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An early algebra approach to pattern generalisation: Actualising the virtual through words, gestures and toilet paper

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Abstract This paper focuses on pattern generalisation as a way to introduce young students to early algebra. We build on research on patterning activities that feature, in their work with algebraic thinking, both looking for sameness recursively in a pattern (especially figural patterns, but also numerical ones) and conjecturing about function-based relationships that relate variables. We propose a new approach to pattern generalisation that seeks to help children (grades 2 and 3) work both recursively and functionally, and to see how these two modes are connected through the notion of variable. We argue that a crucial change must occur in order for young learners to develop a flexible algebraic discourse. We draw on Sfard's (2008) communication approach and on Châtelet's (2000) notion of the *virtual* in order to pursue this argument. We also root our analyses within a new materialist perspective that seeks to describe phenomena in terms of *material entanglement*, which include, in our classroom research context, not just the children and the teacher, but also words, gestures, physical objects and arrangements, as well as numbers, operations and variables.

Keywords Discourse · Generalisation · Gesture · Materialism · Patterns · Variable · Virtual

1 Introduction

This paper centres on pattern generalisation as a way to introduce young students to algebraic thinking. Many researchers have used patterns as a basis for developing algebraic language

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and thinking, despite criticisms of their narrow opportunities for (formal) mathematical generalisation (Carraher, Martinez, & Schliemann, 2008). Indeed, Carraher et al. stress that mathematical generalisation should centrally involve conjecturing as well as working with variables and functions, which a pattern approach can enable. Radford (2008) has shown that the use of patterns can help students move from arithmetic to algebraic generalisations. According to Radford, the latter involves three main processes:

> Generalizing a pattern *algebraically* rests on the capability of *grasping* a commonality noticed on some particulars (say p_1, p_2, p_3, …, p_k); extending or generalizing this commonality to all subsequent terms (p_{k+1}, p_{k+2}, p_{k+3}, …), and being able to use the commonality to provide a direct *expression* of any term of the sequence. (*italics in original*, p. 84)

First, a "local commonality" is noticed, amongst a few terms. Then the local commonality is extended to all the terms of the sequence. Radford underscores a crucial aspect of generalising, which relates to "the manner in which we come to notice the *same* and the *different*" (p. 83). As he and other researchers have shown, students tend to look for sameness recursively, that is, by finding the change that relates each term p_{k+1} of a sequence to the previous term p_k. When using figural sequences, the saming is done by comparing spatial configurations. Whereas the direct expression may account for a structural commonality *across* the terms of a figural sequence, it does not necessarily acknowledge the relationship *between* the position of a term in a sequence and its associated value regarding their explicit covariance.

In their work on developing young learners' mathematical generalisation, Carraher et al. (2008) focus specifically on this *between* relationship. They use single geometric arrangements that are explicitly described in terms of variables (the number of tables or the number of chairs in a given configuration) and ask students to make conjectures about the function-based relationships that relate the variables. Their goal is to help students recognise the independent variable as a variable, rather than just an indicator of position. This is a crucial aspect for being able to talk about the covariance of variables, which is the basis of a dynamic conception of function as two variables change together, one depending on the other.

In this paper, we propose a new approach to generalisation that seeks to help students work with both kinds of generalisation and see how they are connected through the notion of variable. We propose this approach in the context of research undertaken with grades two and three students who were asked to work with figural and numerical sequences. Our aim is to highlight a crucial change that must occur in order for young learners to develop a flexible algebraic discourse. In order to highlight this change, we draw both on Sfard's (2008) communication approach, which enables us to describe the different types of discourse involved in pattern generalisations, and on Châtelet's (2000) notion of the virtual, which accounts for the way in which the children's new mathematical discourse emerges from classroom interactions. More broadly, we adopt the "inclusive materialist" perspective (de Freitas & Sinclair, 2014), which seeks to de-centre the human as the agent of all action and embraces the varieties of 'bodies' involved in a classroom interaction including the body of mathematics. As we elaborate below, this theoretical perspective assumes a material entanglement between the various "bodies", which enables us to understand how the mathematical concept of variable can emerge, and is inextricable from, the words, gestures and actions on the physical objects in the classroom.

2 Research on generalising through patterns and sequences

In the last two decades, research studies in mathematics education have been exploring the use of patterning activities as a way of introducing early algebraic thinking, with particular focus on geometric patterns and figural sequences (e.g., Carraher, Martinez, & Schliemann, 2008; Moss & Beatty, 2010; Radford, 2010; Rivera, 2011). These studies show that students tend to use recursive strategies in order to describe generalisations, rather than the direct functional relationship between the variables involved. In working with grade 2 and 4 students, As Moss and Beatty (2010) point out, "[w]hile recursive strategies allow students to predict what comes in the next couple of positions of a series, it does not foster the ability to perceive the (structural) relationship across the two data sets to find the underlying rule" (p. 16), nor to see these sets as domain and co-domain of a function.

Indeed, according to Radford (2012), the awareness of a structural understanding is a crucial aspect of the emergence of algebraic thinking. Using figural sequence tasks, Radford (2010) investigates the types of algebraic thinking that can be made accessible to grade 2 children before they are introduced to any notation or symbolism. He shows that the children could express the rule for determining the number of objects at different steps in the sequence "in action". For example, they were able to find the number of objects in the figures for steps 12 and 50 in terms of "12 plus 12, plus 1" and "50 plus 50, plus 1", respectively. For Radford, this is an initial level of algebraic thinking, in which the concept of variable is "tacit" in that it is *not explicitly* stated. He then shows how children can make the variable explicit by describing the calculation needed to find an *unknown* step in the sequence in terms of "a number plus a number, plus 1". In their move from a tacit to an *explicit* use of variable, Radford concludes that "the spatial meaning of the unknown is overcome" (p. 79), which means that the children are no longer focused on the figural structure.

Carraher, Martinez, and Schliemann (2008) examine grade 3 students' making of generalisations about geometrical arrangements as they are introduced to linear functions. The tasks involved the problem of seating guests at a dinner party. Students were told the conventions for placing seats around the tables and asked to explore the relationship between the number of tables and the corresponding number of seats. When working with variables, Carraher et al. pursue a functional approach in which the variable is not simply an index of a step number, but is also both *direct and reversible*. In other words, the variable is not simply used to replace particular values (such as 12 and 50), but is also used to move directly from talk about steps to talk about number of objects. This enables the relation between these two quantities to be reversed.

But, even when the "underlying rule" is found, or the step number is seen as a "direct and reversible variable", the understanding of the pattern might rely upon recognition of a correspondence between the set of the sequence terms and the set of their positions. In fact, beyond *recursive patterning*, Smith (2008) outlines two other modes of analysing patterns and relationships: *correspondence relationship* and *covariational thinking*. While the former is based on identifying a correlation between variables, the letter is based on analysing how the two quantities involved vary simultaneously and keeping that change as an explicit, dynamic part of the pattern's description.

Our work was influenced by the research that involves the use of figural sequences, as well as numerical ones, but also sensitive to the complex nature of the variable, which we see as a dynamic, named number that can take on different possible values—and, in particular, to the potential for a more powerful use of the variable as evidenced in the studies described above.

3 Theoretical considerations

In summary, the literature suggests that young children encounter challenges in working algebraically both in terms of how they perceive sequences (structurally or recursively) and how they make use of variables (tacit, explicit, direct, reversible). We would like to conceptualise these challenges within Sfard's (2008) communication approach, which conceptualises learning in terms of a *change* in discourse. In other words, a change in the way that students communicate (using written or spoken language, as well as gestures) implies a change in the way that the student is thinking. Sfard proposes the following four characteristics of discourse that define mathematics distinguished from other discourses: word use, visual mediators, routines and endorsed narratives (see Kim, Ferrini-Mundi, & Sfard 2012; Sfard 2009). In this paper, we will be particularly interested in changes in students' word uses and routines as they relate to describing and predicting patterns.

The review of the literature highlights two aspects of the discourse on algebra that is at play in the kind of generalising described above. The first involves *routines*, which are defined by Sfard as the collection of meta-level rules characterising repetitive patterns in discourse. The routines for generalising sequences seem to involve iteration at first, where one element of a sequence is defined in terms of the preceding one, with little attention to the position of that element in the sequence. This *iterative* or *recursive* routine may change to a *structural* one in which children focus on how the position number of a figure can help describe the shape of the structure. This routine is *one-way* in that it begins always with the position number. It is also *heterogeneous* in that it produces a description (such as "a number plus a number plus 1") based on the position number. The *two-way, homogeneous* routine of generalisation involves using numbers as both inputs and outputs. It also involves a kind of "saming" in that the position numbers become the same kind of number as the attribute numbers. It is two-way in that it permits a direct transformation from a position number to a sequence number and vice versa. Thus, of the three routines described here—recursive, one-way heterogeneous and two-way homogeneous, we see the latter as being the most characteristics of an algebraic discourse[1].

The second aspect of the mathematical discourse involves *word use*, which relates to the particular, mathematical ways in which certain words are used (often differently than in an everyday discourse). In particular, we see "number" being used in a variety of ways by the children described in the literature above. First, there are the numbers used to describe positions (the 3rd figure, Fig. 3 or 12th step); there are the numbers that are used as attributes of component of a figure (as in "12 plus 12 plus 1"); and there are the numbers that are used as attributes of a given figure (the twelfth step has 25 circles). When Radford speaks of a tacit use of variable, he is describing the way in which children use position number as part of a statement about a figure's numerical attribute. Rather than tacit, we prefer to call this a *numerical* use of variable. When children begin to speak in terms of "A number plus a number, plus 1" they are now naming their numbers, which is, in effect, a *parametric* use of

[1] The recursive routine is a legitimate way to look at the pattern, insofar as recursion is a legitimate way to define a pattern or a linear function. But, thinking of the development of early algebraic discourse, the one-way and two-way routines are the only ones that involve the variable not simply as an index of a step number, but as the independent variable that determines a covariance view of the pattern.

variable (insofar as parameters are named numbers). The use of variable that is evoked in Carraher's et al.'s study can be seen as a *functional* one in the sense that it establishes not only the direct link between the position number and the attribute number—which might rest on a reification of function in terms of input/output machine or correspondence between two sets of numbers (see Slavit (1997) for a description of different reifications of function)—but also the manner in which they change together, that is, their covariance as dependent and independent variables (the covariational thinking pointed from Smith). The complexity here is not only about the nature of the variable and the fact of seeing a variable as an ordinal number, but also concerns the nature of generalisation in relation to functional thinking that can emerge associated to the pattern (e.g., Blanton & Kaput, 2011; Carraher, Schliemann, & Schwarz, 2008).

While Sfard's approach provides a powerful way of operationalising thinking and learning, its dialogical assumptions, which see concepts has being discourse-dependent, fail to fully address the role of the body and the physical environment in producing new ways of thinking/talking. This is due, in part, to the strict separation of the mathematical, as a language-based discourse, from the physical. Theorists of embodied cognition, in particular Radford (2014) and Roth (2011), have argued that such bodily actions and physical environments can play a significant role in mathematical cognition. Taking these ideas to the extreme, de Freitas and Sinclair's (2014) *inclusive materialism* posits a flat ontology in which mathematical concepts are also considered to be material.

Inclusive materialism draws heavily on Châtelet's (2000) notion of the *virtual*, which is the key concept that enables us to understand how 'real mathematical knowledge' is fundamentally materially embodied. This concept was developed by Deleuze (1994) as a new ontological category for that which is indeterminate and progressing historically—for that which is becoming. The virtual thus contrasts with the *possible*, which is merely awaiting realization. Indeed, the virtual is the genetic ground of the actual. In order to avoid some kind of identity between the virtual and its actualising, which would bring us back to some kind of Platonism, Deleuze insists on the multiplicity of the virtual.

According to Châtelet, the virtual is involved in any inventive act in mathematics as it heralds the new (a new object like the point at infinity; a new relationship like the distance between two lines). But the virtual is not manufactured by some mechanism in the mind or some deterministic process of abstraction. It partakes of the real; it is engendered through gestures and diagrams that Châtelet describes as "cutting out" new spaces or dimensions in any surface or material site. These gestures and diagrams actualise the virtual, bringing forth the new, unexpected and unscripted (Sinclair, de Freitas, & Ferrara, 2013). Bodily engagement and diagrams do not merely *represent* mathematical ideas that can then be communicated from one person to another—rather, we take these actions as *actualisations*, as ideas and thoughts, as creative interventions. We will show how changes in discourse that occur during the two classroom sessions involves an *actualisation of the virtual* that enables the creation of a new relationship between the position number and the sequence number.

We recognise that the combination of Sfard's approach with inclusive materialism poses some ontological challenges. We use the communicational approach mainly as a method that helps us address changes in language use, while taking from inclusive materialism the ontological assumptions that allow us to account for the way in which the material entanglements can produce mathematical concepts.

4 Methods of research

We chose to undertake our research in the context of a classroom-based intervention in order to achieve the dual aim of improving classroom practice and developing empirically tested *and* theory-based solution to address problems—and well as missed opportunities—in the teaching and learning of early algebra. Indeed, Stylianides and Stylianides (2013) argue that such interventions increase the likelihood that the results of research are applicable while also shedding light on how and why certain situations work. In the next sections, we describe the participants involved in the study, then the implemented teaching sequence.

4.1 Participants

The study is part of a long-term classroom-based intervention (from grade 1 to 5). It took place in a primary school in the suburbs of Torino, in Northern Italy, involving a group of 21 children in grade 3. The children are heterogeneously composed of males and females, coming from a mainly rural settings and humble social context. The class had regular mathematics lessons 2 days per week, for a total of 8 weekly hours. The activities of the study were carried out through two sessions of 4 h each in the spring of grade 3.

During the course of the study, the children worked in pairs, individually and in whole classroom discussions. Three adults were also present: the classroom mathematics teacher, a researcher (the first author) and a graduate student. The teacher and the graduate student were acting as active observers. In this paper, we refer to the researcher as the teacher since she was playing this role in the classroom and the students has already worked with her for approximately 40 h. For one of the sessions, the second author was also present. The children's activities were videotaped by the graduate student and were later transcribed. Data from their written work were collected and also used for analysis.

4.2 Teaching sequence

In the previous grade, the children had already worked on patterns: simple finite sequences of numbers in grade 1 and figural sequences in grade 2. One of the very first sequences used in grade 2 involved numbers of the form $6n-2$ (Fig. 1a). The children had to complete the first six figures. They were able to express generalisation at a local level, through the *recursive* routine of finding a figure starting from the previous or following one. They used expressions like "The rule is: add always 6" or "Adding always 2 rows of 3". As the teacher indicated a figure in the sequence with "Fig.", the children could speak of the Figure in position 3 saying "Fig. 3". They were using "Fig." as a numerical variable.

In other grade 2 activities, the children worked on remote Figures (like 13, 35 and 50, as well as from about a *big* Figure to *any* Figure), through figural sequences of the form $2n+1$

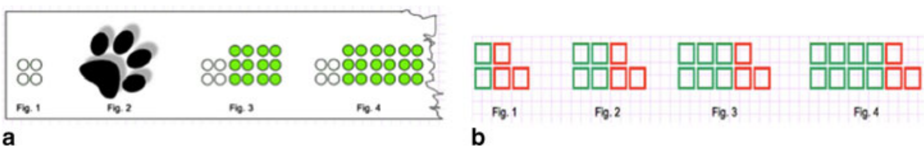

Fig. 1 Figural sequences involving numbers of the form: $6n-2$ (**a**) and $2n+3$ (**b**)

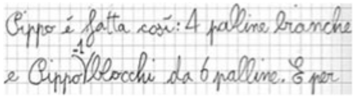

Fig. 2 Statement about Figure Pippo

and $2n+3$. In these tasks, the children started making use of *parameter* variables, though for them, the variable was the number of the Figure and not yet explicitly its position. For example, in the activity involving the sequence in Fig. 1b, they made statements of the kind: "In fig. Pippo, you have to do the double of Pippo plus 3.". There was even a sense of efficiency emerging in these *structural* routines: "To be quicker, forget about the three red for a moment, do the double of the number of the Figure, then I add the three red".

At the beginning of grade 3, the children were again shown the figural sequence of Fig. 1a, but this time were asked to think of how to find *directly* any figure of the sequence. In this case, they used the same kind of *structural* routine as in grade two, making statements such as "Figure Pippo is made this way: 4 white circles and Pippo – 1 blocks of six circles" (Fig. 2).

This type statement, like the grade 2 ones, clearly refer to the spatial disposition of the figures, expressing the generality of the shape for any specific position (Figures 1 to 5, 10, etc.). However, it seems that the children were continuing to use words such as "Pippo" as *parameter* variables, and were using *one-way, heterogeneous* routines that describe the particular figure associated with a particular position. Moreover, the parameter variable was not explicitly associated to the position of the Figure in the sequence, nor was it seen as being variable.

In grade 3, the goal was to shift attention to the *functional* link between the position number and the number of elements in the figure for that position. The teacher started to speak of the "position" of the figures in the sequence, instead of just using the number to index the figure. Further, she introduced numerical sequences (as opposed to figural sequences) to facilitate the work on the direct relationship between a sequence number and its position in the sequence. However, when the children were presented with these sequences, they seemed to return to *recursive* routines. Furthermore, they no longer made use of *parameter* variables in that they no longer spoke of the relationship between the position number and the sequence number. For this reason the teacher introduced the toilet paper, with the aim of evoking the steps or positions of a sequence, and the need for speaking explicitly of "position" for a figure.

5 Developing a discourse of generalisation

In what follows below, we first describe the initial work concerning numerical sequences, which centrally featured the use of a roll of toilet paper. We then analyse in more detail how the classroom discourse changed both in terms of *word use* and *routines*, highlighting the nature of the material entanglement.

5.1 The toilet paper and the position number

Initially, the positions (1, 2, 3, …) were written on each of the toilet paper pieces (Fig. 3a–d). The teacher then prompted the children to attend to two sequences: (A) Start from three. The rule is: Adding always three; (B) Start from six. The rule is: Adding always two.

The children's voices were immediately heard generating the sequences aloud. The teacher introduced two stacks of coloured post-it notes (pink and yellow), on which the sequence numbers could be written (Fig. 4a), There were thus three different sequences visible on the floor. The teacher rolled the toilet paper up and wrote the very first numbers of sequences A and B on the post-it notes (Fig. 4b–c). When asked what the pink "2" post-it note was, the children said, "It's the first number" (Riccardo), "It's the first piece of the toilet paper" (Lara), "If it would be on the toilet paper, it would be the first piece of the toilet paper" (Sara). So, for Lara and Sara, the toilet paper was still there as a way of marking position. After placing the first two numbers of sequence A on the floor (Fig. 4c), the teacher asked where the first number of sequence B should be placed. Francesco answered "here below", pointing with his hand to the specific place on the floor that was directly below the first pink post-it note and the first piece of toilet paper (Fig. 4d).

When the teacher asked about other numbers in sequences A and B, Lara said: "It is as if the cars were the pieces" (the post-it note had the shape of a car) (Fig. 5a). At this point, the teacher decided to unroll the toilet paper, but this time, without labelling it with the counting numbers. When she asked where she should place it, Agnese replied, pointing to its first piece, "For example, we put the first car, which has three, here" (Fig. 5b). The toilet paper was thus placed at the beginning of the sequences ("from here"; Fig. 5c–d).

When the teacher started asking about new numbers in the sequences, the nature of the situation changed again. Sara said that in sequence A, the number nine "should be in position three". At this point, position three was a *blank (uncut) sheet* of toilet paper (the third one), rather than a numbered one. The teacher challenged the children with two problems: (1) to find a way to indicate the position; (2) to find a way of explaining why nine is in position three, "without counting". The teacher suggested that the children use the toilet paper strip that had

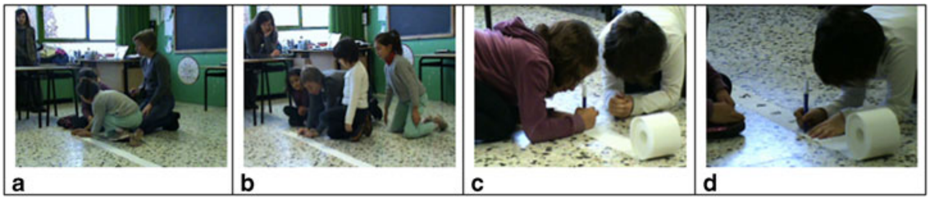

Fig. 3 Introducing the toilet paper and numbers on its strips to convey the idea of position (**a**–**d**)

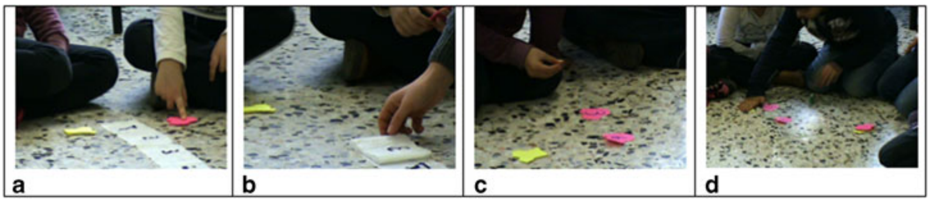

Fig. 4 Trying to connect position numbers on the toilet paper to numbers in two sequences (**a**); Abandoning the toilet paper (**b**); Introducing the numbers for two sequences with the post-its (**c**–**d**)

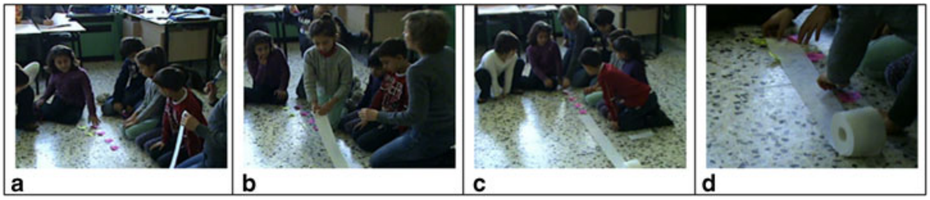

Fig. 5 The places of the post-its are related to the strips of the toilet paper by Lara (**a**) and Agnese (**b**); the toilet paper re-used (**c**–**d**)

the numbers written on it (Fig. 6a). But, the children did not provide other arguments apart from "three, six, nine are three numbers".

The teacher then wrote "P." on each piece of toilet paper. The children began to refer to the toilet paper strip in terms of the "small piece one" and "small piece two", etc. (Fig. 6b). A new 'remote' number question was posed: "If we would go on with sequence A using the pink post-its, where should 21 be?" Sara and others answered by gesturing to a position on the floor. The teacher asked how they could be sure where to put 21 without having all the previous post-its. But the children explained this in terms of "space" between subsequent numbers in the sequence (Fig. 6c).

To the question: "In which position is 21?", many children answered "in the seventh", some reciting the multiplication table for three, and others counting with their fingers or keeping trace of the numbers on the floor. The teacher insisted: "Isn't there something that tells us position seven?" (pointing to the toilet paper). Francesco repeated the term "position" to refer to the piece with "P. 7" on it, while Sara pointed to it. It is here that the children decided to shift the labelled toilet paper strip under the post-its of sequence A, as if to show the association between the number 21 and the position 7 (Fig. 6d).

Analysis The initial introduction of the toilet paper was meant to offer a discrete, ongoing actualising of "position". With both the sequence of numbers on the toilet paper and the two new sequences A and B, the children used a *recursive* routine. Even after the teacher asked the children to place the sequences beside the toilet paper, they did not seem to see the toilet paper numbers as marking positions in relation to the other two sequences. However, once the toilet paper was rolled up, the children could now think of the "pieces" in terms of first, second, third… instead of 1, 2, 3. Indeed, the unrolled, unlabelled toilet paper started to be used as a way of speaking about position, as Sara showed with her *explicit* use of the word "position". Even with this explicit use of the position number, the teacher and the children seemed to have a discursive conflict in the sense that the children only produced *recursive* arguments when asked why nine was in position three. Their *iterative* routine continued even with the second challenging task, when the teacher asked about the position of a remote number 21.

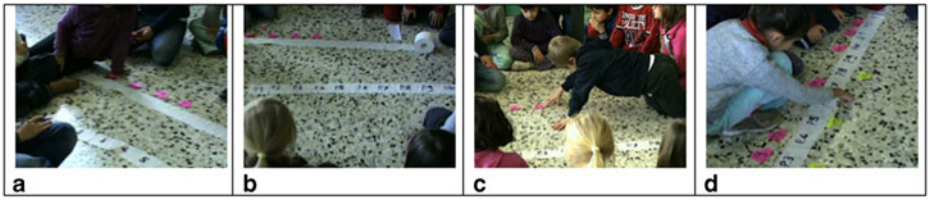

Fig. 6 Establishing a connection between the original part of the toilet paper and the sequences (**a**); Introducing "P." for the position (**b**); Taking care of the "space" between one number and the other in the sequence (**c**); Associating sequence numbers with their position (**d**)

However, when the children physically moved the toilet paper, placing it under the sequence, a new association between the position numbers and the two sequences A and B was forged, evidently establishing a relationship between position numbers and sequence numbers. We see this action as an *actualisation* of the *functional* relationship between post-its and the toilet paper strip. But the teacher's request to find the position number "without counting" was not yet meaningful to the children, as we will show in the next section. As is evident in this excerpt, the way the children talked about and saw the sequences A and B was materially entangled with the toilet paper, which was beginning to emerge as a way to mark position.

5.2 Emergence of a new routine of "without counting"

The next lesson began with a discussion about whether two sequences could both arrive at a given number (in this case, 26) and, if so, whether one arrived before the other. The children were all seated in a semi-circle. Mattia got up and moved to the floor, placing himself at the same location as where the toilet paper had been used in the previous lesson.

> Mattia: For example here there's the two (*running his RH on the floor toward his left as if invoking the unrolling of the toilet paper*)… here there's a sequence (*moving back a bit, pointing to a new position on the floor and running his RH to the left*)… here there's the three (*again moving back, pointing to a new position on the floor and running his RH to the left*)… hum, then, then[2]
> Francesco: But, Mattia, those are the positions, aren't they?
> Mattia: The task says: Look for the sequence that arrives first or that doesn't arrive at number 26… hum… for example, here it arrives (*running the RH on the floor in front of him*) because (*looking at the position on the floor for the starting point of the sequence*) it's 2, 4, 6, 8, 10, 12 (*bending toward the place for 2 s on the floor, moving with his body on the left, always pointing to different positions on the floor;* Fig. 7a–b) 14, 16, 18, 20 (*no longer pointing and just turning to the left gazing at the imaginary continuation of the sequence, rhythmically nodding his head*) 22, 24, 26 (*keeping rhythm and speaking louder*)… (*moving back*). Instead, the multiplication table for three (*looking at the starting point of the corresponding imaginary sequence*), 3, 6, 9 (*moving with his body step by step turning towards the left, always pointing to different positions on the floor, keeping the rhythm with the head*), 12 (*shifting on the left as he needed to be in front of the number*), 15 (*going on to shift as to follow the numbers*)… 3, 6, 9, 12, 15… 18 (the other children counting with him: 18, 21) (*turning his body and following with the RH, gazing at the floor;* Fig. 7c), 21 (*stopping and gazing at the floor;* Fig. 7d).

The teacher then invited the children to think of the two sequences as being a right and left shoe, respectively, that walks along Mattia's imaginary line. She associated the second sequence with "the multiplication table for three", and then posed a new question:

> Teacher: Without counting, I want an answer that explains to me how I can find the position in which right shoe arrives at 26 without counting! (louder) Immediately!
> Mattia: Make the calculation
> Giorgia: Without counting I found the position where number 26 is

[2] In transcriptions, we use *italics* to indicate actions and gestures, underline for sounds, expressions and pauses, RH/LH for right hand and left hand.

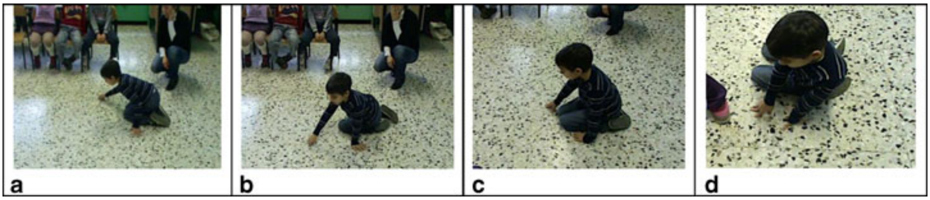

Fig. 7 Mattia following the multiples of 2 (**a**–**b**) and the multiples of three on the floor (**c**–**d**)

Teacher: How?
Giorgia: Hum…'cause
Teacher: Meanwhile, let's answer the question: Which is the position? (*many children stir on their chairs*)
Giorgia: Position 12
Riccardo: No!
Giorgia: Hum, yeah, position 13
Riccardo: 24 is in the 12!
Giorgia: Yeah, it's true, I got confused. In position 13, because 20… I know that I always have to go forward two, don't I? (*RH miming a double jump in the air*) In the multiplication table for two.

There was a long pause, then Veronica spoke up.

Veronica: Since in the sequence right shoe (*touching her RL*) there's, there are the numbers of the multiplication table for two, I know that in the multiplication table for 2 at place 10, in position 10, there's number 20
Teacher: But why?
Veronica: And if I go forward
Giorgia: Always forward two

The teacher then repeated her desire for a direct way of finding the position, adding "if I know it for position 13 directly then I know it for any (with emphasis) position!"

Simone: In position three, there's the six (*keeping the calculation with his hands*), and we already are at six, plus
Teacher: That's the problem!
Veronica: Without counting
Mattia: Without addition, multiplication, division
Teacher: I didn't say without addition, multiplication, division. I said directly (*pushing the hands joined in front*), with a unique operation
Simone: In position three there's the six
Teacher: (with emphasis) Aha, in position three there's the six, why?
Agnese: Two times (uncertain; Filippo with his hand raised)
Filippo: Because you do the double[3]… For example, three (*both hands open just in front of his torso;* Fig. 8a), you double it (*parallel hands rotated toward a higher front position;* Fig. 8b–d) and it makes 6. You double 10 and it makes 20 (*repeating the previous gesture*). You double!

[3] In Italian there are two ways of referring to doubling: "fare il doppio" (literally, to make the double) and "raddoppiare" (to double). We have chosen to translate "fare il doppio" as "to do the double" and "raddopiare" as "to double".

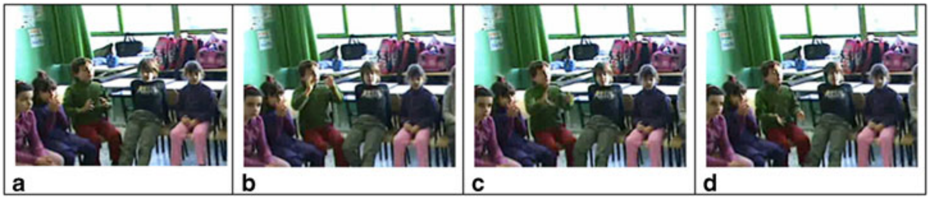

Fig. 8 Filippo's action of doubling the position numbers (**a**–**d**)

The teacher reminded the children that they were originally trying to solve the inverse problem, which is "Why is 26 in position 13?".

Filippo: Because, hum, 13, make 13 and 13 (*repeating twice the gesture used for the double before*) that makes 20…
Others: 26
Filippo: 26… Or you can make 13 times two
Giorgia: Or two times 13 (*crossing her hands*)
Veronica: But if I don't know the position?
Teacher: Fantastic! The problem is precisely: How to find the position? We found it, we said it's 13. Why? The explanation is in that 13 times two makes 26
Filippo: Or you can make 26 divided by two (*miming two with his LH*).

Following this, the teacher tried to summarise the ideas discussed. She said that she didn't want to hear talk like "after 6 there is 8, after 8 there is 10". She said she wanted the children to find "the simplest way, the quickest way" to solve the task.

At this point, Agnese decided to move to the centre of the classroom, where the toilet paper had been, and talked about doubling again (Fig. 9a–d).

Agnese: You try to, you do the double (*shrugging*)… that is (*thumping toward and gazing to the floor*)
Teacher: That is?
Agnese: Hum (*on her knees, moves to the imagined position of the sequence on the floor, gazing to it*)… here there's one (*RH palm touching the imagined first position on the floor*), the double of one is two (*RH turning to the back and touching a position below the previous one, Agnese gazing to the teacher*). Here there's two (*the body moving on the right, RH palm touching a new position on the floor*), hum, position two (*gazing to the teacher, while RH fingers keep the position*), and in position two there's four (*RH palm touching a corresponding position below*), and you do the double (*moving her body on the right*), then
Others: The double of two is four

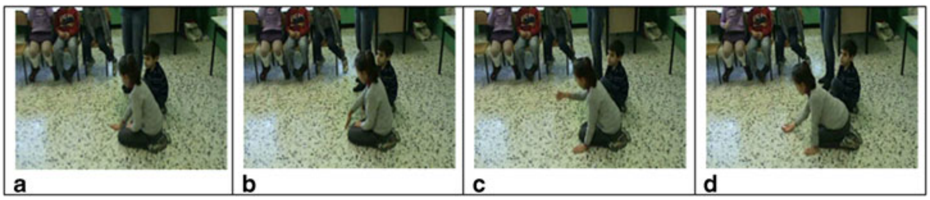

Fig. 9 Agnese's gesture of doubling (**b**–**c**) the position (**a**) to get the number in that position (**d**)

The teacher asked Agnese to repeat what she had just said.

Agnese: You do the double of one (*RH pointing to the starting position of the imagined sequence, gaze to it; then, the teacher moves to point with her RL to the original position on the floor and Agnese reaches out the position with the whole body, touching it with her RH palm*), the double of one and it makes two (*RH palm jumping to touch a below position on the floor*). You do the double of two (*RH palm moving to the new position on the floor;* Fig. 9a), which is the position, and it makes four (*RH palm jumping to touch a below corresponding position on the floor;* Fig. 9b–d), because in position four (*raising the head and looking at the rest of the class*), in position two there's the four. In position three (*RH palm pointing to a new position on the floor*), I do the double (*turning RH on its back*) and there's six (*RH back touching the below corresponding position*)
Teacher: Then, in position 10?
Mattia: In the tenth position there's 20
Agnese: 10, I do the double (*repeating in the air the turning gesture with RH, the palm up, and gazing to the teacher*) and there's the 20 (convinced)
Teacher: And in position 99?
Mattia: Hum, I don't know the double of 99
Riccardo: First you have to make 99 times two
Filippo: Ah, yeah, 99 times 99
Riccardo: Times two, not 99
Someone: Plus
Giorgia: 99 times two!
Teacher: 99 times two, then one makes the calculation, the result doesn't matter now. That's the point. I don't care about the result, because I calculate the result with the calculator if I want. I care to know the operation that I have to make immediately
Mattia: Without counting
Filippo: (his hand raised) I want to say something
Teacher: Tell me
Filippo: Then, you get a result and divide it by two (*repeating the inverse doubling gesture*).

The teacher asked for the position number of 38. Francesco first offered "76", which he justified as "38 times two".

Filippo: 38 divided by two! … Because, hum, before, in position… for example, we said that, in position, in which position it was and we doubled it (*repeating the doubling gesture*). And now we ask in which position the number is (*contrary gesture twice*) and now we divide it by two.

Analysis When Mattia invoked the two sequences on the floor through his gestures and gaze, he reinforced the *recursive* defining of the sequence as he, and then his classmates, chanted the multiples of three. The children were showing that they could imagine the position numbers forming a linear path much like the toilet paper. In this sense, they were attending to the sequential positions of the multiples, but only as geographical locations rather than as variables. Prompted by the shoe idea, the teacher focused on the one sequence that arrived at 26. The teacher wanted to shift the students' discourse by having them solve this “immediately, without counting”. However, the children did not understand the request, which the teacher then repeated. Giorgia and Riccardo both made statements connecting the position number with the sequence number 26, without using a *recursive* routine. However, Giorgia’s follow-up statement showed that her attention was still on the *recursive* unfolding of the sequence (i.e., on $2+2+2\ldots$ rather than 2×13).

The teacher cut Simone off when he spoke of adding, which prompted Veronica to repeat that the children should not be counting. Mattia made it clear that “without counting” meant without using any operations, which helped explains why the children found the teacher’s request so difficult. The teacher then made a third attempt at shifting the discourse by explaining that getting “directly” to the position of a number in the sequence involved “a unique operation”.

Agnese began to infer a *direct* relationship between the position and the sequence numbers when she spoke of “two times.” Her hesitation helped to mark a shift in thinking. Then Filippo named the relationship “you double”, now moving towards a *parametric* use of variable when he provided the various examples of doubling. Actually, he made a numerical use of the variable when, at the beginning, he referred to the single example of the task to be solved. But then, in shifting attention to various examples through an iteration of the process, he adopted a parametric use of the variable. At the end, his statement “you double” no longer referred to any particular number that was being doubled, but to the operation itself, thus introducing a *functional* use of the same variable, although without explicit mention of the position as subject. His gesture (shown in Fig. 8a–d) enacted this operation, as it *actualised* the transformation from position to sequence number (a transformation that could apply to any position number to find directly its associated sequence number).

Filippo returned once again to the *recursive* routine, but this time only momentarily. Then Veronica raised the important issue about the (new) inverse problem. This was an important shift in discourse because Veronica was now thinking about the *homogeneous, two-way* relationship between the position and sequence numbers. At this point, Filippo introduced the idea of division by two.

Both Agnese and Mattia returned to a *recursive* routine, which prompted the teacher to try, once again to speak in terms of a *functional* relationship between the sequence number and the position number. When Agnese moved to the centre, she did *not* speak of the position number *functionally*, even though the functional link between the position number and the sequence number was *actualised* by her gesture. Agnese’s gesture, similar to Filippo’s, involved turning her right hand, which expressed the *transformation* of the position number into the sequence number. This transformation was immediately expressed in words, when “position” was used to distinguish the roles of the two numbers that intervene in the transformation. Agnese was thus invoking not only Filippo’s gesture, which had accompanied his initial explicit relationship, but also, by working on the floor, the toilet paper and the post-it notes that had previously enabled the naming and extension of the sequence numbers.

The teacher then tried to use this transformation that Agnese had evoked to show how it enabled a *general* method of finding the sequence number of a given position. By telling the children that she did not care about the value of the sequence number, she further emphasised the value of the *homogeneous, two-way* routine.

When Mattia said, “without counting” he seemed to finally understand the teacher’s request for an “immediate” solution. But the inverse problem persisted. It was Filippo who introduced the inverse doubling gesture that would enable the solution to this problem. When the teacher pointed out that the problem was really solved, Filippo responded, but this time making use of the variable as a *parameter*. We see a *parametric* use of variable because Filippo used "it", which was the position number (in fact, *any* position number). Indeed, in making reference to the specific "it", for example to 38, he is introducing the parameter, but when he thinks of the "it" in its generality (of doubling it), he is already passing to a *functional* use of the variable. He also used the inverse doubling gesture, a gesture that evoked the relationship between the sequence numbers and the position numbers, as well as between the toilet paper and the post-it notes. Filippo also tried to express the difference between the two problems in terms of their nature ("before", "now"; "in which position it was", "in which position the number is"), as well as in operational terms ("we doubled it", "we divide it by two").

In this excerpt, the toilet paper has remained present, but becomes connected to the sequence in a new way by the transformation gestures of Agnese and Filippo, who actualise the relationship between the position numbers (actualised by the toilet paper) and the sequence numbers. The transformation gesture seems to give rise to its own inverse, which enables the children to work on the inverse problem. The position number thus becomes a variable by first extending along the sequence of the toilet paper and then leaping across from the toilet paper to the sequence in order to forge the covariance of position number and sequence number. The gesture actualises the bridge between these two sets of variables.

6 Discussion

Our analyses showed how the toilet paper was introduced as a way of speaking directly about position numbers, hoping to move from using parameter variables to functional ones, as well as from a one-way, heterogeneous routine to a two-way, homogeneous routine. However, initially, the toilet paper became just another sequence of numbers similar to the sequence of multiples of two or three. The use of the post-it notes allowed the toilet paper to return as a way of associating position numbers with sequence numbers. This was also when the teacher started asking for the *direct way* of finding the position number. The children thus had three challenges: the first was to figure out what this new "without counting" routine was about; the second was to figure out how the recursive routine related to the new one that the teacher was asking for; and, the third was to figure out when to use the recursive routine and when to use the new one. These challenges were embedded in the motive of solving the original inverse problem.

The children began to devise a routine for identifying the sequence number by transforming the position number—that is, they devised a *one-way, heterogeneous* routine, heterogeneous because the position number and sequence number were still seen as different kinds of number (which they talked about in different ways, with or without the article). This began occurring temporarily with Filippo but explicitly later, when Agnese introduced a *new* routine that served to bring forth the *transformation* (doing "the double") that related a sequence number to its position. It was only through this *change* in routine that the children first began to make use of the *position* variable as *parameter*. Two aspects of the classroom process were relevant in bringing forth the transformation. One is related to the need for re-invoking the toilet paper and to the *virtuality* that the use of the toilet paper actualised at different times in the activity. The other is concerned with the way the transformation, evoked by the toilet paper, came to be *actualised* in the classroom discourse through bodily actions and gestures.

Concerning the first aspect, the potential that the toilet paper helped bring forth in the children's routines is the *direct* relationship between sequence number and position number.

From the first attempt to use the toilet paper, the *functional* link was potentially there through the physical inscription of numbers, and then when the post-it notes began to be assigned to specific pieces. The same virtuality came back later, during the next lesson, even though the toilet paper was physically absent.

Regarding the second aspect, when Mattia first imagined the two shoe sequences on the floor and reasoned about whether they would reach a given number, the toilet paper was re-invoked again as a way of being able to refer to position. However, in that moment, the invocation was still reinforcing a *recursive* routine. The potential link was *actualised* by the graspable material entanglement of Agnese's gestures on the floor with the (imagined) toilet paper and the mathematical transformation she described. Agnese's actions were invoking the numbers and their positions at once, and her turning hand *actualised* the transformation from one to the other. In this point, the toilet paper was still there on the floor, in the same place were it had been before. A different *actualisation* of the virtual occurred when Filippo shifted the 'physical' transformation from the floor to the space in front of him (repeating the same movement he used earlier to anticipate the transformation). The nature of the transformation changed once more, and with it the nature of discourse, becoming a way of solving the direct problem posed by the teacher. As a matter of fact, it came to be materialized no longer in the potential presence of the toilet paper on the floor, but by means of Filippo's visible, mobile gestures in the air that *actualised* the calculation through which the transformation is mathematically expressed and, with it, the two-way, homogeneous routine is materially brought forth. From these gestures the transformation spread through the whole classroom, and the children were able to solve collectively the three challenges discussed above. The variable can thus be seen as having one foot in the physical world of the toilet paper, post-it notes, words and gestures and another in the world of the potential, where it can take on any value and determine an associated number sequence.

7 Conclusions

Our research aimed to find out whether we can design patterning tasks that enable the emergence of a concept of the variable as direct, parametric or functional. Further, based on our theoretical perspective, we sought to investigate how the emergence of this concept occurs and, more specifically, how the teacher's questions, the children's gestures, the toilet paper unrolled along the classroom floor, and so on, enabled the actualisation of the virtual and through it the development of a discourse of generalisation.

We recall that Radford's definition of algebraic generalisation of patterns given in the introduction suggests three kinds of steps: noticing a local commonality in a few members of the sequence, which requires making a choice between what counts as the same and the different; extending the commonality to all the terms of the sequence; using it to find a direct expression of the terms of the sequence. Then, according to this definition we have a local commonality and a global commonality. For the third step to occur, the elaboration of a rule based on variables is required. We argue that this does not necessarily mean that one grasps the direct functional relation between one term of the sequence and its position as involving the notions of independent and dependent variables—especially when children are working with numerical sequences. For example, one can rely on a *parametric* use of variable without using it *functionally*. While the parametric use often relies upon the rule that expresses the awareness of a correspondence view between two sets of numbers, the functional use is much more related to a covariance conception of the way that the terms of the sequence change depending on their position. But the covariance view is fundamental for developing functional thinking associated to the development of a flexible algebraic discourse through patterns (Smith, 2008; Carraher, Schliemann, & Schwarz, 2008; Blanton & Kaput, 2011). The crucial aspect has to do instead, for us, with the coordination and connection of the two uses. Therefore, we argue that generalisation involves something more than the three steps outlined by Radford, which has to do with their potential mutual dynamics.

In this paper, we offer an alternative perspective, from which generalisation can be seen in terms of Châtelet's notion of the virtual. To this aim, we return for a moment to the discussion. As soon as the children were able to look at what counts as the same, they were also able to look at difference, which is already given by the existence of sameness, without being explicitly pointed out as such. In so doing, the children can detect some change, and where change occurs, movement is also made present: e.g., passing from one term to the other, and to the successive ones or to any one (no matter how far, how familiar). This is the case for the children when, for example, they used the structural routine of "the double of Pippo plus 3", referring to the spatial disposition of the elements in the "Figure Pippo". "Pippo" is used as *parameter* variable that expresses the generality of the shape for any particular position. However, the particular figure is still associated with a *particular* position, even when the shape is perceived as a whole (because of a correspondence relationship). In so doing, the routine (and, with it, the activity) does *not* leave room for the virtual.

The virtual that is related to generalisation is potentially present in any pattern, be it a figural or a numerical pattern. It is manifest in the direct link between the position number and the sequence number, which is a transformation of the former into the latter. The absence of space for the virtual is evident in the fact that the children almost immediately resort to the *recursive* routine offered by a numerical pattern. Drawing on Châtelet's distinction between the possible and the virtual, this *one-way heterogeneous* routine only allows the children to realize the possible, that is, to see the spatial disposition of the figure in relation to its position in the sequence. However, it fails to forge the bridge that connects the position number and sequence number, which takes two different kinds of number and "sames" them through a *virtual link* that would enable the use of a *two-way, homogeneous* routine.

As our analyses showed, a pattern requires two types of *mobility*: the horizontal one (associated with the *recursive* routine, and related to Radford's first two steps) and the vertical one (associated with the *two-way, homogeneous* routine, and related to Radford's third step). While the parametric use of variable may arise from recursive routines and prompt a functional way of seeing, talking about and acting on variables, this cannot happen if there is only a perceptual recognition of sameness in the figures. More importantly, the relationship between the horizontal and vertical mobilities seems necessary in algebraic discourse, for enabling a covariance conception of the function that involves the independent and dependent variables within the pattern.

Without the virtual, the children were not able to use the variable for position in a *functional* way, thus ignoring the vertical movement. Their actions, if captured in diagram, would yield arrows going from one term to the other, along the horizontal. The use of the toilet paper introduced a significant *change*, materially *actualising* both kinds of mobility, as well as the connection between them, even when it was physically absent. So, the change is not only a change in discourse but, crucially, a change in the sensorimotor engagement of the children. The new gestures and bodily actions, if captured in diagram, would introduce new arrows, this time going *across* the toilet paper, from position number to sequence number (and back).

Tatha (1980), who quotes Gattegno, posits that *algebra is an awareness of dynamics…*" (*italics in original*, p. 6). Instead of focusing on the local and global commonalities, and their use, we propose to define algebraic generalisation in terms of *an awareness of the dynamics/mobility* that can be awakened in and throughout a pattern. Indeed, as Tahta writes, "There cannot be an adequate awareness of dynamics if there is nothing to act dynamically on" (p. 6). In our case, the children became more and more aware of these dynamics thanks to the use of the toilet paper and its constant re-invoking through entangled material ways of acting, moving and talking: Mattia's ways of moving along, posing on and gazing at the floor as if he was following the sequences going on; Agnese's gestures that engendered the virtual transformation of the position number into the corresponding sequence number; and, Filippo's repeated gestures in the air, which changed once again the nature of the transformation making it operational. These are all part of the contracting and mutating bundle of material encounters among bodies that include the classroom floor, post-it notes, gestures, numbers and so on. Indeed, the sequences, the concept of variable and its uses, and the transformation itself, are central materialities at play in bringing about a mathematical generalisation.

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