On the asymptotic density of the support of a Dirichlet convolution

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ON THE ASYMPTOTIC DENSITY OF THE SUPPORT OF A DIRICHLET CONVOLUTION

CARLO SANNA

ABSTRACT. Let ν be a multiplicative arithmetic function with support of positive asymptotic density. We prove that for any not identically zero arithmetic function f such that $\sum_{f(n) \neq 0} 1/n < \infty$, the support of the Dirichlet convolution $f * \nu$ possesses a positive asymptotic density. When f is a multiplicative function, we give also a quantitative version of this claim. This generalizes a previous result of P. Pollack and the author, concerning the support of Möbius and Dirichlet transforms of arithmetic functions.

1. INTRODUCTION

Let f and g be two arithmetic functions, i.e., functions from the set of positive integers to the set of complex numbers. The Dirichlet convolution of f and g is the arithmetic function denoted by $f * g$ and defined as

$$(f * g)(n) := \sum_{d \mid n} f(d) \, g(n/d),$$

for all positive integers n. The set of arithmetic functions together with Dirichlet convolution forms a commutative monoid with identity element ϵ, the arithmetic function that satisfies $\epsilon(1) = 1$ and $\epsilon(n) = 0$ for all integers $n \geq 2$. Furthermore, any arithmetic function f has an inverse f^{-1} with respect to the Dirichlet convolution if and only if $f(1) \neq 0$, in which case f^{-1} can be computed recursively by the identities $f^{-1}(1) = 1/f(1)$ and

$$f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d \mid n \\ d < n}} f^{-1}(d) \, f(n/d), \quad n \geq 2.$$

The Dirichlet transform $\hat{f}$ and the Möbius transform $\check{f}$ of the arithmetic function f are defined by $\hat{f} := f * 1$ and $\check{f} := f * \mu$, where μ is the Möbius function. Notably, the Dirichlet inverse of μ is the identically equal to 1 arithmetic function. Actually, this is the content of the well-know Möbius inversion formula, that is $\check{\hat{f}} = \hat{\check{f}} = f$ (see [Hua09, Ch. 2] for details).

We call (f, g) a Möbius pair if f and g are arithmetic function with $f = \hat{g}$, or equivalently $g = \check{f}$. In a previous paper, P. Pollack and the author studied the asymptotic density of the support of functions f and g in a Möbius pair (f, g); by the support of an arithmetic function h we mean the set of all positive integers n such that $h(n) \neq 0$, we denote it with $\text{supp}(h)$. They give the following result [PS13, Theorem 1.1].

Theorem 1.1. *Suppose that (f, g) is a nonzero Möbius pair. If $\text{supp}(f)$ is thin then $\text{supp}(g)$ possesses a positive asymptotic density. The same result holds with the roles of f and g reversed.*

We recall that a set of positive integers $\mathscr{A}$ is said to be *thin* if $\sum_{a \in \mathscr{A}} 1/a < \infty$. Given a multiplicative arithmetic function ν, we extend the notion of Möbius pair by saying that (f, g) is a ν-pair if f and g are arithmetic function with $g = f * \nu$, or equivalently $f = g * \nu^{-1}$. (Note that ν is Dirichlet invertible since as a multiplicative function it satisfies $\nu(1) = 1$). Here, we prove the following generalization of Theorem 1.1.

Theorem 1.2. *Let ν be a multiplicative arithmetic function with support of positive asymptotic density. Suppose that (f, g) is a nonzero ν-pair. If $\mathrm{supp}(f)$ is thin then $\mathrm{supp}(g)$ possesses a positive asymptotic density.*

This is a true generalization of Theorem 1.1, since 1 and μ are multiplicative arithmetic functions with support of positive asymptotic density. Precisely, $\mathrm{supp}(\mu)$ is the set of squarefree numbers and it has density $6/\pi^2$, as is well known.

The discovery of Theorem 1.1 was initially motivated by the desire to prove a kind of *uncertainty principle for the Möbius transform*, in the sense that f and g cannot both be of thin support if (f, g) is a nonzero Möbius pair, which is in turn a generalization of a previous result of P. Pollack [Pol11]. Note the analogy with the well-know uncertainty principle of harmonic analysis, which states that a function $f \in L^1(\mathbf{R})$ and its Fourier transform cannot both have support of finite Lebesgue measure, unless $f = 0$ a.e. [Ben85]; for a survey on uncertainty principles, see [FS97].

Now, in the same spirit, we show that Theorem 1.2 leads to the following uncertainty principle.

Corollary 1.1. *Let ν be a multiplicative arithmetic function with support of positive asymptotic density. If (f, g) is a nonzero ν-pair then $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ cannot be both thin.*

In the case when f and g are multiplicative arithmetic functions, we give also the following quantitative version of Theorem 1.2.

Theorem 1.3. *Let ν be a multiplicative arithmetic function with support of positive asymptotic density. Then there exists a constant $C_\nu > 0$, depending only on ν, such that for every ν-pair (f, g) of multiplicative arithmetic functions it holds*

$$\mathbf{d}(\mathrm{supp}(g)) \geq \frac{C_\nu}{\sum_{n \in \mathrm{supp}(f)} 1/n},$$

where by convention $C_\nu/\infty := 0$. More specifically, we can choose C_ν as

$$\frac{6}{\pi^2} \prod_{p \notin \mathrm{supp}(\nu)} \left(1 + \frac{1}{p}\right)^{-1},$$

where the product is proved to converges to a positive real number.

P. Pollack and the author have also given a result similar to Theorem 1.1, but weighted by the absolute values of f and g [PS13, Theorem 1.2]. We recall that the mean value of an arithmetic function f is the limit $\lim_{x \to \infty}(1/x) \sum_{n \leq x} f(n)$, whenever it exists.

Theorem 1.4. *Suppose that (f, g) is a nonzero Möbius pair. If*

$$\sum_{n=1}^{\infty} \frac{|f(n)|}{n} < \infty,$$

then $|g|$ possesses a positive mean value. The same result holds with the roles of f and g reversed.

We generalize Theorem 1.4 by proving the following:

Theorem 1.5. *Let ν be a bounded multiplicative arithmetic function with support of positive asymptotic density. Suppose that (f, g) is a ν-pair. If*

$$\sum_{n=1}^{\infty} \frac{|f(n)|}{n} < \infty,$$

then $|g|$ possesses a finite mean value. Moreover, if f do not vanish identically and

$$\inf_{n \in \mathrm{supp}(\nu)} |\nu(n)| > 0,$$

then the mean value of $|g|$ is positive.

Notation. Hereafter, $\mathbf{N}$ denotes the set of positive integers and $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$. For any $\mathscr{A} \subseteq \mathbf{N}$ and $x \geq 0$ we write $\mathscr{A}(x) := \#(\mathscr{A} \cap [1, x])$ for the number of elements of $\mathscr{A}$ not exceeding x. We use $\mathbf{d}(\mathscr{A}) := \lim_{x \to \infty} \mathscr{A}(x)/x$ for the asymptotic density of $\mathscr{A}$, whenever this exists. Similarly, we denote the lower and upper asymptotic density of $\mathscr{A}$ by $\underline{\mathbf{d}}(\mathscr{A})$ and $\overline{\mathbf{d}}(\mathscr{A})$, respectively. The letter p always denotes a prime number. The notation $p^k \,\|\, n$ means that $p^k \mid n$, but $p^{k+1} \nmid n$.

2. Preliminaries

This section is devoted to some lemmas needed for the proof of Theorem 1.2. The first one deals with the existence of the asymptotic density of certain sieved sets of positive integers.

Lemma 2.1. *Let $\mathscr{B}$ be a set of positive integers and suppose that to every $b \in \mathscr{B}$ there corresponds a set $\Omega_b \subseteq \{0, 1, \ldots, b-1\}$. Furthermore, for any $b \in \mathscr{B}$ let*

$$\mathscr{S}_b := \{n \in \mathbf{N} : (n \bmod b) \in \Omega_b\}$$

and assume that $\mathscr{S}_b(x) \leq c_b x$ for all $x \geq 0$, where c_b are positive constants satisfying

$$\sum_{b \in \mathscr{B}} c_b < \infty.$$

Then the set of positive integers n such that $(n \bmod b) \notin \Omega_b$ for all $b \in \mathscr{B}$ possesses an asymptotic density.

Proof. We have to prove that the set

$$\mathscr{S}_{\mathscr{B}} := \{n \in \mathbf{N} : (n \bmod b) \notin \Omega_b \quad \forall b \in B\},$$

has an asymptotic density. If $\mathscr{B}$ is empty then $\mathscr{S}_{\mathscr{B}} = \mathbf{N}$ and the claim is trivial.

Suppose that $\mathscr{B}$ is finite with $k \geq 1$ elements $b_1, b_2, \ldots, b_k$. From the inclusion-exclusion principle it follows

$$(1) \qquad \mathscr{S}_{\mathscr{B}}(x) = \mathbf{N}(x) + \sum_{h=1}^{k} (-1)^h \sum_{1 \leq i_1 < \cdots < i_h \leq k} \left(\mathscr{S}_{b_{i_1}} \cap \cdots \cap \mathscr{S}_{b_{i_h}} \right)(x),$$

for all $x > 0$. Now, fix h and $i_1, \ldots, i_h$ positive integers such that $h \leq k$, $i_1 < \cdots < i_h \leq k$, as in equation (1), and set $\ell := \operatorname{lcm}(b_{i_1}, b_{i_2}, \ldots, b_{i_h})$. As a consequence of the Chinese Remainder Theorem, there exists a subset Θ of $\{0, 1, \ldots, \ell - 1\}$ such that for $n \in \mathbf{N}$ it holds $n \in \mathscr{S}_{b_{i_1}} \cap \cdots \cap \mathscr{S}_{b_{i_h}}$ if and only if $(n \bmod \ell) \in \Theta$. So $\mathscr{S}_{b_{i_1}} \cap \cdots \cap \mathscr{S}_{b_{i_h}}$ has asymptotic density $\#\Theta/\ell$. Dividing (1) by x and letting $x \to \infty$ yields that $\mathscr{S}_{\mathscr{B}}$ has an asymptotic density.

Suppose now that $\mathscr{B}$ is infinite and let $b_1, b_2, \ldots$ be a numbering of $\mathscr{B}$. For all positive integer k define $\mathscr{B}_k := \{b_1, b_2, \ldots, b_k\}$. We have just seen that $\mathscr{S}_{\mathscr{B}_k}$ has an asymptotic density, so put $d_k := \mathbf{d}(\mathscr{S}_{\mathscr{B}_k})$. Since $\mathscr{S}_{\mathscr{B}_1} \supseteq \mathscr{S}_{\mathscr{B}_2} \supseteq \cdots$, it follows that $d_1, d_2, \ldots$ is a nonnegative decreasing sequence, so there exists $d := \lim_{k \to \infty} d_k$. Furthermore

$$(2) \qquad 0 \leq \mathscr{S}_{\mathscr{B}_k}(x) - \mathscr{S}_{\mathscr{B}}(x) \leq \sum_{i=k+1}^{\infty} \mathscr{S}_{b_i}(x) \leq x \sum_{i=k+1}^{\infty} c_{b_i}$$

for all $x > 0$ and $k \in \mathbf{N}$, where the last term in the above inequality is a convergent series (by the hypothesis). Dividing equation (2) by x and letting $x \to \infty$ we obtain that

$$(3) \qquad \limsup_{x \to \infty} \left| \frac{\mathscr{S}_{\mathscr{B}}(x)}{x} - d_k \right| \leq \sum_{i=k+1}^{\infty} c_{b_i}.$$

Finally, letting $k \to \infty$ in equation (3) it follows that $\mathscr{S}_{\mathscr{B}}$ has asymptotic density d. $\qquad \square$

If $\mathscr{A}$ is a set of positive integers, we write $\mathscr{M}(\mathscr{A}) := \{an : a \in \mathscr{A}, \, n \in \mathbf{N}\}$ for the *set of multiples* of $\mathscr{A}$. The interested reader can found many results on sets of multiples in [Hal96]. We need only to state the following lemma about the asymptotic density of $\mathscr{M}(\mathscr{A})$.

Lemma 2.2. *If $\mathscr{A}$ is a thin set of positive integers, then $\mathscr{M}(\mathscr{A})$ has an asymptotic density. Moreover, if $1 \notin \mathscr{A}$ then $\mathbf{d}(\mathscr{M}(\mathscr{A})) < 1$.*

Proof. See [PS13, Lemma 2.2]. Incidentally, note that the existence of $\mathbf{d}(\mathscr{M}(\mathscr{A}))$ is a corollary of Lemma 2.1, setting $\mathscr{B} := \mathscr{A}$, $\Omega_b = \{0\}$ and $c_b := 1/b$ for all $b \in \mathscr{B}$. $\square$

Lemma 2.3. *Let $\mathscr{A}$ be a set of positive integers, $\mathscr{P}$ a set of prime numbers, and suppose that to every $p \in \mathscr{P}$ there corresponds a set of nonnegative integers $\mathscr{K}_p$. Define $\mathscr{S}$ to be the set of all positive integers n such that neither $a \mid n$ for some $a \in \mathscr{A}$, nor $p^k \| n$ for some $p \in \mathscr{P}$ and $k \in \mathscr{K}_p$. If $\mathscr{P}_1 := \{p \in \mathscr{P} : 1 \in \mathscr{K}_p\}$ and $\mathscr{A} \cup \mathscr{P}_1$ if thin then $\mathscr{S}$ possesses an asymptotic density. Moreover, if $0 \notin \mathscr{K}_p$ for all $p \in \mathscr{P}$ and $1 \notin \mathscr{A}$ then $\mathbf{d}(\mathscr{S}) > 0$.*

Proof. Consider first the case when $0 \notin \mathscr{K}_p$ for all $p \in \mathscr{P}$. We want to use Lemma 2.1. For define $\mathscr{B} := \mathscr{A} \cup \{p^{k+1} : p \in \mathscr{P},\ k \in \mathscr{K}_p\}$. We construct Ω_b as follows: start with empty Ω_b for all $b \in \mathscr{B}$, then throw 0 into Ω_a for all $a \in \mathscr{A}$ and throw $p^k, 2p^k, \ldots, (p-1)p^k$ into $\Omega_{p^{k+1}}$ for all $p \in \mathscr{P}$ and $k \in \mathscr{K}_p$. If $a \in \mathscr{A} \setminus \{p^{k+1} : p \in \mathscr{P},\ k \in \mathscr{K}_p\}$ then $\mathscr{S}_a(x) \leq x/a$, so set $c_a := 1/a$. On the other hand, if $p \in \mathscr{P}$ and $k \in \mathscr{K}_p$ then $\mathscr{S}_{p^{k+1}}(x) \leq x/p^k$, so set $c_{p^{k+1}} := 1/p^k$. Since $\mathscr{A} \cup \mathscr{P}_1$ is thin, it follows

$$\sum_{b \in \mathscr{B}} c_b \leq \sum_{a \in \mathscr{A}} \frac{1}{a} + \sum_{p \in \mathscr{P}_1} \sum_{k \in \mathscr{K}_p} \frac{1}{p^k} + \sum_{p \in \mathscr{P} \setminus \mathscr{P}_1} \sum_{k \in \mathscr{K}_p} \frac{1}{p^k}$$

$$\leq \sum_{a \in \mathscr{A}} \frac{1}{a} + \sum_{p \in \mathscr{P}_1} \sum_{k=1}^{\infty} \frac{1}{p^k} + \sum_{p \in \mathscr{P} \setminus \mathscr{P}_1} \sum_{k=2}^{\infty} \frac{1}{p^k}$$

$$\leq \sum_{a \in \mathscr{A}} \frac{1}{a} + \sum_{p \in \mathscr{P}_1} \frac{1}{p-1} + \sum_{p} \frac{1}{p^2 - p} < \infty,$$

and thus Lemma 2.1 implies that $\mathscr{S}$ has an asymptotic density, since $n \in \mathscr{S}$ if and only if $(n \bmod b) \notin \Omega_b$ for all $b \in \mathscr{B}$. In particular, if $1 \notin \mathscr{A}$, then the set $\mathscr{C} := \mathscr{A} \cup \mathscr{P}_1 \cup \{p^2 : p \in \mathscr{P}\}$ is thin and also $1 \notin \mathscr{C}$. Hence, it follows from Lemma 2.2 that $\mathscr{M}(\mathscr{C})$ has asymptotic density less than 1. On the other hand, $(\mathbf{N} \setminus \mathscr{M}(\mathscr{C})) \subseteq \mathscr{S}$, so $\mathbf{d}(\mathscr{S}) > 0$.

Now consider the case when $0 \in \mathscr{K}_p$ for some $p \in \mathscr{P}$ and define $\mathscr{P}_0 := \{p \in \mathscr{P} : 0 \in \mathscr{K}_p\}$. It results that $p \mid n$ for all $p \in \mathscr{P}_0$ and $n \in \mathscr{S}$. If $\mathscr{P}_0$ is infinite, then $\mathscr{S}$ is empty, and hence has asymptotic density zero. On another hand, if $\mathbf{N} \subseteq \mathscr{K}_p$ for some $p \in \mathscr{P}_0$ then $\mathscr{S}$ is empty again and the claim follows. So we are left with the case when $\mathscr{P}_0$ is finite and for each $p \in \mathscr{P}_0$ there exists a positive integer k_p such that $\{0, 1, \ldots, k_p - 1\} \subseteq \mathscr{K}_p$, but $k_p \notin \mathscr{K}_p$. Therefore, any $n \in \mathscr{S}$ is divisible by $\pi := \prod_{p \in \mathscr{P}_0} p^{k_p}$. Define

$$\mathscr{A}' := \left\{ \frac{a}{\gcd(a, \pi)} : a \in \mathscr{A} \right\},$$

and let $\mathscr{K}'_p := \{k - k_p : k \in \mathscr{K}_p,\ k > k_p\}$ for $p \in \mathscr{P}_0$ and $\mathscr{K}'_p := \mathscr{K}_p$ for $p \in \mathscr{P} \setminus \mathscr{P}_0$. Then $n \in \mathscr{S}$ if and only if $n = \pi m$ for a positive integer m such that neither $a \mid m$ for some $a \in \mathscr{A}'$, nor $p^k \| m$ for some $p \in \mathscr{P}$ and $k \in \mathscr{K}'_p$. Note that $\mathscr{A}'$ is thin since $\mathscr{A}$ is thin. Furthermore, let $\mathscr{P}'_1 := \{p \in \mathscr{P} : 1 \in \mathscr{K}'_p\}$, then $\mathscr{P}'_1 \subseteq \mathscr{P}_0 \cup \mathscr{P}_1$, so $\mathscr{P}'_1$ is thin, since $\mathscr{P}_0$ is finite and $\mathscr{P}_1$ is thin by hypothesis. This yields that $\mathscr{A}' \cup \mathscr{P}'_1$ is itself thin. Since $0 \notin \mathscr{K}'_p$ for all $p \in \mathscr{P}$, it then follows from the first part of the proof that $\mathscr{S}$ possesses an asymptotic density. $\square$

Now we show that the support of any multiplicative arithmetic function has an asymptotic density and we give a way to know if this density is zero or positive. This is particularly useful if one needs to apply Theorem 1.2.

Lemma 2.4. *If ν is a multiplicative arithmetic function then* $\mathrm{supp}(\nu)$ *has an asymptotic density and specifically*

$$
(4) \qquad \mathbf{d}(\mathrm{supp}(\nu)) = \prod_p \left(1 - \frac{1}{p}\right) \sum_{\substack{k=0 \\ p^k \in \mathrm{supp}(\nu)}}^{\infty} \frac{1}{p^k}.
$$

In particular, $\mathbf{d}(\mathrm{supp}(\nu)) > 0$ *if and only if* $\sum_{p \notin \mathrm{supp}(\nu)} 1/p < \infty$.

Proof. From a result of G. Tenenbaum [Ten95, Theorem 11, p. 48], if f is a multiplicative arithmetic function with values in $[0,1]$ then

$$
\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} f(n) = \prod_p \left(1 - \frac{1}{p}\right) \sum_{k=0}^{\infty} \frac{f(p^k)}{p^k}.
$$

Equation (4) follows choosing f as the indicator function of $\mathrm{supp}(\nu)$, which is multiplicative. After some calculations, we obtain

$$
\mathbf{d}(\mathrm{supp}(\nu)) = \prod_{p \in \mathrm{supp}(\nu)} \left(1 - \frac{c_p}{p^2}\right) \prod_{p \notin \mathrm{supp}(\nu)} \left(1 - \frac{1}{p} + \frac{c_p}{p^2}\right),
$$

where $c_p \in [0,1]$ for all prime numbers p. In conclusion, $\mathbf{d}(\mathrm{supp}(\nu)) > 0$ if and only if $\sum_{p \notin \mathrm{supp}(\nu)} 1/p < \infty$. Regarding the convergence of infinite products, see [Kra08, Ch. 8]. □

The next lemma is the key to the proof of Theorem 1.2. It is a generalization of [PS13, Lemma 2.3], hence the first parts of their proof are similar.

Lemma 2.5. *Let ν be a multiplicative arithmetic function with support of positive asymptotic density. Let $\mathscr{A}$ be a thin set of positive integers. If $\mathscr{T} \subseteq \mathscr{S} \subseteq \mathscr{A}$, where $\mathscr{S}$ is finite, then the set $\mathscr{C}$ of positive integers n for which both:*

    (i) $\mathscr{S} = \{d \in \mathscr{A} : d \mid n\}$*; and*

    (ii) $\mathscr{T} = \{d \in \mathscr{S} : d \mid n, \ \nu(n/d) \neq 0\}$,

has an asymptotic density.

Proof. Let χ_ν be the indicator function of $\mathrm{supp}(\nu)$. Define the arithmetic function χ by taking

$$
(5) \qquad \chi(n) := \prod_{d \in \mathscr{T}} \chi_\nu(n/d) \prod_{c \in \mathscr{S} \setminus \mathscr{T}} (1 - \chi_\nu(n/c)),
$$

for each n satisfying condition (i), and let $\chi(n) := 0$ otherwise. Then χ is the indicator function of $\mathscr{C}$. Moreover, when n satisfies (i), expanding the second product in (5) we obtain that

$$
\chi(n) = \sum_{\mathscr{T} \subseteq \mathscr{U} \subseteq \mathscr{S}} (-1)^{|\mathscr{U}| - |\mathscr{T}|} \prod_{e \in \mathscr{U}} \chi_\nu(n/e).
$$

So using $'$ to denote a sum restricted to integers n satisfying (i), we find that

$$
(6) \qquad \mathscr{C}(x) = \sum_{n \le x} \chi(n) = \sum_{\mathscr{T} \subseteq \mathscr{U} \subseteq \mathscr{S}} (-1)^{|\mathscr{U}| - |\mathscr{T}|} {\sum_{n \le x}}' \prod_{e \in \mathscr{U}} \chi_\nu(n/e),
$$

for all $x > 0$. Dividing equation (6) by x and letting $x \to \infty$, it suffices to prove that for each set $\mathscr{U}$ with $\mathscr{T} \subseteq \mathscr{U} \subseteq \mathscr{S}$, the set

$$
\mathscr{V} := \{n \in \mathbf{N} : n \text{ satisfies (i)}, \ n/e \in \mathrm{supp}(\nu) \text{ for all } e \in \mathscr{U}\}
$$

has an asymptotic density.

If n satisfies (i) then $L \mid n$, where $L := \mathrm{lcm}\{d \in \mathscr{S}\}$. In fact, (i) holds for n if and only if $n = Lq$ for some $q \in \mathbf{N}$ such that $a/\gcd(a, L) \nmid q$ for all $a \in \mathscr{A} \setminus \mathscr{S}$. On the other hand, $n/e \in \mathrm{supp}(\nu)$ for $e \in \mathscr{U}$ if and only if there must exist no prime p and positive integer k such that $p^k \parallel n/e$ and $\nu(p^k) = 0$. Now, in view of using Lemma 2.3, define

$$
\mathscr{B} := \left\{ \frac{a}{\gcd(a, L)} : a \in \mathscr{A} \setminus \mathscr{S} \right\},
$$

and let $\mathscr{P}$ be the set of prime numbers p such that $\nu(p^k) = 0$ for some $k \in \mathbf{N}$. Being that $\mathscr{A}$ is a thin set it follows that also $\mathscr{B}$ is a thin set. Moreover, define

$$\mathscr{K}_p := \{h - j : h \in \mathbf{N}, \, j \in \mathbf{N}_0, \, e \in \mathscr{U}, \, \nu(p^h) = 0, \, p^j \, || \, L/e\} \cap \mathbf{N}_0.$$

So, it follows that $n = Lq \in \mathscr{V}$ if and only if neither $b \mid q$ for some $b \in \mathscr{B}$, nor $p^k \, || \, q$ for some $p \in \mathscr{P}$ and $k \in \mathscr{K}_p$. Let $\mathscr{P}_1 := \{p \in \mathscr{P} : 1 \in \mathscr{K}_p\}$. If $p \in \mathscr{P}_1$ then there are only two possible cases: $p \notin \operatorname{supp}(\nu)$ and $p \nmid L/e$ for some $e \in \mathscr{U}$; or $\nu(p^{j+1}) = 0$ and $p^j \, || \, L/e$ for some $j \in \mathbf{N}$, $e \in \mathscr{U}$ (there are only a finite number of such primes, because $\mathscr{U}$ is finite). Since ν has support of positive asymptotic density, from Lemma 2.4 it results that $\sum_{p \notin \operatorname{supp}(\nu)} 1/p < \infty$, so by the previous consideration $\mathscr{P}_1$ is a thin set. In conclusion, $\mathscr{B} \cup \mathscr{P}_1$ is thin and from Lemma 2.3 it follows that $\mathscr{V}$ possesses an asymptotic density. This completes the proof. $\qquad\square$

3. Proofs of Theorem 1.2 and Corollary 1.1

The idea in the proof of Theorem 1.2 is the same as for Theorem 1.1, but the claim is now strengthened and the proof is simplified by Lemma 2.3. In particular, it is no longer necessary, with the new approach, to split the proof into two parts, as was done for Theorem 1.1.

We call two elements n_1 and n_2 of $\operatorname{supp}(g)$ equivalent if they share the same set $\mathscr{S}$ of divisors from $\operatorname{supp}(f)$ and for $d \in \mathscr{S}$ it holds $\nu(n_1/d) \neq 0$ if and only if $\nu(n_2/d) \neq 0$. Actually, this is an equivalence relation, with equivalence classes $\mathscr{A}_1, \mathscr{A}_2, \ldots$ Then to any $\mathscr{A}_i$ there correspond a nonempty set $\mathscr{S}_i$ and a subset $\mathscr{T}_i$ of $\mathscr{S}_i$ such that $n \in \mathscr{A}_i$ if and only if $\{d \in \operatorname{supp}(f) : d \mid n\} = \mathscr{S}_i$ and $\{d \in \mathscr{S}_i : \nu(n/d) \neq 0\} = \mathscr{T}_i$. Moreover, $\mathscr{T}_i \subseteq \mathscr{S}_i \subseteq \operatorname{supp}(f)$ with $\operatorname{supp}(f)$ thin by hypotheses and $\mathscr{S}_i$ finite, since it is a set of divisors of a positive integer. It follows from Lemma 2.5 that $\mathscr{A}_i$ possesses an asymptotic density. On the other hand,

$$\operatorname{supp}(g) = \bigcup_{i=1}^{\infty} \mathscr{A}_i,$$

with disjoint union. If there are only finitely many $\mathscr{A}_i$ then it follows immediately that $\mathbf{d}(\operatorname{supp}(g))$ exists, since the asymptotic density if finitely additive. If instead there are infinitely many $\mathscr{A}_i$ then define

$$m_k := \min_{i > k} \max(\mathscr{S}_i)$$

and observe that $m_k \to \infty$ as $k \to \infty$. If $n \in \bigcup_{i > k} \mathscr{A}_i$ then $\{d \in \operatorname{supp}(f) : d \mid n\} = \mathscr{S}_i$ for some integer $i > k$. So n has a divisor $d \in \operatorname{supp}(f)$ with $d \geq m_k$ and as a consequence

$$(7) \qquad \overline{\mathbf{d}}\left(\bigcup_{i > k} \mathscr{A}_i\right) \leq \sum_{\substack{d \in \operatorname{supp}(f) \\ d \geq m_k}} \frac{1}{d}.$$

As $k \to \infty$ the right-hand side of equation (7) tends to zero, since $\operatorname{supp}(f)$ is thin. Therefore, it follows that $\operatorname{supp}(g)$ has an asymptotic density (see [PS13, Lemma 2.1]).

Now, we want to prove that $\mathbf{d}(\operatorname{supp}(g)) > 0$. Since f does not vanish identically, $\operatorname{supp}(f)$ has a minimum d. We claim that a positive portion of positive integers n satisfies the following conditions: n has only d as a divisor from $\operatorname{supp}(f)$ and $n/d \in \operatorname{supp}(\nu)$; to the effect that $g(n) = f(d)\,\nu(n/d) \neq 0$, i.e., $n \in \operatorname{supp}(g)$. Define

$$\mathscr{B} := \left\{ \frac{b}{\gcd(b, d)} : b \in \operatorname{supp}(f), \, b \neq d \right\},$$

and let $\mathscr{P}$ be the set of prime numbers p such that $\nu(p^k) = 0$ for some $k \in \mathbf{N}$, for each $p \in \mathscr{P}$ set $\mathscr{K}_p := \{k \in \mathbf{N} : \nu(p^k) = 0\}$. Let $\mathscr{V}$ be the set of positive integers m such that neither $b \mid m$ for some $b \in \mathscr{B}$, nor $p^k \, || \, m$ for some $p \in \mathscr{P}$ and $k \in \mathbf{N}$. If $m \in \mathscr{V}$ then $n = dm$ has d as his only divisor from $\operatorname{supp}(f)$ and $n/d = m \in \operatorname{supp}(\nu)$. Note that $\mathscr{B}$ is a thin set, since $\operatorname{supp}(f)$ is thin. Thanks to Lemma 2.4 it results that $\mathscr{P}_1 := \{p \in \mathscr{P} : 1 \in \mathscr{K}_p\}$ is thin, so $\mathscr{B} \cup \mathscr{P}_1$ is thin. Moreover, $0 \notin \mathscr{K}_p$ for all $p \in \mathscr{P}$ and $1 \notin \mathscr{B}$, so it follows from Lemma 2.3 that $\mathbf{d}(\mathscr{V}) > 0$ and finally $\mathbf{d}(\operatorname{supp}(g)) > 0$.

At this point, the proof of Corollary 1.1 is immediate. By partial summation one can show that a set of positive asymptotic density is never thin. Let (f, g) be a nonzero ν-pair. If $\operatorname{supp}(f)$ is thin then from Theorem 1.2 it follows that $\operatorname{supp}(g)$ has positive asymptotic density and hence it is not thin. Otherwise, if $\operatorname{supp}(g)$ is thin, note that ν^{-1} is a multiplicative function because it is the Dirichlet inverse of a multiplicative function. Furthermore, $\nu^{-1}(p) = -\nu(p)$ for all primes p. Thus, we get by Lemma 2.4

$$\sum_{p \notin \operatorname{supp}(\nu^{-1})} \frac{1}{p} = \sum_{p \notin \operatorname{supp}(\nu)} \frac{1}{p} < \infty,$$

and hence $\operatorname{supp}(\nu^{-1})$ has positive asymptotic density. From Theorem 1.2, since (g, f) is a v^{-1}-pair, it follows that $\operatorname{supp}(f)$ has positive asymptotic density and so it is not thin.

4. Proof of Theorem 1.3

Since ν is a multiplicative arithmetic function with support of positive asymptotic density, it follows from Lemma 2.4 that $\sum_{p \notin \operatorname{supp}(\nu)} 1/p < \infty$, and so

$$C'_\nu := \frac{6}{\pi^2} \prod_{p \notin \operatorname{supp}(\nu)} \left(1 + \frac{1}{p}\right)^{-1}$$

is a well-defined positive real constant. On the one hand, since f is multiplicative, it is easily seen that

$$\sum_{n \in \operatorname{supp}(f)} \frac{1}{n} \geq \prod_{p \in \operatorname{supp}(f)} \left(1 + \frac{1}{p}\right).$$

On the other hand, since g is multiplicative too, we get, again by Lemma 2.4, that

$$\mathbf{d}(\operatorname{supp}(g)) = \prod_p \left(1 - \frac{1}{p}\right) \sum_{\substack{k=0 \\ p^k \in \operatorname{supp}(g)}}^\infty \frac{1}{p^k} \geq \prod_{p \in \operatorname{supp}(g)} \left(1 - \frac{1}{p^2}\right) \prod_{p \notin \operatorname{supp}(g)} \left(1 - \frac{1}{p}\right)$$

$$= \prod_p \left(1 - \frac{1}{p^2}\right) \prod_{p \notin \operatorname{supp}(g)} \left(1 + \frac{1}{p}\right)^{-1} = \frac{6}{\pi^2} \prod_{p \notin \operatorname{supp}(g)} \left(1 + \frac{1}{p}\right)^{-1}.$$

Finally, $g(p) = \nu(p) + f(p)$ for all prime numbers p, so we obtain

$$\mathbf{d}(\operatorname{supp}(g)) = \frac{6}{\pi^2} \prod_{\substack{p \notin \operatorname{supp}(\nu) \\ p \notin \operatorname{supp}(f)}} \left(1 + \frac{1}{p}\right)^{-1} \prod_{\substack{p \in \operatorname{supp}(f) \\ f(p) \neq -\nu(p)}} \left(1 + \frac{1}{p}\right)^{-1}$$

$$\geq C'_\nu \prod_{p \in \operatorname{supp}(f)} \left(1 + \frac{1}{p}\right)^{-1} \geq \frac{C'_\nu}{\sum_{n \in \operatorname{supp}(f)} 1/n}.$$

This completes the proof.

Remark 4.1. Theorem 1.3 is no longer true if the hypothesis that f is multiplicative, or equivalently that g is multiplicative, is dropped. For example, fix an integer $d \geq 2$ and take f as the indicator function of the singleton $\{d\}$, to the effect that f is not multiplicative. On the one hand, it results $\mathbf{d}(\operatorname{supp}(g)) \leq 1/d$. On the other hand, obviously $\sum_{n \in \operatorname{supp}(f)} 1/n = 1/d$. Thus, it must be $C_\nu \leq 1/d^2$. Due to the arbitrariness of d if follows that C_ν cannot be positive.

An interesting question might be the evaluation of the best constant C_ν in Theorem 1.3, i.e., the infimum

$$C_\nu := \inf_{(f,g)} \mathbf{d}(\operatorname{supp}(g)) \sum_{n \in \operatorname{supp}(f)} \frac{1}{n},$$

over all ν-pairs (f, g) of multiplicative arithmetic functions with f of thin support. Theorem 1.3 gives us a lower bound for C_ν. For an upper bound, notice that setting $f = \epsilon$ we obtain $C_\nu \leq \mathbf{d}(\mathrm{supp}(\nu))$. Thus, in particular, we have $C_\mu = 6/\pi^2$.

5. Proof of Theorem 1.5

For each $y \geq 0$ define the function g_y by setting

$$g_y(n) := \sum_{\substack{d \mid n \\ d \leq y}} f(d)\, \nu(n/d),$$

for all $n \in \mathbf{N}$. We can regard g_y as a sort of "truncated Dirichlet convolution" of f and ν. The following lemma holds.

Lemma 5.1. *If $\sum_{n=1}^\infty |f(n)|/n < \infty$ then:*
  (i) *For all $y \geq 0$ the function $|g_y|$ has a finite mean value λ_y.*
  (ii) *λ_y tends to a finite limit λ as $y \to \infty$.*
  (iii) *$|g|$ has mean value λ.*

Proof. The proof is almost identical to that of [PS13, Lemma 4.1], so we do not give the details. The only differences is that one needs to use Lemma 2.5 instead of [PS13, Lemma 2.3] and that in the proof of (ii) and (iii) one makes use of the boundedness of ν. $\square$

Now, suppose that $\delta := \inf_{n \in \mathrm{supp}(\nu)} |\nu(n)|$ is positive and f is not identically zero. We want to prove that the mean value of $|g|$ is positive. Let d be the least positive integer in $\mathrm{supp}(f)$. In the proof of Theorem 1.2 we have seen that a positive portion of $n \in \mathbf{N}$ has d as their only divisor from $\mathrm{supp}(f)$ and satisfies $n/d \in \mathrm{supp}(\nu)$, so that $|g(n)| = |f(d)|\,|\nu(n/d)| \geq |f(d)|\,\delta$. Let $\mathscr{A}$ denote the set of these integers n, then

$$\frac{1}{x} \sum_{n \leq x} |g(n)| \geq \frac{1}{x} \sum_{\substack{n \leq x \\ n \in \mathscr{A}}} |g(n)| \geq |f(d)|\,\delta\,\frac{\mathscr{A}(x)}{x} > 0$$

for large x, since $\underline{\mathbf{d}}(\mathscr{A}) > 0$. Hence, the mean value of $|g|$ is positive. This completes the proof.

Remark 5.1. In Theorem 1.5, the existence of the mean value of $|g|$ is no longer guaranteed if the hypothesis of boundedness of ν is omitted. For example, consider the arithmetic functions ν defined by $\nu(n) := n$ for all $n \in \mathbf{N}$ and $f = \epsilon$; it results that $|g|$ has not a finite mean value. Furthermore, the positiveness of the mean value of $|g|$ is no longer guaranteed if the hypothesis $\inf_{n \in \mathrm{supp}(\nu)} |\nu(n)| > 0$ is omitted. E.g., consider the arithmetic functions ν defined by $\nu(n) := 1/n$ for all $n \in \mathbf{N}$ and $f = \epsilon$, it results that $|g|$ has mean value zero.

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Università degli Studi di Torino, Department of Mathematics, Turin, Italy
E-mail address: carlo.sanna.dev@gmail.com