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Killing-Yano tensors and multi–hermitian structures

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Abstract

We show that the Euclidean Kerr-NUT-(A)dS metric in $2m$ dimensions locally admits 2^m hermitian complex structures. These are derived from the existence of a non-degenerate closed conformal Killing-Yano tensor with distinct eigenvalues. More generally, a conformal Killing-Yano tensor, provided its exterior derivative satisfies a certain condition, algebraically determines 2^m almost complex structures that turn out to be integrable as a consequence of the conformal Killing-Yano equations. In the complexification, these lead to 2^m maximal isotropic foliations of the manifold and, in Lorentz signature, these lead to two congruences of null geodesics. These are not shear-free, but satisfy a weaker condition that also generalizes the shear-free condition from four dimensions to higher dimensions. In odd dimensions, a conformal Killing-Yano tensor leads to similar integrable distributions in the complexification. We show that the recently discovered 5-dimensional solution of Lü, Mei and Pope also admits such integrable distributions, although this does not quite fit into the story as the obvious associated two-form is not conformal Killing-Yano. We give conditions on the Weyl curvature tensor imposed by the existence of a non-degenerate conformal Killing-Yano tensor; these give an appropriate generalization of the type D condition on a Weyl tensor from four dimensions.

1 Introduction

In the construction of exact solutions to the Einstein equations in four dimensions, a prominent role is played by shear-free congruences of null geodesics. In vacuum, these lead, via the Goldberg-Sachs theorem, to the algebraic degeneracy of the Weyl tensor and considerable simplification of the gravitational field equations. The Kerr-Newman black-hole solutions [1, 2] has degenerate Weyl tensor of type D and such solutions are particularly well endowed in the sense that they admit two such congruences. In higher dimensions, the Kerr-Schild and Kerr-NUT-(A)dS solutions of [3–6] do have preferred null congruences, but they are not shear-free. In [7] it was proposed that the appropriate higher dimensional concept to extend the 4-dimensional results should be that of an integrable complex distribution $\mathcal{D} \subset T_{\mathbb{C}}M$, $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$ that is totally null and of maximal dimension. With this definition, a number of 4-dimensional results were generalized to arbitrary dimension. In Euclidean signature in even dimensions, this is simply a metric compatible complex structure, i.e., a Hermitian structure. In Lorentz signature, $\mathcal{D}\cap\overline{\mathcal{D}}$ is necessarily one-dimensional and defines a null congruence. This is automatically shear-free in four dimensions, but not in higher dimensions, but it Lie derives a complex structure on the tangent space orthogonal and transverse to the null congruence. In this paper we show that it is these null congruences that are relevant in the study of the higher-dimensional Kerr-NUT-(A)dS solutions.

In four dimensions the type D condition on the Weyl curvature of a vacuum space-time is equivalent to the condition that it admits two distinct geodesic shear-free congruences. It is also equivalent to the existence of a *conformal Killing-Yano tensor*, a 2-form $\boldsymbol{\phi}$ that, on a general n-dimensionial manifold, satisfies

$$\nabla_{\boldsymbol{X}}\boldsymbol{\phi} = \frac{1}{3}\boldsymbol{X}\lrcorner\boldsymbol{\tau} + \frac{1}{n-1}\boldsymbol{X}^* \wedge \boldsymbol{K},$$

for all vector fields $\boldsymbol{X}$, where $\boldsymbol{\tau}$, a 3-form and $\boldsymbol{K}$, a 1-form, are determined by the equation. Such a form is said to be a *Killing* 2*-form* if $\boldsymbol{K}\equiv 0$, and a $*$*-Killing* 2*-form* if $\boldsymbol{\tau}\equiv 0$. In four dimensions Killing 2-forms are mapped onto $*$-Killing 2-forms by Hodge duality, but in general dimension, the two concepts are distinct. Killing-Yano tensors and their generalisation to any p-forms were first introduced by Kentaro Yano as a natural generalisation of Killing vectors to forms in [8, 9]. Conformal Killing-Yano tensors as a generalisation of conformal Killing vectors made their first appearance in [10, 11], and are often refered to as conformal Killing forms or twistor forms.

Killing-Yano tensors underly much of the theory of the four-dimensional black hole solutions. In [12], Brandon Carter identifies the fourth conserved quantity in the Kerr-Newman black hole solution, which allows the separation of the Hamilton-Jacobi equations and the complete integrability of geodesic motions. In [13, 14] it is shown that this 'hidden' symmetry can be represented by a Killing tensor, which turns out to be the 'square' of a conformal Killing 2-form. In the same papers, a spinorial approach to the problem sheds light on the null geodesic shear-free congruences in the Kerr geometry: in tensor language, the real eigenvectors of the conformal Killing-Yano tensor define a pair of geodesic shear-free null congruences.

More recently, similar structures have been found for the black hole solutions in higher dimensions. These have been the object of intensive study motivated to a large extent by ideas from string theory and M-theory. The Kerr-NUT-(A)dS metric is a higher-dimensional generalisation of the Kerr metric, generalising also the Plebański-Demiański metric. Explicitly, in Euclideanised form, the n-dimensional Kerr-NUT-(A)dS metric is given by [15]

$$\boldsymbol{g} = \sum_{\mu=1}^{m}\left(\boldsymbol{e}^{\mu}\odot\boldsymbol{e}^{\mu} + \boldsymbol{e}^{m+\mu}\odot\boldsymbol{e}^{m+\mu}\right) + \epsilon\boldsymbol{e}^{2m+1}\odot\boldsymbol{e}^{2m+1}$$

where, in terms of local coordinates $\{x_\mu, \psi_k\}$,

$$\boldsymbol{e}^{\mu} = \left(\frac{U_\mu}{X_\mu}\right)^{1/2}\mathrm{d}x_\mu, \quad \boldsymbol{e}^{m+\mu} = \left(\frac{X_\mu}{U_\mu}\right)^{1/2}\sum_{k=0}^{m-1}A_\mu^{(k)}\mathrm{d}\psi_k, \quad \boldsymbol{e}^{2m+1} = \left(-\frac{c}{A^{(m)}}\right)^{1/2}\left(\sum_{k=0}^{m}A^{(k)}\mathrm{d}\psi_k\right).$$

with

$$X_\mu = (-1)^\epsilon \frac{(g^2 x_\mu^2 - 1)}{x_\mu^{2\epsilon}} \prod_{k=1}^{m-1+\epsilon} (a_k^2 - x_\mu^2) + 2M_\mu (-x_\mu)^{1-\epsilon}, \qquad U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{m} (x_\nu^2 - x_\mu^2),$$

$$c = \prod_{k=1}^{m} a_k^2, \qquad A_\mu^{(k)} = \sum_{\substack{\nu_1<\nu_2<\ldots<\nu_k \\ \nu_i\neq\mu}} x_{\nu_1}^2 x_{\nu_2}^2 \ldots x_{\nu_k}^2, \qquad A^{(k)} = \sum_{\nu_1<\nu_2<\ldots<\nu_k} x_{\nu_1}^2 x_{\nu_2}^2 \ldots x_{\nu_k}^2.$$

Here, $m = [n/2]$, and $\epsilon = n - 2m$. The constants a_k, $-\mathrm{i}^{1+\epsilon} M_m$, M_μ $(\mu \neq m)$ are the rotation coefficients, the mass and the NUT parameters respectively, and $\lambda = -g^2$ is proportional to the cosmological constant. (With appropriate choices of the constants, Lorentzian real slices can also be found.) Like its four-dimensional counterpart, the Kerr-NUT-(A)dS metric admits a closed conformal Killing-Yano tensor

$$\boldsymbol{\phi} = \sum x_\mu \boldsymbol{e}^\mu \wedge \boldsymbol{e}^{m+\mu}. \tag{1.1}$$

Aspects of the four-dimensional theory have been generalised to Kerr-NUT-(A)dS metric in arbitrary dimensions in a series of papers [16–29] in which the separation of the Hamilton-Jacobi, Klein-Gordon and Dirac equations and the complete integrability of geodesic motions are dealt with. In this paper, we turn our attention to the existence of a set of integrable almost complex structures. Defining

$$\boldsymbol{\theta}^\mu = 2^{-1/2}(\boldsymbol{e}^\mu + \mathrm{i}\boldsymbol{e}^{m+\mu}) \qquad \text{and} \qquad \bar{\boldsymbol{\theta}}^{\bar\mu} = 2^{-1/2}(\boldsymbol{e}^\mu - \mathrm{i}\boldsymbol{e}^{m+\mu})$$

puts the Kerr-NUT-(A)dS metric into the form

$$\boldsymbol{g} = \sum_\mu 2\boldsymbol{\theta}^\mu \odot \bar{\boldsymbol{\theta}}^{\bar\mu} + \epsilon \boldsymbol{e}^{2m+1} \odot \boldsymbol{e}^{2m+1}.$$

A straightforward computation of the Levi-Civita connection 1-form implies the integrability of each of the 2^m distributions defined as the annihilator of a set of m 1-forms obtained by choosing one from each pair $\{\boldsymbol{\theta}^\mu, \bar{\boldsymbol{\theta}}^{\bar\mu}\}$, $\mu = 1, \ldots m$. These correspond to integrable almost complex structures in the case $\epsilon = 0$, and to integrable CR structures in the case $\epsilon = 1$. These results are essentially local in nature and although the complex structures will be defined on a dense open set, they will not generally extend over the whole of the regular space-time (thus they will not be global on S^4 or S^6).

The plan of the paper is as follows. We first recall the basic facts concerning conformal Killing-Yano tensors and maximal isotropic distributions while establishing the notation. We then prove our main result on integrability, both in even and odd dimensions and discuss the examples above in more detail. In these examples the Killing Yano tensor is closed. We also study the example of the new 5-dimensional metrics discovered by Lü, Mei and Pope [30] which we show does admit the corresponding integrable distribution, although the most obvious choice for a conformal Killing-Yano tensor does not seem to work.

We go on to show how a Killing-Yano tensor imposes algebraic restrictions on the Weyl tensor. We also study the closely related structure of Hamiltonian 2-forms and show that these also lead to a family of integrable complex structures as for Killing-Yano tensors. In the last section, we re-express our results in terms of spinors. In particular, eigenspinors of the conformal Killing-Yano tensor are shown to be pure and to determine the integrable distributions discussed earlier. Finally, we briefly discuss further issues arising from our discussion, the different possible reality structure, the Kerr-Schild form of the metrics, the Kerr theorem, degenerate Killing-Yano tensors and Killing spinors

2 Preliminaries

By and large we will not use the Einstein summation convention, but will occasionally when there is no ambiguity and will warn the reader of this. We adopt the notation that round and square brackets enclosing a group of indices denote symmetrisation and anti-symmetrisation respectively, e.g.

$$k_{(ab)} = \frac{1}{2!}\left(k_{ab} + k_{ba}\right) \qquad \text{and} \qquad k_{[abc]} = \frac{1}{3!}\left(k_{abc} - k_{bac} + k_{bca} - k_{cba} + k_{cab} - k_{acb}\right).$$

Indices are raised and lowered via the metric. Tensorial quantities will be given in bold symbols, and scalar quantities in regular symbols.

2.1 Conformal Killing 2-forms

Conformal Killing-Yano tensors are now much studied, see [31] for a thorough treatment. We shall only state results pertinent to conformal Killing 2-forms. In what follows, $\boldsymbol{V}^* \equiv \boldsymbol{g}(\boldsymbol{X})$ denotes the dual of a vector $\boldsymbol{X}$, and d^* the adjoint of the exterior derivative d. On p-forms on an n-dimensional (pseudo-) Riemannian it is given by $\mathrm{d}^* = (-1)^{np+n+1} * \mathrm{d} *$, where the $*$ is the Hodge duality operator.

Definition 2.1 A *conformal Killing-Yano tensor* or *conformal Killing* 2*-form* on an n-dimensional (pseudo-) Riemannian manifold M is a 2-form $\boldsymbol{\phi}$ which satisfies the following equation

$$\nabla_{\boldsymbol{X}}\boldsymbol{\phi} = \frac{1}{3}\boldsymbol{X} \lrcorner \boldsymbol{\tau} + \frac{1}{n-1}\boldsymbol{X}^* \wedge \boldsymbol{K}, \qquad \left(\nabla_c \phi_{ab} = \tau_{cab} + \frac{2}{n-1} g_{c[a} K_{b]}\right) \tag{2.1}$$

for all vector fields $\boldsymbol{X}$. It follows at once $\boldsymbol{\tau} = \mathrm{d}\boldsymbol{\phi}$ and $\boldsymbol{K} = -\mathrm{d}^*\boldsymbol{\phi}$. If $\boldsymbol{\phi}$ is co-closed, i.e. $\boldsymbol{K} = 0$, it is called a *Killing* 2*-form.* If $\boldsymbol{\phi}$ is closed, i.e. $\boldsymbol{\tau} = 0$, it is called a $*$*-Killing* 2*-form.*

Equation (2.1) is over-determined and one can show that in the case $n \neq 4$ it is equivalent to a parallel section of the bundle $\mathcal{E}^2(M) = \bigwedge^2 T^*M \oplus \bigwedge^3 T^*M \oplus \bigwedge^1 T^*M \oplus \bigwedge^2 T^*M$ with respect to the Killing connection $\tilde{\nabla}$ as described in [31]. An element $\boldsymbol{\Phi} = (\boldsymbol{\phi}, \boldsymbol{\tau}, \boldsymbol{K}, \boldsymbol{\chi}) \in \mathcal{E}^2(M)$ satisfies $\tilde{\nabla}\boldsymbol{\Phi} = 0$ if and only if $\boldsymbol{\tau} = \mathrm{d}\boldsymbol{\phi}$, $\boldsymbol{K} = -\mathrm{d}^*\boldsymbol{\phi}$, and $\boldsymbol{\chi} = \Delta\boldsymbol{\phi}$ where $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$

is the Beltrami-Laplacian on forms. The case $n = 4$ necessitates a slight modification of the prolongation in which Hodge duality must be taken into account. In flat space with flat coordinates $\{x^a\}$, integration leads to

$$\boldsymbol{\phi} = \left(\frac{1}{2} \|\boldsymbol{x}\|^2 \mathring{\boldsymbol{\chi}} - \boldsymbol{x}^* \wedge \boldsymbol{x} \lrcorner \mathring{\boldsymbol{\chi}} \right) + \boldsymbol{x}^* \wedge \mathring{\boldsymbol{K}} + \boldsymbol{x} \lrcorner \mathring{\boldsymbol{\tau}} + \mathring{\boldsymbol{\phi}}, \tag{2.2}$$

where $\boldsymbol{x} = (x^1, x^2, \ldots, x^n)$ is the position vector field, and $\mathring{\boldsymbol{\chi}}$, $\mathring{\boldsymbol{K}}$, $\mathring{\boldsymbol{\tau}}$ and $\mathring{\boldsymbol{\phi}}$ are constants.

2.2 Maximal isotropic foliations and null congruences

We will be concerned with integrable distributions $\mathcal{D} \subset T_{\mathbb{C}}M$ that are maximal and isotropic, i.e., in $2m$ dimensions, $\mathcal{D}$ will be m-dimensional and the metric vanishes on restriction to $\mathcal{D}$, i.e., for $\boldsymbol{V}, \boldsymbol{W} \in \mathcal{D}$, $\boldsymbol{g}(\boldsymbol{V}, \boldsymbol{W}) = 0$ and the integrability being given by the Frobenius integrability condition $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$. It is always possible to find a frame of 1-forms $\{\boldsymbol{\theta}^a\} = \{\boldsymbol{\theta}^\mu, \boldsymbol{\theta}_\mu\}$, $(a = 1, \ldots, 2m;\ \mu = 1, \ldots m)$ such that $\mathcal{D}$ is the annihilator of the $\boldsymbol{\theta}_\mu$ and $\boldsymbol{g} = \sum_\mu \boldsymbol{\theta}^\mu \odot \boldsymbol{\theta}_\mu$.

In Euclidean signature, $\mathcal{D} \cap \bar{\mathcal{D}} = \{0\}$ because there are no real non-zero null vectors, and so such distributions correspond to complex structures with respect to which the metric is Hermitian, i.e., we can choose $\boldsymbol{\theta}^\mu = \overline{\boldsymbol{\theta}_\mu}$.

In Lorentz signature, $\mathcal{D} \cap \bar{\mathcal{D}}$ is 1-dimensional because, on the one hand, there are no linear subspaces of a Lorentzian lightcone of dimension greater than one, and on the other, if $\mathcal{D} \cap \bar{\mathcal{D}} = \{0\}$, $\mathcal{D}$ would be a complex structure for which the metric is Hermitian, but such metrics must have an even number of positive and negative eigenvalues over the reals, whereas in Lorentz signature there is just one positive eigenvalue.

We have the following lemma

Lemma 2.2 *Suppose that the maximal isotropic distribution $\mathcal{D}$ is integrable, $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$, then, supposing the space-time to be analytic, the integral surfaces of $\mathcal{D}$ in the complexification are totally geodesic.*

Proof. This can be seen as follows. Introduce a basis $\{\boldsymbol{V}_a\} = \{\boldsymbol{V}_\mu, \boldsymbol{V}^\mu\}$ of vector field dual to $\{\boldsymbol{\theta}^a\} = \{\boldsymbol{\theta}^\mu, \boldsymbol{\theta}_\mu\}$ where $\boldsymbol{V}_\mu$ spans $\mathcal{D}$. The Ricci rotation coefficients $\omega_{\mu\nu\lambda}$, $\omega_{\mu\nu}{}^\lambda$, etc..., will be given by $[\boldsymbol{V}_\mu, \boldsymbol{V}_\nu] = \sum_\lambda (\omega_{\mu\nu\lambda} \boldsymbol{V}^\lambda + \omega_{\mu\nu}{}^\lambda \boldsymbol{V}_\lambda)$. The integrability condition $[\boldsymbol{V}_\mu, \boldsymbol{V}_\nu] \in \mathcal{D}$ implies that the Ricci rotation coefficients $\omega_{\mu\nu\lambda} = 0$ and so also the corresponding connection coefficients $\Gamma_{\mu\nu\lambda} = 0$ where $\nabla_{\boldsymbol{V}_\mu} \boldsymbol{V}_\nu = \sum_\lambda (\Gamma_{\mu\nu}{}^\lambda \boldsymbol{V}_\lambda + \Gamma_{\mu\nu\lambda} \boldsymbol{V}^\lambda)$. This allows one to deduce that the form $\boldsymbol{\theta}_1 \wedge \ldots \wedge \boldsymbol{\theta}_m$, which is orthogonal to all the $\boldsymbol{V}_\mu$, is parallel up scale along the $\boldsymbol{V}_\mu$, i.e.

$$\nabla_{\boldsymbol{V}_\mu} \boldsymbol{\theta}_1 \wedge \ldots \wedge \boldsymbol{\theta}_m = \sum_\nu \Gamma_{\mu\nu}{}^\nu \boldsymbol{\theta}_1 \wedge \ldots \wedge \boldsymbol{\theta}_m.$$

Thus the integral surfaces of $\mathcal{D}$ are totally geodesic. □

We have the straightforward corollary

Corollary 2.3 *In Lorentzian signature, the null congruence defined by $\mathcal{D} \cap \bar{\mathcal{D}}$ is geodesic.*

2.3 The normal form of a generic 2-form

Throughout this paper we will restrict attention to the case where the Killing-Yano tensor $\boldsymbol{\phi}$ is generic in the sense that all its eigenvalues will be assumed to be distinct (i.e., the eigenvalues of the endomorphism obtained by raising one index with the metric). The following is a standard result and we only sketch its proof briefly to set notation.

Lemma 2.4 *There exists a basis of 1-forms* $\{\boldsymbol{\theta}^a\} = \{\boldsymbol{\theta}^\mu, \boldsymbol{\theta}_\mu, \epsilon \boldsymbol{e}_{2m+1}\}$ *that are a null basis for the metric, i.e.,* $\boldsymbol{g} = \sum_\mu \boldsymbol{\theta}^\mu \odot \boldsymbol{\theta}_\mu + \epsilon \boldsymbol{e}_{2m+1}^2$ *(so that each of the 1–forms* $\{\boldsymbol{\theta}^\mu, \boldsymbol{\theta}_\mu\}$ *is null) and such that*

$$\boldsymbol{\phi} = \sum_\mu \lambda_\mu \boldsymbol{\theta}^\mu \wedge \boldsymbol{\theta}_\mu \,. \tag{2.3}$$

This normal form is unique up to separate rescalings of the $(\boldsymbol{\theta}^\mu, \boldsymbol{\theta}_\mu) \to (a_\mu \boldsymbol{\theta}^\mu, a_\mu^{-1} \boldsymbol{\theta}_\mu)$ *with* $a_\mu \neq 0$*, and up to permutations of the* μ *and* $(\boldsymbol{\theta}^\mu, \boldsymbol{\theta}_\mu) \to (\boldsymbol{\theta}_\mu, \boldsymbol{\theta}^\mu)$ *for one value of* μ*, with the other forms left invariant.*

[Here as before, $\epsilon = 1$ *in the odd dimensional case and zero otherwise.]*

Proof. [Sketch] The genericity assumption allows us to use a basis of eigen-(co)vectors for $\boldsymbol{\phi}$. It is a standard fact that the eigenvectors with non-zero eigenvalue of a 2-form with respect to a metric are isotropic. This follows from the identity (using the summation convention now until the end of this section)

$$\phi_{ab} l^b = \lambda g_{ab} l^b \qquad \Rightarrow \qquad \lambda l_a l^a = \phi_{ab} l^a l^b = 0 \,.$$

A pair of eigenvectors l^a, n^a with eigenvalues λ, ν respectively satisfy $\phi_{ab} l^a n^b = \nu l^a n_a = -\lambda l^a n_a$ and so will be orthogonal to each-other unless their eigenvalues differ by a sign. Thus, since the eigenvectors span the space, they must come in pairs with eigenvalues of opposite sign with possibly one zero eigenvalue in odd dimensions, and can be normalised such that the claims of the lemma are satisfied. □

We note that for a real $\boldsymbol{\phi}$ in Euclidean signature we must have $\boldsymbol{\theta}^\mu = \overline{\boldsymbol{\theta}_\mu}$ and the λ_μ will all be imaginary. For real $\boldsymbol{\phi}$ in Lorentzian signature, we must have that one eigenvalue, say λ_1 is real, as will therefore be $\boldsymbol{\theta}^1$ and $\boldsymbol{\theta}_1$, with the other eigenvalues imaginary (with $\boldsymbol{\theta}^\mu = \overline{\boldsymbol{\theta}_\mu}$). This follows from the requirement that the metric have just one timelike direction as before in the discussion of maximal isotropic foliations.

3 Main results

3.1 Maximal isotropic distributions associated to $\boldsymbol{\phi}$ and their integrability

Given the representation of $\boldsymbol{\phi}$ in Lemma 2.4, we can write down 2^m maximal isotropic distributions, one for each choice of integers $\mu_1 < \mu_2 < \ldots < \mu_r \subset \{1, \ldots, m\}$. These are defined to be the distribution annihilated by the forms $\{\boldsymbol{\theta}^{\mu_1}, \ldots, \boldsymbol{\theta}^{\mu_r}, \boldsymbol{\theta}_{\nu_1}, \ldots, \boldsymbol{\theta}_{\nu_{m-r}}, \epsilon \boldsymbol{e}_{2m+1}\}$ where the $\nu_1, \ldots, \nu_{m-r}$ are the distinct integers in the complement of the $\mu_1, \ldots, \mu_r$ in $\{1, \ldots, m\}$. The purpose of this section is to prove the following theorem.

Theorem 3.1 *Let M be a $2m$-dimensional Riemannian manifold equipped with a non-degenerate diagonalisable conformal Killing-Yano tensor $\boldsymbol{\phi}$ with distinct eigenvalues. Let the 3-form $\boldsymbol{\tau} = \mathrm{d}\boldsymbol{\phi}$ satisfy*

$$\boldsymbol{\theta}^\mu \wedge \boldsymbol{\tau}(\boldsymbol{V}^\mu, \cdot, \cdot) = 0, \qquad \boldsymbol{\theta}_\mu \wedge \boldsymbol{\tau}(\boldsymbol{V}_\mu, \cdot, \cdot) = 0, \tag{3.1}$$

for each μ (i.e., with no summation). Then the 2^m maximal isotropic distributions associated to $\boldsymbol{\phi}$ are integrable. In Euclidean signature they define 2^m distinct complex structures, whereas in Lorentzian signature they define just two geodesic congruences, each associated with 2^{m-1} integrable maximal isotropic distributions.

Remark 3.2 Condition (3.1) is automatically satisfied

1. when the conformal Killing-Yano tensor is closed, and
2. in four dimensions.

Proof. Let M be a (real) $2m$-dimensional (pseudo-) Riemannian manifold and $\boldsymbol{\phi}$ be a non-degenerate Killing-Yano tensor on M. Suppose that $\boldsymbol{\phi}$ has the form (2.3) in a null basis of one-forms $\{\boldsymbol{\theta}^\mu, \boldsymbol{\theta}_\mu\}$, dual basis $\{\boldsymbol{V}_\mu, \boldsymbol{V}^\mu\}$ and distinct eigenvalues $\{\lambda_\mu, -\lambda_\mu\}$.

In terms of the vector basis $\{\boldsymbol{V}_\mu, \boldsymbol{V}^\mu\}$, the integrability of all these distributions will be implied by the conditions

$$\begin{aligned}
[\boldsymbol{V}_\mu, \boldsymbol{V}_\nu] &= \omega_{\mu\nu}{}^\mu \boldsymbol{V}_\mu + \omega_{\mu\nu}{}^\nu \boldsymbol{V}_\nu, \\
[\boldsymbol{V}^\mu, \boldsymbol{V}^\nu] &= \omega^{\mu\nu}{}_\mu \boldsymbol{V}^\mu + \omega^{\mu\nu}{}_\nu \boldsymbol{V}^\nu, \\
[\boldsymbol{V}_\mu, \boldsymbol{V}^\nu] &= \omega_\mu{}^{\nu\mu} \boldsymbol{V}_\mu + \omega_\mu{}^\nu{}_\nu \boldsymbol{V}^\nu,
\end{aligned} \tag{3.2}$$

satisfied for all distinct μ, ν, and no summation. These are constraints on the Ricci rotation coefficients and hence on the connection. In terms of the connection coefficients we must show

$$\begin{aligned}
&\Gamma_{\kappa\mu\nu} = 0, && \Gamma^{\kappa\mu\nu} = 0, && \text{(for all } \kappa, \mu, \nu), \\
&\Gamma_{\kappa\mu}{}^\nu = 0, && \Gamma^\kappa{}_\nu{}^\mu = 0, && \text{(for all } \nu \neq \mu, \kappa), \\
&\Gamma_\kappa{}^{\mu\nu} = 0, && \Gamma^\kappa{}_{\mu\nu} = 0, && \text{(for all } \kappa \neq \mu, \nu).
\end{aligned} \tag{3.3}$$

which imply equations (3.2).

In terms of basis components, the Killing-Yano equation (2.1) yields (no summation)

$$\partial_\kappa \phi_\mu{}^\nu + (\lambda_\mu - \lambda_\nu)\, \Gamma_{\kappa\mu}{}^\nu = \tau_{\kappa\mu}{}^\nu - \frac{1}{n-1} \delta^\nu_\kappa K_\mu, \tag{3.4a}$$

$$(\lambda_\mu + \lambda_\nu)\, \Gamma^\kappa{}_{\mu\nu} = \tau^\kappa{}_{\mu\nu} + \frac{2}{n-1} \delta^\kappa_{[\mu} K_{\nu]}, \tag{3.4b}$$

$$(\lambda_\mu + \lambda_\nu)\, \Gamma_{\kappa\mu\nu} = \tau_{\kappa\mu\nu}. \tag{3.4c}$$

From equations (3.4a), we then obtain

$$\begin{aligned}
K_\mu &= -(n-1)\partial_\mu\lambda_\mu, && && (3.5a)\\
\tau_{\nu\mu}{}^{\mu} &= \partial_\nu\lambda_\mu, && \text{for all } \nu\neq\mu, && (3.5b)\\
\tau_{\nu\mu}{}^{\nu} - \frac{1}{n-1}K_\mu &= (\lambda_\mu-\lambda_\nu)\Gamma_{\nu\mu}{}^{\nu}, && \text{for all } \nu\neq\mu, && (3.5c)\\
\delta^\mu_\kappa\tau_{\kappa\mu}{}^{\nu} &= (\lambda_\mu-\lambda_\nu)\,\Gamma_{\kappa\mu}{}^{\nu}, && \text{for all } \nu\neq\kappa,\mu. && (3.5d)
\end{aligned}$$

On the other hand, equations (3.4b) give

$$\begin{aligned}
\tau_\kappa{}^{\mu\nu} &= (\lambda_\mu+\lambda_\nu)\Gamma_\kappa{}^{\mu\nu}, && \text{for all } \kappa\neq\mu,\nu, && (3.6a)\\
\tau^{\nu}{}_{\nu\mu} + \frac{1}{n-1}K_\mu &= (\lambda_\mu+\lambda_\nu)\Gamma^{\nu}{}_{\nu\mu}, && \text{for all } \nu\neq\mu. && (3.6b)
\end{aligned}$$

By symmetry, we have similar relations involving $\Gamma^{\kappa}{}_{\mu}{}^{\nu}$, $\Gamma_\kappa{}^{\mu\nu}$, and $\Gamma^{\kappa\mu\nu}$. By assumption all the eigenvalues $\{\lambda_\mu, -\lambda_\mu\}$ are distinct, so that equations (3.4c), (3.5d) and (3.6a) imply the integrability conditions (3.3) if and only if

$$\begin{aligned}
\tau_{\kappa\mu\nu} &= 0, & \tau^{\kappa\mu\nu} &= 0, & &(\text{for all } \kappa,\mu,\nu),\\
\tau_{\kappa\mu}{}^{\nu} &= 0, & \tau^{\kappa}{}_{\nu}{}^{\mu} &= 0, & &(\text{for all } \nu\neq\mu,\kappa),\\
\tau_\kappa{}^{\mu\nu} &= 0, & \tau^{\kappa}{}_{\mu\nu} &= 0, & &(\text{for all } \kappa\neq\mu,\nu),
\end{aligned}$$

which is equivalent to equation (3.1).

At this point, we now have enough information about the connection to obtain the integrability of the maximal isotropic distributions. In particular, we have the condition $\Gamma_{\mu\nu\lambda}=0$ which implies as in the proof of Lemma 2.2,

$$\nabla_{\boldsymbol{V}_\mu}\boldsymbol{\theta}_1\wedge\ldots\wedge\boldsymbol{\theta}_m = \sum_\nu \Gamma_{\mu\nu}{}^{\nu}\boldsymbol{\theta}_1\wedge\ldots\wedge\boldsymbol{\theta}_m.$$

This, in particular, implies for $\boldsymbol{\Omega}=\boldsymbol{\theta}_1\wedge\ldots\wedge\boldsymbol{\theta}_m$ that $\mathrm{d}\boldsymbol{\Omega}=\boldsymbol{\alpha}\wedge\boldsymbol{\Omega}$ for some 1-form $\boldsymbol{\alpha}$, so that the distribution $\mathcal{D}=\langle\boldsymbol{V}_1,\ldots,\boldsymbol{V}_m\rangle$ orthogonal to $\boldsymbol{\Omega}$ is integrable. However, all the maximal isotropic distributions determined by $\boldsymbol{\phi}$ are on an equal footing with $\mathcal{D}$; all such distributions are equivalent to $\mathcal{D}$ by interchanging $(\boldsymbol{\theta}^\mu,\boldsymbol{\theta}_\mu)\to(\boldsymbol{\theta}_\mu,\boldsymbol{\theta}^\mu)$ for different values of μ. Thus, all such distributions are integrable. This can, of course, be checked explicitly by calculating $\nabla_{\boldsymbol{W}}\boldsymbol{\theta}_{\sigma(1)}\wedge\ldots\wedge\boldsymbol{\theta}_{\sigma(p)}\wedge\boldsymbol{\theta}^{\sigma(p+1)}\wedge\ldots\boldsymbol{\theta}^{\sigma(m)}$ where σ is an arbitrary permutation of $1,\ldots,m$ and $\boldsymbol{W}$ is in the kernel of $\boldsymbol{\theta}_{\sigma(1)}\wedge\ldots\wedge\boldsymbol{\theta}_{\sigma(p)}\wedge\boldsymbol{\theta}^{\sigma(p+1)}\wedge\ldots\boldsymbol{\theta}^{\sigma(m)}$. □

For future use we record the expressions for some of the remaining connection components. Combining equations (3.5a), (3.5c) and (3.6b) gives

$$\begin{aligned}
\Gamma_{\nu\mu}{}^{\nu} &= \partial_\mu \ln|\lambda_\mu-\lambda_\nu|, & \Gamma^{\nu\mu}{}_{\nu} &= \partial^\mu \ln|\lambda_\mu-\lambda_\nu|, && (3.7a)\\
\Gamma^{\nu}{}_{\mu\nu} &= \partial_\mu \ln|\lambda_\mu+\lambda_\nu|, & \Gamma_\nu{}^{\mu\nu} &= \partial^\mu \ln|\lambda_\mu+\lambda_\nu|. && (3.7b)
\end{aligned}$$

Remark 3.3 We emphasise that the above result is essentially local. We have made the assumption, for example, that the Killing-Yano tensor has distinct and non-constant eigenvalues, and this assumption will generically break down on some nontrivial subset of codimension at least one. In general, then, the complex structures will not extend globally over such subsets. The Kerr-NUT-(A)dS metric provides such an example. Setting the mass and the NUT parameters to zero, the metric reduces to a Ricci-flat conformally flat metric. But it is a standard result that the round four-sphere does not admit a global hermitian complex structure (there are complex structures on the complement of a point in S^4 that naturally extend to $\mathbb{CP}^2$ or a quadric).

3.2 Odd-dimensional manifolds and integrable CR structures

The above results extend naturally to odd-dimensional manifolds. When M is a $(2m+1)$-dimensional real manifold, the natural analogue of a complex structure is a CR structure. An almost Cauchy-Riemann (CR) structure is an m-dimensional subbundle $\mathcal{D}$ of the complexified tangent bundle $T_{\mathbb{C}}M$, so that $\mathcal{D} \cap \bar{\mathcal{D}} = 0$. It is a CR structure if it is integrable, $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$. This is the structure that a hypersurface in $\mathbb{C}^{m+1}$ inherits from the ambient complex structure; $\mathcal{D}$ are those vectors in the holomorphic tangent bundle $T^{(1,0)}$ that are tangent to the hypersurface. On our (pseudo-) Riemannian manifold, we will also require $\mathcal{D}$ to be isotropic so that we will have $T_{\mathbb{C}}M = \mathcal{D} \oplus \bar{\mathcal{D}} \oplus K$ where K is the orthogonal complement of $\mathcal{D} \oplus \bar{\mathcal{D}}$.

Given a non-degenerate Killing-Yano tensor $\boldsymbol{\phi}$ on M, K will be the kernel of $\boldsymbol{\phi}$. By Lemma 2.4, assuming that $\boldsymbol{\phi}$ has distinct non-zero eigenvalues, we can find a basis of 1-forms $\{\boldsymbol{e}^0, \boldsymbol{\theta}^\mu, \boldsymbol{\theta}_\mu\}$ all null except $\boldsymbol{e}^0$, in which $\boldsymbol{\phi} = \sum_\mu \lambda_\mu \theta^\mu \wedge \theta_\mu$ is diagonal, degenerate on K, but non-degenerate on $K^\perp$. We write $\{\boldsymbol{V}_0, \boldsymbol{V}_\mu, \boldsymbol{V}^\mu\}$, $(\mu = 1, \ldots, m)$, for the corresponding dual vector basis so that $\boldsymbol{V}_0$ spans K, and the 2^m distributions are found by choosing one vector from each of the pairs $(\boldsymbol{V}_\mu, \boldsymbol{V}^\mu)$ for each μ.

The question of whether the 2^m almost Cauchy-Riemann structures $\boldsymbol{\phi}$ are integrable reduces to the integrability of the 2^m maximal isotropic distributions on $K^\perp$, hence, to whether relations (3.2) are satisfied. So, if $\mathcal{D}$ is one of our 2^m maximal isotropic distribution in $K^\perp$, then $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$. More precisely, relations (3.2) tell us that $\omega_{\mu\nu 0}$, $\omega^{\mu\nu}{}_0$, and $\omega_\mu{}^\nu{}_0$ vanish for all $\mu \neq \nu$. In odd dimensions, the conformal Killing-Yano equations (2.1) gives the extra conditions

$$\partial_0 \phi_\mu{}^\nu + (\lambda_\mu - \lambda_\nu)\Gamma_{0\mu}{}^\nu = 0 \tag{3.8a}$$

$$(\lambda_\mu + \lambda_\nu)\Gamma_{0\mu\nu} = \tau_{0\mu\nu} \tag{3.8b}$$

$$\lambda_\mu \Gamma_{00\mu} = \frac{1}{n-1} K_\mu \tag{3.8c}$$

$$-\lambda_\mu \Gamma_{\nu\mu 0} = \tau_{\nu\mu 0} \tag{3.8d}$$

$$\lambda_\mu \Gamma^\nu{}_{\mu 0} = \tau^\nu{}_{\mu 0} + \frac{1}{n-1}\delta^\nu_\mu K_0. \tag{3.8e}$$

Equations (3.8a) lead to

$$\partial_0 \lambda_\mu = 0 \tag{3.9a}$$
$$(\lambda_\mu - \lambda_\nu){\Gamma_{0\mu}}^{\nu} = 0 \qquad \text{for all } \mu \neq \nu, \tag{3.9b}$$

and equations (3.8e) to

$$\lambda_\mu {\Gamma^\nu}_{\mu 0} = {\tau^\nu}_{\mu 0} \qquad \text{for all } \mu \neq \nu, \tag{3.10a}$$
$$\lambda_\mu {\Gamma^\mu}_{\mu 0} = {\tau^\mu}_{\mu 0} + \frac{1}{n-1} K_0. \tag{3.10b}$$

By the assumptions on the eigenvalues, equations (3.8d) and (3.10a) show that

$$\Gamma_{\nu\mu 0} = 0 = {\Gamma^{\nu\mu}}_0 = {\Gamma^\nu}_{\mu 0} \quad \Longleftrightarrow \quad \tau_{\nu\mu 0} = 0 = {\tau^{\nu\mu}}_0 = {\tau^\nu}_{\mu 0}, \qquad \text{for all } \mu \neq \nu,$$

which is subsumed into condition (3.1). Thus, Theorem 3.1 extends to the odd-dimensional case.

Remark 3.4 Given such an integrable distribution $\mathcal{D}$, we can adjoin $\boldsymbol{V}_0$ to form the distribution $\tilde{\mathcal{D}} = \{\boldsymbol{V}_0, \mathcal{D}\}$. The integrability of this distribution requires in addition to the above, that $[\boldsymbol{V}_\mu, \boldsymbol{V}_0] = {\omega_{\mu 0}}^\mu \boldsymbol{V}_\mu + {\omega_{\mu 0}}^0 \boldsymbol{V}_0$ and $[\boldsymbol{V}^\mu, \boldsymbol{V}_0] = {\omega^\mu}_{0\mu} \boldsymbol{V}^\mu + {{\omega^\mu}_0}^0 \boldsymbol{V}_0$ for all μ, but this follows from the above conditions on the connection. Thus, these distributions will also be integrable.

We also note that combining equations (3.5a) and (3.8c) yields

$$\Gamma_{00\mu} = -\partial_\mu \ln |\lambda_\mu|. \tag{3.11}$$

3.3 Examples

3.3.1 The $*$-Killing (or closed conformal Killing-Yano) case

For the Kerr-NUT-(A)dS, we can see these distributions explicitly. The integrability is most evident in terms of the inverse metric which in the even dimensional case is given by [15]

$$\boldsymbol{g}^{-1} = \sum_{\mu=1}^{m} 2\boldsymbol{V}^\mu \odot \boldsymbol{V}_\mu$$

where

$$\boldsymbol{V}_\mu = \frac{2^{-1/2}}{\sqrt{U_\mu}} \left(\sqrt{X_\mu} \frac{\partial}{\partial x_\mu} - \mathrm{i} \frac{1}{\sqrt{X_\mu}} \left(\sum_{k=0}^{m-1} (-1)^k x_\mu^{2(m-1-k)} \frac{\partial}{\partial \psi_k} \right) \right), \qquad \boldsymbol{V}^\mu = \overline{\boldsymbol{V}_\mu}.$$

The key feature here is that, apart from the scale factor $\sqrt{U_\mu}$, in these coordinates, the coefficients of the vectors $\boldsymbol{V}^\mu$ and $\boldsymbol{V}_\mu$ depend only on the coordinate x_μ. Thus a distributions made up of any set of the basis vectors with distinct values of μ will commute amongst themselves so that the distribution is integrable. (The only pairs of these basis vectors that do no commute amongst themselves in this way are $\{\boldsymbol{V}^\mu, \boldsymbol{V}_\mu\}$.) In particular, all the maximal istropic distributions spanned by m such basis vectors with distinct values of μ will form an integrable distribution. Explicitly, we have

$$
\begin{aligned}
[\boldsymbol{V}_\mu, \boldsymbol{V}_\nu] &= 2^{-1/2}\frac{x_\nu\sqrt{Q_\nu}}{x_\nu^2 - x_\mu^2}\boldsymbol{V}_\mu - 2^{-1/2}\frac{x_\mu\sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2}\boldsymbol{V}_\nu \\
[\boldsymbol{V}^\mu, \boldsymbol{V}^\nu] &= 2^{-1/2}\frac{x_\nu\sqrt{Q_\nu}}{x_\nu^2 - x_\mu^2}\boldsymbol{V}^\mu - 2^{-1/2}\frac{x_\mu\sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2}\boldsymbol{V}^\nu \\
[\boldsymbol{V}_\mu, \boldsymbol{V}^\nu] &= 2^{-1/2}\frac{x_\nu\sqrt{Q_\nu}}{x_\nu^2 - x_\mu^2}\boldsymbol{V}_\mu - 2^{-1/2}\frac{x_\mu\sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2}\boldsymbol{V}^\nu \\
[\boldsymbol{V}_\mu, \boldsymbol{V}^\mu] &= 2^{-1/2}\frac{\partial\sqrt{Q_\mu}}{\partial x_\mu}\left(\boldsymbol{V}_\mu - \boldsymbol{V}^\mu\right) + 2\cdot 2^{-1/2}\sum_{\nu\neq\mu}\frac{x_\mu\sqrt{Q_\nu}}{x_\nu^2 - x_\mu^2}\left(\boldsymbol{V}_\nu - \boldsymbol{V}^\nu\right),
\end{aligned}
$$

where $Q_\mu = X_\mu/U_\mu$.

3.3.2 A five-dimensional black hole solution

We consider the recently discovered metric of Lü, Mei and Pope, [30]. The metric is $\boldsymbol{g}_5 = \sum_{i=0}^4 (\boldsymbol{e}^i)^2$ with

$$
\begin{aligned}
\boldsymbol{e}^0 &= \left(\frac{a_0}{xy}\right)^{1/2}(\mathrm{d}\phi + (x+y)\mathrm{d}\psi + xy\mathrm{d}t). \\
\boldsymbol{e}^1 &= \frac{1}{2(1-xy)}\left(\frac{x-y}{X}\right)^{1/2}\mathrm{d}x \qquad \boldsymbol{e}^3 = \frac{1}{1-xy}\left(\frac{X}{x(x-y)}\right)^{1/2}(\mathrm{d}\phi + y\mathrm{d}\psi) \\
\boldsymbol{e}^2 &= \frac{1}{2(1-xy)}\left(\frac{y-x}{Y}\right)^{1/2}\mathrm{d}y \qquad \boldsymbol{e}^4 = \frac{1}{1-xy}\left(\frac{Y}{y(y-x)}\right)^{1/2}(\mathrm{d}\phi + x\mathrm{d}\psi)
\end{aligned}
$$

where X and Y are quartic polynomials in x and y respectively. Then, the dual vector basis is given by

$$
\begin{aligned}
\boldsymbol{e}_0 &= \left(\frac{1}{a_0 xy}\right)^{1/2}\partial_t \\
\boldsymbol{e}_1 &= 2(1-xy)\left(\frac{X}{x-y}\right)^{1/2}\partial_x \qquad \boldsymbol{e}_3 = (1-xy)\left(\frac{1}{x(x-y)X}\right)^{1/2}(x^2\partial_\phi - x\partial_\psi + \partial_t) \\
\boldsymbol{e}_2 &= 2(1-xy)\left(\frac{Y}{y-x}\right)^{1/2}\partial_y \qquad \boldsymbol{e}_4 = (1-xy)\left(\frac{1}{y(y-x)Y}\right)^{1/2}(y^2\partial_\phi - y\partial_\psi + \partial_t)
\end{aligned}
$$

Defining $\boldsymbol{V}_1 = 2^{-1/2}(\boldsymbol{e}_1 - \mathrm{i}\boldsymbol{e}_3)$ and $\boldsymbol{V}_2 = 2^{-1/2}(\boldsymbol{e}_2 - \mathrm{i}\boldsymbol{e}_4)$, we obtain the Ricci rotation coefficients from the commutators

$$[\boldsymbol{V}_1, \boldsymbol{V}_2] = -2^{-1/2}\left(\frac{Y}{y-x}\right)^{1/2}\frac{(-2x^2+xy+1)}{x-y}\boldsymbol{V}_1 + 2^{-1/2}\left(\frac{X}{x-y}\right)^{1/2}\frac{(-2y^2+xy+1)}{y-x}\boldsymbol{V}_2$$
$$\left[\boldsymbol{V}_1, \boldsymbol{V}^2\right] = -2^{-1/2}\left(\frac{Y}{y-x}\right)^{1/2}\frac{(-2x^2+xy+1)}{x-y}\boldsymbol{V}_1 + 2^{-1/2}\left(\frac{X}{x-y}\right)^{1/2}\frac{(-2y^2+xy+1)}{y-x}\boldsymbol{V}^2$$
$$\left[\boldsymbol{V}_1, \boldsymbol{V}^1\right] = 2^{-1/2}\left(2\partial_x\left((1-xy)\left(\frac{X}{x-y}\right)^{1/2}\right) + \left(\frac{X}{x-y}\right)^{1/2}\left(\frac{1-5xy}{x}\right)\right)(\boldsymbol{V}_1 - \boldsymbol{V}^1)$$
$$- 2^{-1/2}\left(\frac{Y}{y-x}\right)^{1/2}\left(\frac{x}{y}\right)^{1/2}\frac{2(1-xy)}{(x-y)}(\boldsymbol{V}_2 - \boldsymbol{V}^2) + 2\mathrm{i}\left(\frac{a_0}{xy}\right)^{1/2}(1-xy)^2 x^{-1/2}\boldsymbol{e}_0$$
$$[\boldsymbol{V}_1, \boldsymbol{e}_0] = -2^{-1/2}\frac{1-xy}{x}\left(\frac{X}{x-y}\right)^{1/2}\boldsymbol{e}_0$$

with the remaining commutators given by complex conjugation and the symmetry $1 \leftrightarrow 2$ accompanied by $x \leftrightarrow y$ and $X \leftrightarrow Y$.

It is also straighforward to check that its associated rank-two and rank-three complex distributions are all integrable .

We now turn our attention to the existence of a conformal Killing-Yano tensor $\boldsymbol{\phi}$ in normal form in this basis. We first note that the four-dimensional metric $\boldsymbol{g}_4 = \sum_{i=1}^4 (\boldsymbol{e}^i)^2$ has a conformal Killing-Yano tensor given by

$$\begin{aligned}\boldsymbol{\phi} &= \mathrm{i}\frac{x^{1/2}}{1-xy}\boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}_1 + \mathrm{i}\frac{y^{1/2}}{1-xy}\boldsymbol{\theta}^2 \wedge \boldsymbol{\theta}_2 \\ &= \frac{1}{2(1-xy)^3}(\mathrm{d}x \wedge (\mathrm{d}\phi + y\mathrm{d}\psi) + \mathrm{d}y \wedge (\mathrm{d}\phi + x\mathrm{d}\psi))\end{aligned}$$

where $\boldsymbol{\theta}^\mu = 2^{-1/2}(\boldsymbol{e}^\mu + \mathrm{i}\boldsymbol{e}^{2+\mu})$ and $\boldsymbol{\theta}_\mu = \overline{\boldsymbol{\theta}^\mu}$. This choice can be justified by the fact that the metric $\boldsymbol{g}_4$ is simply a conformal rescaling of a (euclideanised) Kerr metric in Plebański-Demiański form, and that a conformal Killing-Yano tensor has conformal weight 3. One can also check that, with this choice, the eigenvalues of $\boldsymbol{\phi}$ satisfy equations (3.7a) and (3.7b). On the other hand, on considering the full metric metric $\boldsymbol{g}_5$ and by (3.11), the eigenvalues λ_μ of $\boldsymbol{\phi}$ must also satisfy

$$2\partial_x \ln|\lambda_1| = \frac{1}{x} \qquad\qquad 2\partial_y \ln|\lambda_2| = \frac{1}{y},$$

solutions of which can be taken to be $\lambda_1 = x^{1/2}$ and $\lambda_2 = y^{1/2}$. Hence, in spite of the existence of integrable maximal isotropic distributions, there is no conformal Killing-Yano tensor in normal form in this basis, and the converse of the (odd-dimensional version of) Theorem 3.1 does not hold.

3.4 Conditions on the Weyl conformal tensor

Let us return to the general case of a conformal Killing-Yano tensor $\boldsymbol{\phi}$ on a real $2m$-dimensional (pseudo-) Riemannian manifold M. As before, we consider the complexification of the tangent bundle TM. We can extend $\boldsymbol{\phi}$ to an endomorphism $\hat{\boldsymbol{\phi}}$ on the space of 2-forms $\bigwedge^2 T^*M$. Similarly, we can view the Weyl tensor $\boldsymbol{C}$ as an endomorphism $\hat{\boldsymbol{C}}$ on $\bigwedge^2 T^*M$. It is a standard result [31] that the commutator of $\hat{\boldsymbol{C}}$ and $\hat{\boldsymbol{\phi}}$ vanishes, i.e.

$$\left[\hat{\boldsymbol{C}}, \hat{\boldsymbol{\phi}}\right] = 0. \tag{3.12}$$

If $\boldsymbol{\phi}$ is diagonal in the null basis $\{\boldsymbol{\theta}^a\} = \{\boldsymbol{\theta}^\mu, \boldsymbol{\theta}_\mu\}$, then $\hat{\boldsymbol{\phi}}$ is also diagonal in the canonical basis of 2-forms $\left\{\boldsymbol{\theta}^a \wedge \boldsymbol{\theta}^b\right\}$, and

$$\hat{\boldsymbol{\phi}}\left(\boldsymbol{\theta}^\mu \wedge \boldsymbol{\theta}^\nu\right) = \left(\lambda_\mu + \lambda_\nu\right) \boldsymbol{\theta}^\mu \wedge \boldsymbol{\theta}^\nu, \qquad \hat{\boldsymbol{\phi}}\left(\boldsymbol{\theta}_\mu \wedge \boldsymbol{\theta}_\nu\right) = -\left(\lambda_\mu + \lambda_\nu\right) \boldsymbol{\theta}_\mu \wedge \boldsymbol{\theta}_\nu,$$
$$\hat{\boldsymbol{\phi}}\left(\boldsymbol{\theta}^\mu \wedge \boldsymbol{\theta}_\nu\right) = \left(\lambda_\mu - \lambda_\nu\right) \boldsymbol{\theta}^\mu \wedge \boldsymbol{\theta}_\nu.$$

Assuming that the eigenvalues of $\boldsymbol{\phi}$ are all distinct, $\hat{\boldsymbol{\phi}}$ has $2m(m-1)$ non-zero eigenvalues and has an m-dimensional kernel spanned by $\left\{\boldsymbol{\theta}^\mu \wedge \boldsymbol{\theta}_\mu : \mu = 1, \dots m\right\}$. By the commutation relation (3.12), it then follows that $\hat{\boldsymbol{C}}$ and $\hat{\boldsymbol{\phi}}$ have $m(2m-1)$ common eigen-2-forms, and we can deduce that all components of the Weyl tensor with the possible exception of

$$C_{\mu\nu}{}^{\mu\nu}, \qquad C_{\mu}{}^{\nu}{}_{\nu}{}^{\mu}, \qquad C_{\mu}{}^{\mu}{}_{\nu}{}^{\nu}, \qquad C_{\mu}{}^{\mu}{}_{\mu}{}^{\mu},$$

for all distinct μ, ν, vanish in the canonical basis. Consequently,

Theorem 3.5 *Let $\boldsymbol{\phi}$ be a non-degenerate conformal Killing-Yano tensor with distinct eigenvalues, diagonal in the null basis $\{\boldsymbol{\theta}^a\} = \{\boldsymbol{\theta}^\mu, \boldsymbol{\theta}_\mu\}$. Then the Weyl tensor $\boldsymbol{C}$ satisfies (no summation)*

$$\boldsymbol{C}(\boldsymbol{V}_a, \boldsymbol{V}_b, \boldsymbol{V}_c, \boldsymbol{V}_d) = 0, \qquad \boldsymbol{C}(\boldsymbol{V}_\mu, \boldsymbol{V}^\mu, \boldsymbol{V}_a, \boldsymbol{V}_b) = 0,$$
$$\boldsymbol{C}(\boldsymbol{V}_\mu, \boldsymbol{V}_a, \boldsymbol{V}^\mu, \boldsymbol{V}_b) = 0, \qquad \boldsymbol{C}(\boldsymbol{V}_\mu, \boldsymbol{V}_a, \boldsymbol{V}_\mu, \boldsymbol{V}_b) = 0,$$

for all distinct μ, a, b, c, d, where $\{\boldsymbol{V}_a\} = \{\boldsymbol{V}_\mu, \boldsymbol{V}^\mu\}$ is the dual basis.

Remark 3.6 Using the notation set in Section 3.2, we note that $\boldsymbol{\theta}^\mu \wedge \boldsymbol{e}^0$ and $\boldsymbol{\theta}_\mu \wedge \boldsymbol{e}^0$ are also eigenspinors of $\hat{\boldsymbol{\phi}}$, and thus, of $\hat{\boldsymbol{C}}$ too, which implies that $C_{\mu 0}{}^{\mu 0}$ may not vanish. Hence, the above theorem also holds in odd dimensions, where the lower-case Roman indices may now take the value 0.

In four dimensions, the Weyl tensor $\hat{\boldsymbol{C}}$ splits into a SD part $\hat{\boldsymbol{C}}^+$ and an ADS part $\hat{\boldsymbol{C}}^-$. Each of $\hat{\boldsymbol{C}}^\pm$ has a pair of degenerate eigenvalues, and their eigen-2-forms are precisely the SD and ASD isotropic 2-planes

$$\left\{\boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2,\, \boldsymbol{\theta}_1 \wedge \boldsymbol{\theta}_2,\, \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}_1 + \boldsymbol{\theta}^2 \wedge \boldsymbol{\theta}_2\right\} \quad \text{and} \quad \left\{\boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}_2,\, \boldsymbol{\theta}_1 \wedge \boldsymbol{\theta}^2,\, \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}_1 - \boldsymbol{\theta}^2 \wedge \boldsymbol{\theta}_2\right\}.$$

In the language of general relativity, this is the defining property for the manifold to be of Petrov[1] type D. In fact, the existence of a conformal Killing 2-form $\boldsymbol{\phi}$ on a four-dimensional

[1] Although this classification applies mostly to Lorentzian manifolds, it can easily be extended to four-dimensional proper Riemmanian manifolds [32].

(Lorentzian) manifold implies [33–35] that the spacetime is of Petrov type D or N according to whether $\boldsymbol{\phi}$ is of rank 4 or of rank 2. A classification of the Weyl tensor in higher-dimensional Lorentzian spacetimes has been undertaken in [36–40], wherein the four-dimensional concept of *(gravitational) principal null direction* (GPND) is generalised to that of *Weyl aligned null directions* (WAND). It is shown in [25, 40] that the Kerr-NUT-(A)dS metric is of Petrov type D in an appropriate sense. More generally, the following statement, which answers a conjecture put forward in [24], is a direct consequence of Theorem 3.5.

Corollary 3.7 *Let $\boldsymbol{\phi}$ be a non-degenerate conformal Killing-Yano tensor with distinct eigenvalues, diagonal in the null basis $\{\boldsymbol{\theta}^a\}$. Then each of the basis vectors $\{\boldsymbol{V}_a\}$ is a WAND of the Weyl tensor $\boldsymbol{C}$. In particular, $\boldsymbol{C}$ is of type D.*

3.5 Relation to Hamiltonian 2-forms

Reference [41] introduces the notion of a *Hamiltonian* 2*-form* on a Kähler manifold, a $(1,1)$-form $\boldsymbol{\psi}$ which satisfies

$$\nabla_{\boldsymbol{X}}\boldsymbol{\psi} = \frac{1}{2}\left(\mathrm{d}\sigma \wedge \boldsymbol{J}(\boldsymbol{X}^*) - \boldsymbol{J}(\mathrm{d}\sigma)\wedge \boldsymbol{X}^*\right), \quad \left(\nabla_c \psi_{ab} = -\frac{1}{2}\left(\omega_{c[a}\nabla_{b]}\sigma + \sum_d g_{c[a}\, J_{b]}{}^{d}\nabla_d\sigma\right)\right) \tag{3.13}$$

for all vector fields $\boldsymbol{X}$. Here, ∇ is the Levi-Civita covariant derivative, and $\boldsymbol{J}$ the complex structure. Contracting equation (3.13) with the Kähler form $\boldsymbol{\omega}$ yields $\sigma = \mathrm{tr}_{\boldsymbol{\omega}}\,\boldsymbol{\psi}$, the trace of $\boldsymbol{\psi}$ with respect to $\boldsymbol{\omega}$.

In [42], a new class of five-dimensional toric Einstein-Sasaki manifolds is constructed by taking the BPS (supersymmetric) limit of a four-dimensional black hole solution similar to the Kerr-NUT-(A)dS metric. These limiting cases were later generalised to arbitrary dimensions in [15, 25]. Broadly, under the change of coordinates $x_\mu \to x_\mu = 1 + \varepsilon\xi_\mu$ and after linear redefinitions of the coordinates ψ_k and the constants, in the limit $\varepsilon \to 0$, the $2m$-dimensional Kerr-NUT-(A)dS metric becomes

$$\boldsymbol{g} = \sum_{\mu=1}^{m}\left(\tilde{\boldsymbol{e}}^\mu \odot \tilde{\boldsymbol{e}}^\mu + \tilde{\boldsymbol{e}}^{m+\mu}\odot \tilde{\boldsymbol{e}}^{m+\mu}\right)$$

where, in terms of the local coordinates $\{\xi_\mu, t_k\}$,

$$\tilde{\boldsymbol{e}}^\mu = \left(\frac{\Delta_\mu}{\Theta_\mu}\right)^{1/2}\mathrm{d}\xi_\mu, \qquad\qquad \tilde{\boldsymbol{e}}^{m+\mu} = \left(\frac{\Theta_\mu}{\Delta_\mu}\right)^{1/2}\sum_{k=1}^{m}\sigma_\mu^{(k)}\mathrm{d}t_k,$$

with

$$\Delta_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{m}(\xi_\nu - \xi_\mu), \qquad \sigma_\mu^{(k)} = \sum_{\substack{\nu_1<\nu_2<\ldots<\nu_k\\ \nu_i\neq\mu}} \xi_{\nu_1}\xi_{\nu_2}\ldots\xi_{\nu_k}, \qquad \sigma^{(k)} = \sum_{\nu_1<\nu_2<\ldots<\nu_k}\xi_{\nu_1}\xi_{\nu_2}\ldots\xi_{\nu_k},$$

and where $\Theta_\mu = \Theta_\mu(\xi_\mu)$ are functions of one variable. In this limiting case the almost-hermitian structure $\boldsymbol{\omega} = \sum_\mu \boldsymbol{e}^\mu \wedge \boldsymbol{e}^{m+\mu}$ becomes

$$\boldsymbol{\omega} = \sum_{k=1}^{m} \mathrm{d}\sigma^{(k)} \wedge \mathrm{d}t_k,$$

which is closed, and hence, $\boldsymbol{\omega}$ is Kähler.

The metric $\boldsymbol{g}$ is thus Kähler, and turns out to be Ricci-flat. Further, as pointed out in [25, 42], it is identical to the *orthotoric* metric that has been found independently in [41]. Such metrics are characterised by the existence of a Hamiltonian 2-form. The Hamiltonian 2-form for the above metric is given explicitly by

$$\boldsymbol{\psi} = \sum_\mu \xi_\mu \tilde{\boldsymbol{e}}^\mu \wedge \tilde{\boldsymbol{e}}^{m+\mu}. \tag{3.14}$$

The striking parallel between $*$-Killing 2-forms on Einstein manifolds and Hamiltonian 2-forms on Kähler manifolds leads us to the natural question of whether the latter play a rôle similar to that of the former in the context of integrable isotropic distributions. The answer to this question is yes, and the result follows directly by a computation of the components of the connection 1-form as in the proof of Theorem 3.1.

Theorem 3.8 *Let $(M, \boldsymbol{g}, \boldsymbol{J}, \boldsymbol{\omega})$ be a $2m$-dimensional Kähler manifold equipped with a non-degenerate Hamiltonian 2-form $\boldsymbol{\psi}$ with distinct eigenvalues. Then, the 2^m maximal isotropic distributions associated to $\boldsymbol{\psi}$ are integrable and define 2^m distinct complex structures.*

Proof. The defining equation (3.13) for the Hamilton 2-form gives in terms of the null basis

$$\partial_\kappa \psi_\mu{}^\nu + (\lambda_\mu - \lambda_\nu)\Gamma_{\kappa\mu}{}^\nu = -\frac{\mathrm{i}}{2}\delta_\kappa^\nu \partial_\mu \sigma \tag{3.15a}$$

$$(\lambda_\mu + \lambda_\nu)\Gamma^\kappa{}_{\mu\nu} = 0 \tag{3.15b}$$

$$(\lambda_\mu + \lambda_\nu)\Gamma_{\kappa\mu\nu} = 0 \tag{3.15c}$$

Equation (3.15a) implies further

$$\partial_\mu \lambda_\mu = -\frac{\mathrm{i}}{2}\partial_\mu \sigma \tag{3.16a}$$

$$\partial_\nu \lambda_\mu = 0 \qquad \text{for all } \nu \neq \mu \tag{3.16b}$$

$$(\lambda_\mu - \lambda_\nu)\Gamma_{\nu\mu}{}^\nu = -\frac{\mathrm{i}}{2}\partial_\mu \sigma \qquad \text{for all } \nu \neq \mu \tag{3.16c}$$

$$(\lambda_\mu - \lambda_\nu)\Gamma_{\kappa\mu}{}^\nu = 0 \qquad \text{for all distinct } \kappa,\ \mu,\ \nu. \tag{3.16d}$$

Since by assumption the eigenvalues are distinct, the integrability conditions (3.3) are satisfied. □

We also know [41, 43] that given a Hamiltonian 2-form $\boldsymbol{\psi}$, the 2-form $\boldsymbol{\phi}$ defined by

$$\boldsymbol{\phi} \equiv \boldsymbol{\psi} - \frac{1}{2}\sigma\boldsymbol{\omega}$$

is a conformal Killing 2-form. Such a $\boldsymbol{\phi}$ will not be closed in general. In fact,

$$\mathrm{d}\boldsymbol{\phi} = -\frac{3}{n-1}\boldsymbol{\omega} \wedge \boldsymbol{J}(\mathrm{d}^*\boldsymbol{\phi}), \tag{3.17}$$

so that $\boldsymbol{\phi}$ is closed iff it is co-closed iff it is parallel. On examining equation (3.7b) and the eigenvalues of $\boldsymbol{\phi}$, one can see that equation (3.17) is equivalent to the vanishing of the connection components $\Gamma^{\nu}{}_{\mu\nu}$ and $\Gamma_{\nu}{}^{\mu\nu}$ as implied by equation (3.16d). This thus provides an example of a non-closed conformal Killing-Yano tensor which gives rise to 2^m integrable complex structures.

4 Foliating spinors

In four dimensions, maximal isotropic planes correspond to spinors up to scale, [44], and so spinors provide an efficient and convenient calculus for studying such isotropic planes. In higher dimensions, spinors are less efficient as spin spaces grow in dimension exponentially, and the condition that a spinor is 'pure', i.e., that it corresponds to a maximal isotropic plane becomes non-trivial. Nevertheless, they form a natural formalism for understanding these structures. In particular, the generalized Kerr theorem [7] shows that maximally isotropic foliations of complexified flat space-time are in 1 : 1 correspondence with holomorphic m-surfaces in twistor space and this can be identified with the bundle of pure spinors over a Euclidean signature real slice. In the following we give a spinor formulation of our previous results. This will perhaps also be of benefit in answering other spinorial questions, such as, for example, the separation of variables in spinor equations in a space-time with a Killing-Yano tensor.

In this section, we first recall the basic facts of spin geometry. Our exposition is based on various sources [44–46], and we have harmonised the different approaches as far as possible. We then recast the previous results in the language of spinors.

4.1 The split signature model

Let V be an m-dimensional real vector space with dual V^*, and consider the direct sum $V \oplus V^*$ endowed with an inner product $\boldsymbol{g}$ of split signature (m, m). One can always find a null basis $\{\boldsymbol{\theta}_a\}_{a=1}^{2m} = \{\boldsymbol{\theta}_\mu, \boldsymbol{\theta}^\mu\}_{\mu=1}^{m}$ of $V \oplus V^*$, i.e.

$$\boldsymbol{g}(\boldsymbol{\theta}^\mu, \boldsymbol{\theta}_\nu) = \delta^\mu_\nu \qquad \text{and} \qquad \boldsymbol{g}(\boldsymbol{\theta}^\mu, \boldsymbol{\theta}^\nu) = 0 = \boldsymbol{g}(\boldsymbol{\theta}_\mu, \boldsymbol{\theta}_\nu),$$

for all μ, ν, so that the inner product is given by

$$\boldsymbol{g} = \sum 2\boldsymbol{\theta}^\mu \odot \boldsymbol{\theta}_\mu.$$

The canonical basis for the exterior algebra $\bigwedge^{\bullet}(V \oplus V^*)$ is induced from the basis of $V \oplus V^*$, and we will often use the notation

$$\begin{aligned}\boldsymbol{\theta}^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} &\equiv \boldsymbol{\theta}^{\mu_1} \wedge \ldots \wedge \boldsymbol{\theta}^{\mu_p} \wedge \boldsymbol{\theta}_{\nu_1} \wedge \ldots \wedge \boldsymbol{\theta}_{\nu_q}, \\ \boldsymbol{\theta}^{a_1 \ldots a_{p+q}} &\equiv \boldsymbol{\theta}^{a_1} \wedge \ldots \wedge \boldsymbol{\theta}^{a_{p+q}},\end{aligned}$$

where $1 \leq \mu_i, \nu_i, a_i \leq m$. We denote by $\mathbf{1}$ the basis element of $\bigwedge^0(V \oplus V^*) \cong \mathbb{R} \cong \bigwedge^0 V \cong \bigwedge^0 V^*$.

Remark 4.1 Any even-dimensional real vector space with a positive definite metric $\boldsymbol{g}$, once complexified, admits a splitting $V \oplus \bar{V}$ where the anti-holomorphic subspace $\bar{V}$ is isotropic and can be identified with the dual space V^* via the Hermitian inner product induced by $\boldsymbol{g}$. If $\{\boldsymbol{\theta}^a\}_{a=1}^{2m} \equiv \left\{\boldsymbol{\theta}^\mu, \bar{\boldsymbol{\theta}}^{\bar{\mu}}\right\}_{\mu,\bar{\mu}=1,\ldots,m}$ are the complex basis 1-forms, then

$$\boldsymbol{g} : \bar{\boldsymbol{\theta}}_{\bar{\mu}} \to \boldsymbol{\theta}^\mu,$$

where

$$\boldsymbol{g} = \sum 2\boldsymbol{\theta}^\mu \odot \bar{\boldsymbol{\theta}}^{\bar{\mu}}.$$

Similarly we can see that all of the subsequent results on $V \oplus V^*$ apply to all signatures on the understanding that they will need to be applied to the complexification of $V \oplus V^*$.

A subspace N of $V \oplus V^*$ such that $N \subseteq N^\perp$ is called *isotropic* and *maximal isotropic* when strict equality holds. Under the action of the Hodge duality operator, the space of all maximal isotropic subspaces splits into *self-dual* (SD) and *anti-self-dual* (ASD) components. When $V \oplus V^*$ is complexified we have a one-to-one correspondence between SD (ASD) maximal isotropics and orthogonal complex structures with positive (negative) orientation.

4.2 Spin representation

The spin representation $\mathbb{S}$ of the special orthogonal group $\mathrm{SO}(V \oplus V^*)$ is a 2^m-dimensional vector space, which splits into two 2^{m-1}-dimensional irreducible representations $\mathbb{S}^+$ and $\mathbb{S}^-$. These are the chiral spin representations of $\mathrm{SO}(V \oplus V^*)$. We shall give two alternative approaches to the theory of spinors, both of which will be used in the present paper according to the context.

The Clifford algebra can be regarded as a matrix algebra consisting of *γ-matrices* satisfying the Clifford equation

$$\gamma_a \gamma_b + \gamma_b \gamma_a = -2 g_{ab} \mathbf{I},$$

where $\mathbf{I}$ is the identity on $\mathbb{S}$. Introduce a basis $\{\boldsymbol{\theta}_\alpha\} = \{\boldsymbol{\theta}_A, \boldsymbol{\theta}_{A'}\}$ of $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$, so that lower-case Greek indices (beginning of the alphabet) running from 1 to 2^m refer to $\mathbb{S}$, and unprimed and primed upper-case Roman indices running from 1 to 2^{m-1} to $\mathbb{S}^+$ and $\mathbb{S}^-$

respectively, with $\alpha = A \oplus A'$, and similarly for the dual spin spaces. The action on each of the chiral spin spaces $\mathbb{S}^{\pm}$ can similarly be expressed in terms of 'reduced' $\hat{\gamma}$- and $\check{\gamma}$-matrices

$$\gamma_{a\alpha}{}^{\beta} = \begin{pmatrix} 0 & \hat{\gamma}_{aA}{}^{B'} \\ \check{\gamma}_{aA'}{}^{B} & 0 \end{pmatrix}$$

satisfying the relations

$$\hat{\gamma}_a \check{\gamma}_b + \hat{\gamma}_a \check{\gamma}_b = -2 g_{ab} \mathbf{I}^+ \qquad \text{and} \qquad \check{\gamma}_a \hat{\gamma}_b + \check{\gamma}_b \hat{\gamma}_a = -2 g_{ab} \mathbf{I}^-,$$

where $\mathbf{I}^{\pm}$ are the identity endomorphisms on $\mathbb{S}^{\pm}$. In the following statements, $\hat{\gamma}$- and $\check{\gamma}$-matrices may be substituted for γ-matrices in an appropriate way. Thus, one can express Clifford multiplication $\cdot$, i.e. the action of the Clifford group on $\mathbb{S}$, as follows: given a vector $\boldsymbol{V} = V^a \boldsymbol{\theta}_a$ and a spinor $\boldsymbol{\zeta}$, then

$$\boldsymbol{V} \cdot \boldsymbol{\zeta} = V^a \gamma_a \boldsymbol{\zeta}.$$

On the other hand, we note that $\mathbb{S}$ is isomorphic as a vector space to the exterior algebra $\bigwedge^{\bullet} V^*$. More precisely, $\mathbb{S}^+$ and $\mathbb{S}^-$ are isomorphic to $\bigwedge^{\text{even}} V^*$ and $\bigwedge^{\text{odd}} V^*$ with canonical bases $\{\mathbf{1}, \boldsymbol{\theta}^{\mu_1 \cdots \mu_p} : p \text{ even}\}$ and $\{\boldsymbol{\theta}^{\mu_1 \cdots \mu_p} : p \text{ odd}\}$ respectively. In this setting, Clifford multiplication is given explicitly by[2]

$$(\boldsymbol{X} + \boldsymbol{\xi}) \cdot \boldsymbol{\zeta} = -\boldsymbol{X} \lrcorner \boldsymbol{\zeta} + \boldsymbol{\xi} \wedge \boldsymbol{\zeta}$$

for all vectors $\boldsymbol{X} + \boldsymbol{\xi} \in V \oplus V^*$.

The advantage of the former approach is conciseness of notation when dealing with purely spinorial quantities. However, the latter provides more practical tools when it comes to computations in arbitrary dimensions. We shall set up a convenient dictionary between the two formalisms by identifying each of the basis elements $\{\boldsymbol{\theta}_\alpha\}$ with each of the basis elements $\{\mathbf{1}, \boldsymbol{\theta}^{\mu_1 \cdots \mu_p}\}$ of $\mathbb{S} \cong \bigwedge^{\bullet}$ as follows.

$$\begin{aligned} \boldsymbol{\theta}_\alpha &\leftrightarrow \mathbf{1}, \boldsymbol{\theta}^{\mu_1 \cdots \mu_p} && \text{for any } 1 \leq p \leq m \\ \boldsymbol{\theta}_A &\leftrightarrow \mathbf{1}, \boldsymbol{\theta}^{\mu_1 \cdots \mu_p} && \text{for any even } 1 \leq p \leq m \\ \boldsymbol{\theta}_{A'} &\leftrightarrow \boldsymbol{\theta}^{\mu_1 \cdots \mu_p} && \text{for any odd } 1 \leq p \leq m. \end{aligned}$$

One may regard the indices α, A, and A' as labels for a group of indices $\mu_0 \ldots \mu_p$. Both types of bases will be regarded as the canonical bases of $\mathbb{S}$, $\mathbb{S}^+$ and $\mathbb{S}^-$ induced from the basis of V.

Since the Clifford algebra is isomorphic to the exterior algebra $\bigwedge^{\bullet}(V \oplus V^*)$, one can extend the Clifford multiplication to any elements of $\bigwedge^{\bullet}(V \oplus V^*)$. Writing $\gamma_{a_1 \ldots a_p} = \gamma_{[a_1} \cdots \gamma_{a_p]}$, and employing the summation convention from hereon until the end of the subsection, then, for any p-form $\boldsymbol{\phi} = \phi_{a_1 \ldots a_p} \boldsymbol{\theta}^{a_1 \ldots a_p}$ and any spinor $\boldsymbol{\zeta}$, we have $\boldsymbol{\phi} \cdot \boldsymbol{\zeta} = \phi_{a_1 \ldots a_p} \gamma^{a_p \cdots a_1} \boldsymbol{\zeta}$. Of particular

[2]Our convention differs from [45] where the Clifford multiplication squares to *plus* the norm squared.

importance is the Lie algebra $\mathfrak{so}(V\oplus V^*)$, which is isomorphic to $\bigwedge^2(V\oplus V^*)$. Any element $\boldsymbol{\phi}$ of $\mathfrak{so}(V\oplus V^*)$ admits the decomposition

$$\boldsymbol{\phi}=\begin{pmatrix}\boldsymbol{A} & \boldsymbol{\beta}\\ \boldsymbol{B} & -\boldsymbol{A}^*\end{pmatrix}, \tag{4.1}$$

where $\boldsymbol{A}\in\operatorname{End}V$, and $\boldsymbol{\beta}\in\operatorname{Hom}(V^*,V)$ and $\boldsymbol{B}\in\operatorname{Hom}(V,V^*)$ are skew. The action of $\boldsymbol{\phi}$ on spin space $\mathbb{S}$ is then given by

$$\begin{aligned}
\boldsymbol{B}\cdot\boldsymbol{\zeta} &= B_{\mu\nu}\boldsymbol{\theta}^{\nu}\wedge(\boldsymbol{\theta}^{\mu}\wedge\boldsymbol{\zeta})=-\boldsymbol{B}\wedge\boldsymbol{\zeta} && (B_{\mu\nu}=B_{[\mu\nu]})\\
\boldsymbol{\beta}\cdot\boldsymbol{\zeta} &= \boldsymbol{\beta}^{\mu\nu}\boldsymbol{\theta}_{\nu}\lrcorner(\boldsymbol{\theta}_{\mu}\lrcorner\boldsymbol{\zeta})=\boldsymbol{\beta}\lrcorner\boldsymbol{\zeta} && (\beta^{\mu\nu}=\beta^{[\mu\nu]})\\
\boldsymbol{A}\cdot\boldsymbol{\zeta} &= A^{\nu}{}_{\mu}\boldsymbol{\theta}^{\mu}\wedge(\boldsymbol{\theta}_{\nu}\lrcorner\boldsymbol{\zeta})-\frac{1}{2}\operatorname{tr}\boldsymbol{A}\boldsymbol{\zeta}=\boldsymbol{A}^*\boldsymbol{\zeta}-\frac{1}{2}\operatorname{tr}\boldsymbol{A}\boldsymbol{\zeta} && (A^{\nu}{}_{\mu}=-A_{\mu}{}^{\nu}),
\end{aligned} \tag{4.2}$$

for any spinor $\boldsymbol{\zeta}$.

Spin space $\mathbb{S}$ is equipped with an inner product $<\cdot,\cdot>$ which is symmetric or antisymmetric according to m. This inner product descends to an inner product on each of the chiral spin spaces $\mathbb{S}^{\pm}$ when m is even, but it is degenerate on $\mathbb{S}^{\pm}$ when m is odd, in which case it gives rise to an isomorphism between a space of one chirality and the dual of the space of the opposite chirality, i.e. $\mathbb{S}^{\pm}\cong(\mathbb{S}^{\mp})^*$. In general, given any two spinors $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ one can define a p-form $\boldsymbol{\phi}$ by

$$\boldsymbol{\phi}=<\boldsymbol{\eta},\boldsymbol{\theta}^{a_1\dots a_p}\gamma_{a_1\dots a_p}\boldsymbol{\zeta}>.$$

4.3 Maximal isotropic planes and pure spinors

To any non-zero spinor $\boldsymbol{\zeta}$ one can associate an isotropic subspace given by

$$N(\boldsymbol{\zeta})=\{\boldsymbol{X}+\boldsymbol{\xi}\in V\oplus V^*:(\boldsymbol{X}+\boldsymbol{\xi})\cdot\boldsymbol{\zeta}=0\},$$

and any element $\boldsymbol{X}+\boldsymbol{\xi}\in N(\boldsymbol{\zeta})$ has the form

$$\boldsymbol{X}+\boldsymbol{\xi}=<\boldsymbol{\eta},\boldsymbol{\theta}_a\gamma^a\boldsymbol{\zeta}>$$

for some spinor $\boldsymbol{\eta}$. If $N(\boldsymbol{\zeta})$ is maximal we say that $\boldsymbol{\zeta}$ is a *pure* spinor. In particular, the 2^m basis elements $\{\boldsymbol{1},\boldsymbol{\theta}^{\mu_1\dots\mu_p}\}$ of $\mathbb{S}$ are pure with associated maximal isotropic planes

$$\begin{aligned}
N(\boldsymbol{1}) &= \{\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_m\}\oplus\{\boldsymbol{0}\},\\
N(\boldsymbol{\theta}^{\mu_1\dots\mu_p}) &= \left\{\boldsymbol{\theta}_{\mu_{p+1}},\dots,\boldsymbol{\theta}_{\mu_m}\right\}\oplus\{\boldsymbol{\theta}^{\mu_1},\dots,\boldsymbol{\theta}^{\mu_p}\},\\
N(\boldsymbol{\theta}^{1\dots m}) &= \{\boldsymbol{0}\}\oplus\left\{\boldsymbol{\theta}^1,\dots,\boldsymbol{\theta}^m\right\},
\end{aligned}$$

where $\mu_i\neq\mu_j$ for all $i\neq j$.

One can show that a spinor is pure if and only if it is chiral. We denote the space of $\pm$ chiral pure spinors by $\mathbb{T}^{\pm}$. The projective pure spinors $\mathbb{PT}^{\pm}$ are the spaces $\mathbb{T}^{\pm}$ defined up to scalings. There is a one-to-one correspondence between projective pure spinors of a given chirality and maximal isotropic planes of a given duality. The next proposition is a direct consequence of equation (4.2).

Proposition 4.2 *Let $\boldsymbol{\phi}$ be an element of $\mathfrak{so}(V \oplus V^*)$ such that $\boldsymbol{\phi}$ is diagonal in the null basis $\{\boldsymbol{\theta}^\mu, \boldsymbol{\theta}_\mu\}$, i.e.*

$$\boldsymbol{\phi} = \sum_\mu \lambda_\mu \boldsymbol{\theta}^\mu \wedge \boldsymbol{\theta}_\mu, \tag{4.3}$$

for some λ_μ. Then the eigenspinors of $\boldsymbol{\phi}$ regarded as an endomorphism on $\mathbb{S}$ are simply the basis elements of $\bigwedge V^* (= \mathbb{S})$, i.e.

$$\boldsymbol{\phi} \cdot \mathbf{1} = \tilde{\lambda}_0 \mathbf{1}, \qquad \boldsymbol{\phi} \cdot \boldsymbol{\theta}^{\mu_1 \ldots \mu_p} = \tilde{\lambda}_{\mu_1 \ldots \mu_p} \boldsymbol{\theta}^{\mu_1 \ldots \mu_p}$$

where the eigenvalues $\tilde{\lambda}_0$, $\tilde{\lambda}_{\mu_1 \ldots \mu_p}$ are given in terms of the eigenvalues λ_μ by

$$\tilde{\lambda}_0 = -\frac{1}{2} \sum_\mu \lambda_\mu, \qquad \tilde{\lambda}_{\mu_1 \ldots \mu_p} = -\frac{1}{2} \sum_\mu (-1)^\epsilon \lambda_\mu$$

where $\epsilon = \sum_{i=1}^p \delta_\mu^{\mu_i}$. It follows that the eigenspinors of $\boldsymbol{\phi}$ are pure.

4.4 Twistor bundle and integrability condition

Let M be a real $2m$-dimensional (pseudo-) Riemannian spin manifold so that at each point p, its complexified tangent space $\mathbb{C} \otimes T_p M$ can be given the structure of $\mathbb{C} \otimes (V \oplus V^*)$. The preceding sections translate into the language of bundles in the obvious way so that V, $\mathbb{S}$, $\mathbb{T}$, etc... will now refer to bundles over the complexification $M_\mathbb{C}$ of M. We extend the Levi-Civita covariant derivative ∇ to a covariant derivative on the spin bundle $\mathbb{S}$ – also denoted ∇. For any basis spinor field $\boldsymbol{\theta}$ we have

$$\nabla \boldsymbol{\theta} = -\frac{1}{2} \sum \boldsymbol{\Gamma}_{ab} \boldsymbol{\theta}^a \wedge \boldsymbol{\theta}^b \cdot \boldsymbol{\theta}, \tag{4.4}$$

where ${\boldsymbol{\Gamma}_a}^b$ is the Levi-Civita connection 1-form on T^*M, and $\boldsymbol{\Gamma}_{ab} = \boldsymbol{\Gamma}_{[ab]}$.

Maximal isotropic distributions of TM are in one-to-one correspondence with orthogonal almost complex structures on $\mathbb{C} \otimes TM$ and sections of the projective twistor bundle $\mathbb{PT}$ over $M_\mathbb{C}$ (i.e. projective pure spinor fields on M). The Frobenius integrability condition can then be articulated as follows.

Proposition 4.3 *A maximal isotropic distribution or its associated orthogonal almost complex structure is integrable if and only if the associated projective pure spinor field $\boldsymbol{\zeta}$ satisfies*

$$(\nabla_{\boldsymbol{X}} \boldsymbol{\zeta}) \wedge \boldsymbol{\zeta} = 0, \quad \text{i.e.} \quad \nabla_{\boldsymbol{X}} \boldsymbol{\zeta} = f_{\boldsymbol{X}} \boldsymbol{\zeta}, \tag{4.5}$$

for all vector fields $\boldsymbol{X} \in \Gamma(N_{\boldsymbol{\zeta}})$ and for some function f on the manifold depending on $\boldsymbol{X}$. In other words, integrable orthogonal almost complex structures correspond to holomorphic sections of the projective twistor bundle.

If one applies equation (4.5) to all the (projective) basis elements of the spin bundle and adopts the following convention

$$\Gamma_{ca}{}^{b} = \boldsymbol{\theta}_c \lrcorner \boldsymbol{\Gamma}_a{}^{b}$$

for the components of $\boldsymbol{\Gamma}_a{}^{b}$, one then obtains

Proposition 4.4 *All* 2^m *projective basis elements* $\mathbf{1}$, $\boldsymbol{\theta}^{\mu_1 \cdots \mu_p}$ *of the projective twistor bundle* $\mathbb{PT} \subset \mathbb{PS} \cong \mathbb{P}(\bigwedge^\bullet TV^*)$ *are integrable if and only if the connection components*

$$\begin{aligned} &\Gamma_{\kappa\mu\nu},\, \Gamma^{\kappa\mu\nu}, && (\text{for all } \mu, \nu, \kappa), \\ &\Gamma_{\kappa}{}^{\mu\nu},\, \Gamma^{\kappa}{}_{\mu\nu}, && (\text{for all } \kappa \neq \mu, \nu), \\ &\Gamma_{\kappa\mu}{}^{\nu},\, \Gamma^{\kappa}{}_{\nu}{}^{\mu}, && (\text{for all } \nu \neq \kappa, \mu), \end{aligned} \tag{4.6}$$

all vanish.

Remark 4.5 We note that equations (4.6) are equivalent to equations (3.2) obtained by the Frobenius integrability condition.

By Proposition 4.2, we have

Corollary 4.6 *The eigenspinors of any spin endomorphism on* $\mathbb{S}$ *of the form* (4.3) *are integrable if and only if the components of the connection* 1*-form* (4.6) *all vanish.*

We can then reformulate Theorem 3.1 as follows.

Theorem 4.7 *Let* M *be a* $2m$*-dimensional spin manifold equipped with a conformal Killing-Yano tensor* $\boldsymbol{\phi}$ *as in Theorem 3.1. Then the* 2^m *eigenspinors of* $\boldsymbol{\phi}$ *are integrable.*

4.5 Weyl curvature restrictions revisited

We can also give a spinorial articulation of Theorem 3.5 in the same vein as [44]. Denote by $\boldsymbol{\Psi}$ and by $\boldsymbol{\Psi}^{\pm}$ the completely traceless elements of $\bigodot^2 \mathrm{End}(\mathbb{S})$ and $\bigodot^2 \mathrm{End}(\mathbb{S}^{\pm})$ corresponding to the Weyl tensor $\boldsymbol{C}$ viewed as an element of $\bigodot^2 \mathfrak{so}(n)$. In spin components,

$$\Psi^{+}{}_{AC}^{BD} \equiv \hat{\gamma}^{a}{}_{A}{}^{E'} \check{\gamma}^{b}{}_{E'}{}^{B} C_{abcd} \hat{\gamma}^{c}{}_{C}{}^{F'} \check{\gamma}^{d}{}_{F'}{}^{D} \quad \text{and} \quad \Psi^{-}{}_{A'C'}^{B'D'} \equiv \check{\gamma}^{a}{}_{A'}{}^{E} \hat{\gamma}^{b}{}_{E}{}^{B'} C_{abcd} \check{\gamma}^{c}{}_{C'}{}^{F} \hat{\gamma}^{d}{}_{F}{}^{D'},$$

where we have used the summation convention. Theorem 3.5 can then be reformulated in spinorial terms:

Theorem 4.8 *Let* $\boldsymbol{\phi}$ *be a non-degenerate conformal Killing-Yano tensor with distinct eigenvalues diagonal in the null basis* $\{\boldsymbol{\theta}_a\}$*. Then each of the eigenspinors* $\boldsymbol{\theta}_A$, $\boldsymbol{\theta}_{A'}$ $(A, A' = 1, \ldots, 2^{m-1})$ *of the corresponding spin endomorphisms satisfies (no summation)*

$$\boldsymbol{\Psi}^{+}(\boldsymbol{\theta}_A, \boldsymbol{\theta}_A) \wedge \boldsymbol{\theta}_A = 0 \qquad \text{and} \qquad \boldsymbol{\Psi}^{-}(\boldsymbol{\theta}_{A'}, \boldsymbol{\theta}_{A'}) \wedge \boldsymbol{\theta}_{A'} = 0. \tag{4.7}$$

In four dimensions, each spin space $\mathbb{S}^{\pm}$ is a two-dimensional complex vector space equipped with a symplectic inner product, so that any symmetric spinor of valence p is fully decomposable as a symmetric product of p spinors of valence 1. In particular, the $\boldsymbol{\Psi}^{\pm}$ admit such a decomposition, and equations (4.7) are equivalent to

$$\boldsymbol{\Psi}^{+} = \Psi_2 \boldsymbol{\theta}^1 \odot \boldsymbol{\theta}^1 \odot \boldsymbol{\theta}^2 \odot \boldsymbol{\theta}^2,$$

where $\Psi_2 \equiv \Psi_{1122}$, and similarly for $\boldsymbol{\Psi}^{-}$. By the Goldberg-Sachs theorem, it follows that each of the spinors $\boldsymbol{\theta}_A$, $\boldsymbol{\theta}_{A'}$, $(A, A' = 1, 2)$, satisfies the integrability condition (4.5).

Remark 4.9 A possible classification of $\boldsymbol{\Psi}^{\pm}$ in six dimensions and its relation with integrable spinors were first investigated in [47–49].

4.6 Spin bundle over odd-dimensional manifolds

Let M be a $(2m+1)$-dimensional Riemannian spin manifold. Then, the complexification $T_{\mathbb{C}}M$ of the tangent bundle admits a splitting $V \oplus V^* \oplus K$, where V and V^* are m-dimensional vector bundles dual (and conjugate) to each other, and K is a complex line bundle. The spin bundle $\mathbb{S}$ over M is now irreducible and isomorphic to $\bigwedge V^*$. Maximal isotropic distributions of $V \oplus V^*$ correspond to sections of the projective twistor (or pure spinor) bundle $\mathbb{PT}$ over M, or equivalently, to orthogonal almost complex structures on $V \oplus V^*$. When such an isotropic distribution is integrable, the corresponding section of $\mathbb{PT}$ is holomorphic, and the corresponding almost CR structure of M integrable. The eigenspinors of a conformal Killing-Yano tensor $\boldsymbol{\phi}$ are precisely the basis elements of the spin bundle $\mathbb{S}$ induced from the basis of V^*. Now, assuming that $\boldsymbol{\phi}$ has distinct eigenvalues, Theorem 4.7 extends naturally to odd-dimensional manifolds.

5 Concluding remarks and applications

5.1 Intersection of foliations and reality conditions

To obtain similar results on a real pseudo-Riemannian manifold M, it suffices to impose suitable reality conditions on the complexified tangent bundle $T_{\mathbb{C}}M$. Given a real manifold equipped with a pseudo-Riemannian metric of signature s, the intersection of an integrable maximal isotropic distribution $\mathcal{D}$ and its (integrable) conjugate $\bar{\mathcal{D}}$ gives rise [7] to an integrable real isotropic distribution $K = \mathcal{D} \cap \bar{\mathcal{D}}$ whose rank can be any of $(2m - |s|)/2$ modulo 2 (in [7] it was claimed in error that the rank is always $(2m - |s|)/2$). As we have shown earlier, in the complexification, the integral surfaces of $\mathcal{D}$ are totally geodesic, and so therefore are those of K. In the case of Lorentzian manifolds, where $|s| = 2m - 2$, the foliation K is 1-dimensional and tangent to a congruence of null geodesics and the *screen space* $K^{\perp}/K$ of K inherits the complex structure on $T_{\mathbb{C}}M$ from $\mathcal{D}$ that is Lie derived along the congruence [7, 50]. In four dimensions, the preservation fo the complex structure on the screen space is equivalent to the *shear-free* condition, i.e., the preservation of the *conformal* structure of

$K^{\perp}/K$ along K. This is a consequence of the fact that complex structures and conformal metrics on a surface are the same. However, this is not true in higher dimensions, and a six-dimensional counter-example invalidating this equivalence between being shear-free and preserving a complex structure in higher dimensions is given in [51].

Spinorially, the real structure of M induces a complex conjugation $\mathcal{C}$ on the spin bundle, which preserves each of the chiral spin bundles when $s/2$ is even, and interchanges them when $s/2$ is odd. Depending on s, $\mathcal{C}$ may be quaternionic [44], i.e. $\mathcal{C}^2 = -1$. In the Lorentzian case, a real vector field $\boldsymbol{k}$ as a section of the isotropic line bundle K as defined above can then be expressed as[3]

$$\boldsymbol{k} = < \bar{\boldsymbol{\zeta}}, \boldsymbol{e}_a \gamma^a \boldsymbol{\zeta} >, \tag{5.1}$$

where $\boldsymbol{\zeta}$ is an integrable pure spinor, and $\bar{\boldsymbol{\zeta}}$ its conjugate under $\mathcal{C}$. In dimensions greater than six, the choice of the spinor $\boldsymbol{\zeta}$ for $\boldsymbol{k}$ is no longer unique. A more general treatment of real structures and spinors is given in [52].

5.2 The Kerr-Schild ansatz

The Kerr-NUT-(A)dS metric given in [15] has a long history that originates in the four-dimensional Kerr-Schild ansatz found in [1] in 1963. The original aim of Kerr's paper was to construct a solution to Einstein's equations which is of Petrov type D with a twisting (i.e. hypersurface orthogonal) GPND $\boldsymbol{k}$. It turns out that the newly-found metric is an *exact* first-order perturbation of the flat metric, and describes a rotating black hole of mass M:

$$\boldsymbol{g} = \tilde{\boldsymbol{g}} + \frac{2M}{U} (\boldsymbol{k})^2 . \tag{5.2}$$

where $\tilde{\boldsymbol{g}}$ is the Minkowski background metric, U some function, and $\boldsymbol{k}$ is a shear-free isotropic geodesic (real) vector field with respect to both $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$. Since $\boldsymbol{k}$ is shear-free, it belongs to a maximal isotropic integrable distribution of complexified Minkowski spacetime, and thus has the form (5.1).

The Kerr-Schild ansatz (5.2) has been generalised in higher dimensions in Lorentzian signature $(1, 2m-1)$ in [5]. The background metric $\tilde{\boldsymbol{g}}$ is now allowed to be a pure (A)dS $2m$-dimensional metric. The vector field $\boldsymbol{k}$ is again isotropic and geodesic with respect to both $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$. However, as shown in [39], it fails to retain its shear-free property in dimensions greater than four. There does, however, remain the question of whether $\boldsymbol{k}$ arises from one of our integrable maximal isotropic distributions (i.e., pure spinors). For the Kerr-NUT-(A)dS metric one can follow the various coordinates transformations leading from the Kerr-Schild ansatz to the Kerr-NUT-(A)dS metric in [5, 15], and we find that the *real* vector $\boldsymbol{k}$ of the former is the same (up to factor) as one of the *complex* basis vectors $\{\boldsymbol{\theta}_\mu, \boldsymbol{\theta}^\mu\}$ of the latter. By Theorem 3.1, the eigenspinors of the closed conformal Killing-Yano tensor (1.1) on the

[3] This is *always* possible and guaranteed by the various properties of the spin inner product and the conjugation in different dimensions and signatures.

Kerr-NUT-(A)dS metric are integrable, so that each $\boldsymbol{\theta}_\mu$ (and each $\boldsymbol{\theta}^\mu$) will belong to (an intersection of) some integrable maximal isotropic distributions, and we can write

$$\boldsymbol{\theta}_\mu = < \boldsymbol{\eta}, \boldsymbol{\theta}_a \gamma^a \boldsymbol{\zeta} >,$$

for some integrable (pure) basis spinors $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$. Again, there is some freedom in the choice of spinors. Since there is no fundamental difference between the complexified Kerr-Schild metric and Kerr-NUT-(A)dS metric, it follows that by imposing a suitable reality condition, $\boldsymbol{k}$ will have the form (5.1) in all (even) dimensions.

Remark 5.1 It is shown in [6] how one can obtain an m-Kerr-Schild ansatz in split signature (m, m) from the Kerr-NUT-(A)dS metric by Wick rotating the coordinates ψ_k. After some work, we have

$$\boldsymbol{g} = \tilde{\boldsymbol{g}} - \sum_{\mu=1}^{m} \frac{2 b_\mu x_\mu}{U_\mu} \left(\boldsymbol{k}_{(\mu)} \right)^2,$$

where $\tilde{\boldsymbol{g}}$ is the background pure (A)dS metric, the b_μ are the mass and the NUT parameters, and the functions U_μ and coordinates x_μ are as for the Kerr-NUT-(A)dS metric. The m real vectors $\boldsymbol{k}_{(\mu)}$ are linearly independent, mutually orthogonal, isotropic and geodesic with respect to both $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$. Thus, they span a *real* maximal isotropic distribution. In fact, all 2^m maximal isotopic distributions arising from the closed conformal Killing-Yano tensors are real. The spin bundles over M are also real, and, using the same argument as above, each $\boldsymbol{k}_{(\mu)}$ can be expressed as

$$\boldsymbol{k}_{(\mu)} = < \boldsymbol{\eta}_{(\mu)}, \boldsymbol{e}_a \gamma^a \boldsymbol{\zeta}_{(\mu)} >,$$

for some appropriate choice of spinors $\boldsymbol{\zeta}_{(\mu)}$ and $\boldsymbol{\eta}_{(\mu)}$.

The odd-dimensional versions of the above metrics are similar and share the same properties as their even-dimensional counterparts.

5.3 The Kerr Theorem

The Kerr Theorem provides a systematic method of finding shear-free null geodesic vector fields arising from an integrable almost complex structure in complexified Minkowski spacetime. The original theorem consists in solving $F = 0$ where F is a certain holomorphic function of the *complexified* isotropic flat coordinates [1, 53]. Penrose gave the Kerr theorem a new and more geometric formulation by realizing F as a function on twistor space in his original paper on twistor geometry [54]. A generalisation to higher dimensions was given in [7]. Essentially, it states that a pure spinor field on a (complexified) flat $2m$-dimensional manifold M equipped with a conformal metric is integrable if and only if it can be determined by the intersection of an m-dimensional analytic surface and the set of projective pure spinor spaces in twistor space representing a region of M. This surface is defined by

$m(m-1)/2$ homogeneous holomorphic functions on twistor space. In the context of the four-dimensional type D, e.g., the Lorentzian Kerr-NUT metric, the integrable spinor field is determined by a single quadratic function constructed from the angular momentum twistor. In higher dimensions, the co-dimension and hence the number of such functions increases quadratically with the dimension (being the dimension of the space of pure spinors), and a characterization of the structure has yet to emerge.

5.4 Degenerate conformal Killing 2-forms and conformal Killing spinors

As pointed out above, a degenerate conformal Killing-Yano tensor on a four-dimensional Lorentzian manifold implies that the Weyl tensor is of Petrov type N. By the Goldberg-Sachs theorem, this spinor is integrable. In fact, on an n-dimensional Riemannian manifold, such a conformal Killing-Yano tensor arises as the 'squaring' of a *conformal Killing spinor* or *twistor spinor*, i.e. a spinor $\boldsymbol{\zeta}$ which satisfies the twistor equation

$$\nabla_{\boldsymbol{X}}\boldsymbol{\zeta} + \frac{1}{n}\boldsymbol{X}\cdot \not{D}\,\boldsymbol{\zeta} = 0,$$

for all vector fields $\boldsymbol{X}$, where $\not{D}$ is the Dirac operator [43, 55]. When $\boldsymbol{\zeta}$ satisfies the Dirac equation $\not{D}\,\boldsymbol{\zeta} = \lambda\boldsymbol{\zeta}$ for some function λ, $\boldsymbol{\zeta}$ is called a *Killing spinor*. In the special case where $\lambda \equiv 0$, $\boldsymbol{\zeta}$ is a *parallel spinor*. Clearly, a pure Killing spinor automatically satifies the integrability condition (4.5). A conformal Killing spinor must also satisfy the integrablity condition

$$\boldsymbol{C}(\boldsymbol{X},\boldsymbol{Y})\cdot\boldsymbol{\zeta} = 0, \tag{5.3}$$

for all vector fields $\boldsymbol{X}$, $\boldsymbol{Y}$.

One can show [43, 56] that given two conformal Killing spinors $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$, the 2-form defined by

$$\boldsymbol{\phi} =< \boldsymbol{\eta}, \boldsymbol{\theta}^{ab}\gamma_{ab}\boldsymbol{\zeta} >$$

is a conformal Killing-Yano tensor[4]. Conformal Killing spinors have been extensively studied, and we refer the reader to the literature (e.g. [55] and references therein) for details.

References

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