A note on Marked Point Processes and multivariate subordination

Published version:

DOI:10.1016/j.spl.2016.11.008

A note on Marked Point Processes and multivariate subordination

Petar Jevtić,
McMaster University
Department of Mathematics and Statistics
Ontario L8S 4K1, Canada
pjevtic@mcmaster.ca

Marina Marena
University of Torino
Department of Economics and Statistics
C.so Unione Sovietica, 218 bis
10134 Torino, Italy
marina.marena@unito.it

Patrizia Semeraro
Politecnico di Torino
Viale Pier Andrea Mattioli, 39
10125 Torino, Italy
patrizia.semeraro@polito.it

November 16, 2016

Abstract

The aim of this paper is to state a correspondence between marked Poisson processes and multivariate subordinated Lévy processes. We prove that, under suitable conditions, marked Poisson processes are in law subordinated Brownian motions and we provide their Lévy triplet and characteristic function. We introduce the class of multivariate Gaussian marked Poisson processes and prove that - in law - they are multivariate subordinated Brownian motions.

MSC2010 Classification: 60G15, 60G51, 60G55, 60G57

Keywords: marked Poisson processes, subordinated Lévy processes, multivariate Poisson random measure, multivariate subordinators, multivariate generalized asymmetric Laplace motion.

Introduction

The aim of this note is to state a correspondence between multivariate marked Poisson processes and multivariate subordinated Lévy processes. The motivation of this note is to further the work of Barndorff-Nielsen et al. (2001) who give characterization of multivariate subordinated Lévy processes which are relevant in modern finance and related fields. A standard reference for Poisson processes, Lévy processes and their relationship is Çınlar (2011).

Let $\mathbf{\Pi}$ be a Poisson random measure on a measurable space $(E, \mathcal{E})$ with a σ-finite mean measure $\mu_{\mathbf{\Pi}}$. By slight abuse of notation, with $\mathbf{\Pi} = \{\Pi_i, i \in I\}$ we indicate both the random measure and the collection of its atoms indexed by some countable set I. Marked Poisson processes are constructed by attaching a random variable to each atom of the random measure $\mathbf{\Pi}$. Formally, let $\boldsymbol{Z} = \{Z_i, i \in I\}$ be a family of random variables (marks) on a measurable space $(F, \mathcal{F})$ indexed by the same countable set I. Assume that the variables Z_i are conditionally independent given $\mathbf{\Pi}$ with distributions $Q(\Pi_i, \cdot)$, where $Q(\boldsymbol{s}, B)$ is a transition probability kernel from $(E, \mathcal{E})$ into $(F, \mathcal{F})$. Each variable Z_i can be considered as an indicator of some property associated with the atom Π_i. Then, as proved in Theorem 3.2 in Çınlar (2011), $\boldsymbol{M} := (\mathbf{\Pi}, \boldsymbol{Z})$ forms a Poisson random measure on $(E \times F, \mathcal{E} \otimes \mathcal{F})$ with mean $\mu_{\mathbf{\Pi}} \times Q$, where $(\mu_{\mathbf{\Pi}} \times Q)(d\boldsymbol{x}, d\boldsymbol{y}) = \mu_{\mathbf{\Pi}}(d\boldsymbol{x})Q(\boldsymbol{x}, d\boldsymbol{y})$. The new measure $\boldsymbol{M}$ is called marked Poisson random measure.

Let us recall that the subordination of a Lévy process $\boldsymbol{L} = \{\boldsymbol{L}(t), t \geq 0\}$ by a univariate subordinator $\tau(t)$, i.e. a Lévy process on $\mathbb{R}_+ = [0, \infty)$ with increasing trajectories, defines a new process $\boldsymbol{X} = \{\boldsymbol{X}(t), t \geq 0\}$ by the composition $\boldsymbol{X}(t) := (L_1(\tau(t)), \ldots, L_n(\tau(t)))^T$. Theorem 30.1 in Sato (1999) characterizes the subordinated process $\boldsymbol{X}$ in terms of its Lévy triplet. Barndorff-Nielsen et al. (2001) generalize the above construction by allowing the introduction of multivariate subordinators, i.e. Lévy processes on $\mathbb{R}_+^n = [0, \infty)^n$, whose trajectories are increasing in each coordinate. For purposes of introduction of multivariate subordination, the notion of $\mathbb{R}_+^d$-parameter process, as introduced in Barndorff-Nielsen et al. (2001), is required. Consider the multiparameter $\boldsymbol{s} = (s_1, ..., s_d)^T \in \mathbb{R}_+^d$ and the partial order on $\mathbb{R}_+^d$

$$\boldsymbol{s}^1 \preceq \boldsymbol{s}^2 \quad \Leftrightarrow \quad s_j^1 \leq s_j^2,\ j = 1, \ldots, d.$$

Let now $\boldsymbol{L}(\boldsymbol{s}) = (L_1(\boldsymbol{s}), L_2(\boldsymbol{s}), \ldots, L_n(\boldsymbol{s}))^T$ be a process with parameters in $\mathbb{R}_+^d$ and values in $\mathbb{R}^n$. It is called an $\mathbb{R}_+^d$-parameter Lévy process on $\mathbb{R}^n$ if the following holds

- for any $m \geq 3$ and for any choice of $\boldsymbol{s}^1 \preceq ... \preceq \boldsymbol{s}^m$, $\boldsymbol{L}(\boldsymbol{s}^j) - \boldsymbol{L}(\boldsymbol{s}^{j-1})$, $j = 2, ..., m$, are independent,
- for any $\boldsymbol{s}^1 \preceq \boldsymbol{s}^2$ and $\boldsymbol{s}^3 \preceq \boldsymbol{s}^4$ satisfying $\boldsymbol{s}^2 - \boldsymbol{s}^1 = \boldsymbol{s}^4 - \boldsymbol{s}^3$, $\mathcal{L}(\boldsymbol{L}(\boldsymbol{s}^2) - \boldsymbol{L}(\boldsymbol{s}^1)) = \mathcal{L}(\boldsymbol{L}(\boldsymbol{s}^4) - \boldsymbol{L}(\boldsymbol{s}^3))$ where $\mathcal{L}(\cdot)$ denotes the law of the random variable,
- $\boldsymbol{L}(\mathbf{0}) = \mathbf{0}$ almost surely, and
- almost surely, $\boldsymbol{L}(\boldsymbol{s})$ is right continuous with left limits in $\boldsymbol{s}$ in the partial ordering of $\mathbb{R}_+^d$.

Let $\{\boldsymbol{L}(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{R}_+^d\}$ be a multiparameter Lévy process on $\mathbb{R}^n$ with Lévy triplet $(\boldsymbol{\gamma_L}, \Sigma_{\boldsymbol{L}}, \nu_{\boldsymbol{L}})$, and let $\boldsymbol{\tau}(t)$ be a d dimensional subordinator independent of $\boldsymbol{L}(\boldsymbol{s})$ having Lévy triplet $(\boldsymbol{\gamma_\tau}, 0, \nu_{\boldsymbol{\tau}})$. The subordinated process $\boldsymbol{X} = \{\boldsymbol{X}(t), t \geq 0\}$ defined by

$$\boldsymbol{X}(t) := \boldsymbol{L}(\boldsymbol{\tau}(t)) = \begin{pmatrix} L_1(\tau_1(t), \dots, \tau_d(t)) \\ \vdots \\ L_n(\tau_1(t), \dots, \tau_d(t)) \end{pmatrix}, \ t \geq 0$$

is a Lévy process, as proved in Theorem 4.7 in Barndorff-Nielsen et al. (2001), who also provide its characteristic function and Lévy triplet. Our main result provides a link between marked Poisson processes and multivariate subordinated Lévy processes. In particular we give conditions for marks and underlying Poisson measure such that marked Poisson process are in law subordinated Lévy process as defined in Barndorff-Nielsen et al. (2001). In addition we provide their Lévy triplet.

As an example we introduce the class of Gaussian marked Poisson processes and prove that in law they belong to the class of multivariate subordinated Brownian motions. We show that, under suitable conditions, the processes in this class have characteristic functions in closed form. In particular, we focus on a multivariate Laplace process. The Laplace distribution (Laplace (1774)) is infinitely divisible and able to account for heavier than Gaussian tails. For this reason, the multivariate associated Laplace process become popular for multivariate modeling in several areas, as Engineering and Finance (Kotz et al. (2012)).

1 Lévy Marked Poisson processes

Here we construct a Marked Poisson process of Lévy type. Let $\boldsymbol{\Pi}$ be a Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}^d_+, \mathcal{B}_{d+1})$, where $\mathcal{B}_{d+1}$ is the Borel σ algebra, with mean measure $\mu_{\boldsymbol{\Pi}} = Leb \times \nu_{\boldsymbol{\Pi}}$, where $\nu_{\boldsymbol{\Pi}}$ is a Lévy measure, such that $\nu_{\boldsymbol{\Pi}}(\{\boldsymbol{0}\}) = 0$ and $\int_{\mathbb{B}} |\boldsymbol{x}| \nu_{\boldsymbol{\Pi}}(dx) < \infty$, where $\mathbb{B} = \{\boldsymbol{x} \in \mathbb{R}^d, |\boldsymbol{x}| \leq 1\}$ is the unit ball. The process defined by

$$\boldsymbol{\pi}(t) := \int_{(0,t] \times \mathbb{R}^d_+} \boldsymbol{x} \boldsymbol{\Pi}(ds, d\boldsymbol{x}), \tag{1.1}$$

is a zero drift multivariate subordinator with Lévy measure $\nu_{\boldsymbol{\Pi}}$. The atoms of $\boldsymbol{\Pi}$ are family of random variables $\boldsymbol{\Pi} = \{(\Pi_1, \boldsymbol{\Pi}_2) = \{(\Pi_{1i}, \boldsymbol{\Pi}_{2i}), i \in I\}\}$ on $\mathbb{R}_+ \times \mathbb{R}^d_+$, where Π_{1i} are the jump times and $\boldsymbol{\Pi}_{2i}$ are the jump sizes.

If $\boldsymbol{L}(\boldsymbol{s})$ is an $\mathbb{R}^d_+$- multiparameter process on $\mathbb{R}^n$ and $\lambda^{\boldsymbol{s}} = \mathcal{L}(\boldsymbol{L}(\boldsymbol{s}))$, and B is any set belonging to $\mathcal{B}_n$ i.e. $B \in \mathcal{B}_n$, we introduce the transition probability kernel Q defined by $Q(\boldsymbol{s}, B) = \lambda^{\boldsymbol{s}}(B)$, i.e.

$$\begin{aligned} Q(\boldsymbol{0}, B) &:= P(\boldsymbol{L}(\boldsymbol{0}) \in B) = \mathbb{1}_B(\boldsymbol{0}) \\ Q(\boldsymbol{s}, B) &:= P(\boldsymbol{L}(\boldsymbol{s}) \in B) \end{aligned} \tag{1.2}$$

and name it *multiparameter Lévy kernel*. The first expression of equation (1.2) is a consequence of being $\boldsymbol{L}(\boldsymbol{0}) = \boldsymbol{0}$ with probability one.

The following theorem provide a connection between marked Poisson processes and multivariate subordinated Lévy processes.

Theorem 1.1. *Let* $\boldsymbol{Z} = \{\boldsymbol{Z}_i, i \in I\}$ *be a family of marks of* $\boldsymbol{\Pi} = \{(\Pi_{1i}, \boldsymbol{\Pi}_{2i}), i \in I\}$ *on* $(\mathbb{R}^n, \mathcal{B}_n)$, *with distribution* $Q(\boldsymbol{\Pi}_{2i}, \cdot)$. *The family* $\boldsymbol{N} = (\Pi_1, \boldsymbol{Z})$ *forms a Poisson random measure on* $(\mathbb{R}_+ \times \mathbb{R}^n, \mathcal{B}_{n+1})$ *with mean measure* $\mu_{\boldsymbol{N}}(dt, d\boldsymbol{y}) = dt \int_{\mathbb{R}^d_+} \nu_{\boldsymbol{\Pi}}(d\boldsymbol{s}) Q(\boldsymbol{s}, d\boldsymbol{y})$. *If we choose* $\boldsymbol{\gamma}_{\boldsymbol{Y}} = \int_{\mathbb{R}^d_+} \nu_{\boldsymbol{\Pi}}(d\boldsymbol{s}) \int_{\mathbb{B}} \boldsymbol{x} Q(\boldsymbol{s}, d\boldsymbol{x})$, *the process* $\boldsymbol{Y}(t)$ *defined as*

$$\boldsymbol{Y}(t) := \boldsymbol{\gamma}_{\boldsymbol{Y}} t + \int_{(0,t]\times\mathbb{B}} \boldsymbol{y}[\boldsymbol{N}(ds, d\boldsymbol{y}) - \mu_{\boldsymbol{N}}(ds, d\boldsymbol{y})] + \int_{(0,t]\times\mathbb{B}^c} \boldsymbol{y}\boldsymbol{N}(ds, d\boldsymbol{y}), \tag{1.3}$$

is (in law) a subordinated Lévy process constructed by subordination of a multiparameter Lévy process $\boldsymbol{L}(\boldsymbol{s})$ *such that* $\mathcal{L}(\boldsymbol{L}(\boldsymbol{s}))$=$Q(\boldsymbol{s}, \cdot)$, *with the subordinator* $\boldsymbol{\pi}(t)$. *The Lévy triplet of* $\boldsymbol{Y}$ *is* $(\boldsymbol{\gamma}_{\boldsymbol{Y}}, \Sigma_{\boldsymbol{Y}}, \nu_{\boldsymbol{Y}})$, *where*

$$\Sigma_{\boldsymbol{Y}} = \boldsymbol{0},$$
$$\nu_{\boldsymbol{Y}}(B) = \int_{\mathbb{R}^d_+} \nu_{\boldsymbol{\Pi}}(d\boldsymbol{s}) Q(\boldsymbol{s}, B).$$

Proof. Let $\hat{Q}$ be a transition probability kernel from $(\mathbb{R}_+ \times \mathbb{R}^d_+, \mathcal{B}_{d+1})$ into $(\mathbb{R}^n, \mathcal{B}_n)$, so that for each $t \in \mathbb{R}_+$ we have $\hat{Q}(t, \boldsymbol{s}, d\boldsymbol{y}) = Q(\boldsymbol{s}, d\boldsymbol{y})$. From Theorem 3.2 in Çınlar (2011) it follows that the pair $\boldsymbol{T} := (\boldsymbol{\Pi}, \boldsymbol{Z})$ forms a Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}^d_+ \times \mathbb{R}^n, \mathcal{B} \otimes \mathcal{B}_d \otimes \mathcal{B}_n)$ with mean $\mu_{\boldsymbol{T}} = Leb \times \nu_{\boldsymbol{\Pi}} \times \hat{Q}$, i.e. $\mu_{\boldsymbol{T}}(dt, d\boldsymbol{s}, d\boldsymbol{y}) = dt\nu_{\boldsymbol{\Pi}}(d\boldsymbol{s})\hat{Q}(t, \boldsymbol{s}, d\boldsymbol{y}) = dt\nu_{\boldsymbol{\Pi}}(d\boldsymbol{s})Q(\boldsymbol{s}, d\boldsymbol{y})$. Define now $\boldsymbol{N} := (\Pi_1, \boldsymbol{Z})$. Then $\boldsymbol{N}$ forms a Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}^n, \mathcal{B}_{n+1})$ with mean measure $\mu_{\boldsymbol{N}}(dt, d\boldsymbol{y}) = dt \int_{\mathbb{R}^d_+} \nu_{\boldsymbol{\Pi}}(d\boldsymbol{s}) Q(\boldsymbol{s}, d\boldsymbol{y})$ equal to the margin measure of the measure $\boldsymbol{T}$ on $(\mathbb{R}_+ \times \mathbb{R}^n, \mathcal{B}_{n+1})$. Let $\nu_{\boldsymbol{N}}(d\boldsymbol{y}) = \int_{\mathbb{R}^d_+} \nu_{\boldsymbol{\Pi}}(d\boldsymbol{s}) Q(\boldsymbol{s}, d\boldsymbol{y})$, it holds $\mu_{\boldsymbol{N}} = Leb \times \nu_{\boldsymbol{N}}$. Let now $\boldsymbol{L}(\boldsymbol{s})$ be a multiparameter process with $\mathcal{L}(\boldsymbol{L}(\boldsymbol{s})) = Q(\boldsymbol{s}, \cdot)$ and let it be independent from the subordinator $\boldsymbol{\pi}(t)$. Consider the process $\boldsymbol{X}(t) := \boldsymbol{L}(\boldsymbol{\pi}(t))$. By Theorem 4.7 in Barndorff-Nielsen et al. (2001) $\boldsymbol{X}(t)$ is a Lévy process with the following Lévy triplet

$$\boldsymbol{\gamma}_{\boldsymbol{X}} = \int_{\mathbb{R}^d_+} \nu_{\boldsymbol{\Pi}}(d\boldsymbol{s}) \int_{\mathbb{B}} \boldsymbol{x} Q(\boldsymbol{s}, d\boldsymbol{x}),$$
$$\Sigma_{\boldsymbol{X}} = \boldsymbol{0},$$
$$\nu_{\boldsymbol{X}}(B) = \int_{\mathbb{R}^d_+} \nu_{\boldsymbol{\Pi}}(d\boldsymbol{s}) Q(\boldsymbol{s}, B).$$

The Poisson random measure $\boldsymbol{N}$ has mean measure $\mu_{\boldsymbol{N}} = Leb \times \nu_{\boldsymbol{N}}$, where $\nu_{\boldsymbol{N}} = \nu_{\boldsymbol{X}}$. Therefore the process $\boldsymbol{Y}$ defined in equation (1.3) is a Lévy process with Lévy measure $\nu_{\boldsymbol{Y}} = \nu_{\boldsymbol{X}}$. By construction $\Sigma_{\boldsymbol{Y}} = 0$, thus $\boldsymbol{Y}$ is pure jump. By assumption we choose

$$\boldsymbol{\gamma}_{\boldsymbol{Y}} = \int_{\mathbb{R}^d_+} \nu_{\boldsymbol{\Pi}}(d\boldsymbol{s}) \int_{\mathbb{B}} \boldsymbol{x} Q(\boldsymbol{s}, d\boldsymbol{x}) = \boldsymbol{\gamma}_{\boldsymbol{X}}.$$

Therefore $\boldsymbol{Y}$ has the same Lévy triplet of $\boldsymbol{X}$ and the assert is proved. □

We call the process $\boldsymbol{Y}$ in (1.3) *Lévy marked Poisson process* (*LmPp*) and its corresponding random measure $\boldsymbol{N}$ is called *Lévy marked Poisson random measure*. Since $\boldsymbol{Y}$ is a subordinated Lévy process, its straightforward to derive its characteristics function by applying Theorem 4.7 in Barndorff-Nielsen et al. (2001).

Corollary 1.1. *The characteristic function of* $\boldsymbol{Y}(t)$ *is:*

$$E[e^{i\langle \boldsymbol{z}, \boldsymbol{Y}(t)\rangle}] = \exp(t\Psi_{\boldsymbol{\pi}}(\log \psi_{\boldsymbol{L}}(\boldsymbol{z}))), \;\; \boldsymbol{z} \in \mathbb{R}^n_+,$$

where for any $\boldsymbol{w} = (w_1, ..., w_d)^T \in \mathbb{C}^n$ *with* $Re(w_j) \leq 0,\ j = 1, ..., d$

$$\Psi_{\boldsymbol{\pi}}(\boldsymbol{w}) = \int_{\mathbb{R}^d_+} (e^{\langle \boldsymbol{w}, \boldsymbol{s} \rangle} - 1)\nu_{\Pi}(d\boldsymbol{s}) \text{ and } \log(\psi_{\boldsymbol{L}}(\boldsymbol{z})) = (\log(\psi_1(\boldsymbol{z})), ..., \log(\psi_d(\boldsymbol{z})))^T$$

ψ_j *being the characteristic function of* $\boldsymbol{L}(\boldsymbol{\delta}_j)$*,* $\boldsymbol{\delta}_j = (\delta_{j1}, ..., \delta_{jd})^T$*, where* $\boldsymbol{\delta}_{jk}$ *is Kronecker's delta.*

2 Trajectories

Theorem 1.1 allows for the discussion of regularity of trajectories of process $\boldsymbol{Y}$.

Proposition 2.1. *If the Poisson random measure* $\boldsymbol{\Pi}$ *in* (1.1) *satisfies* $\int_{\mathbb{B}} |\boldsymbol{x}|^{1/2} \nu_{\boldsymbol{\Pi}}(\boldsymbol{x}) < \infty$*, the marked Poisson process* $\boldsymbol{Y}(t)$ *has bounded variations on any finite time interval, i.e.* $\int_{\mathbb{B}} |\boldsymbol{x}| \nu_{\boldsymbol{Y}}(d\boldsymbol{x}) < \infty$*.*

Proof. Since $\boldsymbol{Y}(t) =_{\mathcal{L}} \boldsymbol{L}(\boldsymbol{\pi}(t))$ the assert follows from Theorems 3.3 and 4.7 in Barndorff-Nielsen et al. (2001). □

Concerning the issue of finite/infinite activity an immediate consequence of

$$\nu_{\boldsymbol{Y}}(\mathbb{R}^n) = \int_{\mathbb{R}^n_+} Q(\boldsymbol{s}, \mathbb{R}^n)\nu_{\boldsymbol{\Pi}}(d\boldsymbol{s}) = \int_{\mathbb{R}^n_+} \nu_{\boldsymbol{\Pi}}(d\boldsymbol{s}) = \nu_{\boldsymbol{\Pi}}(\mathbb{R}^n_+),$$

is that process $\boldsymbol{Y}$ has finite activity ($\nu_{\boldsymbol{Y}}(\mathbb{R}^n) < \infty$) if and only if $\boldsymbol{\Pi}$ does ($\nu_{\boldsymbol{\Pi}}(\mathbb{R}^n_+) < \infty$). Since the marginal Lévy measures are defined as

$$\nu_{\boldsymbol{Y}_j}(A_j) := \nu_{\boldsymbol{Y}}(\mathbb{R} \times \ldots \times \mathbb{R} \times A_j \times \mathbb{R} \times \ldots \times \mathbb{R}),\ A_j \in \mathcal{B}(\mathbb{R}),\ j \in \{1, \ldots, n\},$$

we have $\nu_{\boldsymbol{Y}_j}(\mathbb{R}) < \infty$ for all $j \in \{1, \ldots, n\}$ iff $\nu_{\boldsymbol{Y}}(\mathbb{R}^n) < \infty$.

Given that subordinator $\boldsymbol{\pi}(t)$ introduced in equation (1.1) has zero drift, if $\boldsymbol{Y}(t)$ has bounded variations on any finite time interval ($\int_{\mathbb{B}} |\boldsymbol{x}| \nu_{\boldsymbol{Y}}(d\boldsymbol{x}) < \infty$) it is a pure jump zero drift Lévy process and can be written as the sum of its jumps

$$\boldsymbol{Y}(t) = \sum_{s \in D \cap [0,t]} \Delta \boldsymbol{Y}(s)$$

where $D = \{s > 0 : \Delta \boldsymbol{Y}(s) \neq 0\}$. The measure $\boldsymbol{N}$ in Theorem 1.1 counts the number of jumps in time interval $(0, t]$ whose size belongs to A and its mean measure is $Leb \times \nu_{\boldsymbol{Y}}$, where $\nu_{\boldsymbol{Y}}$ is exactly the Lévy measure of $\boldsymbol{Y}$. Equation 1.3 becomes

$$\boldsymbol{Y}(t) = \int_{(0,t] \times \mathbb{R}^n} \boldsymbol{y} \boldsymbol{N}(ds, d\boldsymbol{y}). \tag{2.1}$$

Intuitively, Theorem 1.1 states that Lévy marked Poisson random measures define subordinated Lévy motions.

3 Gaussian-marked Poisson processes

In this section we specify a multiparameter Gaussian kernel, defined by means of the law of a multiparameter Brownian motion, which we introduce below.

Let $\boldsymbol{B}(t)$ be a Brownian motion on $\mathbb{R}^d$ with independent components, drift $\boldsymbol{\mu}$ and Gaussian covariance matrix $\boldsymbol{\Sigma} = diag(\sigma_1^2, \ldots, \sigma_d^2)$ and let $\{\boldsymbol{B}(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{R}_+^d\}$ be the independent multiparameter Lévy process defined from $\boldsymbol{B}(t)$, i.e. $\boldsymbol{B}(\boldsymbol{s}) := (B_1(s_1), \ldots, B_d(s_d))^T$.

For a given $\boldsymbol{A} = (a_{ij})_{n\times d}$, $a_{ij} \in \mathbb{R}$, we can define the process

$$\boldsymbol{B}^{\rho}(t) = (B_1^{\rho}(t), \ldots, B_n^{\rho}(t))^T := \boldsymbol{A}\boldsymbol{B}(t), \;\; t \in \mathbb{R}_+ \tag{3.1}$$

with drift $\boldsymbol{\mu}^{\rho} = \boldsymbol{A}\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}^{\rho} = \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^T$, where the superscript ρ indicates that $\boldsymbol{B}^{\rho}$ has correlated margins with correlation matrix $\boldsymbol{\rho} = (\rho_{ij})_{n\times n}$. Semeraro (2008) proved that the process defined by

$$\boldsymbol{B}^{\rho}(\boldsymbol{s}) = \boldsymbol{A}\boldsymbol{B}(\boldsymbol{s}), \;\; \boldsymbol{s} \in \mathbb{R}_+^d \tag{3.2}$$

is an $\mathbb{R}_+^d$-parameter Lévy process on $\mathbb{R}^n$. We call the $\mathbb{R}_+^d$-parameter Lévy process $\boldsymbol{B}^{\rho}(\boldsymbol{s})$ in (3.2) $\mathbb{R}_+^d$-*parameter Brownian motion.* Notice that an $\mathbb{R}_+^d$-parameter Brownian motion is uniquely defined by the matrix A and not by the correlation matrix Σ^{ρ}. At this point we can define the multiparameter Gaussian kernel corresponding to $\boldsymbol{B}^{\rho}(\boldsymbol{s})$.

Definition 3.1. *A multiparameter Gaussian kernel $\boldsymbol{G}$ is a multiparameter Lévy kernel from $(\mathbb{R}_+^d, \mathcal{B}_n)$ into $(\mathbb{R}^n, \mathcal{B}_n)$ such that*

$$\begin{aligned} \boldsymbol{G}(\boldsymbol{0}, B) &:= P(\boldsymbol{B}^{\rho}(\boldsymbol{0}) \in B) = \mathbb{1}_B(\boldsymbol{0}) \\ \boldsymbol{G}(\boldsymbol{s}, B) &:= P(\boldsymbol{B}^{\rho}(\boldsymbol{s}) \in B) \end{aligned}$$

where $B \in \mathcal{B}_n$, and $\boldsymbol{B}^{\rho}(\boldsymbol{s})$ as defined above, is an $\mathbb{R}_+^d$-parameter Brownian motion with drift $\boldsymbol{\mu}^{\rho}$ and Gaussian covariance matrix $\boldsymbol{\Sigma}^{\rho}$.

Definition 3.2. *A Gaussian-marked multivariate Poisson process $\boldsymbol{Y}(t)$ is a Lévy marked multivariate Poisson process, where the conditional distribution of marks is defined by a multiparameter Gaussian kernel.*

Theorem 1.1 applies to Gaussian-marked multivariate Poisson processes. As a consequence they are multivariate subordinated Brownian motions. Notice that from (3.2), it holds

$$\boldsymbol{Y}(t) =_{\mathcal{L}} \boldsymbol{B}^{\rho}(\boldsymbol{\pi}(t)) := A\boldsymbol{B}(\boldsymbol{\pi}(t)),$$

where $\boldsymbol{B}(\boldsymbol{\pi}(t)) = (B_1(\pi_1(t)), ..., B_d(\pi_d(t)))$ has mutually independent components and $\boldsymbol{\pi}(t)$ is the subordinator in (1.1).

Proposition 3.1. *The characteristic function of a Gaussian-marked Poisson process has the following form:*

$$\begin{aligned} &\mathbb{E}[e^{i<\boldsymbol{z},\boldsymbol{Y}(t)>}] = \exp\{t\Psi_{\boldsymbol{\pi}}(\log \psi_{\boldsymbol{B}^{\rho}}(\boldsymbol{z}))\} \\ &= \exp\left\{t\Psi_{\boldsymbol{\pi}}\left(i\mu_1 \sum_{k=1}^{n} a_{k1}z_k - \frac{1}{2}\sigma_1^2\Big(\sum_{k=1}^{n} a_{k1}z_k\Big)^2, \ldots, i\mu_d \sum_{k=1}^{n} a_{kd}z_k - \frac{1}{2}\sigma_d^2\Big(\sum_{k=1}^{n} a_{kd}z_k\Big)^2\right)\right\}, \end{aligned}$$

where $\Psi_{\boldsymbol{\pi}}$ is provided in (1.1).

Proof. Since,

$$\boldsymbol{B}^{\rho}(\boldsymbol{s}) := \begin{pmatrix} B_1^{\rho}(s_1,\ldots,s_d) \\ \vdots \\ B_n^{\rho}(s_1,\ldots,s_d) \end{pmatrix} = \begin{pmatrix} a_{11}B_1(s_1)+\ldots+a_{1d}B_d(s_d) \\ \vdots \\ a_{n1}B_1(s_1)+\ldots+a_{nd}B_d(s_d) \end{pmatrix},$$

we have

$$\boldsymbol{B}^{\rho}(\boldsymbol{\delta}_j) = \boldsymbol{B}^{\rho}(0,\ldots,\underbrace{1}_{\text{j-th}},\ldots,0) = (a_{1j}B_j(1),\cdots,a_{nj}B_j(1))^T.$$

Thus,

$$\begin{aligned}\psi_j(\boldsymbol{z}) &= \mathbb{E}[\exp\{i < \boldsymbol{B}^{\rho}(\boldsymbol{\delta}_j),\boldsymbol{z} >\}] = \mathbb{E}\Big[\exp\Big\{i\sum_{k=1}^{n}a_{kj}B_j(1)z_k\Big\}\Big] \\ &= \mathbb{E}\Big[\exp\Big\{iB_j(1)\sum_{k=1}^{n}a_{kj}z_k\Big\}\Big] = \exp\Big\{i\mu_j\sum_{k=1}^{n}a_{kj}z_k - \frac{1}{2}\sigma_j^2\Big(\sum_{k=1}^{n}a_{kj}z_i\Big)^2\Big\},\end{aligned}$$

for $j = 1,\ldots,d$. Hence

$$\begin{aligned}&\log(\psi_{\boldsymbol{B}^{\rho}}(\boldsymbol{z})) = (\log\psi_1(\boldsymbol{z}),\ldots,\log\psi_d(\boldsymbol{z})) \\ &= \Big(i\mu_1\sum_{k=1}^{n}a_{k1}z_k - \frac{1}{2}\sigma_1^2\Big(\sum_{k=1}^{n}a_{k1}z_k\Big)^2,\ldots,i\mu_d\sum_{k=1}^{n}a_{kd}z_k - \frac{1}{2}\sigma_d^2\Big(\sum_{k=1}^{n}a_{kd}z_k\Big)^2\Big),\end{aligned}$$

giving

$$\begin{aligned}&\psi_{\boldsymbol{Y}(t)}(\boldsymbol{z}) = \exp\{t\Psi_{\boldsymbol{\pi}}(\log\psi_{\boldsymbol{B}^{\rho}}(\boldsymbol{z}))\} \\ &= \exp\Big\{t\Psi_{\boldsymbol{\pi}}\Big(i\mu_1\sum_{k=1}^{n}a_{k1}z_k - \frac{1}{2}\sigma_1^2\Big(\sum_{k=1}^{n}a_{k1}z_k\Big)^2,\ldots,i\mu_d\sum_{k=1}^{n}a_{kd}z_k - \frac{1}{2}\sigma_d^2\Big(\sum_{k=1}^{n}a_{kd}z_k\Big)^2\Big)\Big\}.\end{aligned}$$

□

According to Proposition 3.1, if the subordinator $\boldsymbol{\pi}(t)$ has characteristic function in closed form, the process $\boldsymbol{Y}(t)$ has.

4 Laplace Gaussian-marked Poisson process

We consider here the multivariate generalized asymmetric Laplace (GAL) motion studied in Kozubowski et al. (2013) and widely used in Finance as Variance Gamma process (see Madan and Seneta (1990)). The GAL process can be interpreted as a subordinated Gaussian motion, where the subordinator is a one dimensional Gamma subordinator. A GAL process $\boldsymbol{Y}(t)$ with parameters $\boldsymbol{\mu} \in \mathbb{R}^n,\ \boldsymbol{\Sigma} = (\sigma_{ij}), \sigma_{ij} > 0, \gamma > 0, \beta > 0$ - shortly $GAL(\boldsymbol{\mu},\boldsymbol{\Sigma},\gamma,\beta)$ - has the following characteristic function

$$\psi_{\boldsymbol{Y}}(\boldsymbol{z}) = \left(1 - \frac{(i\boldsymbol{z}^T\boldsymbol{\mu}^{\rho} - \frac{1}{2}\boldsymbol{z}^T\boldsymbol{\Sigma}^{\rho}\boldsymbol{z})}{\beta}\right)^{-\gamma},\ \ \boldsymbol{z} \in \mathbb{R}_+^n$$

The following proposition is a direct consequence of Theorem 1.1.

Proposition 4.1. *Let $\boldsymbol{Y}(t)$ be a $GAL(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \gamma, \beta)$ process. Then $\boldsymbol{Y}(t)$ is bounded variation Gaussian-marked Poisson process as in* (2.1) *where $\nu_{\boldsymbol{\Pi}}$ is a Gamma measure with parameters (γ, β).*

We now construct a Gaussian-marked Poisson process of Laplace type which generalizes the multivariate Laplace motion and has a multivariate underlying Poisson measure $\nu_{\boldsymbol{\Pi}}$. We use a factor-based Poisson measure, i.e. the Poisson measure associated to the factor-based subordinator introduced in Semeraro (2008). A factor based subordinator $\boldsymbol{\pi}(t)$ is defined by

$$\boldsymbol{\pi}(t) := (\pi_1^I(t) + \alpha_1 \pi^C(t), \ldots, \pi_n^I(t) + \alpha_n \pi^C(t))$$

where $\pi_j^I(t)$ and $\pi^C(t)$, for $j = 1, \ldots, n$, are independent subordinators with Lévy measures ν_j^I and ν^C respectively. The multivariate Poisson random measure $\boldsymbol{\Pi}$ associated to $\boldsymbol{\pi}(t)$ is a Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^n))$ with mean $Leb \times \nu_{\boldsymbol{\Pi}}$, that we call factor-based Poisson random measure. We recall below the measure $\nu_{\boldsymbol{\Pi}}$, which is derived in Semeraro (2008). Consider a set $A \in \mathcal{B}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ and $\Delta_\alpha = \{(\alpha_1 s, \ldots, \alpha_n s)^T : s \in \mathbb{R}_+\}$ where $\alpha_j \in \mathbb{R}$ for $j \in \{1, \ldots, n\}$, and $A_j^\alpha = Pr_j(A \cap \Delta_\alpha)$, Pr_j being the projection of A on the j-th coordinate axis. Since $\frac{A_j^\alpha}{\alpha_j} = \{s \in \mathbb{R} : \alpha_j s \in A_j^\alpha\}$, and by construction $\frac{A_j^\alpha}{\alpha_j} = \frac{A_k^\alpha}{\alpha_k}$ for each $j, k \in \{1, \ldots, n\}$, we define $A_\Delta := \frac{A_j^\alpha}{\alpha_j}$ for each j. Finally, let $A_j := A \cap D_j$ having $D_j = \{\boldsymbol{x} \in \mathbb{R}^n : x_k = 0, k \neq j, k = 1, \ldots, n\}$. The Lévy measure $\nu_{\boldsymbol{\Pi}}$ is as follows

$$\nu_{\boldsymbol{\Pi}}(A) = \sum_{j=1}^n \nu_j^I(A_j) + \nu^C(A_\Delta), \quad A \in \mathcal{B}(\mathbb{R}^n \setminus \{\mathbf{0}\}). \tag{4.1}$$

Finally recall the characteristic exponent $\Psi_{\boldsymbol{\pi}(t)}$ of $\boldsymbol{\pi}(t)$, or any $\boldsymbol{w} = (w_1, \ldots, w_n)^T \in \mathbb{C}^n$ with $Re(w_j) \leq 0$ having $j = 1, \ldots, n$ is given by

$$\Psi_{\boldsymbol{\pi}}(\boldsymbol{w}) = \sum_{j=1}^n \Psi_{\pi_j^I}(w_j) + \Psi_{\pi^C}\left(\sum_{j=1}^n \alpha_j w_j\right)$$

where $\Psi_{\pi_j^I}(w_j) = \int_{\mathbb{R}_+} (e^{w_j s} - 1) \nu_{\pi_j^I}(ds)$ and $\Psi_{\pi^C}(z) = \int_{\mathbb{R}_+} (e^{zs} - 1) \nu_{\pi_C}(ds)$, $z \in \mathbb{C}$.

We now specify the factor-based Poisson measure to be of Gamma type. Let us assume that the factor based Poisson measure $\boldsymbol{\Pi}$ has Lévy measure $\nu_{\boldsymbol{\Pi}}$ defined in (4.1) such that ν_j^I are Gamma measures with parameters (γ_j, β_j) and ν^C is a Gamma measure with parameters (γ, β). Considering the properties of Gamma distribution, the j-th marginal distribution of the subordinator $\boldsymbol{\pi}(t)$ has Gamma distribution if $\beta_j = \frac{\beta}{\alpha_j}$. In this case the j-th marginal distribution of $\boldsymbol{\pi}(1)$ becomes a Gamma distribution with parameters $(\gamma_j + \gamma, \frac{\beta}{\alpha_j})$. Under these assumptions the process $\boldsymbol{Y}$ defined in (1.3) is named *multivariate Laplace marked Poisson (mLmP)* process. By applying Corollary 1.1 the characteristic function of the *mLmP* process is

$$\psi_{\boldsymbol{Y}(t)}^{mLmP}(\boldsymbol{z}) = \prod_{j=1}^n \left[1 - \frac{\alpha_j \left(i\mu_j \sum_{k=1}^n a_{kj} z_k - \frac{1}{2}\sigma_j^2 (\sum_{k=1}^n a_{kj} z_k)^2\right)}{\beta}\right]^{-\gamma_j t} \cdot \tag{4.2}$$

$$\left[1 - \frac{\sum_{j=1}^n \alpha_j \left(i\mu_j \sum_{k=1}^n a_{kj} z_k - \frac{1}{2}\sigma_j^2 (\sum_{k=1}^n a_{kj} z_k)^2\right)}{\beta}\right]^{-\gamma t}.$$

From the characteristic function (4.2) it follows that the mLmP process generalizes the GAL process, which can be derived if the idiosyncratic Poisson measures (or subordinators) degenerate. Furthermore, under suitable restrictions of the model parameters, from (4.2) follows that if the Gaussian marks have independent marginal distributions we recover the α-Variance Gamma process introduced in Semeraro (2008).

References

Barndorff-Nielsen, O. E., Pedersen, J., and Sato, K. (2001). Multivariate subordination, self-decomposability and stability. *Advances in Applied Probability*, pages 160–187.

Çınlar, E. (2011). *Probability and stochastics*, volume 261. Springer Science & Business Media.

Kotz, S., Kozubowski, T., and Podgorski, K. (2012). *The Laplace distribution and generalizations: a revisit with applications to communications, economics, engineering, and finance*. Springer Science & Business Media.

Kozubowski, T. J., Podgórski, K., and Rychlik, I. (2013). Multivariate generalized laplace distribution and related random fields. *Journal of Multivariate Analysis*, 113:59–72.

Laplace, P. (1774). Mémoire sur la probabilité des causes par les évènemens.[translated in stigler (1986).].

Madan, D. B. and Seneta, E. (1990). The variance gamma (VG) model for share market returns. *Journal of business*, pages 511–524.

Sato, K. (1999). *Lévy processes and infinitely divisible distributions*. Cambridge University Press Cambridge.

Semeraro, P. (2008). A multivariate variance gamma model for financial applications. *International Journal of Theoretical and Applied Finance*, 11(01):1–18.