligned} N_A(\mathbf{s}) = & N_1^{(p_1)}\left[\mathbf{\Sigma}^1(\mathbf{I}_l^1(A)), \mathbf{\Omega}_u^1(\mathbf{I}_l^1(A)), \mathbf{\Omega}_v^1(\mathbf{I}_l^1(A))\right](s_1) \\ & \times N_1^{(p_2)}\left[\mathbf{\Sigma}^2(\mathbf{I}_l^2(A)), \mathbf{\Omega}_u^2(\mathbf{I}_l^2(A)), \mathbf{\Omega}_v^2(\mathbf{I}_l^2(A))\right](s_2)... \\ & \times N_1^{(p_d)}\left[\mathbf{\Sigma}^d(\mathbf{I}_l^d(A)), \mathbf{\Omega}_u^d(\mathbf{I}_l^d(A)), \mathbf{\Omega}_v^d(\mathbf{I}_l^d(A))\right](s_p), \end{aligned} \tag{31}$$

where $\mathbf{s} = (s_1, ..., s_p)$. In [5], also the definitions of partially overlapping and dual-compatible are generalized, as follows.

Definition 4.6. *Two anchors A_1 and A_2 partially overlap if there exists a direction k with $1 \leq k \leq d$ such that $\mathbf{I}_l^k(A_1)$ overlaps with $\mathbf{I}_l^k(A_2)$.*

Definition 4.7. *A d-variate T-mesh M is dual-compatible (DC) with respect to the order vector $(p_1, ...p_d)$ if any two anchors A_1, A_2 partially overlap.*

Then, if we assume that the T-mesh is dual-compatible, the fact that each couple of anchors partially overlap in at least one direction implies, just like in the proof of Proposition 4.1 for the bivariate case, that the linear functionals of type

$$\begin{aligned} \lambda_A = & \lambda_{i^1(A)}^{(p_1)}\left[\mathbf{\Sigma}^1(\tilde{\mathbf{I}}^1(A)), \mathbf{\Omega}_u^1(\tilde{\mathbf{I}}^1(A)), \mathbf{\Omega}_v^1(\tilde{\mathbf{I}}^1(A))\right] \\ & \otimes \lambda_{i^2(A)}^{(p_2)}\left[\mathbf{\Sigma}^2(\tilde{\mathbf{I}}^2(A)), \mathbf{\Omega}_u^2(\tilde{\mathbf{I}}^2(A)), \mathbf{\Omega}_v^2(\tilde{\mathbf{I}}^2(A))\right]... \\ & \otimes \lambda_{i^d(A)}^{(p_d)}\left[\mathbf{\Sigma}^d(\tilde{\mathbf{I}}^d(A)), \mathbf{\Omega}_u^d(\tilde{\mathbf{I}}^d(A)), \mathbf{\Omega}_v^d(\tilde{\mathbf{I}}^d(A))\right], \end{aligned} \tag{32}$$

are the dual basis for the splines $\{N_A(\mathbf{s})\}_{A \in \mathcal{A}_{\mathbf{p}}(M)}$ of the space of d-variate GT-splines. As a consequence, also the approximation results obtained by using the dual basis can be generalized.

5 Conclusions

We presented a construction for the dual basis of the GT-splines associated to a DC T-mesh. Then, by such dual basis, we defined a quasi-interpolation projection operator Π in order to study the approximation power of the GT-spline space. Whenever for the norms of the elements of the dual basis we can give a bound which does not diverge as the T-mesh is refined, we can guarantee that the approximation order depends only on the order of the considered GT-splines. Moreover, we showed some meaningful examples where such assumption about the norms of the elements of the dual basis is certainly satisfied.

Acknowledgements

For this work Cesare Bracco was supported by the program “Finanziamento Giovani Ricercatori 2015” of the “Gruppo Nazionale per il Calcolo Scientifico – Istituto Nazionale di Alta Matematica Francesco Severi”, and Durkbin Cho was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (2015R1A1A1A05001109).

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