Constructing meanings of fraction with MLD[1] students

Elisabetta Robotti, e.robotti@univda.it

Università della Valle d'Aosta-Université de la Vallée d'Aoste

Abstract

There is consensus among researches that fractions are among the most complex mathematical concepts that children encounter in their years in primary education. Factors contributing to the complexities of teaching and learning of fractions lies in the fact that they comprise a multifaceted construct (Charalambous and Pitta-Pantazi, 2005) encompassing five interrelated sub-constructs: part-whole, ratio, operator, quotient and measure. However, while part-whole sub-construct is strongly linked to ratio and operator, it is weakly linked to the measure and quotient sub-construct. Our aim is to present a didactical sequence that fosters the development of meanings of fractions related to the relationship between part-whole and measure and between part-whole and ratio, in order to conceive fractions as numbers that can be placed on the number line. The didactical sequence is addressed to elementary school classes that are "inclusive" with respect to students with mathematical learning disorders (MLD).

Résumé

Il y a un consensus parmi les recherches que les fractions sont parmi les concepts mathématiques les plus complexes que les enfants rencontrent dans l'enseignement primaire. Les facteurs qui contribuent à la complexité de l'enseignement et de l'apprentissage des fractions réside dans le fait qu'ils comprennent une construction à multiples facettes (Charalambous et Pitta-Pantazi, 2005) comprennent cinq interdépendants sous-constructions: partie-tout, rapport, opérateur, quotients et mesure. Néanmoins, alors que la sous-construction de partie-tout est fortement liée à pourcentage et l'opérateur, elle est faiblement liée aux sous-constructions de mesure et quotient. Notre objectif est de présenter une séquence didactique qui favorise le développement du sens de fraction lié à la relation entre partie-tout et mesure et entre partie-tout et rapport, afin de concevoir les fractions en tant que nombres rationnelles qui peuvent être placés sur la ligne de nombres. La séquence didactique est adressée aux élèves de l'école primaire y compris les élèves ayant des troubles d'apprentissage mathématiques.

Conceptual framework

This research is based on a range of different perspectives, from mathematics education to neuroscience and cognitive psychology. I discuss how such perspectives can be combined and provide the theoretical bases to design the didactical sequence, which allows implementing inclusive education. The main ideas I taken into account from cognitive psychology is that mathematical achievement depends on short-term memory (STM) and working memory (WM) (Raghubar et al. 2010). Moreover, it depends on non-verbal intelligence, addressed to general cognition without reference to the language ability (DeThorne & Schaefer, 2004). These findings suggest that non-verbal intelligence may partially depend on spatial skills (Rourke & Conway, 1997) and, these last, can be potentially important in mathematical performances, where explicit or implicit visualization is required. Moreover, research in cognitive science (Stella & Grandi, 2012) has identified specific and preferential channels of access and elaboration of information. For students with MLD these are the visual non-verbal, the kinaesthetic-tactile and/or the auditory channels at the expense of the verbal channel. Students who come to prefer the visual non-verbal channel, tend to appreciate and elaborate visual-spatial representations, and are best at memorizing images, symbols, graphs, diagrams (Miller, 1987). Studies in mathematics education as well,

[1] Mathematical Learning Disorders: a discrepancy between low arithmetical abilities and overall intelligence level and chronological age; difficulty in acquiring formal arithmetic operations and arithmetic facts (Classification systems of developmental disorders: the ICD-10 and theDSM-IV)

although with different conceptual frameworks, have highlighted how sensory-motor, perceptive, and kinaesthetic experiences are fundamental for the formation of mathematical concepts – even highly abstract ones (Arzarello, 2006; Radford, 2003). For example, Arzarello (2006) points to how recent research in math education underline that the construction of mathematical knowledge, as cognitive activity, is supported by the sensori-motor system activated in suitable contexts. Also Radford (2006) highlights that the understanding of the relationship among body, actions carried out through artefacts (objects, technological tools, etc.), and linguistic and symbolic activity is essential in order to get the human cognition and mathematical thinking in particular.

Main difficulties in teaching-learning fractions

As highlighted by Hannula (2003), the multiplicity of interpretations and applications of fractions is generally not reflected in a corresponding variety of representations and problems presented within educational activities. The most used approaches are: regularly shaped regions divided into equal parts of which some are distinguished, and the number line. This highlight that the sub-constructs linked to the notion of fraction are not really interrelated and there are some of them more developed of the other (for instance, part-whole sub-construct) as well as some stereotype representations (such as shaped regions divided into equal parts). As we will show in the following, this kind of representation can be a useful starting point to develop the notion of fractions but it cannot be considered the only one representation of fraction. For this reason we will take into account other kinds of representations and other sub-construct linked to part-whole sub-construct and to measure. As matter of fact, a limited set of representations can lead to difficulties. Hunnula (2003) found that the number line was more problematic then other kind of representation and postulated that the earlier experience of students led them to look for something divided into m parts of which they could take n, but without having a clear idea of what the appropriate "whole" would be. Moreover, "Many students indiscriminately identify a presentation of m indicated out of n parts of a region as a representation of m/n. Such a limited "expertise" is manifest in widely documented behaviors such as accepting a representation of m out of n unequal parts (Newstead, Olivier, 1999), not accepting that m out of n equal parts can also represent any equivalent fraction (Carraher, Schluemann, 1991), failure to grasp the interpretation of fractions as numbers (Amato 2005)" (Verschaffel, Green and Torbeyns, 2006, pag 76).

This leads students to have difficulties concerning the ordering of fractions on number line. For instance, assuming that the properties of ordering natural numbers can be extended to ordering fractions (e.g. assuming that the product/quotient of two fractions makes a greater/smaller fraction), or positioning fractions on the number line using the pattern of whole numbers (Iuculano & Butterworth, 2011). As matter of fact, frequently, at least in Italian education, the conception of fraction is not explicitly identified as a rational number. Only when it is transformed into a decimal number is it placed on the number line. Fractions constitute an important leap within domain of arithmetic because they represent a first approach towards the idea of extension of the set of Natural Numbers. In this sense, fractions need to assume a specific position on the number line (Bobis et al., 2013).

Methodology, sequence of activities and main results

The sequence of activities was designed by 22 primary school teachers and 1 supervisor (the author) composing a study group. The activities were carried out during a pilot experimentation, which involved 22 classes (nine 5th grade classes, six 4th grade classes and seven 3rd grade classes), before being revised for an upcoming full-blown study. In this paper I will report on the pilot experimentation carried out in the 3rd grade classes. The activities asked to work with different artefacts: A4 sheets of paper, squared-paper strips or represented squared strips in notebooks, the number line represented in notebook and a string on the wall. As described below, the teacher guided the use of these artefacts by focused tasks. The activities are clustered in three main groups: partitioning of the A4 sheet of paper, partitioning of a strip of squared paper, and placing fractions on the number line. I focus here on the first and second activities.

Activity 1: Partitioning of the A4 sheet of paper

The aim of this activity was the introduction of "equivalent fractions" as equivalent surfaces, and of "sum of unit fractions" for obtaining the whole (the chosen unit, that is, the A4 sheet).

Teacher asks students to:

- Partitioning colored A4 sheets in equal parts by folding and using the ruler. Each color corresponds to a unit fraction (Fig. 1);

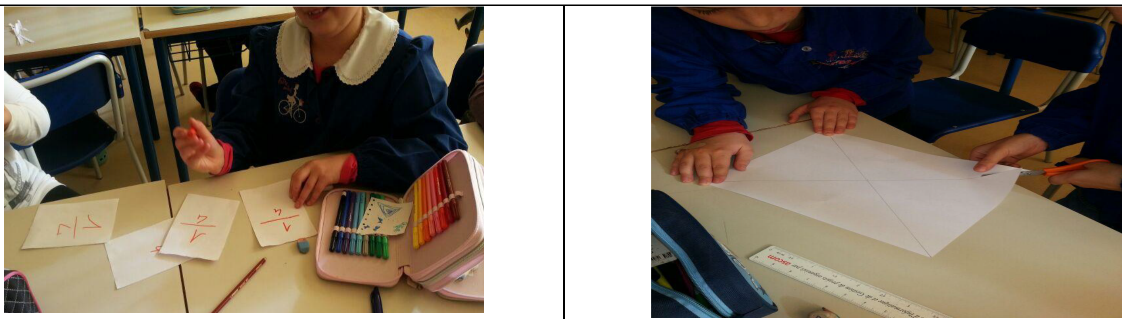


Fig. 1 Partitioning A4 sheets in equal parts by folding and using the ruler. In these images, the A4 sheet is not a coloured sheet. This educational choice is adopted in the educational activities of other classes.

- Put in a box marked with the unit fraction's label (box of the 1/2 unit fractions, box of the 1/4 unit fractions,…) the unit fractions obtained by each student;

- Compare the different shapes of each unit fraction and verifying the equivalence, introducing the "equivalent fractions" as equivalent surfaces, by a "cutting and recomposing" strategy. This activity allows students to overcome the idea that regions congruent are the only representations of equivalent fractions considering also representations of equivalent regions (Fig. 2).

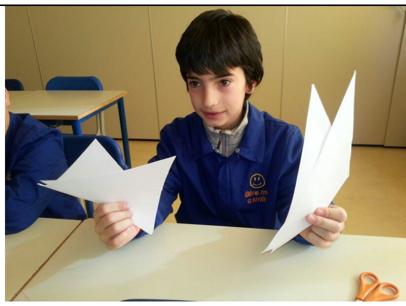
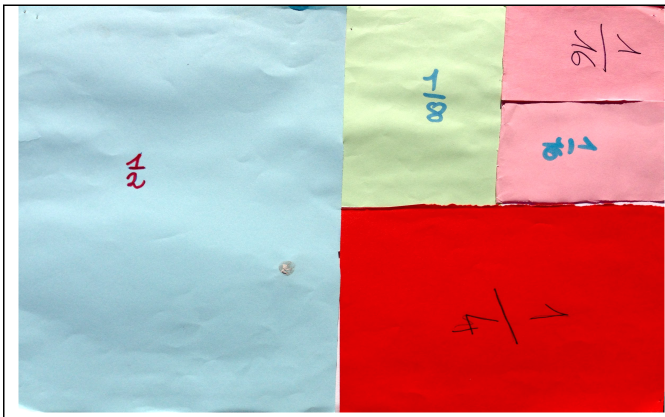


Fig. 2 Equivalent regions represent equivalent unit fractions.

- Cover a A4 white sheet with different unit of fractions (Fig.3) taken from the unit fractions' boxes. The task requires the use of a procedure in which the fraction is conceived as part-whole, where the "whole" is the A4 sheet of paper and the part is the unit fraction.



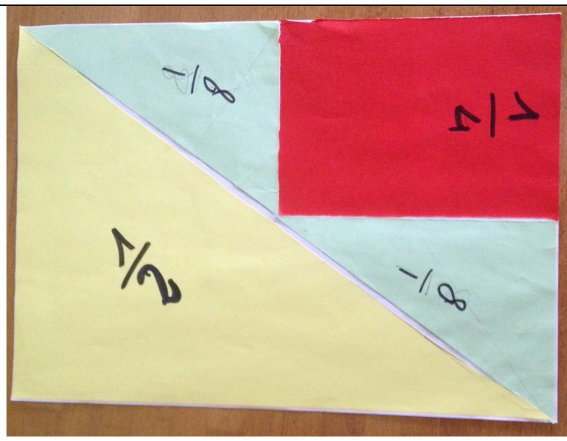


Fig. 3 The A4 sheets of paper is covered by different unit of fractions. Thus, summing unit fractions, students obtain the whole. Note that *equivalent fractions* are *equivalent surfaces*

The "sum of unit fractions" for obtaining the whole (A4 sheet) is a crucial task both to introduce the meaning of sum of unit measures, and to foster the conceptualization of unit fractions as independent to their shape (as congruent regions). Note that the artifacts used in this activity exploit the preferential channels of access to information for MLD students: visual non-verbal and kinaesthetic-tactile.

Activity 2: Partitioning of a strip of squared paper

These activities are clustered into three sessions.

In *Session A*, the aim was comparing unit fractions by representing them on different squared strips. Thus, given a certain unit of measure, the teacher asks students to position it on different strips (concrete strips, Figure 3, or represented on the notebook, Figure 4) and to position the unit fractions 1/4, 1/2, 1/8 each on a strip. The task requires the use of a procedure in which the fraction is conceived as measure (distance from zero): considered the unit of measure, it is divided into 2 or 4 or 8 equall parts; each of them is considered as unit fraction. The manipulation of these artifacts is prevalently a perceptive experience, developed by kinaesthetic-tactile and non-verbal visual channel.

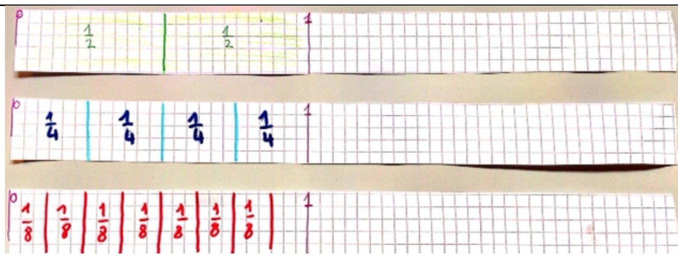


Fig. 3 Three unit fractions 1/4, 1/2, 1/8, represented each one on a different concrete strips of squared paper, are compared in visual and perceptive way.

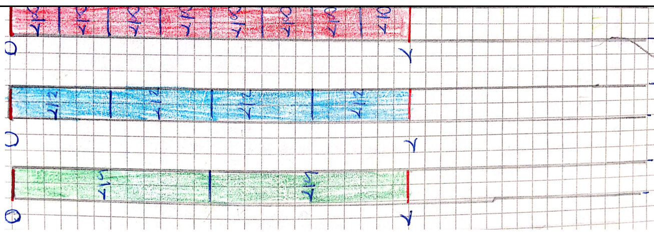


Fig. 4 Three unit fractions 1/4, 1/2, 1/8, represented each one on a different strips drawn on notebook, are compared in visual way.

Students produce linguistic signs associated to the name of the fraction expressed in verbal language (“Un mezzo” – tr. “One half”), in verbal visual language (the writing “un mezzo”- tr. One haf) and arithmetical language (1/2). The teacher institutionalizes the relationship between the different signs (partitions of the strips, visual verbal, visual non verbal, and arithmetical signs) in terms of rational numbers. Note that the task was completed by all groups of students

The aim of the *Session B* was to introduce the dependence of the fraction on the unit of measure. The students are clustered in groups and the teacher asks them to choose a unit of measure and to reproduce it on their own strip. Then, she asks to place the fraction ½ on the strip; Students observe the dependence of the fraction on the unit of measure by comparing the results of the different groups (Figure 5). Note that the previous kinaestetyc-tactile approach is no longer an effective strategy in order to compare the results. Now is necessary managing the meaning of fraction as measure.

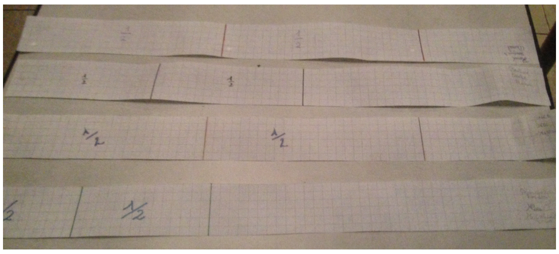


Fig. 5 Strips of squared paper where students have choosen different unit of measure and designed the unit fraction ½.

In the *Session C* the main aim was to introduce *lcm* of denominators and ordering unit fractions on the same strip. Thus, chosen an appropriate unit of measure (in this case, 24 squares), the teacher asks students to position it on the strip and to represent the following unit fractions 1/3, 1/6, 1/8, 1/2, 1/4 (Figure 6). We observed that students did not simply looked for the unit of measure spontaneously, generally using trial and error methods (cm), but they also checked the efficiency of their choice (lcm). Moreover, positioning on a single strip different fractions, makes the ordering of fractions exactly like that of the other perceptively evident numbers.

The task supports the overcoming of the epistemological and cognitive obstacle concerning the positioning of fractions on the number line using the pattern of whole numbers (Iuculano & Butterworth, 2011, Bartolini Bussi et al., 2013). Note that here the labels are referred to points on the strip and not to area as before. The color becomes a tool supporting working memory and possibly also long term memory, through which the meanings developed can be recalled and used.

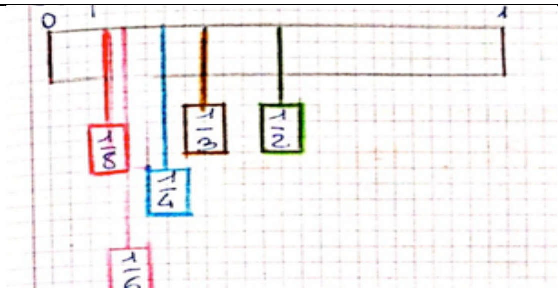


Fig. 6 Strip drown by students choosing the appropriate unit of measure in order to represent on it

the unit of fractions 1/3, 1/6, 1/8, 1/2, 1/4

This activity is functional to the ordering of fractions on the number line, as we can observe in the following figure (Figure 7).

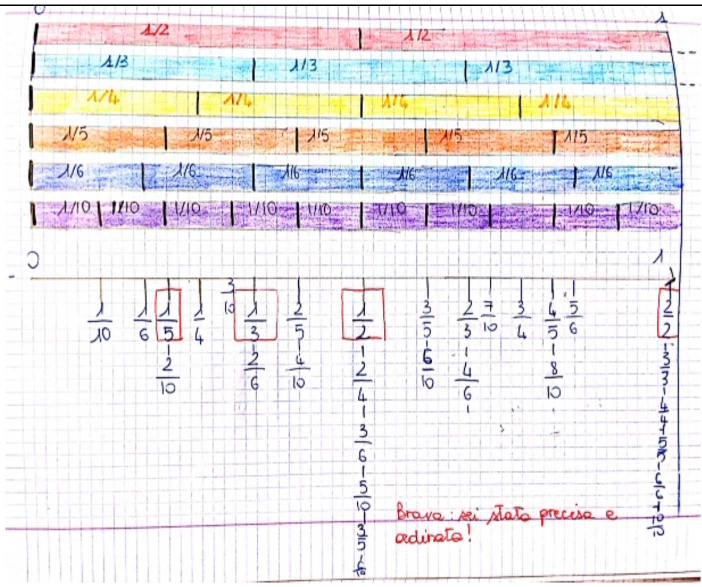


Fig. 7 The ordering of fractions on the strips (on top of the sheet) is functional to the positioning and, consequently, to the ordering of fractions on the number line (on the bottom of the sheet).

In order to overcome the unit of measure (the whole) and comparing fractions greater than 1, teacher asks students to consider four strips of paper. The strips are hanged on the wall, putting them one over the other. Thus, given a unit of measure, teacher asks to position it on each strip and to represent the fractions 4/5, 2/3, 5/3, 7/5 each on a strip (Figure 8).

Fig. 8 Four strips are hanged on the wall, putting them one over the other. In each of them there is indicated the same unit of measure and there are represented different fractions. Among them the fractions 5/3 and 7/5 that are greater than 1.

Students can compare fractions supported by the colour of fractions. Indeed, we can observe that both the green fraction (5/3) and the blue fraction (7/5) overcome the unit of measure 1 and 5/3 is greater than 7/5. The need to overcome the unit of measure is essential in order to visualize fractions on the strip (and not only unit fractions) and, then, on the number line, as visualized in the following figure (Figure 9).

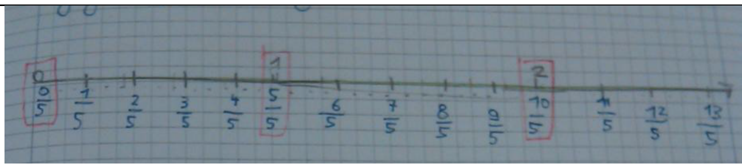


Fig. 9 Different fractions are positioned on the number line.

Conclusion

I have described a sequence of activities, designed on the base of a range of different perspectives, from mathematics education to cognitive psychology, which allow implementing inclusive education. I have outlined particularly significant and relevant passages of the sequence of activities, showing how different sub-constructs of the concept of fraction are activated and how the transition among them was guided. Exploiting preferential channels of access and elaboration of information for students with MLD, described by Miller (1987) and Stella & Grandi (2012), we have designed educational activities which, starting to the use of visual non-verbal, kinaesthetic-tactile and the auditory channels, allow student to access sub-construct of fraction concerning part-whole, as equivalent surfaces, and measure. In the second part of the educational sequence, the transition towards fraction as rational number on the number line is performed exploiting visual verbal channel as well. The analysis of the teaching intervention has shown that students have elaborated personal meanings consistent with the mathematical meanings related to fractions and they have overcome main difficulties highlighted by the educational research. In particular, partitioning colored A4 sheets in equal parts allows student to compare the different shapes of each unit fraction and verifying the equivalence, introducing the “equivalent fractions” as equivalent surfaces, by a "cutting and recomposing" strategy. This activity allows students to overcome the idea that regions congruent are the only representations of equivalent fractions considering also representations of equivalent regions.

Moreover, covering a A4 white sheet with different unit of fractions taken from the unit fractions’ boxes, requires the use of a procedure in which the fraction is conceived as part-whole, where the “whole” is the A4 sheet of paper and the part is the unit fraction. The “sum of unit fractions” for obtaining the whole (A4 sheet) is a crucial task both to introduce the meaning of sum of unit measures, and to foster the conceptualization of unit fractions as independent to their shape (as congruent regions). Afterwards, the strip was used as instrument to develop the meanings related to fractions as operators on unit of measure and, then, to the ordering of fractions, to equivalent fractions and finally to equivalence classes. The use of the strip, the string and color (for a certain period of time), has had a key role in favoring the construction of the number line as a mathematical object. On the number line fractions, associated with points, could assume the role of rational numbers being representatives of equivalence classes. Finally, it is possible that this kind of construction of meanings related to fractions might also support the management of procedural aspects involved in operations with fractions, as various researches both in mathematical education and in cognitive science have already suggested (Siegler, 2013; Robotti, Antonini, Baccaglini Frank, 2015, Robotti, 2013). Further studies are needed to explore and to confirm this hypothesis that we consider significant both for research and for teaching.

We would like to greatly thank all the teachers of the "Questione di numeri: mediatori e didattica della matematica efficace"[2] project who have realized, together with the author, this research study.

References

Amato, S. A. (2005). Developing students' understanding of the concept of fractions as numbers. In H.L., Chick & J. L. Vincent (Eds.), *Proceedings of the 29th PME International Conference, 2,* 49-56.

Arzarello, F. (2006). Semiosis as a multimodal process. *Relime*, Numero Especial, 267-299.

Bartolini, M.G, Baccaglini-Frank, A., & Ramploud, A. (2013). Intercultural dialogue and the geography and history of thought. *For the Learning of Mathematics, 34*(1), 31-33.

Bobis, J., Mulligan, J, & Lowrie, T. (2013). *Mathatics for children challenging children to think mathematically*, 4th edition. Malaysia: Pearson Australia.

Carraher, D. W., Schliemann, A. D. (1991). Children's understanding of fractions as expressions of relative magnitude. In F. Furinghetti (Ed.), *Proceedings of the 15th PME International Conference, 1,* 184-191.

Charalambous, C. Y., & Pitta-Pantaz, D. (2005). Revisiting a theoretical model on fractions: Implications for teaching and research. In H. L. Chick & J. L. Vincent (Eds.) *Proceedings of the 29th PME International Conference,* 2, *233-240.*

DeThorne L.S. & Schaefer B.A. (2004). A guide to child nonverbal IQ measures. *American Journal of Speech-Language Pathology*, *13*, 275–290.

Hannula M.S. (2003). Locating fractionon number line. In N.A. Paterman, B. J. Dougherty, J. T. Ziliox (eds), *Proceedings of the 27th PME International Conference,* 3, 17-24.

Iuculano, T. & Butterworth, B. (2011). Understanding the real value of fractions and decimals. *The Quarterly Journal of Experimental Psychology*, *64*(11), 2088–2098. DOI: 10.1080/17470218.2011.604785

Miller A. (1987), *Cognitive Style: An Integrated Model.* In Educational Psychology, vol.7 (4), pp. 251-68.

Mewstead, K., Olivier, A. (1999). Addressing students' conceptions of common fractions. In O. Zaslavsky (Ed.), *Proceedings of the 23rd PME International Conference, 3,* 329-336.

Robotti, E. (2013). Dynamic representations for algebraic objects available in AlNuSet: how develop meanings of the notions involved in the equation solution. In C. Margolinas (Ed.), *Task Design in Mathematics Education. Proceedings of ICMI Study 22*: Vol. 1, (pp. 101-110). Oxford: ICMI.

Robotti, E., Antonini, S., Baccaglini Frank, A. E., (2015). Coming to see fractions on the number line. *Proceedings of the 9th CERME European Conference* (in press).

Radford, L. (2003). Gestures, speech, and the sprouting of signs: a semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning, 5*(1), 37–70. DOI: 10.1207/S15327833MTL0501_02

Radford, L (2006). The anthropology of meaning. *Educational Studies in Mathematics*, *61*, 39-65.

Raghubar, K.P., Barnes, M.A., & Hecht, S.A. (2010). Working memory and mathematics: a review of developmental, individual difference, and cognitive approaches. *Learning and Individual Differences, 20*, 110–122. DOI: 10.1016/j.indif.2009.10.005

Rourke, B.P., & Conway, J.A. (1997). Disabilities of arithmetic and mathematical reasoning: perspectives from neurology and neuropsychology. *Journal of Learning Disabilities, 30,* 34–46. DOI: 10.1177/002221949703000103

Stella G., Grandi L., (2011). *Come leggere la dislessia e i DSA*, Milano: Giunti Editore.

Verschaffel, L., Green, B., and Torbeyns, J. (2006) Numerical thinking. In A. Gutiérrez, P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education. Past, Present and*

[2] "Questione di numeri: mediatori e didattica della matematica efficace", project coordinated by Università della Valle d'Aosta- Université de la Valle d'Aoste, Assessorato Istruzione e Cultura - Sovraintendenza agli studi Ufficio Supporto all'Autonomia Scolastica, Région Autonome de la Vallée d'Aoste.

Future, 51-82.