Semilinear pseudodifferential equations in spaces of tempered ultradistributions

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Semilinear pseudodifferential equations in spaces of tempered ultradistributions

Marco Cappiello [a], Stevan Pilipović [b] and Bojan Prangoski [c]

Abstract

We study a class of semilinear elliptic equations on spaces of tempered ultradistributions of Beurling and Roumieu type. Assuming that the linear part of the equation is a pseudodifferential operator of infinite order satisfying a suitable ellipticity condition we prove a regularity result in the functional setting above for weak Sobolev type solutions.

0 Introduction

In this paper we consider a class of semilinear equations and prove a result of regularity in the spaces of tempered ultradistributions of Beurling and Roumieu type. These distributions can be regarded as a global counterpart on $\mathbb{R}^d$ of the local ultradistributions studied by Komatsu [16, 17, 19] and they represent a natural generalization of non-quasi-analytic Gelfand-Shilov type ultradistributions, cf. [15, 22, 23]. As well as the Gelfand-Shilov spaces, they are also a good functional setting for pseudodifferential operators of infinite order, namely with symbol $a(x,\xi)$ admitting exponential growth in both x and ξ, see [2, 3, 25]. Here we want to apply the pseudodifferential operators introduced by the third author in [25] to the study of semilinear equations. In the recent paper [10], we considered the case of linear equations and proved a result of hypoellipticity via the construction of a parametrix. To treat semilinear equations, we need to adopt a more sophisticated method based on suitable commutator and nonlinear estimates. The same method had been previously used in [1, 4, 5, 6, 7, 8, 9, 13] to obtain results of regularity in Gelfand-Shilov spaces and in spaces of analytic functions for differential and pseudodifferential operators of finite order. With respect to the previous results, here we consider more in general equations in which the linear part is a pseudodifferential operator of infinite order. Moreover we allow very general nonlinear terms given by infinite sums of powers of the unknown function.

Before stating our results, let us fix some notation and introduce the functional setting where they are obtained. In the sequel, the sets of integer, non-negative integer, positive integer, real and complex numbers are denoted as standard by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{C}$. We denote $\langle x\rangle = (1+|x|^2)^{1/2}$ for $x\in\mathbb{R}^d$, $D^\alpha = D_1^{\alpha_1}\dots D_d^{\alpha_d}$, $\quad D_j^{\alpha_j} =$

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$i^{-1}\partial^{\alpha_j}/\partial x^{\alpha_j}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$. Fixed $B > 0$, we shall denote by Q_B^c the set of all $(x, \xi) \in \mathbb{R}^{2d}$ for which we have $\langle x \rangle \geq B$ or $\langle \xi \rangle \geq B$. Finally, for $s \in \mathbb{R}$, we shall denote by $H^s(\mathbb{R}^d)$ the Sobolev space of all $u \in \mathcal{S}'(\mathbb{R}^d)$ for which $\langle \xi \rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^d)$, where $\hat{u}$ denotes the Fourier transform of u. Following [16], in the sequel we shall consider sequences M_p of positive numbers such that $M_0 = M_1 = 1$ and satisfying all or some of the following conditions:

$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$, $p \in \mathbb{Z}_+$;

$(M.2)$ $M_p \leq c_0 H^p \min\limits_{0 \leq q \leq p} \{M_{p-q}M_q\}$, $p, q \in \mathbb{N}$, for some $c_0, H \geq 1$;

$(M.3)$ $\displaystyle\sum_{p=q+1}^{\infty} \frac{M_{p-1}}{M_p} \leq c_0 q \frac{M_q}{M_{q+1}}$, $q \in \mathbb{Z}_+$,

$(M.4)$ $\displaystyle\left(\frac{M_p}{p!}\right)^2 \leq \frac{M_{p-1}}{(p-1)!} \cdot \frac{M_{p+1}}{(p+1)!}$, for all $p \in \mathbb{Z}_+$,

In some assertions in the sequel we could replace $(M.3)$ by the weaker assumption

$(M.3)'$ $\displaystyle\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$,

cf. [16]. We observe moreover that $(M.4)$ implies $(M.1)$.
As an example of sequence satisfying all the conditions above we can take $M_p = p!^s$, $s > 1$.

For a multi-index $\alpha \in \mathbb{N}^d$, M_α will mean $M_{|\alpha|}$, $|\alpha| = \alpha_1 + ... + \alpha_d$. We can associate to any sequence M_p as above the function

$$M(\rho) = \sup_{p \in \mathbb{N}} \log_+ \frac{\rho^p}{M_p}, \quad \rho > 0.$$

This is a non-negative, continuous, monotonically increasing function which vanishes for sufficiently small $\rho > 0$ and increases more rapidly than $\ln \rho^p$ when ρ tends to infinity, for any $p \in \mathbb{N}$ (cf. [16]).

We shall denote by $\mathfrak{R}$ the set of positive sequences which monotonically increase to infinity. For $(r_p) \in \mathfrak{R}$, consider the sequence $N_0 = 1$, $N_p = M_p \prod_{j=1}^p r_j$, $p \in \mathbb{Z}_+$. It is easy to verify that this sequence satisfies $(M.1)$ and $(M.3)'$. Its associated function will be denoted by $N_{r_p}(\rho)$, i.e. $N_{r_p}(\rho) = \sup\limits_{p \in \mathbb{N}} \log_+ \dfrac{\rho^p}{M_p \prod_{j=1}^p r_j}$, $\rho > 0$. Note, for given (r_p) and every $k > 0$ there is $\rho_0 > 0$ such that $N_{r_p}(\rho) \leq M(k\rho)$ for $\rho > \rho_0$.

Now we can introduce the space of tempered ultradistributions and its test function space. For $m > 0$ and a sequence M_p satisfying the conditions $(M.1) - (M.3)$, we shall denote by $\mathcal{S}_\infty^{M_p,m}(\mathbb{R}^d)$ the Banach space of all functions $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_m := \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{m^{|\alpha|} |D^\alpha \varphi(x)| e^{M(m|x|)}}{M_\alpha} < \infty, \tag{0.1}$$

endowed with the norm in (0.1) and we denote $\mathcal{S}^{(M_p)}(\mathbb{R}^d) = \varprojlim\limits_{m \to \infty} \mathcal{S}_\infty^{M_p,m}(\mathbb{R}^d)$ and $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d) = \varinjlim\limits_{m \to 0} \mathcal{S}_\infty^{M_p,m}(\mathbb{R}^d)$. In the sequel we shall consider simultaneously the two latter spaces by using the common notation $\mathcal{S}^*(\mathbb{R}^d)$. For each space we will consider a suitable symbol class. Definitions and statements will be formulated first for the (M_p) case and then for the $\{M_p\}$ case, using the notation $*$. We shall denote by $\mathcal{S}^{*\prime}(\mathbb{R}^d)$ the strong dual space of $\mathcal{S}^*(\mathbb{R}^d)$. We refer to [14, 22, 23] for the properties of $\mathcal{S}^*(\mathbb{R}^d)$ and $\mathcal{S}^{*\prime}(\mathbb{R}^d)$. Here we just recall that the Fourier transformation is an automorphism on $\mathcal{S}^*(\mathbb{R}^d)$ and on $\mathcal{S}^{*\prime}(\mathbb{R}^d)$ and that for $M_p = p!^s$, $s > 1$, we have $M(\rho) \sim \rho^{1/s}$. In this case $\mathcal{S}^*(\mathbb{R}^d)$ coincides respectively with the Gelfand-Shilov spaces $\Sigma_s(\mathbb{R}^d)$ (resp. $\mathcal{S}_s(\mathbb{R}^d)$) of all functions $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

$$\sup_{\alpha,\beta\in\mathbb{N}^d} h^{-|\alpha|-|\beta|}(\alpha!\beta!)^{-s} \sup_{x\in\mathbb{R}^d} |x^\beta \partial^\alpha \varphi(x)| < \infty$$

for every $h > 0$ (resp. for some $h > 0$), cf. [15, 22].

Following [25] we now introduce the class of pseudodifferential operators involved in the sequel. Let M_p, A_p be two sequences of positive numbers. We assume that M_p satisfies $(M.1)$, $(M.2)$, $(M.3)$ and $(M.4)$ and that A_p satisfies $A_0 = A_1 = 1$, $(M.1), (M.2), (M.3)'$ and $(M.4)$. Moreover we suppose that $A_p \subset M_p$ i.e. there exist $c_0 > 0, L > 0$ such that $A_p \leq c_0 L^p M_p$ for all $p \in \mathbb{N}$. Let $\rho_0 = \inf\{\rho \in \mathbb{R}_+ |\, A_p \subset M_p^\rho\}$. Obviously $0 < \rho_0 \leq 1$. Let $\rho \in \mathbb{R}_+$ be arbitrary but fixed such that $\rho_0 \leq \rho \leq 1$ if the infimum can be attained, or otherwise $\rho_0 < \rho \leq 1$. For any fixed $h > 0, m > 0$ we denote by $\Gamma^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h, m)$ the space of all functions $a(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ such that

$$\sup_{\alpha,\beta\in\mathbb{Z}^d_+} \sup_{(x,\xi)\in\mathbb{R}^{2d}} \frac{|D^\alpha_\xi D^\beta_x a(x,\xi)| \langle (x,\xi)\rangle^{\rho|\alpha+\beta|} e^{-M(m|x|)-M(m|\xi|)}}{h^{|\alpha+\beta|} A_\alpha A_\beta} < \infty,$$

where $M(\cdot)$ is the associated function for the sequence M_p. Then we define

$$\begin{aligned}
\Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}; m) &= \varprojlim_{h\to 0} \Gamma^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h, m);\\
\Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}) &= \varinjlim_{m\to\infty} \Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}; m);\\
\Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h) &= \varprojlim_{m\to 0} \Gamma^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h, m);\\
\Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}) &= \varinjlim_{h\to\infty} \Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h).
\end{aligned}$$

We associate to any symbol $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ a pseudodifferential operator $a(x,D)$ defined, as it is usual, by

$$a(x,D)f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} a(x,\xi) \hat{f}(\xi) d\xi, \qquad f \in \mathcal{S}^*(\mathbb{R}^d), \tag{0.2}$$

where $\hat{f}$ denotes the Fourier transform of f. Operators of the form (0.2) act continuously on $\mathcal{S}^*(\mathbb{R}^d)$ and on $\mathcal{S}^{*\prime}(\mathbb{R}^d)$. Moreover, a symbolic calculus for $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ (denoted there by $\Gamma^{*,\infty}_{A_p,A_p,\rho}(\mathbb{R}^{2d})$) has been constructed. As a consequence it was proved that the class of pseudodifferential operators with symbols in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ is closed with respect to composition and adjoints, cf. [25] and the next section for details. Moreover, in [10] we consider hypoelliptic symbols in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ and we proved the existence of parametrices for the associated operators. Now we need to introduce a notion of elliptic symbol in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$. For this purpose let $\tilde{M}_p$ be another sequence which satisfies $\tilde{M}_0 = \tilde{M}_1 = 1$, $(M.1)$, $(M.2)$, $(M.3)'$ and $(M.4)$ and $M_p \subset \tilde{M}_p$, i.e. there exists $\tilde{c}, \tilde{L} > 0$ such that $M_p \leq \tilde{c}\tilde{L}^p \tilde{M}_p$ (observe that $\tilde{M}_p$ can be the same as M_p). Obviously, without losing generality, we can assume that the constant H from $(M.2)$ is the same for the sequences A_p, M_p and $\tilde{M}_p$. For $(k_p) \in \mathfrak{R}$ we denote by $\tilde{N}_{k_p}(\cdot)$ the associated function to the sequence $\tilde{M}_p \prod_{j=1}^{p} k_j$. One easily obtains the following inequalities

$$\tilde{M}(\lambda/\tilde{L}) \leq M(\lambda) + \ln_+ \tilde{c} \text{ and } \tilde{N}_{k_p}(\lambda/\tilde{L}) \leq N_{k_p}(\lambda) + \ln_+ \tilde{c}, \ \forall \lambda > 0. \tag{0.3}$$

Definition 0.1. *A symbol $a \in \Gamma^{*,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$ is said to be $(\tilde{M}_p)$-elliptic, (resp. $\{\tilde{M}_p\}$-elliptic) if*

i) there exist $m, B, c > 0$ (resp. there exist $(k_p) \in \mathfrak{R}$ and $B, c > 0$) such that

$$|a(x,\xi)| \geq c e^{\tilde{M}(m|\xi|)} e^{\tilde{M}(m|x|)}, \left(\text{ resp. } |a(x,\xi)| \geq c e^{\tilde{N}_{k_p}(|\xi|)} e^{\tilde{N}_{k_p}(|x|)} \right)$$

for all $(x,\xi) \in Q^c_B$;

ii) for every $h > 0$ there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$\left| D^\alpha_\xi D^\beta_x a(x,\xi) \right| \leq C \frac{h^{|\alpha|+|\beta|} A_{\alpha+\beta} |a(x,\xi)|}{\langle (x,\xi) \rangle^{\rho(|\alpha|+|\beta|)}}.$$

for all $(x,\xi) \in Q^c_B$.

Finally we introduce the class of nonlinear terms involved in our equations.

For $\beta \in \mathbb{N}^d$, let $p_\beta(x)$ be smooth functions on $\mathbb{R}^d$ such that for every $h > 0$ there exists $C > 0$ such that

$$|D^\alpha_x p_\beta(x)| \leq C \frac{h^{|\alpha|+|\beta|} A_\alpha e^{\tilde{M}(h|x|)}}{\tilde{M}_\alpha} \text{ for all } \alpha, \beta \in \mathbb{N}^d, \tag{0.4}$$

in the (M_p) case (resp.

$$|D^\alpha_x p_\beta(x)| \leq C \frac{h^{|\alpha|+|\beta|} A_\alpha e^{\tilde{N}_{k_p}(h|x|)}}{\tilde{M}_\alpha \prod_{j=1}^{|\alpha|} k_j} \text{ for all } \alpha, \beta \in \mathbb{N}^d, \tag{0.5}$$

in the $\{M_p\}$ case). For such a family of functions $p_\beta(x)$ and $u \in H^s(\mathbb{R}^d), s > d/2$ we can consider the function

$$F[u] = \sum_{|\beta|=2}^{\infty} p_\beta u^{|\beta|}, \tag{0.6}$$

The condition $s > d/2$ implies that $F[u]$ is well defined and continuous on $\mathbb{R}^d$ and $\left\| F[u] e^{-\tilde{M}(h|\cdot|)} \right\|_{L^\infty(\mathbb{R}^d)} < \infty$ (resp. $\left\| F[u] e^{-\tilde{N}_{k_p}(h|\cdot|)} \right\|_{L^\infty(\mathbb{R}^d)} < \infty$) for some h. This (together with (0.3)) implies that $F[u] \in \mathcal{S}^{*\prime}$.

The main result of the paper is the following

Theorem 0.2. *Let $a \in \Gamma^{*,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$ be $(\tilde{M}_p)$-elliptic (resp. $\{\tilde{M}_p\}$-elliptic) and let $f \in \mathcal{S}^*\left(\mathbb{R}^d\right)$. Let $u \in H^s\left(\mathbb{R}^d\right)$, $s > d/2$, be a solution of the equation*

$$Au = f + F[u], \tag{0.7}$$

with $F[u]$ defined by (0.4) and (0.6) (resp. (0.5) and (0.6)). Then the following properties hold:

i) For every $h > 0$ there exists $C > 0$ (resp. there exist $h, C > 0$) such that $|u(x)| \leq Ce^{-M(h|x|)}$. Moreover, $u \in \mathcal{C}^\infty(\mathbb{R}^d)$ with the following estimate on its derivatives: there exists $\tilde{h}, C > 0$ such that

$$\sup_\alpha \frac{\tilde{h}^{|\alpha|}\|D^\alpha u\|_{L^\infty}}{\tilde{M}_\alpha} < \infty, \left(\text{resp. } \sup_\alpha \frac{\tilde{h}^{|\alpha|}\|D^\alpha u\|_{L^\infty}}{\tilde{M}_\alpha \prod_{j=1}^{|\alpha|} k_j} < \infty \right).$$

ii) Furthermore, if $F[u]$ is a finite sum, then $u \in \mathcal{S}^*\left(\mathbb{R}^d\right)$.

The paper is organised as follows. In Section 1 we recall some basic facts about the pseudodifferential operators studied in [25]. Moreover we prove some precise estimates for the norms of some composed operators which will be instrumental in the proof of Theorem 0.2. Finally, Section 2 is devoted to the proof of Theorem 0.2 which will be divided in two parts, one corresponding to the proof of the decay properties of the solution and the other related to its regularity.

1 Pseudodifferential operators on $\mathcal{S}^*(\mathbb{R}^d), \mathcal{S}^{*\prime}(\mathbb{R}^d)$

In this section we recall some results contained in [25] and concerning the calculus for pseudodifferential operators with symbols in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$. For the purposes of this paper we need to modify slightly some statements with respect to [25]. The proof of these new assertions is completely analogous to the original ones and do not deserve to be repeated here. We start by giving the following definition which will be useful in the sequel.

Definition 1.1. *A measurable function f on $\mathbb{R}^d$ is said to be of ultrapolynomial growth of class $*$ if $\|f(\cdot)e^{-M(h|\cdot|)}\|_{L^\infty(\mathbb{R}^d)} < \infty$ for some $h > 0$ (resp. for every $h > 0$).*

Now we recall the notion of asymptotic expansion for symbols in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$, cf. [25, Definition 2].

Definition 1.2. *Let M_p and A_p be as in the definition of $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ and let $m_0 = 0, m_p = M_p/M_{p-1}, p \in \mathbb{Z}_+$. We denote by $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ the space of all formal sums $\sum_{j\in\mathbb{N}} a_j$ such that for some $B > 0$, $a_j \in \mathcal{C}^\infty(int Q^c_{Bm_j})$ and satisfy the following condition: there exists $m > 0$ such that for every $h > 0$ (resp. there exists $h > 0$ such that for every $m > 0$) we have*

$$\sup_{j\in\mathbb{N}} \sup_{\alpha,\beta\in\mathbb{N}^d} \sup_{(x,\xi)\in Q^c_{Bm_j}} \frac{|D^\alpha_\xi D^\beta_x a_j(x,\xi)|\langle(x,\xi)\rangle^{\rho(|\alpha+\beta|+2j)}e^{-M(m|x|)-M(m|\xi|)}}{h^{|\alpha+\beta|+2j}A_\alpha A_\beta A_j^2} < \infty.$$

Notice that any symbol $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ can be regarded as an element $\sum\limits_{j\in\mathbb{N}} a_j$ of $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ with $a_0 = a, a_j = 0$ for $j \geq 1$.

Definition 1.3. *A symbol $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ is equivalent to $\sum_{j\in\mathbb{N}} a_j \in FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ (we write $a \sim \sum_{j\in\mathbb{N}} a_j$ in this case) if there exist $m, B > 0$ such that for every $h > 0$ (resp. there exist $h, B > 0$ such that for every $m > 0$) the following condition holds:*

$$\sup_{N\in\mathbb{Z}_+} \sup_{\alpha,\beta\in\mathbb{N}^d} \sup_{(x,\xi)\in Q^c_{Bm_N}} \frac{\left|D^\alpha_\xi D^\beta_x \big(a(x,\xi) - \sum\limits_{j<N} a_j(x,\xi)\big)\right| e^{-M(m|x|)-M(m|\xi|)}}{h^{|\alpha+\beta|+2N} A_\alpha A_\beta A_N^2 \langle (x,\xi)\rangle^{-\rho(|\alpha+\beta|+2N)}} < \infty.$$

An operator $a(x,D)$ with symbol $a \sim 0$ is $*$-regularizing, namely it extends to a linear and continuous map from $\mathcal{S}^{*\prime}(\mathbb{R}^d)$ to $\mathcal{S}^*(\mathbb{R}^d)$, see [25, Theorem 3]. Moreover, for every sum $\sum_{j\in\mathbb{N}} a_j \in FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ one can find a symbol $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ such that $a \sim \sum_{j\in\mathbb{N}} a_j$, cf. [25, Theorem 4]. Actually by the same argument one can prove the following more precise assertion.

Proposition 1.4. *Let g be a positive continuous function such that $g(w)$ and $1/g(w)$ have ultrapolynomial growth of class $*$ and let U be a subset of $FS^{*,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$ for which there exists $B > 0$ such that for every $h > 0$ there exists $C > 0$ (resp. there exist $B, h, C > 0$) such that*

$$\sup_{j\in\mathbb{N}} \sup_{\alpha,\beta} \sup_{(x,\xi)\in Q^c_{Bm_j}} \frac{\left|D^\alpha_\xi D^\beta_x a_j(x,\xi)\right| \langle (x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2j\rho}}{h^{|\alpha|+|\beta|+2j} A_\alpha A_\beta A_j^2 g(x,\xi)} \leq C$$

for all $\sum_{j\in\mathbb{N}} a_j \in U$. Then for every $\tilde{h} > 0$ there exists $C > 0$ (resp. there exist $\tilde{h}, C > 0$) such that the following condition holds: for every sum $\sum a_j \in U$ there exists a symbol $a \sim \sum_{j\in\mathbb{N}} a_j$ satisfying the following estimate:

$$\left|D^\alpha_\xi D^\beta_x a(x,\xi)\right| \leq C \frac{\tilde{h}^{|\alpha|+|\beta|} A_\alpha A_\beta g(x,\xi)}{\langle (x,\xi)\rangle^{\rho(|\alpha|+|\beta|)}}.$$

In [25, Theorem 7] it was proved that the composition of two operators $b(x,D)$ and $a(x,D)$ with symbols in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ is the sum of an operator $f_{a,b}(x,D)$ with symbol $f_{a,b} \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$, with $f_{a,b} \sim \sum_j f_{a,b,j}$, where

$$f_{a,b,j}(x,\xi) = \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial^\alpha_\xi b(x,\xi) D^\alpha_x a(x,\xi), \tag{1.1}$$

and of a $*$-regularizing operator $T_{a,b}$. In fact, $f_{a,b} = \sum_j (1-\chi_j) f_{a,b,j}$ where χ_j is defined in the following way (cf. the proof of [25, Theorem 4]). Take $\varphi, \psi \in \mathcal{D}^{(A_p)}\left(\mathbb{R}^d\right)$, in the (M_p) case, resp. $\varphi, \psi \in \mathcal{D}^{\{A_p\}}\left(\mathbb{R}^d\right)$ in the $\{M_p\}$ case, such that $0 \leq \varphi, \psi \leq 1$, $\varphi(x) = 1$ when $\langle x\rangle \leq 2$, $\psi(\xi) = 1$ when $\langle \xi\rangle \leq 2$ and $\varphi(x) = 0$ when $\langle x\rangle \geq 3$, $\psi(\xi) = 0$ when $\langle \xi\rangle \geq 3$. Then define $\chi(x,\xi) = \varphi(x)\psi(\xi)$, $\chi_n(x,\xi) = \chi\left(\dfrac{x}{Rm_n}, \dfrac{\xi}{Rm_n}\right)$ for $n \in \mathbb{Z}_+$ and $\chi_0(x,\xi) = 0$, where $m_n = M_n/M_{n-1}$ and $R > 0$ is large enough.

Proposition 1.5. *Let U_1 and U_2 be bounded subsets of $\Gamma^{(M_p),\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}, m'\right)$ (resp. $\Gamma^{\{M_p\},\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}, h'\right)$), for some $m' > 0$ (resp. for some $h' > 0$). Then for every $a \in U_1$ and $b \in U_2$ we have $b(x,D)a(x,D) = f_{a,b}(x,D) + T_{a,b}$ where $f_{a,b} = \sum_j (1-\chi_j) f_{a,b,j}$ and χ_j are the cut-off functions defined above which can be chosen uniformly for $a \in U_1, b \in U_2$, and with $f_{a,b,j}$ given by (1.1). Moreover, the family $T_{a,b}$ of *-regularizing operators is an equicontinuous subset of $\mathcal{L}_b\left(\mathcal{S}'^*\left(\mathbb{R}^d\right), \mathcal{S}^*\left(\mathbb{R}^d\right)\right)$.*

From the results above we notice that in general the composition of two operators with symbols in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ is still an operator of infinite order. In the sequel we will be interested to the case when the composition is a finite order operator with bounded symbol, hence the related operator is bounded on Sobolev spaces. With this purpose we give the following definition.

Definition 1.6. *Let $V, W \subseteq \Gamma^{*,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$ and let $f(w)$ be a positive continuous function on $\mathbb{R}^{2d}$ such that $f(w)$ and $1/f(w)$ are of ultrapolynomial growth of class *. The sets V and W are said to be $(f,*)$-conjugate if for every $h > 0$ there exists $C > 0$ (resp. there exist $h, C > 0$) such that*

$$|D^\alpha a(w)| \le C \frac{h^{|\alpha|} A_\alpha}{\langle w\rangle^{\rho|\alpha|} f(w)} \quad \text{and} \quad |D^\alpha b(w)| \le C \frac{h^{|\alpha|} A_\alpha f(w)}{\langle w\rangle^{\rho|\alpha|}} \quad \text{for all } a \in V,\, b \in W.$$

Obviously if V and W are $(f,*)$-conjugate then they are bounded subsets of $\Gamma^{(M_p),\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}; m'\right)$ for some $m' > 0$ (resp. $\Gamma^{\{M_p\},\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}; h'\right)$, for some $h' > 0$).

Proposition 1.7. *Let V and W be $(f,*)$-conjugate. Then, there exists $C > 0$ such that*

$$\|b(x,D)a(x,D)\|_{\mathcal{L}_b(H^s)} \le C, \quad \text{for all } a \in V,\, b \in W.$$

Proof. Let $f_{a,b}$ be the symbol of the operator $b(x,D)a(x,D)$ defined as above. Then $b(x,D)a(x,D) = f_{a,b}(x,D) + T_{a,b}$, where $T_{a,b}$ form an equicontinuous subset of $\mathcal{L}_b\left(\mathcal{S}'^*\left(\mathbb{R}^d\right), \mathcal{S}^*\left(\mathbb{R}^d\right)\right)$. Then $f_{a,b} \sim \sum_j f_{a,b,j}$, where

$$f_{a,b,j}(w) = \sum_{|\nu|=j} \frac{1}{\nu!} \partial_\xi^\nu b(w) D_x^\nu a(w).$$

Observe that

$$\begin{aligned} |D_w^\alpha f_{a,b,j}(w)| &\le \sum_{\beta\le\alpha} \sum_{|\nu|=j} \frac{1}{\nu!} \left|D_w^{\alpha-\beta} D_\xi^\nu b(w)\right| \left|D_w^\beta D_x^\nu a(w)\right| \\ &\le C_1 \sum_{\beta\le\alpha} \sum_{|\nu|=j} \frac{1}{\nu!} \frac{h^{|\alpha|+2|\nu|} A_{|\alpha|+2|\nu|}}{\langle w\rangle^{\rho(|\alpha|+2|\nu|)}} \le C_2 \frac{(2hH)^{|\alpha|+2j} A_\alpha A_{2j}}{\langle w\rangle^{\rho(|\alpha|+2j)}}. \end{aligned}$$

Now, since $f_{a,b} = \sum_j (1-\chi_j) f_{a,b,j}$ with χ_j defined as above, one easily obtains that for every $h > 0$ there exists $C > 0$, resp. there exist $h, C > 0$ such that $|D_w^\alpha f_{a,b}(w)| \le C h^{|\alpha|} A_\alpha \langle w\rangle^{-\rho|\alpha|}$. From this it follows that $f_{a,b}(x,D)$, $a \in V$, $b \in W$, form a bounded subset of $\mathcal{L}_b(H^s)$ (cf. theorem 1.7.14 and theorem 2.1.11 and its proof of [21]), the claim follows. □

The next result has been proved in [10] for more general hypoelliptic operators. It is immediate to verify that it holds in particular for symbols satisfying the ellipticity conditions in Definition 0.1.

Theorem 1.8. *Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be $(\tilde{M}_p)$-elliptic (resp. $\{\tilde{M}_p\}$-elliptic). Then there exists a *-regularizing operator T and a symbol $b \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ such that $b(x,D)a(x,D) = \mathrm{Id} + T$. Moreover, the symbol b satisfies the following condition: there exists $B' > 0$ such that for every $h > 0$ there exist $C > 0$ (resp. there exist $h, C > 0$) such that*

$$\left|D^\alpha_\xi D^\beta_x b(x,\xi)\right| \leq C\frac{h^{|\alpha|+|\beta|}A_{\alpha+\beta}}{|a(x,\xi)|\langle(x,\xi)\rangle^{\rho(|\alpha|+|\beta|)}}, \qquad (x,\xi) \in Q^c_{B'}. \tag{1.2}$$

Lemma 1.9. *Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be $(\tilde{M}_p)$-elliptic (resp. $\{\tilde{M}_p\}$-elliptic) and let b be the symbol of the parametrix of $a(x,D)$. Then the sets $\{b\}$ and $\left\{\frac{h^{|\alpha|}}{M_\alpha}D^\alpha_w a \,\middle|\, \alpha \in \mathbb{N}^{2d}\right\}$ are $(|a(w)|,*)$-conjugate for every $h > 0$ (resp. for some $h > 0$). Hence, for every $h > 0$ there exists $C > 0$ (resp. there exist $h, C > 0$) such that $\|b(x,D) \circ (D^\alpha_w a)(x,D)\|_{\mathcal{L}_b(H^s)} \leq Ch^{|\alpha|}M_\alpha$. Moreover, in the (M_p) case, there exist $(r_p) \in \mathfrak{R}$ with $r_1 = 1$ and $C > 0$ such that $\|b(x,D) \circ (D^\alpha_w a)(x,D)\|_{\mathcal{L}_b(H^s)} \leq CM_\alpha/R_\alpha$.*

Proof. We have

$$\begin{aligned}
\frac{h^{|\alpha|}}{M_\alpha}\left|D^{\alpha+\beta}_w a(w)\right| &\leq C_1\frac{h^{|\alpha|}h_1^{|\alpha|+|\beta|}A_{\alpha+\beta}|a(w)|}{M_\alpha\langle w\rangle^{\rho(|\alpha|+|\beta|)}} \\
&\leq C_2\frac{(LHhh_1)^{|\alpha|}(Hh_1)^{|\beta|}A_\beta|a(w)|}{\langle w\rangle^{\rho|\beta|}}.
\end{aligned}$$

The $\{M_p\}$ case trivially follows from this by choosing h small enough, since h_1 is fixed. In the (M_p) case, for each fixed h we can take h_1 arbitrary small. One easily sees that this implies that the sets under consideration are $(|a(w)|,(M_p))$-conjugate. The inequality $\|b(x,D) \circ (D^\alpha_w a)(x,D)\|_{\mathcal{L}_b(H^s)} \leq Ch^{|\alpha|}M_\alpha$ follows by Proposition 1.7. It remains to prove the last part of the lemma. Since, in the (M_p) case, we already proved that for every $h > 0$, $\sup_{\alpha\in\mathbb{N}^{2d}} \frac{\|b(x,D) \circ (D^\alpha_w a)(x,D)\|_{\mathcal{L}_b(H^s)}}{h^{|\alpha|}M_\alpha} < \infty$, by Lemma 3.4 of [19] we can conclude that there exists $(\tilde{r}_p) \in \mathfrak{R}$ and $C > 0$ such that $\|b(x,D) \circ (D^\alpha_w a)(x,D)\|_{\mathcal{L}_b(H^s)} \leq CM_\alpha/\tilde{R}_\alpha$. If we take $r_p = \max\{\tilde{r}_p, 1\}$, then $(r_p) \in \mathfrak{R}$, $r_1 = 1$ and the desired estimate holds for this (r_p), possibly with another C. □

Lemma 1.10. *There exists $l \geq 1$ such that for any $h > 0$, the sets $\{b\}$ and $\left\{\frac{h^{|\beta|}}{l^{|\gamma|}\tilde{M}_\gamma}D^\alpha_x p_\beta \xi^\gamma \,\middle|\, \alpha,\beta,\gamma \in \mathbb{N}^d\right\}$ are $\left(e^{\tilde{M}(m|x|)}e^{\tilde{M}(m|\xi|)}, (M_p)\right)$-conjugate, (resp. $\{b\}$ and $\left\{\frac{h^{|\beta|}}{l^{|\gamma|}K_\gamma\tilde{M}_\gamma}D^\alpha_x p_\beta \xi^\gamma \,\middle|\, \alpha,\beta,\gamma \in \mathbb{N}^d\right\}$ are $\left(e^{\tilde{N}_{k_p}(|x|)}e^{\tilde{N}_{k_p}(|\xi|)}, \{M_p\}\right)$-conjugate). In* *particular for each $\tilde{h}>0$ there exists $C>0$ such that $\|(B\circ D^{\alpha}p_{\beta}(x)\partial^{\gamma})(x,D)\|_{\mathcal{L}(H^s)}\leq C\tilde{h}^{|\beta|}l^{|\gamma|}\tilde{M}_{\gamma}$ (resp. $\|(B\circ D^{\alpha}p_{\beta}(x)\partial^{\gamma})(x,D)\|_{\mathcal{L}(H^s)}\leq C\tilde{h}^{|\beta|}l^{|\gamma|}K_{\gamma}\tilde{M}_{\gamma}$).*

Proof. We consider first the (M_p) case. Pick $l\geq 1$ such that $H^2/l\leq m/12$. Let $h,h'>0$ be arbitrary but fixed. Pick $0<h_1<1$ such that $H\sqrt{h_1}\leq h'$, $L\tilde{L}Hh_1\leq 1$, $hh_1\leq 1$ and $H^2\sqrt{h_1}\leq m/6$. Then

$$
\begin{aligned}
&\frac{h^{|\beta|}}{l^{|\gamma|}\tilde{M}_{\gamma}}\left|D^{\mu}_{\xi}D^{\nu}_{x}\left(D^{\alpha}_{x}p_{\beta}(x)\xi^{\gamma}\right)\right|\\
&\leq \frac{h^{|\beta|}\gamma!}{l^{|\gamma|}\tilde{M}_{\gamma}(\gamma-\mu)!}|\xi|^{|\gamma|-|\mu|}\left|D^{\alpha+\nu}_{x}p_{\beta}(x)\right|\\
&\leq C_1\frac{2^{|\gamma|}h^{|\beta|}h_1^{|\alpha|+|\beta|+|\nu|}A_{\alpha+\nu}\mu!\langle(x,\xi)\rangle^{|\mu|+|\nu|}}{l^{|\gamma|}\tilde{M}_{\gamma}\tilde{M}_{\alpha+\nu}\langle(x,\xi)\rangle^{\rho(|\mu|+|\nu|)}}|\xi|^{|\gamma|-|\mu|}e^{\tilde{M}(h_1|x|)}\\
&\leq C_2\frac{(2H/l)^{|\gamma|}(H^2h_1)^{|\nu|}(Hh_1)^{|\alpha|}(hh_1)^{|\beta|}A_{\alpha}A_{\nu}\mu!\langle(x,\xi)\rangle^{|\gamma|+|\nu|}}{\tilde{M}_{\gamma+\nu}\tilde{M}_{\alpha}\langle(x,\xi)\rangle^{\rho(|\mu|+|\nu|)}}e^{\tilde{M}(h_1|x|)}\\
&\leq C_3\frac{(H\sqrt{h_1})^{|\nu|}(L\tilde{L}Hh_1)^{|\alpha|}(hh_1)^{|\beta|}h'^{|\mu|}A_{\nu+\mu}}{\langle(x,\xi)\rangle^{\rho(|\mu|+|\nu|)}}e^{\tilde{M}(h_1|x|)}e^{\tilde{M}((H\sqrt{h_1}+2H/l)\langle(x,\xi)\rangle)}.
\end{aligned}
$$

Since $\tilde{M}_p$ satisfies $(M.2)$, by Proposition 3.6 of [16], we have

$$
\begin{aligned}
e^{\tilde{M}(h_1|x|)}e^{\tilde{M}((H\sqrt{h_1}+2H/l)\langle(x,\xi)\rangle)} &\leq C_4e^{\tilde{M}(h_1|x|)}e^{\tilde{M}(3(H\sqrt{h_1}+2H/l)|x|)}e^{\tilde{M}(3(H\sqrt{h_1}+2H/l)|\xi|)}\\
&\leq C_5e^{\tilde{M}(3(H\sqrt{h_1}+2H/l)|\xi|)}e^{\tilde{M}(3(H^2\sqrt{h_1}+2H^2/l)|x|)}.
\end{aligned}
$$

If we use this in the above estimate, by the way we defined h_1, we have

$$
\frac{h^{|\beta|}}{l^{|\gamma|}\tilde{M}_{\gamma}}\left|D^{\mu}_{\xi}D^{\nu}_{x}\left(D^{\alpha}_{x}p_{\beta}(x)\xi^{\gamma}\right)\right|\leq C\frac{h'^{|\mu|+|\nu|}A_{\mu+\nu}}{\langle(x,\xi)\rangle^{\rho(|\mu|+|\nu|)}}e^{\tilde{M}(m|\xi|)}e^{\tilde{M}(m|x|)},
$$

which proofs the (M_p) case. In the $\{M_p\}$ case one can use the same technique as above (observe that the sequence $K_p\tilde{M}_p$ satisfies $(M.2)$). The last part follows by Proposition 1.7. □

Lemma 1.11. *Let $h>0$ and for each $\beta\in\mathbb{N}^d$, let $p_{\beta}(x)$ be a smooth function satisfying* (0.4) *in the (M_p) case (resp. satisfying* (0.5) *in the $\{M_p\}$ case) and let $j_{\beta}\in\{1,...,d\}$. Then the following properties hold:*

a) The sets $\{b\}$ and $\left\{h^{|\beta|}x_{j_{\beta}}p_{\beta}(x)\middle|\,\beta\in\mathbb{N}^d\right\}$ are $\left(e^{\tilde{M}(m|x|)}e^{\tilde{M}(m|\xi|)},(M_p)\right)$-conjugate (resp. $\left(e^{\tilde{N}_{k_p}(|x|)}e^{\tilde{N}_{k_p}(|\xi|)},\{M_p\}\right)$-conjugate). In particular, for every $\tilde{h}>0$ there exists $C>0$, such that $\left\|(B\circ x_{j_{\beta}}p_{\beta}(x))(x,D)\right\|_{\mathcal{L}_b(H^s)}\leq C\tilde{h}^{|\beta|}$. Moreover, there exist $(r_p)\in\mathfrak{R}$ with $r_1=1$ and $C>0$ such that $\left\|(B\circ x_{j_{\beta}}p_{\beta}(x))(x,D)\right\|_{\mathcal{L}_b(H^s)}\leq C/R_{\beta}$.

b) The sets $\{b\}$ and $\left\{h^{|\beta|}\xi_jp_{\beta}(x)\middle|\,\beta\in\mathbb{N}^d\right\}$ are $\left(e^{\tilde{M}(m|x|)}e^{\tilde{M}(m|\xi|)},(M_p)\right)$-conjugate (resp. $\left(e^{\tilde{N}_{k_p}(|x|)}e^{\tilde{N}_{k_p}(|\xi|)},\{M_p\}\right)$-conjugate). In particular, for every $\tilde{h}>0$

there exists $C>0$ such that $\|(B\circ p_\beta(x)\partial_j)(x,D)\|_{\mathcal{L}_b(H^s)}\leq C\tilde{h}^{|\beta|}$. Moreover, there exist $(r_p)\in\mathfrak{R}$ with $r_1=1$ and $C>0$ such that $\|(B\circ p_\beta(x)\partial_j)(x,D)\|_{\mathcal{L}_b(H^s)}\leq C/R_\beta$.

Proof. We prove *a*), the proof of *b*) being completely analogous. In the (M_p) case, let $h,h'>0$ be arbitrary but fixed. We have

$$\begin{aligned} h^{|\beta|}\left|D^\alpha\left(x^{e_{j_\beta}}p_\beta(x)\right)\right| &\leq h^{|\beta|}\left|x_{j_\beta}\right||D^\alpha p_\beta(x)|+|\alpha|h^{|\beta|}\left|D^{\alpha-e_{j_\beta}}p_\beta(x)\right| \\ &= S_1(x)+S_2(x). \end{aligned}$$

Take $h_2<1$ such that $3h_2H\leq m/2$ and take $h_1<1$ such that $2Hh_1/h_2\leq h'$, $Hh_1\leq m/2$ and $h_1\leq 1/h$. To estimate $S_1(x)$ we have

$$\begin{aligned} S_1(x) &\leq C_1h^{|\beta|}|x|\frac{(Hh_1)^{|\alpha|}h_1^{|\beta|}A_\alpha e^{\tilde{M}(h_1|x|)}\langle(x,\xi)\rangle^{|\alpha|}}{\tilde{M}_{|\alpha|+1}\langle(x,\xi)\rangle^{\rho|\alpha|}} \\ &\leq C_1\frac{(Hh_1)^{|\alpha|}A_\alpha e^{\tilde{M}(h_1|x|)}e^{\tilde{M}(h_2\langle(x,\xi)\rangle)}}{h_2^{|\alpha|+1}\langle(x,\xi)\rangle^{\rho|\alpha|}} \\ &\leq C_2\frac{h'^{|\alpha|}A_\alpha e^{\tilde{M}((h_1+3h_2)H|x|)}e^{\tilde{M}(3h_2|\xi|)}}{\langle(x,\xi)\rangle^{\rho|\alpha|}}\leq C_2\frac{h'^{|\alpha|}A_\alpha e^{\tilde{M}(m|\xi|)}e^{\tilde{M}(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\alpha|}}. \end{aligned}$$

Similar estimates can be obtained for $S_2(x)$ in the same way and the $\{M_p\}$ case can be treated similarly. The estimate $\|(B\circ x_jp_\beta(x))(x,D)\|_{\mathcal{L}_b(H^s)}\leq C\tilde{h}^{|\beta|}$ follows from Proposition 1.7. The last part can be proved similarly as in the proof of Lemma 1.9, by using Lemma 3.4 of [19]. □

2 The proof of Theorem 0.2

The proof of Theorem 0.2 needs some preparation. First of all it is useful to characterise the space $\mathcal{S}^*(\mathbb{R}^d)$ in terms of suitable scales of Sobolev norms.

Namely, let

$$\begin{aligned} \|\varphi\|_{s,h} &= \sum_{\alpha\in\mathbb{N}^d}\frac{h^{|\alpha|}}{M_\alpha}\|x^\alpha\varphi(x)\|_{H^s}, \\ \|\varphi\|_{\{s,h\}} &= \sum_{\alpha\in\mathbb{N}^d}\frac{h^{|\alpha|}}{M_\alpha}\|D^\alpha\varphi(x)\|_{H^s}. \end{aligned}$$

Moreover, for $h>0$ and $(r_p)\in\mathfrak{R}$, set

$$\begin{aligned} H_N^{s,h}[\varphi] &= \sum_{|\alpha|\leq N}\frac{h^{|\alpha|}}{M_\alpha}\|x^\alpha\varphi(x)\|_{H^s} \quad , \quad H_N^{s,h,(r_p)}[\varphi]=\sum_{|\alpha|\leq N}\frac{h^{|\alpha|}R_\alpha}{M_\alpha}\|x^\alpha\varphi(x)\|_{H^s}, \\ E_N^{s,h}[\varphi] &= \sum_{|\alpha|\leq N}\frac{h^{|\alpha|}}{M_\alpha}\|D^\alpha\varphi(x)\|_{H^s} \quad , \quad E_N^{s,h,(r_p)}[\varphi]=\sum_{|\alpha|\leq N}\frac{h^{|\alpha|}R_\alpha}{M_\alpha}\|D^\alpha\varphi(x)\|_{H^s}. \end{aligned}$$

We have the following result, see ....

Lemma 2.1. *Let $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$. The following conditions are equivalent:*

i) $\varphi \in \mathcal{S}^*\left(\mathbb{R}^d\right)$;

ii) there exists $s > d/2$ such that for every $h > 0$ (resp. there exists $h > 0$) such that $\|\varphi\|_{s,h} < \infty$ and $\|\varphi\|_{\{s,h\}} < \infty$.

By Lemma 2.1 we can prove that a function $u \in \mathcal{S}^*(\mathbb{R}^d)$ by proving the decay and the regularity properties separately. This allows to simplify considerably the proofs, see also [5, 6, 7].

Next we state a preliminary technical result which will be used in the subsequent proofs.

Lemma 2.2. *Let M_p be a sequence which satisfies $(M.3)'$, $(M.4)$ and $M_0 = M_1 = 1$. Let $(k'_p), (k''_p) \in \mathfrak{R}$, $k'_1 = k''_1 = 1$. There exists $(r'_p) \in \mathfrak{R}$ such that $r'_1 = 1$, $(r'_p) \leq (k'_p)$, $(r'_p) \leq (k''_p)$ and the sequence $N_p = M_p/R'_p$, for $p \in \mathbb{Z}_+$ and $N_0 = 1$ satisfies $(M.3)'$ and $(M.4)$.*

Proof. Let $a_p > 0$, $p \in \mathbb{Z}_+$, are such that $\sum_{p=1}^{\infty} a_p$ is convergent. Then one easily verifies that $\sum_{p=1}^{\infty} \frac{a_p}{s_p}$ is also convergent, where $s_p = \sqrt{\sum_{j=p}^{\infty} a_j}$, $p \in \mathbb{Z}_+$ (one easily obtains that the partial sums of the series $\sum a_p/s_p$ are a Cauchy sequence). Put $\tilde{c} = \sqrt{\sum_{j=1}^{\infty} 1/m_j}$ and define $\tilde{r}'_p = \tilde{c}\left(\sum_{j=p}^{\infty} \frac{1}{m_j}\right)^{-1/2}$, $p \in \mathbb{Z}_+$. Then we obtain that $\tilde{r}'_1 = 1$, $(\tilde{r}'_p) \in \mathfrak{R}$ and $\sum \tilde{r}'_p/m_p$ converges. Let $r_p = \min\{k'_p, k''_p, \tilde{r}'_p\}$, for $p \in \mathbb{Z}_+$. Then, obviously, $r_1 = 1$, $(r_p) \in \mathfrak{R}$, $(r_p) \leq (k'_p)$, $(r_p) \leq (k''_p)$ and $(r_p) \leq (\tilde{r}'_p)$. Also $\sum r_p/m_p$ converges. Define the sequence (r'_p) by $r'_1 = 1$ and inductively

$$r'_{p+1} = \min\left\{r_{p+1}, \frac{pm_{p+1}}{(p+1)m_p} r'_p\right\},$$

for $p \in \mathbb{Z}_+$. We will prove that this (r'_p) satisfies the desired conditions. First, note that $r'_p \leq r_p$, for all $p \in \mathbb{Z}_+$. Since $r_{p+1} \geq r_p$ and $pm_{p+1} \geq (p+1)m_p$ (which is equivalent to $(M.4)$ for M_p) it follows that

$$r'_{p+1} = \min\left\{r_{p+1}, \frac{pm_{p+1}}{(p+1)m_p} r'_p\right\} \geq \min\{r_p, r'_p\} = r'_p,$$

for all $p \in \mathbb{Z}_+$. To prove that r'_p tends to infinity, assume the contrary. Since we already proved that r'_p is monotonically increasing, there exists $C > 1$ such that $r'_p \leq C$ for all $p \in \mathbb{Z}_+$. Since $(r_p) \in \mathfrak{R}$, there exists $p_0 \in \mathbb{Z}_+$ such that $r_p > C + 1$ for all $p \geq p_0$. But then, $r'_{p+1} = \frac{pm_{p+1}}{(p+1)m_p} r'_p$ for all $p \geq p_0$. Then, for $p \geq p_0$, we have

$$r'_{p+1} = \frac{pm_{p+1}}{(p+1)m_p} r'_p = \frac{pm_{p+1}}{(p+1)m_p} \cdot \frac{(p-1)m_p}{pm_{p-1}} r'_{p-1}$$

$$
\begin{aligned}
&= \quad ... = \frac{pm_{p+1}}{(p+1)m_p} \cdot \frac{(p-1)m_p}{pm_{p-1}} \cdot ... \cdot \frac{p_0 m_{p_0+1}}{(p_0+1)m_{p_0}} r'_{p_0} \\
&= \quad \frac{p_0 m_{p+1}}{(p+1)m_{p_0}} r'_{p_0}
\end{aligned}
$$

which tends to infinity when $p \to \infty$ because of $(M.3)'$ for M_p. Hence $(r'_p) \in \mathfrak{R}$. The claim that N_p satisfies $(M.4)$ is equivalent to $r'_{p+1} \leq \frac{pm_{p+1}}{(p+1)m_p} r'_p$, which is true by the way we defined the sequence (r'_p). Moreover, if we put $n_p = N_p/N_{p-1}$, then $n_p = m_p/r'_p \geq m_p/r_p$ and we know that $\sum r_p/m_p$ converges. Hence N_p satisfies $(M.3)'$. $\square$

After these preliminaries we can prove the following two results.

Theorem 2.3. *Let $a \in \Gamma^{*,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$ be $(\tilde{M}_p)$-elliptic (resp. $\{\tilde{M}_p\}$-elliptic) and let $f \in \mathcal{S}^*\left(\mathbb{R}^d\right)$. Assume that $u \in H^s\left(\mathbb{R}^d\right)$, $s > d/2$, is a solution of $Au = f + F[u]$, where $F[u]$ is defined by* (0.4) *and* (0.6) *(resp.* (0.5) *and* (0.6)*). Then we have $\|u\|_{s,h} < \infty$ for every $h > 0$ (resp. for some $h > 0$).*

Theorem 2.4. *Let $a \in \Gamma^{*,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$ be $(\tilde{M}_p)$-elliptic (resp. $\{\tilde{M}_p\}$-elliptic) and let $f \in \mathcal{S}^*\left(\mathbb{R}^d\right)$. Assume that $u \in H^s\left(\mathbb{R}^d\right)$, $s > d/2$, is a solution of $Au = f + F[u]$, where $F[u]$ is defined by* (0.4) *and* (0.6) *(resp.* (0.5) *and* (0.6)*).*

i) If $F[u]$ is a finite sum, then we have $\|u\|_{\{s,h\}} < \infty$ for every $h > 0$ (resp. for some $h > 0$).

ii) If $F[u]$ is infinite sum, then $\sum_\alpha \frac{h^{|\alpha|}}{\tilde{M}_\alpha}\|\partial^\alpha u\|_{H^s} < \infty$ for some $h > 0$ in the (M_p) case (resp. $\sum_\alpha \frac{h^{|\alpha|}\|\partial^\alpha u\|_{H^s}}{\tilde{M}_\alpha \prod_{j=1}^{|\alpha|} k_j} < \infty$ for some $h > 0$ in the $\{M_p\}$ case).

Notice that by Lemma 2.1, Theorem 0.2 follows directly from the combination of Theorems 2.3 and 2.4. Namely, the assertion i) in Theorem 0.2 follows from Theorem 2.3 whereas for nonlinear terms given by polynomials in u, Theorem 2.4 yields the regularity properties of the solution. Then we can conclude by invoking Lemma 2.1. Let us prove the two latter results.

Lemma 2.5. *Let $A = a(x, D)$ be $(\tilde{M}_p)$-elliptic (resp. $\{\tilde{M}_p\}$-elliptic) operator and let B be its parametrix. Then the following properties hold:*
i) In the (M_p) case, let $(r_p) \in \mathfrak{R}$ be the sequence in Lemma 1.9. Then there exists $(r'_p) \in \mathfrak{R}$ such that $(r'_p) \leq (r_p)$, $r'_1 = 1$ and the sequence M_p/R'_p satisfies $(M.3)'$ and $(M.4)$. Moreover, for each $0 < \varepsilon < 1$ there exists $h_0 = h_0(\varepsilon)$ such that for every $0 < h < h_0$

$$\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|} R'_\alpha}{M_\alpha} \|B[A, x^\alpha]u\|_{H^s} \leq \varepsilon H^{s,h,(r'_p)}_{N-1}[u].$$

ii) In the $\{M_p\}$ case, for each $0 < \varepsilon < 1$ there exists $h_0 = h_0(\varepsilon)$ such that for all $0 < h < h_0(\varepsilon)$

$$\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{M_\alpha} \|B[A, x^\alpha]u\|_{H^s} \leq \varepsilon H^{s,h}_{N-1}[u].$$

Proof. First we prove the (M_p) case. The existence of the sequence (r'_p) is given by Lemma 2.2. For shorter notation, put $N_p = M_p/R'_p$, for $p \in \mathbb{Z}_+$ and $N_0 = 1$. Observe that

$$x^\alpha Au(x) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (-1)^{|\beta|} (D_\xi^\beta a)(x, D)(x^{\alpha-\beta} u(x)).$$

So, we obtain

$$B[A, x^\alpha]u = \sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}} \binom{\alpha}{\beta} (-1)^{|\beta|+1} B(D_\xi^\beta a)(x, D)(x^{\alpha-\beta} u(x)).$$

By Lemma 1.9, there exists $C > 0$ such that $\left\|B(D_\xi^\beta a)(x, D)\right\|_{\mathcal{L}_b(H^s)} \leq CN_\beta$. Let $0 < \varepsilon < 1$ be fixed. Choose $0 < h_0 < 1/2$ such that $h_0 < \varepsilon \left(2C \sum_{|\beta|=1}^\infty 2^{-|\beta|+1}\right)^{-1}$. For $0 < h < h_0$ we obtain

$$\begin{aligned}
&\sum_{|\alpha|=1}^N \frac{h^{|\alpha|}}{N_\alpha} \|B[A, x^\alpha]u\|_{H^s}\\
&\leq \sum_{|\alpha|=1}^N \frac{h^{|\alpha|}}{N_\alpha} \sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}} \binom{\alpha}{\beta} \left\|B(D_\xi^\beta a)(x, D)x^{\alpha-\beta} u\right\|_{H^s}\\
&\leq C \sum_{|\alpha|=1}^N \sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}} \frac{h^{|\alpha|} N_\beta}{N_\alpha} \binom{\alpha}{\beta} \left\|x^{\alpha-\beta} u\right\|_{H^s} \leq C \sum_{|\beta|=1}^N h^{|\beta|} \sum_{\substack{\alpha\geq\beta\\ |\alpha|\leq N}} \frac{h^{|\alpha|-|\beta|}}{N_{\alpha-\beta}} \left\|x^{\alpha-\beta} u\right\|_{H^s}\\
&\leq \varepsilon H_{N-1}^{s,h,(r'_p)}[u],
\end{aligned}$$

where in the third inequality, we used $(M.4)$ for N_p and the fact $\binom{\alpha}{\beta} \leq \binom{|\alpha|}{|\beta|}$. This completes the proof in the (M_p) case. For the $\{M_p\}$ case, let $\varepsilon > 0$. By Lemma 1.9, there exist $h_1, C > 0$ such that $\left\|B(D_\xi^\beta a)(x, D)\right\|_{\mathcal{L}_b(H^s)} \leq Ch_1^{|\beta|} M_\beta$. Choose $h_0 > 0$ such that $h_0 h_1 < 1/2$ and $h_0 h_1 \leq \varepsilon \left(2C \sum_{|\beta|=1}^\infty 2^{-|\beta|+1}\right)^{-1}$. Then, for $0 < h < h_0$, similarly as before, we obtain

$$\begin{aligned}
\sum_{|\alpha|=1}^N \frac{h^{|\alpha|}}{M_\alpha} \|B[A, x^\alpha]u\|_{H^s} &\leq C \sum_{|\beta|=1}^N (hh_1)^{|\beta|} \sum_{\substack{\alpha\geq\beta\\ |\alpha|\leq N}} \frac{h^{|\alpha|-|\beta|}}{M_{\alpha-\beta}} \left\|x^{\alpha-\beta} u\right\|_{H^s}\\
&\leq \varepsilon H_{N-1}^{s,h}[u],
\end{aligned}$$

which completes the proof. $\square$

Lemma 2.6. *Let $A = a(x, D)$ be $(\tilde{M}_p)$-elliptic (resp. $\{\tilde{M}_p\}$-elliptic) operator and let B be its parametrix. Let $F[u]$ be defined by* (0.4), (0.6) *in the* (M_p) *case (resp. by* (0.5), (0.6) *in the* $\{M_p\}$ *case). Then the following properties hold:*

i) In the (M_p) case, let $(r_p) \in \mathfrak{R}$ be the sequence in Lemma 1.11. Then, there exists $(r'_p) \in \mathfrak{R}$ such that $(r'_p) \leq (r_p)$, $r'_1 = 1$ and the sequence M_p/R'_p satisfies $(M.3)'$ and $(M.4)$. Moreover, for each $0 < \varepsilon < 1$ there exists $h_0 = h_0(\varepsilon)$ such that for every $0 < h < h_0$

$$\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|} R'_\alpha}{M_\alpha} \|Bx^\alpha F[u]\|_{H^s} \leq \varepsilon H_{N-1}^{s,h,(r'_p)}[u].$$

ii) In the $\{M_p\}$ case, for each $0 < \varepsilon < 1$ there exists $h_0 = h_0(\varepsilon)$ such that for every $0 < h < h_0$

$$\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{M_\alpha} \|Bx^\alpha F[u]\|_{H^s} \leq \varepsilon H_{N-1}^{s,h}[u].$$

Proof. i) The existence of the sequence (r'_p) is given by Lemma 2.2. Let now $\alpha \in \mathbb{N}^d$ with $|\alpha| \geq 1$ and let $j = j_\alpha \in \{1, ..., d\}$ such that $\alpha_j > 0$. By Lemma 1.11, there exists $C_1 > 0$ such that

$$\|(B \circ x_j p_\beta(x))(x, D)\|_{\mathcal{L}_b(H^s)} \leq C_1/R'_\beta.$$

We obtain that $\left\|B\left(x_j p_\beta(x) x^{\alpha - e_j} u^{|\beta|}\right)\right\|_{H^s} \leq \frac{C_1}{R'_\beta}\left\|x^{\alpha-e_j}u^{|\beta|}\right\|_{H^s}$. Moreover

$$\left\|x^{\alpha-e_j}u^{|\beta|}\right\|_{H^s} \leq C_s^{|\beta|-1}\left\|x^{\alpha-e_j}u\right\|_{H^s}\|u\|_{H^s}^{|\beta|-1}.$$

Hence

$$\left\|B\left(p_\beta(x)x^\alpha u^{|\beta|}\right)\right\|_{H^s} \leq C_1\frac{(C_s\|u\|_{H^s})^{|\beta|-1}}{R'_\beta}\left\|x^{\alpha-e_j}u\right\|_{H^s}.$$

Let $C_2 = \sum_{|\beta|=2}^{\infty} \frac{(C_s\|u\|_{H^s})^{|\beta|-1}}{R'_\beta}$. We obtain

$$\begin{aligned}
\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{N_\alpha}\|Bx^\alpha F[u]\|_{H^s} &\leq \sum_{|\alpha|=1}^{N}\sum_{|\beta|=2}^{\infty}\frac{h^{|\alpha|}}{N_\alpha}\left\|B(x^\alpha p_\beta(x)u^{|\beta|})\right\|_{H^s} \\
&\leq C_1C_2h\sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|-1}}{N_{\alpha-e_j}}\left\|x^{\alpha-e_{j_\alpha}}u\right\|_{H^s} \leq C_3hH_{N-1}^{s,h,(r'_p)}[u].
\end{aligned}$$

Moreover, for fixed $0 < \varepsilon < 1$, since C_3 does not depend on h, we can find $h_0 = h_0(\varepsilon)$ such that for all $0 < h < h_0$

$$\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{N_\alpha}\|Bx^\alpha F[u]\|_{H^s} \leq \varepsilon H_{N-1}^{s,h,(r'_p)}[u],$$

which complete the proof in the (M_p) case.
ii) In the $\{M_p\}$ case by using Lemma 1.11, one similarly obtains that for every $\tilde{h} > 0$ there exists $C_1 > 0$ such that

$$\left\| B\left(p_\beta(x) x^\alpha u^{|\beta|}\right)\right\|_{H^s} \leq C_1 \left(\tilde{h} C_s \|u\|_{H^s}\right)^{|\beta|-1} \left\|x^{\alpha - e_{j_\alpha}} u\right\|_{H^s}.$$

Fix $\tilde{h}$ such that $\tilde{h} C_s \|u\|_{H^s} < 1/2$. We have

$$\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{M_\alpha} \|B x^\alpha F[u]\|_{H^s} \leq C_3' h H_{N-1}^{s,h}[u],$$

for a constant C_3' which is the same for all h. Hence, we obtain the claim in the $\{M_p\}$ case. □

Proof of Theorem 2.3. Fixed $\alpha \in \mathbb{N}^d$ let us multiply both members of (0.7) by x^α. We have $x^\alpha A u = x^\alpha f + x^\alpha F[u]$. Then, introducing commutators we get

$$A(x^\alpha u) = [A, x^\alpha] u + x^\alpha f + x^\alpha F[u]. \tag{2.1}$$

By applying the parametrix B of A to both sides of (2.1) we have

$$x^\alpha u = B[A, x^\alpha] u + B(x^\alpha f) + B(x^\alpha F[u]) + T(x^\alpha u) \tag{2.2}$$

for some *-regularizing operator T. We first consider the (M_p) case. Since $f \in \mathcal{S}^{(M_p)}$, then for every $\tilde{h} > 0$ we have $\sup_\alpha \dfrac{\|x^\alpha f\|_{H^s}}{\tilde{h}^{|\alpha|} M_\alpha} < \infty$. Hence, by Lemma 3.4 of [19], there exist $(\tilde{r}_p) \in \mathfrak{R}$ and $C' > 0$ such that $\|x^\alpha f\|_{H^s} \leq C' M_\alpha / \tilde{R}_\alpha$. Obviously, without loss of generality, we can assume that $\tilde{r}_1 = 1$. By Lemma 2.2 we can find $(r_p') \in \mathfrak{R}$ such that $r_1' = 1$, $(r_p') \leq (\tilde{r}_p)$, (r_p') is smaller than the sequences in Lemmas 1.9 and 1.11, and the sequence $N_p = M_p / R_p'$, for $p \in \mathbb{Z}_+$ and $N_0 = 1$, satisfies $(M.3)'$, $(M.4)$ and $N_1 = 1$. If we multiply (2.2) by $h^{|\alpha|}/N_\alpha$, take Sobolev norms and sum up for $|\alpha| \leq N$, we obtain

$$\begin{aligned} H_N^{s,h,(r_p')}[u] \quad &\leq \quad \|u\|_{H^s} + \sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{N_\alpha} \|B[A, x^\alpha] u\|_{H^s} + \sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{N_\alpha} \|B(x^\alpha f)\|_{H^s} \\ &\quad + \sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{N_\alpha} \|B(x^\alpha F[u])\|_{H^s} + \sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{N_\alpha} \|T(x^\alpha u)\|_{H^s}. \end{aligned}$$

We will estimate each of the terms above. First, since B is bounded on H^s, there exists $C'' > 0$ such that $\|B(x^\alpha f)\|_{H^s} \leq C'' \|x^\alpha f\|_{H^s}$, from what we obtain $\displaystyle\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{N_\alpha} \|B(x^\alpha f)\|_{H^s} \leq C''' \sum_{|\alpha|=1}^{\infty} \frac{1}{2^{|\alpha|}} = C_1$ for all $0 < h < 1/2$. To estimate the sum with $T(x^\alpha u)$, since $|\alpha| > 0$ there exists $j = j_\alpha \in \{1, ..., d\}$ such that $\alpha_j \geq 1$. Hence, there exists $C_2 > 0$ such that $\|T \circ x_j\|_{\mathcal{L}_b(H^s)} \leq C_2$. Then we obtain

$$\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{N_\alpha} \|T(x^\alpha u)\|_{H^s} \leq C_2 h \sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|-1}}{N_{|\alpha|-1}} \left\|x^{\alpha - e_{j_\alpha}} u\right\|_{H^s} \leq C_3 h H_{N-1}^{s,h,(r_p')}[u].$$

Since C_3 does not depend on h, for fixed $0 < \varepsilon < 1$ we can find $h_0 = h_0(\varepsilon) < 1/2$ such that for all $0 < h < h_0$

$$\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{N_\alpha} \|T(x^\alpha u)\|_{H^s} \leq \varepsilon H_{N-1}^{s,h,(r'_p)}[u].$$

Now, if we use Lemmas 2.6 and 2.5 for fixed $0 < \varepsilon < 1$ there exists $h_0 = h_0(\varepsilon)$ such that for all $0 < h < h_0$ we obtain

$$H_N^{s,h,(r'_p)}[u] \leq \|u\|_{H^s} + \varepsilon H_{N-1}^{s,h,(r'_p)}[u] + C_1 + \varepsilon H_{N-1}^{s,h,(r'_p)}[u] + \varepsilon H_{N-1}^{s,h,(r'_p)}[u].$$

By iterating this estimate and possibly shrinking ε we obtain that $\sum_{|\alpha|=0}^{\infty} \frac{h^{|\alpha|}}{N_\alpha} \|x^\alpha u\|_{H^s}$ is finite for some sufficiently small h. If $\tilde{h} > 0$ is arbitrary but fixed, there exists $\tilde{C} > 0$ such that $\tilde{h}^p \leq \tilde{C} h^p R'_p$, for all $p \in \mathbb{Z}_+$. Hence the sum $\sum_{|\alpha|=0}^{\infty} \frac{\tilde{h}^{|\alpha|}}{M_\alpha} \|x^\alpha u\|_{H^s}$ converges. This completes the proof in the (M_p) case. The $\{M_p\}$ case is completely similar. We leave the details to the reader. $\square$

Now we prove Theorem 2.4.

Lemma 2.7. *Let $A = a(x, D)$ be $(\tilde{M}_p)$-elliptic (resp. $\{\tilde{M}_p\}$-elliptic) operator and let B be its parametrix. Then the following properties hold:*
i) In the (M_p) case, let $(r_p) \in \mathfrak{R}$ be the sequence in Lemma 1.9. Then there exists $(r'_p) \in \mathfrak{R}$ such that $(r'_p) \leq (r_p)$, $r'_1 = 1$ and the sequence M_p/R'_p satisfies $(M.3)'$ and $(M.4)$. Moreover, for each $0 < \varepsilon < 1$ there exists $h_0 = h_0(\varepsilon)$ such that for every $0 < h < h_0$

$$\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|} R'_\alpha}{M_\alpha} \|B[A, \partial^\alpha]u\|_{H^s} \leq \varepsilon E_{N-1}^{s,h,(r'_p)}[u].$$

ii) In the $\{M_p\}$ case, for each $0 < \varepsilon < 1$ there exists $h_0 = h_0(\varepsilon)$ such that for all $0 < h < h_0(\varepsilon)$

$$\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{M_\alpha} \|B[A, \partial^\alpha]u\|_{H^s} \leq \varepsilon E_{N-1}^{s,h}[u].$$

Proof. First we prove the (M_p) case. As before we put $N_p = M_p/R'_p$, for $p \in \mathbb{Z}_+$ and $N_0 = 1$ (the existence of (r'_p) is given by Lemma 2.2). Observe that

$$B[A, \partial^\alpha]u = -\sum_{\substack{\beta \leq \alpha \\ \beta \neq 0}} \binom{\alpha}{\beta} B(\partial_x^\beta a)(x, D) \partial_x^{\alpha-\beta} u.$$

By Lemma 1.9, there exists $C > 0$ such that $\left\| B(\partial_x^\beta a)(x, D) \right\|_{\mathcal{L}_b(H^s)} \leq C N_\beta$. Let $0 < \varepsilon < 1$ be fixed. Choose $0 < h_0 < 1/2$ such that $h_0 < \varepsilon \left(2C \sum_{|\beta|=1}^{\infty} 2^{-|\beta|+1} \right)^{-1}$.

For $0 < h < h_0$ we obtain

$$\begin{aligned}
&\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{N_\alpha} \|B[A,\partial^\alpha]u\|_{H^s}\\
&\quad\leq \sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{N_\alpha} \sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}} \binom{\alpha}{\beta} \left\| B(\partial_x^\beta a)(x,D)\partial_x^{\alpha-\beta}u\right\|_{H^s}\\
&\quad\leq C\sum_{|\alpha|=1}^{N} \sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}} \frac{h^{|\alpha|}N_\beta}{N_\alpha}\binom{\alpha}{\beta} \left\|\partial_x^{\alpha-\beta}u\right\|_{H^s} \leq C\sum_{|\beta|=1}^{N} h^{|\beta|} \sum_{\substack{\alpha\geq\beta\\ |\alpha|\leq N}} \frac{h^{|\alpha|-|\beta|}}{N_{\alpha-\beta}} \left\|\partial_x^{\alpha-\beta}u\right\|_{H^s}\\
&\quad\leq \varepsilon E_{N-1}^{s,h,(r'_p)}[u],
\end{aligned}$$

where in the third inequality, we used $(M.4)$ for N_p and the fact $\binom{\alpha}{\beta} \leq \binom{|\alpha|}{|\beta|}$. This completes the proof in the (M_p) case. For the $\{M_p\}$ case, let $\varepsilon > 0$. By Lemma 1.9, there exist $h_1, C > 0$ such that $\left\|B(\partial_x^\beta a)(x,D)\right\|_{\mathcal{L}_b(H^s)} \leq C h_1^{|\beta|} M_\beta$. Choose $h_0 > 0$ such that $h_0 h_1 < 1/2$ and $h_0 h_1 \leq \varepsilon \left(2C\sum_{|\beta|=1}^{\infty} 2^{-|\beta|+1}\right)^{-1}$. Then, for $0 < h < h_0$, similarly as before, we obtain

$$\begin{aligned}
\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{M_\alpha} \|B[A,\partial^\alpha]u\|_{H^s} &\leq C\sum_{|\beta|=1}^{N} (hh_1)^{|\beta|} \sum_{\substack{\alpha\geq\beta\\ |\alpha|\leq N}} \frac{h^{|\alpha|-|\beta|}}{M_{\alpha-\beta}} \left\|\partial_x^{\alpha-\beta}u\right\|_{H^s}\\
&\leq \varepsilon E_{N-1}^{s,h}[u],
\end{aligned}$$

which completes the proof. $\square$

Lemma 2.8. *Let $A = a(x,D)$ be $(\tilde{M}_p)$-elliptic (resp. $\{\tilde{M}_p\}$-elliptic) operator and let B be its parametrix. Let $F[u] = p(x)u^l$, for some $l \geq 2$, $l \in \mathbb{N}$. Then the following properties hold:*
i) In the (M_p) case, let $(r_p) \in \mathfrak{R}$ be the sequence in Lemma 1.11. Then there exists $(r'_p) \in \mathfrak{R}$ such that $(r'_p) \leq (r_p)$, $r'_1 = 1$ and the sequence M_p/R'_p satisfies $(M.3)'$ and $(M.4)$. Moreover, for each $0 < \varepsilon < 1$ there exists $h_0 = h_0(\varepsilon)$ such that for every $0 < h < h_0$

$$\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}R'_\alpha}{M_\alpha} \left\|B\left(\partial^\alpha(p(x)u^l)\right)\right\|_{H^s} \leq \varepsilon \left(E_{N-1}^{s,h,(r'_p)}[u]\right)^l.$$

ii) In the $\{M_p\}$ case, for each $0 < \varepsilon < 1$ there exists $h_0 = h_0(\varepsilon)$ such that for every $0 < h < h_0$

$$\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{M_\alpha} \left\|B\left(\partial^\alpha(p(x)u^l)\right)\right\|_{H^s} \leq \varepsilon \left(E_{N-1}^{s,h}[u]\right)^l.$$

Proof. Observe that

$$B\left(\partial^\alpha(p(x)u^l)\right) = B\left(p(x)\partial^\alpha u^l\right) + \sum_{\substack{\gamma\le\alpha\\ \gamma\ne 0}} \binom{\alpha}{\gamma} B\left(\partial^\gamma p(x)\partial^{\alpha-\gamma}u^l\right)$$

First we consider the (M_p) case. As before we put $N_p = M_p/R'_p$, for $p \in \mathbb{Z}_+$ and $N_0 = 1$. Since $|\alpha| \ge 1$, there exists $j = j_\alpha \in \{1, ..., d\}$ such that $\alpha_j > 0$. By Lemma 1.11, there exists $C_1 > 0$ such that $\|(B \circ p(x)\partial_j)(x, D)\|_{\mathcal{L}_b(H^s)} \le C_1$. Then we have

$$\begin{aligned}\left\|B\left(p(x)\partial^\alpha u^l\right)\right\|_{H^s} &\le C_1 \left\|\partial^{\alpha-e_j}u^l\right\|_{H^s}\\ &\le C_2 \sum_{\nu^{(1)}+...+\nu^{(l)}=\alpha-e_j} \frac{(\alpha-e_j)!}{\nu^{(1)}!\cdot...\cdot\nu^{(l)}!}\prod_{k=1}^l \left\|\partial^{\nu^{(k)}}u\right\|_{H^s}.\end{aligned}$$

Observe that, by $(M.4)$,

$$\frac{(\alpha-e_j)!}{\nu^{(1)}!\cdot...\cdot\nu^{(l)}!}\cdot\frac{h^{|\alpha|}}{N_\alpha} \le \frac{hN_{|\alpha|-1}}{N_\alpha}\prod_{k=1}^l \frac{h^{|\nu^{(k)}|}}{N_{\nu^{(k)}}} \le h\prod_{k=1}^l \frac{h^{|\nu^{(k)}|}}{N_{\nu^{(k)}}}.$$

We obtain

$$\begin{aligned}\sum_{|\alpha|=1}^N \frac{h^{|\alpha|}}{N_\alpha}\left\|B\left(p(x)\partial^\alpha u^l\right)\right\|_{H^s} &\le C_2 h\sum_{|\alpha|=1}^N \sum_{\nu^{(1)}+...+\nu^{(l)}=\alpha-e_{j_\alpha}} \prod_{k=1}^l \frac{h^{|\nu^{(k)}|}}{N_{\nu^{(k)}}}\left\|\partial^{\nu^{(k)}}u\right\|_{H^s}\\ &\le dC_2 h\left(E^{s,h,(r'_p)}_{N-1}[u]\right)^l.\end{aligned}$$

Since C_2 does not depend on h, for fixed $0 < \varepsilon < 1$ we can take $h_0 = \varepsilon/(dC_2)$. Then for all $h < h_0$ we obtain $\sum_{|\alpha|=1}^N \frac{h^{|\alpha|}}{N_\alpha}\left\|B\left(p(x)\partial^\alpha u^l\right)\right\|_{H^s} \le \varepsilon\left(E^{s,h,(r'_p)}_{N-1}[u]\right)^l$. One easily verifies that the functions $p'_\beta(x) = \partial_x^\beta p(x)$, $\beta \in \mathbb{N}^d$ satisfy (0.4). Lemma 1.11 implies that there exists $C_1 > 0$ such that $\|B \circ \partial^\gamma p(x)\|_{\mathcal{L}_b(H^s)} \le C_1/R'_\gamma$. For $h < 1$ we obtain

$$\begin{aligned}&\sum_{|\alpha|=1}^N \frac{h^{|\alpha|}}{N_\alpha}\sum_{\substack{\gamma\le\alpha\\ \gamma\ne 0}}\binom{\alpha}{\gamma}\left\|B\left(\partial^\gamma p(x)\partial^{\alpha-\gamma}u^l\right)\right\|_{H^s}\\
&\le C_1 h\sum_{|\alpha|=1}^N\sum_{\substack{\gamma\le\alpha\\ \gamma\ne 0}}\binom{\alpha}{\gamma}\frac{h^{|\alpha|-1}}{N_\alpha R'_\gamma}\left\|\partial^{\alpha-\gamma}u^l\right\|_{H^s}\\
&\le C'_2 h\sum_{|\alpha|=1}^N\sum_{\substack{\gamma\le\alpha\\ \gamma\ne 0}}\binom{\alpha}{\gamma}\frac{h^{|\alpha|-1}}{N_\alpha R'_\gamma}\sum_{\nu^{(1)}+...+\nu^{(l)}=\alpha-\gamma}\frac{(\alpha-\gamma)!}{\nu^{(1)}!\cdot...\cdot\nu^{(l)}!}\prod_{k=1}^l\left\|\partial^{\nu^{(k)}}u\right\|_{H^s}\\
&\le C'_2 h\sum_{|\gamma|=1}^N\frac{1}{R'_\gamma}\sum_{\substack{\alpha\ge\gamma\\ |\alpha|\le N}}\frac{h^{|\alpha|-1}}{N_\alpha}\sum_{\nu^{(1)}+...+\nu^{(l)}=\alpha-\gamma}\frac{\alpha!}{\nu^{(1)}!\cdot...\cdot\nu^{(l)}!\gamma!}\prod_{k=1}^l\left\|\partial^{\nu^{(k)}}u\right\|_{H^s}\end{aligned}$$

$$\leq C_2'h\sum_{|\gamma|=1}^{N}\frac{1}{N_\gamma R_\gamma'}\sum_{\substack{\alpha\geq\gamma\\|\alpha|\leq N}}\sum_{\nu^{(1)}+\ldots+\nu^{(l)}=\alpha-\gamma}\prod_{k=1}^{l}\frac{h^{|\nu^{(k)}|}}{N_{\nu^{(k)}}}\left\|\partial^{\nu^{(k)}}u\right\|_{H^s}$$

$$\leq C_2'h\sum_{|\gamma|=1}^{N}\frac{1}{M_\gamma}\sum_{|\nu^{(1)}|+\ldots+|\nu^{(l)}|=0}^{N-1}\prod_{k=1}^{l}\frac{h^{|\nu^{(k)}|}}{N_{\nu^{(k)}}}\left\|\partial^{\nu^{(k)}}u\right\|_{H^s}\leq C_3'h\left(E_{N-1}^{s,h,(r_p')}[u]\right)^l.$$

Since C_3' does not depend on h, for fixed $0<\varepsilon<1$ we can choose $h_0=\varepsilon/C_3'$. Then, for all $0<h<h_0$, we have

$$\sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|}}{N_\alpha}\sum_{\substack{\gamma\leq\alpha\\|\gamma|\neq 0}}\binom{\alpha}{\gamma}\left\|B\left(\partial^\gamma p(x)\partial^{\alpha-\gamma}u^l\right)\right\|_{H^s}\leq\varepsilon\left(E_{N-1}^{s,h,(r_p')}[u]\right)^l,$$

which, combined with the above estimate for $\sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|}}{N_\alpha}\left\|B\left(p(x)\partial^\alpha u^l\right)\right\|_{H^s}$, completes the proof in the (M_p) case by shrinking h_0 if necessary. In the $\{M_p\}$ case the proof is similar. □

Proof of Theorem 2.4. *i*) When $F[u]$ is a finite sum it is clear that it is enough to prove the theorem when $F[u]=p(x)u^l$, $l\geq 2$, $l\in\mathbb{N}$. Differentiating both terms of (0.7), we have $\partial^\alpha Au=\partial^\alpha f+\partial^\alpha F[u]$, from which we obtain

$$A(\partial^\alpha u)=[A,\partial^\alpha]u+\partial^\alpha f+\partial^\alpha F[u].$$

Hence, we have

$$\partial^\alpha u=B[A,\partial^\alpha]u+B(\partial^\alpha f)+B(\partial^\alpha F[u])+T(\partial^\alpha u). \tag{2.3}$$

We consider the (M_p) case. Since $f\in\mathcal{S}^{(M_p)}$, for every $\tilde{h}>0$, $\sup_\alpha\frac{\|\partial^\alpha f\|_{H^s}}{\tilde{h}^{|\alpha|}M_\alpha}$ is bounded. Hence, by Lemma 3.4 of [19], there exist $(\tilde{r}_p)\in\mathfrak{R}$ and $C'>0$ such that $\|\partial^\alpha f\|_{H^s}\leq C'M_\alpha/\tilde{R}_\alpha$. Obviously, without losing generality, we can assume that $\tilde{r}_1=1$. By Lemma 2.2 we can find $(r_p')\in\mathfrak{R}$ such that $r_1'=1$, $(r_p')\leq(\tilde{r}_p)$, (r_p') is smaller than the sequences in Lemmas 1.9 and 1.11 and the sequence $N_p=M_p/R_p'$, for $p\in\mathbb{Z}_+$ and $N_0=1$, satisfies $(M.3)'$, $(M.4)$ and $N_1=1$. If we multiply (2.3) by $h^{|\alpha|}/N_\alpha$, take Sobolev norms and sum up for $|\alpha|\leq N$, we obtain

$$\begin{aligned}E_N^{s,h,(r_p')}[u]\ \leq\ &\|u\|_{H^s}+\sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|}}{N_\alpha}\|B[A,\partial^\alpha]u\|_{H^s}+\sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|}}{N_\alpha}\|B(\partial^\alpha f)\|_{H^s}\\&+\sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|}}{N_\alpha}\|B(\partial^\alpha F[u])\|_{H^s}+\sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|}}{N_\alpha}\|T(\partial^\alpha u)\|_{H^s}.\end{aligned}$$

We estimate each of the terms above. By the growth estimate for the symbol of B (1.2) there exists $C''>0$ such that $\|B(\partial^\alpha f)\|_{H^s}\leq C''\|\partial^\alpha f\|_{H^s}$ (cf. theorem

1.7.14 and theorem 2.1.11 and its proof of [21]). Hence $\sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|}}{N_\alpha}\|B(\partial^\alpha f)\|_{H^s} \leq C''' \sum_{|\alpha|=1}^{\infty}\frac{1}{2^{|\alpha|}} = C_1$ for all $0 < h < 1/2$. To estimate the sum with $T(\partial^\alpha u)$, since $|\alpha| > 0$ there exists $j_\alpha \in \{1, ..., d\}$ such that $\alpha_{j_\alpha} \geq 1$. Hence there exists $C_2 > 0$ such that $\|T \circ \partial^{e_{j_\alpha}}\|_{\mathcal{L}_b(H^s)} \leq C_2$. Then we obtain

$$\sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|}}{N_\alpha}\|T(\partial^\alpha u)\|_{H^s} \leq C_2 h \sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|-1}}{N_{|\alpha|-1}}\left\|\partial^{\alpha-e_{j_\alpha}}u\right\|_{H^s} \leq C_3 h E_{N-1}^{s,h,(r'_p)}[u].$$

Since C_3 does not depend on h, for fixed $0 < \varepsilon < 1$ we can find $h_0 = h_0(\varepsilon) < 1/2$ such that for all $0 < h < h_0$

$$\sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|}}{N_\alpha}\|T(\partial^\alpha u)\|_{H^s} \leq \varepsilon E_{N-1}^{s,h,(r'_p)}[u].$$

For fixed $0 < \varepsilon < 1$, by Lemmas 2.7 and 2.8 for the chosen (r'_p), we can find $h_0 = h_0(\varepsilon) < 1/2$ such that for all $0 < h < h_0$, we have

$$E_N^{s,h,(r'_p)}[u] \quad \leq \quad \|u\|_{H^s} + \varepsilon E_{N-1}^{s,h,(r'_p)}[u] + C_1 + \varepsilon\left(E_{N-1}^{s,h,(r'_p)}[u]\right)^l + \varepsilon E_{N-1}^{s,h,(r'_p)}[u].$$

By iterating this estimate one obtains that $\|\partial^\alpha u\|_{H^s}$, $\alpha \in \mathbb{N}^d$, are finite and by shrinking ε if necessary, that the sum $\sum_{|\alpha|=0}^{\infty}\frac{h^{|\alpha|}}{N_\alpha}\|\partial^\alpha u\|_{H^s}$ converges, for some, small enough, h. If $\tilde{h} > 0$ is arbitrary but fixed there exists $\tilde{C} > 0$ such that $\tilde{h}^p \leq Ch^p R'_p$ for all $p \in \mathbb{Z}_+$. Hence $\sum_{|\alpha|=0}^{\infty}\frac{\tilde{h}^{|\alpha|}}{M_\alpha}\|\partial^\alpha u\|_{H^s}$ converges. This completes the proof in the (M_p) case. The proof in the $\{M_p\}$ case is similar and we omit it.

To prove $ii)$ we consider first the (M_p) case. Proceed as in the proof $i)$ to obtain

$$\begin{aligned}\sum_{|\alpha|\leq N}\frac{h^{|\alpha|}}{\tilde{M}_\alpha}\|\partial^\alpha u\|_{H^s} \quad &\leq \quad \|u\|_{H^s} + \sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|}}{\tilde{M}_\alpha}\|B[A,\partial^\alpha]u\|_{H^s} + \sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|}}{\tilde{M}_\alpha}\|B(\partial^\alpha f)\|_{H^s}\\ &+ \sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|}}{\tilde{M}_\alpha}\|B(\partial^\alpha F[u])\|_{H^s} + \sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|}}{\tilde{M}_\alpha}\|T(\partial^\alpha u)\|_{H^s}.\end{aligned}$$

By Lemma 1.10, there exists $l \geq 1$ such that for each $\tilde{h} > 0$ there exists $C_1 > 0$ such that $\|(B \circ D^\alpha p_\beta(x)\partial^\gamma)(x, D)\|_{\mathcal{L}(H^s)} \leq C_1 \tilde{h}^{|\beta|} l^{|\gamma|}\tilde{M}_\gamma$. We have for $0 < h < 1/(4l)$

$$\sum_{|\alpha|=1}^{N}\frac{h^{|\alpha|}}{\tilde{M}_\alpha}\|B(\partial^\alpha F[u])\|_{H^s} \quad \leq \quad \sum_{|\beta|=2}^{\infty}\sum_{|\alpha|=1}^{N}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\frac{h^{|\alpha|}}{\tilde{M}_\alpha}\left\|B(\partial^\gamma p_\beta \partial^{\alpha-\gamma}u^{|\beta|})\right\|_{H^s}$$

$$\leq \quad C_1 \sum_{|\beta|=2}^{\infty} \sum_{|\alpha|=1}^{N} \frac{\tilde{h}^{|\beta|} \left\| u^{|\beta|} \right\|_{H^s}}{2^{|\alpha|}} \leq C_2,$$

where in the last inequality we used that $\left\| u^{|\beta|} \right\|_{H^s} \leq C_s^{|\beta|-1} \|u\|_{H^s}$ and chose $\tilde{h} \leq 1/(2C_s)$. The sequence $\tilde{M}_p$ satisfies $(M.4)$, so by analogous technique as in the proof of Lemma 2.7 one can prove that for each $0 < \varepsilon < 1$ there exists $h_0 = h_0(\varepsilon) < 1/2$ such that for every $0 < h < h_0$

$$\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{\tilde{M}_\alpha} \|B[A, \partial^\alpha]u\|_{H^s} \leq \varepsilon \sum_{|\alpha| \leq N-1} \frac{h^{|\alpha|}}{\tilde{M}_\alpha} \|\partial^\alpha u\|_{H^s} .$$

Also, similarly as in the proof of i), we have that for $0 < h < h_0(\varepsilon)$, $\displaystyle\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{\tilde{M}_\alpha} \|B(\partial^\alpha f)\|_{H^s} \leq C_3$ and

$$\sum_{|\alpha|=1}^{N} \frac{h^{|\alpha|}}{\tilde{M}_\alpha} \|T(\partial^\alpha u)\|_{H^s} \leq \varepsilon \sum_{|\alpha| \leq N-1} \frac{h^{|\alpha|}}{\tilde{M}_\alpha} \|\partial^\alpha u\|_{H^s}.$$

Hence, for $0 < h < h_0(\varepsilon)$, for sufficiently small $h_0(\varepsilon)$, we have

$$\sum_{|\alpha| \leq N} \frac{h^{|\alpha|}}{\tilde{M}_\alpha} \|\partial^\alpha u\|_{H^s} \leq \|u\|_{H^s} + C_4 + 2\varepsilon \sum_{|\alpha| \leq N-1} \frac{h^{|\alpha|}}{\tilde{M}_\alpha} \|\partial^\alpha u\|_{H^s}.$$

By iterating this estimate and possibly shrinking ε we obtain that $\displaystyle\sum_{|\alpha|=0}^{\infty} \frac{h^{|\alpha|}}{\tilde{M}_\alpha} \|\partial^\alpha u\|_{H^s}$ is finite for some sufficiently small h, which finishes the proof in the (M_p) case. The $\{M_p\}$ case is completely analogous. $\square$

3 Examples

We will give interesting examples where Theorem 0.2 is applicable when $M_p = p!^l$, $l > 1$. Let $A_p = p!^v$, with $1 < v < l$ and $0 < \rho < 1$ is such that $v \leq l\rho$. Let $a_0 : (0, \infty) \to (0, \infty)$ be given by $a_0(\lambda) = \displaystyle\sum_{n=0}^{\infty} \frac{h^n \lambda^n}{n!^{l+l'}}$. Then

$$\begin{aligned} a_0^{(k)}(\lambda) \quad &= \quad \frac{1}{\lambda^k} \sum_{n \geq k} \frac{n!}{(n-k)!} \cdot \frac{h^n \lambda^n}{n!^{l+l'}} \\ &= \quad \frac{h^{k/(l+l')}}{\lambda^{k(l+l'-1)/(l+l')}} \sum_{n \geq k} \left(\frac{h^n \lambda^n}{n!^{l+l'}} \right)^{(l+l'-1)/(l+l')} \cdot \left(\frac{h^{n-k} \lambda^{n-k}}{(n-k)!^{l+l'}} \right)^{1/(l+l')} \\ &\leq \quad \frac{h^{k/(l+l')}}{\lambda^{k(l+l'-1)/(l+l')}} \left(\sum_{n \geq k} \frac{h^n \lambda^n}{n!^{l+l'}} \right)^{(l+l'-1)/(l+l')} \cdot \left(\sum_{n \geq k} \frac{h^{n-k} \lambda^{n-k}}{(n-k)!^{l+l'}} \right)^{1/(l+l')} \end{aligned}$$

$$\leq \quad \frac{h^{k/(l+l')}}{\lambda^{k(l+l'-1)/(l+l')}} a_0(\lambda),$$

where, in the first inequality we used Holder's inequality with $p = (l+l')/(l+l'-1)$ and $q = l+l'$. Define $a(w) = a_0(\langle w \rangle)$, $w \in \mathbb{R}^{2d}$. Then $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$. We will need the following multidimensional variant of the Faá di Bruno formula (Corollary 2.10 of [12]).

Proposition 3.1. *([12]) Let $|\alpha| = n \geq 1$ and $h(x_1, ..., x_d) = f(g(x_1, ..., x_d))$ with $g \in \mathcal{C}^n$ in a neighbourhood of x^0 and $f \in \mathcal{C}^n$ in a neighbourhood of $y^0 = g(x^0)$. Then*

$$\partial^\alpha h(x^0) = \sum_{r=1}^{n} f^{(r)}(y^0) \sum_{p(\alpha,r)} \alpha! \prod_{j=1}^{n} \frac{\left(\partial^{\alpha^{(j)}} g(x^0)\right)^{k_j}}{k_j! \left(\alpha^{(j)}!\right)^{k_j}},$$

where

$$\begin{aligned} p(\alpha, r) \quad = \quad & \Big\{ \left(k_1, ..., k_n; \alpha^{(1)}, ..., \alpha^{(n)}\right) \Big| \, \textit{for some}\, 1 \leq s \leq n, k_j = 0 \textit{ and } \alpha^{(j)} = 0 \\ & \textit{for } 1 \leq j \leq n-s;\ k_j > 0 \textit{ for } n-s+1 \leq j \leq n; \textit{ and} \\ & 0 \prec \alpha^{(n-s+1)} \prec ... \prec \alpha^{(n)} \textit{ are such that} \\ & \sum_{j=1}^{n} k_j = r, \sum_{j=1}^{n} k_j \alpha^{(j)} = \alpha \Big\}. \end{aligned}$$

The relation $\prec$ used in this proposition is linear order on $\mathbb{N}^d$ defined in the following way (cf. [12]). We say that $\beta \prec \alpha$ when one of the following holds:

(i) $|\beta| < |\alpha|$;

(ii) $|\beta| = |\alpha|$ and $\beta_1 < \alpha_1$;

(iii) $|\beta| = |\alpha|$, $\beta_1 = \alpha_1, ..., \beta_k = \alpha_k$ and $\beta_{k+1} < \alpha_{k+1}$ for some $1 \leq k < d$.

If we apply the Faá di Bruno formula to the composition of a_0 and $w \mapsto \langle w \rangle$ and use the well known estimate $|\partial^\alpha \langle w \rangle| \leq C' 2^{|\alpha|+1} |\alpha|! \langle w \rangle^{1-|\alpha|}$, $\alpha \in \mathbb{N}^{2d}$, $w \in \mathbb{R}^{2d}$, we have

$$\begin{aligned} |D^\alpha a(w)| \quad & \leq \quad \sum_{r=1}^{|\alpha|} |(D^r a_0)(\langle w \rangle)| \sum_{p(\alpha,r)} \alpha! \prod_{j=1}^{|\alpha|} \frac{\left| D^{\alpha^{(j)}} \langle w \rangle \right|}{k_j! (\alpha^{(j)}!)^{k_j}} \\ & \leq \quad \sum_{r=1}^{|\alpha|} \frac{h^{r/(l+l')} a(w)}{\langle w \rangle^{r(l+l'-1)/(l+l')}} \sum_{p(\alpha,r)} \alpha! \prod_{j=1}^{|\alpha|} \frac{C'^{k_j} 2^{|\alpha^{(j)}| k_j + k_j} \left(|\alpha^{(j)}|! \right)^{k_j} \langle w \rangle^{k_j - |\alpha^{(j)}| k_j}}{k_j! (\alpha^{(j)}!)^{k_j}} \\ & \leq \quad \frac{2^{|\alpha|} a(w)}{\langle w \rangle^{|\alpha|(l+l'-1)/(l+l')}} \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} \alpha! \prod_{j=1}^{|\alpha|} \frac{\left(|\alpha^{(j)}|! \right)^{k_j}}{(\alpha^{(j)}!)^{k_j}} \prod_{j=1}^{|\alpha|} \frac{\left(2C' h^{1/(l+l')} \right)^{k_j}}{k_j!}. \end{aligned}$$

Let $C_0 = 2C'h^{1/(l+l')}$. Then $\prod_{j=1}^{|\alpha|} \frac{C_0^{k_j}}{k_j!} \leq e^{C_0|\alpha|}$. One can easily proof that for $\beta^{(1)}, ..., \beta^{(n)} \in \mathbb{N}^{2d}$,

$$\frac{\left(\beta^{(1)} + ... + \beta^{(n)}\right)!}{\beta^{(1)}! \cdot ... \cdot \beta^{(n)}!} \leq \frac{\left|\beta^{(1)} + ... + \beta^{(n)}\right|!}{\left|\beta^{(1)}\right|! \cdot ... \cdot \left|\beta^{(n)}\right|!}.$$

In fact one easily verifies this inequality for $n = 2$ and the general case follows by induction. We obtain

$$\alpha! \prod_{j=1}^{|\alpha|} \frac{\left(\left|\alpha^{(j)}\right|!\right)^{k_j}}{(\alpha^{(j)}!)^{k_j}} \leq |\alpha|!.$$

We have

$$|D^\alpha a(w)| \leq \frac{\left(2e^{C_0}\right)^{|\alpha|} |\alpha|! a(w)}{\langle w\rangle^{|\alpha|(l+l'-1)/(l+l')}} \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} 1.$$

Now, observe that the set $p(\alpha, r)$ can be canonically injected into the set $\{\gamma \in \mathbb{N}^{|\alpha|(d+1)} | \, |\gamma| \leq |\alpha| + r\}$. Hence $\sum_{p(\alpha,r)} 1 \leq \binom{|\alpha| + r + |\alpha|(d+1)}{|\alpha| + r}$. We can conclude

$$\begin{aligned} |D^\alpha a(w)| &\leq \frac{\left(2e^{C_0}\right)^{|\alpha|} |\alpha|! a(w)}{\langle w\rangle^{|\alpha|(l+l'-1)/(l+l')}} \sum_{r=1}^{|\alpha|} \binom{|\alpha| + r + |\alpha|(d+1)}{|\alpha| + r} \\ &\leq \frac{\left(2e^{C_0}\right)^{|\alpha|} 2^{|\alpha|(d+3)} |\alpha|! a(w)}{\langle w\rangle^{|\alpha|(l+l'-1)/(l+l')}} \end{aligned}$$

Hence, if we take $l' > 0$ large enough such that $(l+l'-1)/(l+l') \geq \rho$, we obtain $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}$ and the growth condition $ii)$ of Definition 0.1 is satisfied for a. By the definition of a the lower bound $i)$ of Definition 0.1 trivially holds for $\tilde{M}_p = p!^{l+l'}$ and some $m > 0$ in the (M_p) case (resp. for $\tilde{M}_p = p!^{l+l'/2}$ and $k_p = p^{l'/2}$ in the $\{M_p\}$ case).

An interesting nontrivial example of the nonlinear term corresponding to this a can be given in the following way. Define p_β by

$$p_\beta(x) = \sum_{\alpha \in \mathbb{N}^d} c_{\alpha,\beta} x^{\alpha+\beta}$$

where $c_{\alpha,\beta}$ satisfy the estimate: for every $h > 0$ there exists $C > 0$ such that $|c_{\alpha,\beta}| \leq Ch^{|\alpha|+|\beta|}/\tilde{M}_{\alpha+\beta}$ (resp. $|c_{\alpha,\beta}| \leq Ch^{|\alpha|+|\beta|}/(\tilde{M}_{\alpha+\beta} \prod_{j=1}^{|\alpha|+|\beta|} k_j)$).

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[a] Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino, Italy
email: marco.cappiello@unito.it

[b] Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Novi Sad, Serbia
email: stevan.pilipovic@gmail.com

[c] Department of Mathematics, Faculty of Mechanical Engineering, University Ss. Cyril and Methodius, Skopje, Macedonia
email: bprangoski@yahoo.com