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# A localization of $\Gamma$-measurability

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**Abstract**

We introduce the notion of $\Gamma$ continuity at a point $x$—where $\Gamma$ is any pointclass—and give conditions under which $\Gamma$ continuity at every $x$ is equivalent to $\Gamma$ measurability. Using this we extend the notion of the integral of a measurable function. Also we examine the case $\Gamma = \Sigma^0_\xi$, where $(\Sigma^0_\xi)_{\xi<\omega_1}$ is the usual ramification of the class of Borel sets, see [A.S. Kechris, Classical Descriptive Set Theory, Springer-Verlag, 1994].


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## 1. Introduction

With the letters $X$ and $Y$ we always mean metric spaces, unless stated otherwise. Also with the term "pointclass" we mean a collection of sets in arbitrary spaces, for example, the class of open sets. If $\Gamma$ is a pointclass and $X$ is a metric space we denote with $\Gamma(X)$ the family of the subsets of $X$ which are in $\Gamma$.

Recall that if $\Gamma$ is an arbitrary pointclass and $f$ is a function from $X$ to $Y$, we call $f$ $\Gamma$ *measurable* iff for each open $G \subseteq Y$ the inverse image $f^{-1}[G]$ is also in $\Gamma$ [1].

We will consider some special cases for $\Gamma$. First define the family $\Sigma^0_1(X)$ as the collection of all open subsets of $X$. Put also $\Pi^0_1(X) = \{A \subseteq X / X \setminus A \in \Sigma^0_1(X)\}$, i.e. the family $\Pi^0_1(X)$ is the collection of closed subsets of $X$.

By transinfinite recursion we define for each $\xi < \omega_1$ [1]

$$\Sigma^0_\xi(X) = \left\{\bigcup_{n\in\omega} A_n / \text{ where } A_n \in \Pi^0_{\xi_n}(X) \text{ for some } \xi_n < \xi, \forall n \in \omega\right\}$$

and

$$\Pi^0_\xi(X) = \left\{A \subseteq X / X \setminus A \in \Sigma^0_\xi(X)\right\}.$$

Put also $\Delta^0_\xi(X) = \Sigma^0_\xi(X) \cap \Pi^0_\xi(X)$, for each $\xi < \omega_1$.

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It is well known that $\Delta^0_\xi(X) \subseteq \Sigma^0_\xi(X) \subseteq \Delta^0_{\xi+1}(X)$ for each $\xi < \omega_1$ and that $\bigcup_{\xi<\omega_1} \Sigma^0_\xi(X) = \bigcup_{\xi<\omega_1} \Pi^0_\xi(X) = \mathcal{B}(X)$, where $\mathcal{B}(X)$ is the Borel $\sigma$-algebra on $X$.

Recall that a topological space $X$ is called *Polish* iff it is separable and metrizable by some metric $d$ such that $(X, d)$ is complete. If $X$ is a perfect Polish space (i.e. a Polish space with no isolated points), then for each $\eta < \xi < \omega_1$ there exists a set $A \in \Delta^0_\xi(X) \setminus \Delta^0_\eta(X)$. With $\Sigma^0_\xi$ we mean the class of all sets which belong to $\Sigma^0_\xi(X)$ for some $X$.

## 2. $\Gamma$ continuity

We now give a notion which is closely connected to $\Gamma$ measurability.

**Definition 2.1.** Let $\Gamma$ be an arbitrary class of sets, a function $f : X \to Y$ and $x \in X$. The function $f$ is called $\Gamma$ *continuous at* $x$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that the set $f^{-1}[S(f(x), \varepsilon)] \cap S(x, \delta)$ is in $\Gamma$.

Also say that $f$ is $\Gamma$ *continuous* iff $f$ is $\Gamma$ continuous at every $x \in X$.

[Of course with $S(f(x), \varepsilon)$ we mean the set $\{y \in Y / p(f(x), y) < \varepsilon\}$ where $p$ is the metric of $Y$. The same for $S(x, \delta)$.]

We will denote the set $f^{-1}[S(f(x), \varepsilon)] \cap S(x, \delta)$ with $A(x, \varepsilon, \delta)$ or simpler with $A(\varepsilon, \delta)$.

We first concentrate on the case where $\Gamma = \Sigma^0_\xi$, for some $\xi < \omega_1$. The following are easy consequences of the definitions.

**Remark 2.2.**

(I) Assume that $\Gamma$ contains the class of open sets and is closed under finite intersections. If $f : X \to Y$ is $\Gamma$ measurable then $f$ is $\Gamma$ continuous.

(II) A function $f : X \to Y$ is $\Sigma^0_1$ continuous exactly when it is continuous, or equivalently iff it is $\Sigma^0_1$ measurable.

(III) Let $\Sigma^0_\xi$ measurable functions $f, g : X \to \mathbb{R}$ and $\lambda \in \mathbb{R}$. Then the functions $f + g$, $\lambda \cdot f$, $f \cdot g$, $|f|$, $\max\{f, g\}$ and $\min\{f, g\}$ are also $\Sigma^0_\xi$ measurable.

(IV) If $\eta < \xi < \omega_1$ and $f : X \to Y$ is $\Sigma^0_\eta$ measurable (or $\Sigma^0_\xi$ continuous at some $x \in X$), then $f$ is also $\Sigma^0_\xi$ measurable (respectively, $\Sigma^0_\xi$ continuous at $x$).

**Example 2.3.**

(I) Define $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 0, & \text{if } x \leqslant 0, \\ 1, & \text{if } x > 0. \end{cases}$$

For each $\varepsilon > 0$ put $\delta = 1 > 0$. Then $A(0, \varepsilon, 1) = (x - 1, 0]$ for each $x \in \mathbb{R}$. Hence $A(0, \varepsilon, 1) \in \Sigma^0_2(\mathbb{R})$. Therefore $f$ is $\Sigma^0_2$ continuous at 0, but not continuous at 0.

(II) Let $f : \mathbb{R} \to \mathbb{R}$ be the Dirichlet function, i.e.

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

For $x \in \mathbb{Q}$ and $\varepsilon > 0$ the set $A(x, \varepsilon, 1)$ is one of the sets $(x - 1, x + 1) \cap \mathbb{Q}$, $\mathbb{R}$. In either case $A(x, \varepsilon, 1)$ is an $F_\sigma$ set and therefore $f$ is $\Sigma^0_2$-continuous on every rational $x$. Also for each rational $x$ and for each $\delta > 0$ the set $A(x, \frac{1}{2}, \delta) = (x - \delta, x + \delta) \cap \mathbb{Q}$ is not a $G_\delta$ set. Hence $f$ is not $\Pi^0_2$-continuous at $\mathbb{Q}$.

With analogous arguments the Dirichlet function is $\Pi^0_2$ but not $\Sigma^0_2$ continuous at every $x \notin \mathbb{Q}$.

It is interesting to distinguish $\Sigma^0_\xi$ continuity from $\Sigma^0_\eta$ continuity. This is fairly easy to do at this point. However, we will obtain this as a result of Theorem 2.5.

According to Remark 2.2, $\Sigma^0_\xi$ measurability implies $\Sigma^0_\xi$ continuity. Of course the inverse is also true in the case of $\xi = 1$. This is because the class of $\Sigma^0_1$ sets is closed under arbitrary unions. However the latter does not hold for the general case; so we cannot refer to an analogous proof in order to establish the inverse in the case of an arbitrary $\xi$.

On the other hand, in some topological spaces we can replace arbitrary unions of open sets with countable ones, i.e. those spaces which satisfy the *Lindelöf property*. It is well known that separable metric spaces are among those spaces. This remark will help us prove that $\Sigma^0_\xi$ measurability is equivalent to $\Sigma^0_\xi$ continuity in separable metric spaces.

**Lemma 2.4.** *Let $(X,d)$ and $(Y,p)$ be separable metric spaces and a function $f : X \to Y$. For each open $G \subseteq Y$ which is bounded (i.e. $\sup_{x,y\in G} p(x,y) < \infty$) there exists a family $(\varepsilon_x)_{x\in f^{-1}[G]}$ of positive reals, such that for every family $(\delta_x)_{x\in f^{-1}[G]}$ of positive reals there exists a countable $I \subseteq f^{-1}[G]$ with*

$$f^{-1}[G] = \bigcup_{x\in I} A(x,\varepsilon_x,\delta_x)$$

*(where $A(x,\varepsilon_x,\delta_x) = f^{-1}[S(f(x),\varepsilon_x)] \cap S(x,\delta_x)$).*

Roughly speaking this lemma transfers arbitrarily large unions in which we may concern, into countable unions.

**Proof.** Let $G \subseteq Y$ open and bounded. Then for each $x \in f^{-1}[G]$ there exists $r > 0$ such that $S(f(x),r) \subseteq G$. Put $\varepsilon_x = \sup\{r > 0 / S(f(x),r) \subseteq G\} > 0$. Then $\varepsilon_x \in \mathbb{R}$ since $G$ is bounded.

One can check that $S(f(x),\varepsilon_x) \subseteq G$ for each $x \in f^{-1}[G]$.

Take $(\delta_x)_{x\in f^{-1}[G]}$ any family of positive reals.

Put $\mathcal{A} = \{S(f(x),\frac{\varepsilon_x}{4}) / x \in f^{-1}[G]\}$. From the Lindelöf property of $Y$ there exists a sequence $(x_n)_{n\in\omega}$ of elements of $f^{-1}[G]$ such that $\bigcup_{n\in\omega} S(f(x_n),\frac{\varepsilon_{x_n}}{4}) = \bigcup \mathcal{A}$.

For each $n \in \omega$ put $B_n = \{x \in X / f(x) \in S(f(x_n),\frac{\varepsilon_{x_n}}{4})\} \subseteq f^{-1}[G]$ and $\mathcal{B}_n = \{S(x,\delta_x) / x \in B_n\}$. From the Lindelöf property of $X$ there exists a sequence $(x^n_k)_{k\in\omega}$ of elements of $B_n$ such that $\bigcup_{k\in\omega} S(x^n_k,\delta_{x^n_k}) = \bigcup \mathcal{B}_n$, for each $n \in \omega$.

Define $I = \{x^n_k / n,k \in \omega\}$.

Observe that if $x = x^n_k$ for some $n,k \in \omega$, then from the definition of $\varepsilon_x$ it follows that $\varepsilon_{x_n} \leqslant 2\varepsilon_x$. Using this remark it is easy to verify that

$$f^{-1}[G] = \bigcup_{x\in I} A(x,\varepsilon_x,\delta_x). \qquad \square$$

**Theorem 2.5.** *Let $X$, $Y$ be separable metric spaces and a function $f : X \to Y$.*

(I) *If there exists a function $\xi : X \to \omega_1$ such that for each $x \in X$ the function $f$ is $\Sigma^0_{\xi(x)}$ continuous at $x$, then $f$ is $\Sigma^0_\zeta$ measurable for some countable ordinal $\zeta \leqslant \sup_{x\in X} \xi(x)$.*

(II) *For each $\xi < \omega_1$, $f$ is $\Sigma^0_\xi$ measurable exactly when it is $\Sigma^0_\xi$ continuous at every $x \in X$.*

**Proof.** It is enough to prove (I) since the second assertion follows immediately from the first.

Let $\{G_n / n \in \omega\}$ be a countable basis for the topology of $Y$. Since $Y$ satisfies the Lindelöf property we may assume that each $G_n$ is an open ball and hence bounded (with respect to the metric of $Y$).

For each $n \in \omega$ choose a family of positive reals $(\varepsilon^n_x)_{x\in f^{-1}[G_n]}$ as in Lemma 2.4.

Now fix some $n \in \omega$. For each $x \in f^{-1}[G_n]$, $f$ is $\Sigma^0_{\xi(x)}$ continuous at $x$. So for the $\varepsilon^n_x > 0$, there exists $\delta_x > 0$ with $A(x,\varepsilon^n_x,\delta_x) \in \Sigma^0_{\xi(x)}$. Choose a family $(\delta_x)_{x\in f^{-1}[G_n]}$ of those $\delta$'s. From Lemma 2.4 there exists a countable set $I \equiv I_n \subseteq f^{-1}[G_n]$ such that

$$f^{-1}[G_n] = \bigcup_{x\in I_n} A\big(x,\varepsilon^n_x,\delta_x\big). \qquad (*)$$

If we let $\zeta_n = \sup_{x\in I_n} \xi(x) \leqslant \sup_{x\in X} \xi(x)$ then $\zeta_n$ is a countable ordinal and $A(x,\varepsilon^n_x,\delta_x) \in \Sigma^0_{\zeta_n}$ for each $x \in I_n$. So $f^{-1}[G_n] \in \Sigma^0_{\zeta_n}$.

Putting $\zeta = \sup_{n\in\omega} \zeta_n$, where $\zeta_n$ is as above, we obtain the result. $\square$

The preceding theorem allows to view—in separable metric spaces—the notion of $\Sigma_\xi^0$ measurability in a "local way", since $\Sigma_\xi^0$ continuity is a local meaning.

This proof applies also to other pointclasses. Let us first recall the class of sets with the *Baire property*. If $A$ is a subset of a topological space $X$, we say that $A$ has the Baire property iff there exists some open $U$ such that the symmetric difference $(A \setminus U) \cup (U \setminus A)$ is meager in $X$.

We denote the class of the sets with the Baire property with *BP*. It is well known that for any topological space $X$, the family $BP(X)$ forms the least $\sigma$-algebra on $X$ which contains all open and all meager sets.

**Corollary 2.6.** *Let $X, Y$ be separable metric spaces and a function $f : X \to Y$. Also let a pointclass $\Gamma$ which contains the open sets and it is closed under countable unions and finite intersections.*

*Then $f$ is $\Gamma$ measurable exactly when it is $\Gamma$ continuous at every $x \in X$.*

*In particular $f$ is BP measurable, i.e. Baire measurable (or $\Sigma_\xi^0$ measurable) exactly when it is BP continuous at every $x \in X$ (respectively, $\Sigma_\xi^0$ continuous at every $x \in X$).*

**Proof.** Proceed as in the proof of Theorem 2.5. □

We now give an example distinguishing $\Sigma_\xi^0$ continuity from $\Sigma_\eta^0$ continuity, for each $\eta < \xi < \omega_1$, using Theorem 2.5.

**Example 2.7.** Let $X$ be a perfect Polish space, $\eta < \xi < \omega_1$ and $A$ in $\Delta_\xi^0(X)$ but not in $\Delta_\eta^0(X)$.

Then the characteristic function of $A$, $\chi_A$ is easily $\Sigma_\xi^0$ measurable (and hence $\Sigma_\xi^0$ continuous) but not $\Sigma_\eta^0$ measurable. From the previous theorem there exists some $x \in X$ such that $f$ is not $\Sigma_\eta^0$ continuous at $x$. Therefore $f$ is $\Sigma_\xi^0$ continuous but not $\Sigma_\eta^0$ continuous.

We conclude with an application of Corollary 2.6. Let us begin with some notations. If $X$ is a metric space, $A$ is a subset of $X$ and $\Gamma$ is a pointclass denote with $\Gamma_A$ the family $\{G \cap A / G \text{ is a subset of } X \text{ and in } \Gamma\}$. Also for a function $f : X \to Y$ put $X(f, \Gamma) = \{x \in X / \text{ the function } f \text{ is } \Gamma \text{ continuous at } x\}$. Finally for $A \subseteq X$ denote with $\chi_A$ the characteristic function of $A$.

Let $X$ be a separable metric space and a function $f : X \to \mathbb{R}$. We will consider the case where $\Gamma$ is a $\sigma$-algebra $\mathcal{M}$ which contains the open sets. Also let some $A \subseteq X(f, \mathcal{M}) \equiv X_f$. Then for each $x \in A$ the function $f$ is $\mathcal{M}$ continuous at $x$. It is clear that the function $f \restriction A$ (i.e. the restriction of $f$ on $A$), is $\mathcal{M}_A$ continuous at every $x \in A$. Now regard $A$ as a metric space. From Corollary 2.6 it follows that the function $f \restriction A$ is $\mathcal{M}_A$ measurable. If furthermore $A \in \mathcal{M}$ one can verify that the function $f \cdot \chi_A$ is $\mathcal{M}$ measurable.

So if $\mu$ is a measure on $(X, \mathcal{M})$ the integral $\int_A f \, d\mu$ is well defined for each $A \in \mathcal{M}$ with $A \subseteq X_f$.

**Definition 2.8.** Let $X$ be a separable metric space, $\mathcal{M}$ be a $\sigma$-algebra on $X$ which contains the open sets and $\mu$ a measure on $(X, \mathcal{M})$. Let also a non-negative function $f : X \to \mathbb{R}$. Define the *partial integral* of the function $f$ as follows:

$$\int^p f \, d\mu = \sup\left\{\int_A f \, d\mu / A \in \mathcal{M} \,\&\, A \subseteq X_f\right\}.$$

**Remark 2.9.** Notice that if $X_f$ is in $\mathcal{M}$ since $f$ is non-negative we have that $\int^p f \, d\mu = \int_{X_f} f \, d\mu$. The set $X_f$ is sort of speak the "largest" set on which the function $f$ behaves like a measurable function. It would be interesting to find conditions under which the set $X_f$ is in $\mathcal{M}$.

If furthermore $X_f = X$ then the partial integral coincides with the usual integral. Thus this new notion is indeed a generalization of the classic one.

Of course we may extend the previous notion to a not necessarily non-negative function with the usual way. For an arbitrary function $f : X \to \mathbb{R}$ let $\int^p f \, d\mu = \int^p f^+ \, d\mu - \int^p f^- \, d\mu$, where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. (In case of $\infty - \infty$ we define the partial integral to be 0.)

It would be interesting also to examine if the classic theorems of the usual integral can be transferred to the partial integral, see [2].

**Example 2.10.** Here we give an example of a partial integral which is reduced to the Lebesgue integral on a Cantor-type set. Define $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$ and (using the Axiom of Choice) let $A$ be a set which contains exactly one member of each equivalence class. Let $\mathcal{M}$ be the Lebesgue $\sigma$-algebra on $\mathbb{R}$ and $\mu$ be the Lebesgue measure on $(\mathbb{R}, \mathcal{M})$. It is well known that $A \notin \mathcal{M}$, in fact for every set $U \subseteq \mathbb{R}$, if $U \cap A \in \mathcal{M}$ then $\mu(U \cap A) = 0$. This is because if $\mu(U \cap A) > 0$ then from the Steinhaus theorem there exists $\delta > 0$ such that $(-\delta, \delta) \subseteq (U \cap A) - (U \cap A) \subseteq A - A$, a contradiction.

Let $\mathcal{B}$ be the family of all open intervals $(x - \delta, x + \delta)$ for which $A^c \cap (x - \delta, x + \delta) \in \mathcal{M}$, where $A^c$ stands for the complement of $A$ in $\mathbb{R}$. If $\mathcal{B} = \emptyset$ put $J = \emptyset$, otherwise choose a countable family $(I_n)_{n\in\omega}$ of the previous intervals which covers $\mathcal{B}$ and put $J = \bigcup_{n\in\omega} I_n$. In any case we have that $A^c \cap J \in \mathcal{M}$ and if $A^c \cap (x - \delta, x + \delta) \in \mathcal{M}$ then $(x - \delta, x + \delta) \subseteq J$.

Define $X = \mathbb{R} \setminus J$. Then $X$ is a closed set and furthermore $\mu(X) > 0$. Otherwise the set $A^c \cap J^c$ would $\mu$-null and thus in $\mathcal{M}$. Hence $A^c = (A^c \cap J) \cup (A^c \cap J^c) \in \mathcal{M}$, a contradiction. It is well known that we can find a Cantor type set $C \subseteq X$ for which $\mu(C) > 0$.

Let $\mathcal{M}_X$ be the restriction of $\mathcal{M}$ on $X$, i.e. $\mathcal{M}_X = \{M \cap X / M \in \mathcal{M}\}$. Also let $\mu_X$ be the restriction of $\mu$ on $\mathcal{M}_X$. Since $X$ is closed and thus in $\mathcal{M}$ it is clear that some $B \subseteq X$ is in $\mathcal{M}_X$ if and only if $B \in \mathcal{M}$.

Define the function $g : X \to \mathbb{R}$ by $g(x) = \inf\{|x - y| / y \in C\}$. The function $\Gamma$ is continuous non-negative and $g^{-1}[\{0\}] = C$.

Now define $f : X \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} g(x) + 1, & \text{if } x \in A, \\ 1, & \text{if } x \notin A. \end{cases}$$

If $x \in C$, i.e. $g(x) + 1 = 1$ it is clear that $f$ is continuous (and hence $\mathcal{M}_X$ continuous) at $x$. For simplicity put $X_f = X(f, \mathcal{M}_X)$. Let now some $x \in X_f \setminus C$. Assume furthermore that $x \in A$. Then $g(x) > 0$ and $f(x) = g(x) + 1$. Put $\varepsilon = \frac{g(x)}{2}$. Notice that if $y \notin A$ then $f(y) = 1 < g(x) + 1 - \varepsilon = f(x) - \varepsilon$. Hence if $f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon)$ then $y \in A$. It follows that

$$\begin{aligned} f^{-1}\big[\big(f(x) - \varepsilon, f(x) + \varepsilon\big)\big] &= f^{-1}\big[\big(f(x) - \varepsilon, f(x) + \varepsilon\big)\big] \cap A \\ &= h^{-1}\big[\big(h(x) - \varepsilon, h(x) + \varepsilon\big)\big] \cap A, \end{aligned}$$

where $h = g + 1$.

Since $x \in X_f$ there exists some $\delta > 0$ such that $f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)] \cap (x - \delta, x + \delta) \in \mathcal{M}_X \subseteq \mathcal{M}$. Also we have that

$$\begin{aligned} & f^{-1}\big[\big(f(x) - \varepsilon, f(x) + \varepsilon\big)\big] \cap (x - \delta, x + \delta) \\ &\quad = h^{-1}\big[\big(h(x) - \varepsilon, h(x) + \varepsilon\big)\big] \cap A \cap (x - \delta, x + \delta) \\ &\quad = U_x \cap A, \end{aligned}$$

where $U_x$ is open in $X$ such that $x \in U_x$. Therefore $U_x \cap A \in \mathcal{M}_X \subseteq \mathcal{M}$ and thus $\mu(U_x \cap A) = 0$. Repeat the same procedure for each $x \in (X_f \setminus C) \cap A$ in order to get the previous set $U_x$. Using the Lindelöf property of $X$ we find a sequence $(x_n)_{n\in\omega}$ in $(X_f \setminus C) \cap A$ such that $(X_f \setminus C) \cap A = \bigcup_{n\in\omega} U_{x_n} \cap A$. It follows that the set $(X_f \setminus C) \cap A$ is $\mu$-null and thus it belongs in $\mathcal{M}$.

Now towards a contradiction assume that there exists some $x \in X_f \setminus C$ which is not in $A$. Then $f(x) = 1$ and $g(x) > 0$. Put $\varepsilon = \frac{g(x)}{2} > 0$. Since $g$ is continuous there exists some $\delta_0$ such that for all $y \in (x - \delta_0, x + \delta_0) \cap X$ we have that $g(y) > \frac{g(x)}{2}$. Also $x \in X_f$ hence there exists some $\delta > 0$ such that $f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)] \cap (x - \delta, x + \delta) \in \mathcal{M}_X \subseteq \mathcal{M}$. We may assume that $\delta \leqslant \delta_0$. Thus if $y \in (x - \delta, x + \delta) \cap X$ then $f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon) \Leftrightarrow y \notin A$. It follows that $f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)] \cap (x - \delta, x + \delta) = A^c \cap (x - \delta, x + \delta) \cap X = A^c \cap J^c \cap (x - \delta, x + \delta) \in \mathcal{M}$.

Since $A^c \cap J \in \mathcal{M}$ it follows that $A^c \cap J \cap (x - \delta, x + \delta) \in \mathcal{M}$ as well. Therefore $A^c \cap (x - \delta, x + \delta) = [A^c \cap J^c \cap (x - \delta, x + \delta)] \cup [A^c \cap J \cap (x - \delta, x + \delta)] \in \mathcal{M}$. From the definition of $J$ it follows that $(x - \delta, x + \delta) \subseteq J$. This is a contradiction since $x \in X = \mathbb{R} \setminus J$.

Therefore $X_f \setminus C = (X_f \setminus C) \cap A$ is $\mu$-null and thus in $\mathcal{M}$. Since $X_f \setminus C \subseteq X$ we have that $X_f \setminus C \in \mathcal{M}_X$. Furthermore $C$ is closed in $X$ and it is also a subset of $X_f$, hence $X_f \in \mathcal{M}_X$.

Thus $\int^p f \, \mathrm{d}\mu_X = \int_{X_f} f \, \mathrm{d}\mu_X = \int_C f \, \mathrm{d}\mu_X = \int_C 1 \, \mathrm{d}\mu_X = \mu_X(C) = \mu(C) > 0$.

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