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# ON THE $p$ -ADIC VALUATION OF HARMONIC NUMBERS

CARLO SANNA

ABSTRACT. For any prime number  $p$ , let  $J_p$  be the set of positive integers  $n$  such that  $p$  divides the numerator of the  $n$ -th harmonic number  $H_n$ . An old conjecture of Eswarathasan and Levine states that  $J_p$  is finite. We prove that for  $x \geq 1$  the number of integers in  $J_p \cap [1, x]$  is less than  $129p^{2/3}x^{0.765}$ . In particular,  $J_p$  has asymptotic density zero. Furthermore, we show that there exists a subset  $S_p$  of the positive integers, with logarithmic density greater than 0.273, and such that for any  $n \in S_p$  the  $p$ -adic valuation of  $H_n$  is equal to  $-\lfloor \log_p n \rfloor$ .

## 1. INTRODUCTION

For each positive integer  $n$ , let

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

be the  $n$ -th harmonic number. The arithmetic properties of harmonic numbers have been studied since a long time. For example, Wolstenholme [7] proved in 1862 that for any prime number  $p \geq 5$  the numerator of  $H_{p-1}$  is divisible by  $p^2$ ; while in 1915, Taisinger [6, p. 3115] showed that  $H_n$  is never an integer for  $n > 1$ .

For each prime number  $p$ , let  $J_p$  be the set of positive integers  $n$  such that the numerator of  $H_n$  is divisible by  $p$ . Eswarathasan and Levine [4] conjectured that  $J_p$  is finite for all primes  $p$ , and provided a method to compute the elements of  $J_p$ . If  $J_p$  is finite, then, after sufficient computation, their method gives a proof that it is finite. They computed  $J_2 = \emptyset$ ,  $J_3 = \{2, 7, 22\}$ ,  $J_5 = \{4, 20, 24\}$ , and

$$J_7 = \{6, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16731, 16735, 102728\}.$$

Boyd [2], using some  $p$ -adic expansions, improved the algorithm of Eswarathasan and Levine, and determined  $J_p$  for all primes  $p \leq 547$ , except 83, 127, and 397; confirming that  $J_p$  is finite for those prime numbers. Notably, he showed that  $J_{11}$  has 638 elements, the largest being an integer of 31 digits. Boyd gave also an heuristic model predicting that  $J_p$  is always finite and that its cardinality is  $\#J_p = O(p^2(\log \log p)^{2+\varepsilon})$ . However, the conjecture of Eswarathasan and Levine is still open.

We write  $J_p(x) := J_p \cap [1, x]$ , for  $x \geq 1$ . Our first result is the following.

**Theorem 1.1.** *For any prime number  $p$  and any  $x \geq 1$ , we have*

$$\#J_p(x) < 129p^{2/3}x^{0.765}.$$

*In particular,  $J_p$  has asymptotic density zero.*

For any prime number  $p$ , let  $\nu_p(\cdot)$  be the usual  $p$ -adic valuation over the rational numbers. Boyd [2, Proposition 3.3] proved the following lemma.

**Lemma 1.2.** *For any prime  $p$ , the set  $J_p$  is finite if and only if  $\nu_p(H_n) \rightarrow -\infty$ , as  $n \rightarrow +\infty$ .*

Therefore, the study of  $J_p$  is strictly related to the negative growth of the  $p$ -adic valuation of  $H_n$ . It is well-known and easy to prove that  $\nu_2(H_n) = -\lfloor \log_2 n \rfloor$ . (Hereafter,  $\lfloor x \rfloor$  denotes the greatest integer not exceeding the real number  $x$ .) Moreover, Kamano [5, Theorem 2] proved

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that  $\nu_3(H_n)$  can be determined easily from the expansion of  $n$  in base 3. Note that, since obviously  $\nu_p(k) \leq \lfloor \log_p n \rfloor$  for any  $k \in \{1, \dots, n\}$ , we have the lower bound

$$(1) \quad \nu_p(H_n) \geq -\lfloor \log_p n \rfloor.$$

Our next result shows that in (1) the equality holds quite often. We recall that the logarithmic density of a set of positive integers  $S$  is defined as

$$\delta(S) := \lim_{x \rightarrow +\infty} \frac{1}{\log x} \sum_{n \in S \cap [1, x]} \frac{1}{n},$$

whenever this limit exists.

**Theorem 1.3.** *For any prime number  $p$ , there exists a set  $S_p$  of positive integers, with logarithmic density  $\delta(S_p) > 0.273$ , and such that  $\nu_p(H_n) = -\lfloor \log_p n \rfloor$  for each  $n \in S_p$ .*

## 2. PROOF OF THEOREM 1.1

For any prime  $p$ , define the sequence of sets  $J_p^{(1)}, J_p^{(2)}, \dots$  as follow:

$$\begin{aligned} J_p^{(1)} &:= \{n \in \{1, \dots, p-1\} : p \mid H_n\}, \\ J_p^{(k+1)} &:= \{pn + r : n \in J_p^{(k)}, r \in \{0, \dots, p-1\}, p \mid H_{pn+r}\} \quad \forall k \geq 1. \end{aligned}$$

First, we need the following lemma.

**Lemma 2.1.** *For all prime numbers  $p$ , we have  $J_p^{(k)} = J_p \cap [p^{k-1}, p^k[$ , for each integer  $k \geq 1$ . In particular,  $J_p = \bigcup_{k=1}^{\infty} J_p^{(k)}$ .*

*Proof.* From [4, Eq. 2.5] we know that if  $n$  is a positive integer and  $r \in \{0, \dots, p-1\}$ , then  $pn + r \in J_p$  implies that  $n \in J_p$ . Therefore, the claim follows quickly by induction on  $k$ .  $\square$

Now we prove a result regarding the number of elements of  $J_p$  in a short interval.

**Lemma 2.2.** *For any prime  $p$ , and any real numbers  $x$  and  $y$ , with  $1 \leq y < p$ , we have*

$$\#(J_p \cap [x, x+y]) < \frac{3y^{2/3}}{2} + 1.$$

*Proof.* Set  $c := \#(J_p \cap [x, x+y])$ . If  $c \leq 1$ , then there is nothing to prove. Hence, suppose  $c \geq 2$  and let  $n_1 < \dots < n_c$  be the elements of  $J_p \cap [x, x+y]$ . Moreover, define  $d_i := n_{i+1} - n_i$ , for any  $i = 1, \dots, c-1$ . Given a positive integer  $d$ , consider the polynomial

$$(2) \quad f_d(X) := (X+1)(X+2) \cdots (X+d).$$

Taking the logarithms of both sides of (2) and differentiating, we obtain the identity

$$\frac{f'_d(X)}{f_d(X)} = \frac{1}{X+1} + \frac{1}{X+2} + \cdots + \frac{1}{X+d}.$$

Thus for any  $i = 1, \dots, c-1$  we have

$$\frac{f'_{d_i}(n_i)}{f_{d_i}(n_i)} = \frac{1}{n_i+1} + \frac{1}{n_i+2} + \cdots + \frac{1}{n_{i+1}} = H_{n_{i+1}} - H_{n_i} \equiv 0 \pmod{p},$$

so that  $f'_{d_i}(n_i) \equiv 0 \pmod{p}$ . Since  $f'_d(X)$  is a non-zero polynomial of degree  $d-1$ , there are at most  $d-1$  solutions modulo  $p$  of the equation  $f'_d(X) \equiv 0 \pmod{p}$ . Therefore, for any  $z \geq 1$ , on the one hand we have

$$(3) \quad \#\{i : d_i \leq z\} = \sum_{1 \leq d \leq z} \#\{i : d_i = d\} \leq \sum_{1 \leq d \leq z} (d-1) < \frac{z^2}{2}.$$

On the other hand,

$$(4) \quad \#\{i : d_i > z\} < \frac{1}{z} \sum_{i=1}^{c-1} d_i = \frac{n_c - n_1}{z} \leq \frac{y}{z}$$

In conclusion, by summing (3) and (4), we get

$$c - 1 = \#\{i : d_i \leq z\} + \#\{i : d_i > z\} < \frac{z^2}{2} + \frac{y}{z}$$

and the claim follows taking  $z = y^{1/3}$ .  $\square$

We are ready to prove Theorem 1.1. If  $p < 83$  then, from the values of  $\#J_p$  computed by Boyd [2, Table 2], one can check that  $\#J_p/p^{2/3} < 129$ , so the claim is obvious. Hence, suppose  $p \geq 83$ , and put  $A := \frac{3}{2}(p-1)^{2/3} + 1$ . By the definition of the sets  $J_p^{(k)}$ , and Lemma 2.2, we get that

$$\#J_p^{(1)} = \#(J_p \cap [1, p-1]) < A,$$

while

$$\#J_p^{(k+1)} = \sum_{n \in J_p^{(k)}} \#(J_p \cap [pn, pn+p-1]) < \#J_p^{(k)} \cdot A,$$

hence it follows by induction that  $\#J_p^{(k)} < A^k$ .

Now let  $s$  be the positive integer determined by  $p^{s-1} \leq x < p^s$ . Note that  $p^s \notin J_p$ , indeed  $\nu_p(H_{p^s}) = -s$  (this is a particular case of Lemma 3.2 in the next section). Thanks to Lemma 2.1 and the previous considerations, we have

$$\begin{aligned} \#J_p(x) &\leq \#J_p(p^s) = \#J_p(p^s - 1) = \sum_{k=1}^s \#(J_p \cap [p^{k-1}, p^k]) = \sum_{k=1}^s \#J_p^{(k)} \\ &< \sum_{k=1}^s A^k < \frac{A^2}{A-1} \cdot A^{s-1} = \frac{A^2}{A-1} \cdot (p^{s-1})^{\log_p A} < \frac{A^2}{A-1} \cdot x^{0.765} < 129p^{2/3}x^{0.765}, \end{aligned}$$

since  $p^{s-1} \leq x$ , while it can be checked quickly that  $\log_p A < 0.765$ . The proof is complete.

### 3. PROOF OF THEOREM 1.3

For any integer  $b \geq 2$  and any  $d \in \{1, \dots, b-1\}$ , let  $F_b(d)$  be the set of positive integers that have the most significant digit of their base  $b$  expansion equal to  $d$ . The set  $F_b(d)$  does not have an asymptotic density, however  $F_b(d)$  has a logarithmic density. In fact,  $F_b(d)$  satisfies a kind of Benford's law [1], as shown by the following lemma.

**Lemma 3.1.** *For all integers  $b \geq 2$  and  $d \in \{1, \dots, b-1\}$ , we have  $\delta(F_b(d)) = \log_b(1 + 1/d)$ .*

*Proof.* See [3].  $\square$

Write  $J_p^* := \{1, \dots, p-1\} \setminus J_p^{(1)}$ .

**Lemma 3.2.** *For  $p$  prime,  $d \in J_p^*$ , and  $n \in F_p(d)$ , we have  $\nu_p(H_n) = -\lfloor \log_p n \rfloor$ .*

*Proof.* Since  $n \in F_p(d)$ , we can write  $n = p^k d + r$ , where  $k := \lfloor \log_p n \rfloor$  and  $r < p^k$  is a non-negative integer. Hence,

$$(5) \quad H_n = \sum_{\substack{m=1 \\ p^k \nmid m}}^n \frac{1}{m} + \sum_{j=1}^d \frac{1}{p^k j} = \sum_{\substack{m=1 \\ p^k \nmid m}}^n \frac{1}{m} + \frac{H_d}{p^k}.$$

On the one hand, it is clear that the last sum in (5) has  $p$ -adic valuation greater than  $-k$ . On the other hand, we have  $\nu_p(H_d/p^k) = -k$ , since  $d \in J_p^*$  and so  $p \nmid H_d$ .

In conclusion,  $\nu_p(H_n) = -k$  as desired.  $\square$

Now we can prove Theorem 1.3. Define the set  $S_p$  as

$$S_p := \bigcup_{d \in J_p^*} F_p(d).$$

It follows immediately from Lemma 3.2 that  $\nu_p(H_n) = -\lfloor \log_p n \rfloor$ , for each  $n \in S_p$ . Moreover, since the sets  $F_p(d)$  are disjoint, and thanks to Lemma 3.1, we have

$$(6) \quad \begin{aligned} \delta(S_p) &:= \sum_{d \in J_p^*} \delta(F_p(d)) = \sum_{d \in J_p^*} \log_p \left( 1 + \frac{1}{d} \right) \geq \sum_{d = \#J_p^{(1)} + 1}^{p-1} \log_p \left( 1 + \frac{1}{d} \right) \\ &= \log_p \left( \frac{p}{\#J_p^{(1)} + 1} \right) = 1 - \frac{\log(\#J_p^{(1)} + 1)}{\log p}. \end{aligned}$$

Suppose  $p \geq 1013$ . By Lemma 2.2 we have

$$\#J_p^{(1)} = \#(J_p \cap [1, p-1]) < \frac{3}{2}(p-2)^{2/3} + 1,$$

hence from (6) we get

$$\delta(S_p) > 1 - \frac{\log\left(\frac{3}{2}(p-2)^{2/3} + 2\right)}{\log p} > 0.273.$$

At this point, the proof is only a matter of computation. The author used the Python programming language (since it has native support for arbitrary-sized integers) to compute the numerators of the harmonic numbers  $H_n$ , up to  $n = 1012$ . Then he determined  $\#J_p^{(1)}$  for each prime number  $p < 1013$ , and using (6) he checked that the inequality  $\delta(S_p) > 0.273$  holds. This required only a few seconds on a personal computer.

#### 4. CONCLUDING REMARKS

In the proof of Theorem 1.1, we used some of the values of  $\#J_p$  from the tables of Boyd's paper. Boyd communicated to the referee that Alekseyev discovered 3 errors in those tables. Namely,  $\#J_{19} = 25$ ,  $\#J_{47} = 24$  and  $\#J_{59} = 23$  rather than the values 19, 11 and 17 given by Boyd. Luckily, this does not affect our result. In addition, the prime  $p = 509$  with  $\#J_p = 13$  was omitted from Boyd's Table 2. It has been checked that the other values in Boyd's tables are correct.

From the proof of Theorem 1.1, it is clear that with our methods one cannot obtain an upper bound better than  $\#J_p(x) < Cp^{2/3}x^{2/3+\varepsilon}$ , for some  $C, \varepsilon > 0$ . Similarly, in the statement of Theorem 1.3 a logarithmic density greater than  $1/3 - \varepsilon$  cannot be achieved.

One way to obtain better results could be an improvement of Lemma 2.2, we leave this as an open question for the readers.

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