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From $\mathcal{PT}$-symmetric quantum mechanics to conformal field theory

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Abstract. One of the simplest examples of a $\mathcal{PT}$-symmetric quantum system is the scaling Yang–Lee model, a quantum field theory with cubic interaction and purely imaginary coupling. We give a historical review of some facts about this model in $d \leq 2$ dimensions, from its original definition in connection with phase transitions in the Ising model and its relevance to polymer physics, to the role it has played in studies of integrable quantum field theory and $\mathcal{PT}$-symmetric quantum mechanics. We also discuss some more general results on $\mathcal{PT}$-symmetric quantum mechanics and the ODE/IM correspondence, and mention applications to magnetic systems and cold atom physics.

Keywords. Parity; time-reversal; $\mathcal{PT}$ symmetry; integrable models; quantum mechanics.

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1. Introduction

The scaling Yang–Lee model in two dimensions is one of the simplest examples of an interacting integrable quantum field theory. It has often been used by the integrable models community as a testing ground for new mathematical and numerical tools. Aspects of the theory of exact S-matrices [1,2], the thermodynamic Bethe ansatz (TBA) [3], the truncated conformal space approximation (TCSA) [4,5], the form factor approach for correlation functions [6,7], the exact off-critical g-function [8,9], the excited state and the boundary TBA methods [5,8,10,11] are all techniques which were very successfully applied in their early stages to the study of non-perturbative phenomena in this simple model.

Deep in the ultraviolet regime the properties of the scaling Yang–Lee model are governed by the conformal field theory $\mathcal{M}_{2,5}$ [12,13] which has a negative central charge and a single relevant spin-zero field ϕ, with negative conformal dimensions. The off-critical integrable version of the model corresponds to the perturbation of the conformally-invariant $\mathcal{M}_{2,5}$ action by the operator ϕ with a purely imaginary coupling constant [1].

Despite its apparent lack of unitarity, the Yang–Lee model in d space-time dimensions is at least relevant in condensed matter physics: it is related to the theory of non-intersecting branched polymers in $d+2$ dimensions [14].

Considerations of the Yang–Lee model led Bessis and Zinn-Justin to conjecture the reality of the spectrum of the Schrödinger equation with cubic potential and purely-imaginary coupling, which in turn prompted Bender and his collaborators to a more general study of Schrödinger problems with complex potentials [15,16]. The paper [15] marks the beginning of $\mathcal{PT}$-symmetric quantum mechanics and quantum field theory as an area of intensive research. This short review and the corresponding introductory lecture given in Mumbai are especially addressed to students approaching the field of quantum mechanics in the complex domain. However, the brief discussions on conformal field theory, condensed matter physics, integrable models and their correspondence with the theory of ordinary differential equations may also stimulate more experienced researchers to look into some of the many open problems related to non-unitary integrable quantum field theories.

2. The Yang–Lee edge singularity

Consider the ferromagnetic Ising model with non-zero external magnetic field H

$$\mathcal{H}[\mathbf{s},H] = -\sum_{\langle ij\rangle} s_i s_j + H\sum_i s_i, \quad s_i = \pm 1, \tag{2.1}$$

where the sum is over the nearest-neighbour sites in a $2d$ square lattice. The partition function is

$$Z(T,H) = \sum_{s_i=\pm 1} \mathrm{e}^{-\mathcal{H}[\mathbf{s},H]/T}. \tag{2.2}$$

In the thermodynamic limit, at $H=0$ with T above a certain critical temperature T_{c}, the system is in a disordered phase with zero magnetization $M(T,H)|_{H=0} = \langle s_i\rangle = 0$. At $T < T_{\mathrm{c}}$ the system is instead in an ordered phase where the $\mathbb{Z}_2$ symmetry is spontaneously broken and $M(T,H)|_{H=0} \neq 0$. Finally at $H=0$ and $T = T_{\mathrm{c}}$ the model undergoes a second-order phase transition where typical spin configurations exhibit fluctuations at all length scales and the continuum limit version of the model is invariant under conformal transformations.

A result that dates back to Yang and Lee in 1952 [17,18] states that the partition function $Z(T,H)$ of the finite lattice version of the Ising model, at fixed T, is an entire function of H with all the zeros located on the purely imaginary H-axis. In the thermodynamic limit the zeros become dense, covering the entire imaginary axis from $-\infty$ to ∞ apart from a possible single finite gap centred at $H=0$. When $T < T_{\mathrm{c}}$, there is no gap in the zero distribution and the magnetization $M(T,H)$ has a finite discontinuity as H crosses the origin along the real axis. Correspondingly the system undergoes a first-order phase transition.

For $T > T_{\mathrm{c}}$ there is a gap for $|H| < h_c(T)$. The edges $H = \pm i h_c(T)$ of this gap are branch points for the magnetization $M(T,H)$ and according to Fisher [19]

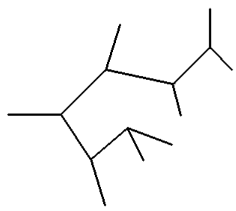

Figure 1. A branched-polymer tree diagram.

they can be considered as conventional critical points. In higher dimensions they correspond to the infra-red behaviour of a scalar field theory with action

$$A = \int \mathrm{d}^d x \left(\frac{1}{2}(\nabla\phi)^2 + i(h-h_c)\phi + i\frac{g}{3}\phi^3 \right). \tag{2.3}$$

The action (2.3) is related to the Parisi–Sourlas theory of non-intersecting branched polymers in $D = d+2$ dimensions [14].

3. The Yang–Lee model and branched polymers

Following [14,20,21], non-intersecting branched polymers in D dimensions correspond to the classical field equation

$$\nabla_D^2\phi(x) + V'(\phi) + i\xi(x) = 0, \tag{3.1}$$

where the field potential $V(\phi)$ is

$$V(\phi) = -(h-h_c)\phi(x) + \frac{g}{3}\phi^3(x), \tag{3.2}$$

and $\xi(x)$ is a stochastic (white noise) variable with

$$\langle \xi(x)\xi(y)\rangle = \delta^D(x-y). \tag{3.3}$$

The polymers can be visualized using tree Feynman diagrams (see figure 1) with corresponding Feynman rules:

- A segment in the graph corresponds to the Green's function of $-\nabla_D^2$.
- Each vertex carries a factor g.
- Each free end carries a factor $-(h-h_c)+i\xi$.

The n-point correlation functions are defined as

$$\langle \phi(x_1)\ldots\phi(x_n)\rangle = \int \mathcal{D}\xi\ \phi(x_1)\ldots\phi(x_n)\mathrm{e}^{-\frac{1}{2}\int \mathrm{d}z^D \xi^2(z)}, \tag{3.4}$$

where the $\phi(x_i)$'s solve the classical field equation (3.1).

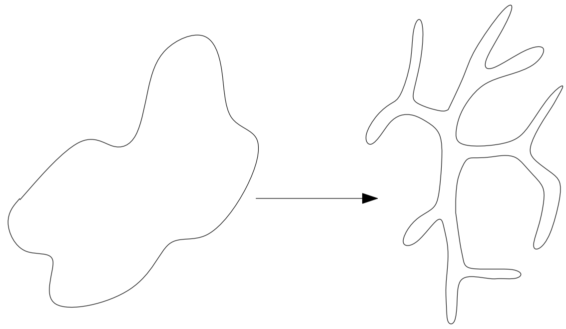

Figure 2. The self-avoiding loops to branched polymers cross-over transition.

Resolving the constraints using the Faddeev–Popov method introduces a Grassmann ghost field and reveals that the theory has an unexpected invariance under supersymmetric transformations. A consequence of this hidden supersymmetry is a surprising dimensional reduction property: correlation functions whose arguments are restricted to a $d = D - 2$ subspace are the same as those for the Yang–Lee theory in d dimensions.

The simplest case corresponds to the branched polymer problem in $D = 2$ dimensions. In $d = D - 2 = 0$ the gradient term in the Yang–Lee action (2.3) is absent and the vacuum–vacuum functional integral becomes, after a complex rotation,

$$\frac{2\pi}{g^{1/3}}\mathrm{Ai}\left(\frac{(h-h_c)}{g^{1/3}}\right) = \int_{-\infty}^{\infty} \mathrm{e}^{i(h-h_c)\phi + i\frac{g}{3}\phi^3}\mathrm{d}\phi. \tag{3.5}$$

This is one of the very few exactly known scaling functions in $D = 2$. This model describes the cross-over from self-avoiding loops (vesicles) to branched polymers [20–22] (see figure 2) where $p \sim g^{-1}$ is the pressure difference between the inside and the outside and $(x_c - x) \sim (h - h_c)g^{-1}$ is related to the monomer fugacity in the discretized version of the model. This result can be interpreted from another perspective: since the original polymer problem is defined in $D = 2$ dimensions and the Airy function is the solution of a 1d Schrödinger equation with linear potential, we have a beautiful physically motivated confirmation of an ODE/IM correspondence [23] linking the spectral theory of simple ordinary differential equations and the theory of integrable models.

4. Conformal field theory in two dimensions: The Ising and the Yang–Lee models

From §2 we know that the Ising model at $H = 0$ and $T = T_\mathrm{c}$ is critical and invariant under conformal transformations. The edge points $H = \pm ih_c(T)$ can also be considered as conventional critical points [19] where the model exhibits conformal invariance [12]. From a renormalization group perspective the point $(H = 0, T = T_\mathrm{c})$ and the pair of equivalent points $(H = \pm ih_c(T),\ T > T_\mathrm{c})$ correspond to two distinct universality classes.

Two-dimensional conformal transformations coincide with analytic transformations

$$z \to f(z), \quad \bar{z} \to \bar{f}(\bar{z}) \tag{4.1}$$

with $z = x + iy$, $\bar{z} = x - iy$. The corresponding infinitesimal generators are

$$l_n = -z^{n+1}\partial_z, \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}, \tag{4.2}$$

and satisfy the Witt algebra

$$[l_m, l_n] = (m-n)l_{m+n}, \quad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}, \quad [l_n, \bar{l}_m] = 0. \tag{4.3}$$

At the quantum level the geometry is important and commutation relations involving infinitesimal generators that change it become anomalous. The Witt algebra is replaced by the Virasoro algebra $l_m \to L_m$, $\bar{l}_m \to \bar{L}_m$, where $[L_n, \bar{L}_m] = 0$ and

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \\ [\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \end{aligned} \tag{4.4}$$

with c the Casimir coefficient describing the coupling of the quantum fluctuations to the finite geometry of the system. The quantity c is also called the central charge, or conformal anomaly.

4.1 *The Kac table*

In two dimensions a universality class is completely identified by its central charge and a complete set of operators, or fields, with their operator product expansions. Among all the operators the primary operators play a central role: they are the ones that transform in the following simple way under a conformal transformation (4.1):

$$\phi(z, \bar{z}) \to (\partial_z f(z))^h (\partial_{\bar{z}} \bar{f}(\bar{z}))^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})), \tag{4.5}$$

where h and $\bar{h}$ are the holomorphic and the anti-holomorphic conformal dimensions respectively [24]. The remaining operators – called descendants – can be obtained from the primary operators by acting on them with combinations of L_{-n} and $\bar{L}_{-n}$. For generic values of c there are infinitely-many primary operators, but in their pioneering paper [25] on conformal field theory, Belavin *et al* showed that for each pair of coprime integers $\mathsf{p} < \mathsf{q}$, a 'minimal' conformal field theory $\mathcal{M}_{\mathsf{p},\mathsf{q}}$ can be defined, with

$$c = 1 - 6\frac{(\mathsf{p}-\mathsf{q})^2}{\mathsf{pq}} < 1 \tag{4.6}$$

and a finite number of primary operators, with conformal dimensions

$$h_{r,s} = \frac{(\mathsf{p}s - \mathsf{q}r)^2 - (\mathsf{p}-\mathsf{q})^2}{4\mathsf{p}\mathsf{q}}, \quad (1 \le r < \mathsf{p},\ 1 \le s < \mathsf{q}r/\mathsf{p}\ ). \tag{4.7}$$

The set of conformal dimensions forms the so-called Kac table of the model. The critical Ising model corresponds to the minimal model $\mathcal{M}_{3,4}$ with central charge $c = 1/2$ and Kac table

$$\mathbf{1} \leftrightarrow h_{1,1} = \bar{h}_{1,1} = 0, \quad \epsilon \leftrightarrow h_{1,3} = \bar{h}_{1,3} = 1/2, \quad \sigma \leftrightarrow h_{1,2} = \bar{h}_{1,2} = 1/16, \tag{4.8}$$

where $\mathbf{1}$ is the identity operator, ϵ and σ are respectively the energy and the magnetic field operators, while the Yang–Lee edge singularity corresponds to $\mathcal{M}_{2,5}$, with central charge $c = -22/5$ and Kac table

$$\mathbf{1} \leftrightarrow h_{1,1} = \bar{h}_{1,1} = 0, \quad \phi \leftrightarrow h_{1,2} = \bar{h}_{1,2} = -1/5. \tag{4.9}$$

Notice that the second of these conformal dimensions is negative, a reflection of the non-unitary nature of the Yang–Lee model.

5. The scaling Yang–Lee model in two dimensions

A 2d conformal field theory is a particular example of an integrable quantum field theory, the integrability of the model being a direct consequence of the existence of the infinite number of commuting conserved charges built from $\{L_n, \bar{L}_n\}$. It was an idea of Zamolodchikov [26] that a conformal field theory can be perturbed or driven away from the critical point by one of the relevant spin-zero fields of the theory, in such a way that integrability would be preserved. For particular primary fields he was able to deduce that an infinite set of commuting conserved charges survives after the perturbation and that in these cases the model remains integrable. It is now known that both ϕ_{12} and ϕ_{13} perturbations of the minimal models lead to integrable off-critical quantum field theories. The perturbation of $\mathcal{M}_{2,5}$ by the ϕ operator

$$A = A_{\mathrm{CFT}} + i(h - h_c) \int \mathrm{d}^2x\, \phi(x) \tag{5.1}$$

is therefore integrable, and can be studied using powerful techniques from the theory of integrable models. Results that have been obtained include:

- The exact 2-body S-matrix was proposed by Cardy and Mussardo [1] and independently by Smirnov [2] in 1989. The resulting theory for $(h - h_c) > 0$ contains a single particle species with a ϕ^3 interaction. No other bound states exist.
- Based on the S-matrix of [1,2], Zamolodchikov derived a set of thermodynamic Bethe ansatz (TBA) equations describing the exact ground-state energy of the model on a cylinder with finite circumference R [3].
- In 1996 the thermodynamic Bethe ansatz equations were generalized to the full set of excited states [10,11].

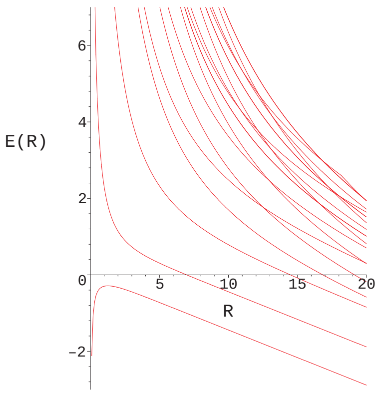

Figure 3. Yang–Lee energy levels for $(h - h_c) > 0$.

- In 1990 Zamolodchikov found an exact form-factor expansion for particular two-point correlation functions [6].

Finally, the low-lying energy levels of the theory can be studied numerically using another method, proposed by Yurov and Zamolodchikov [4], called the truncated conformal space approximation. This relies on the diagonalization of the perturbed Hamiltonian defined on a circle of circumference R on a suitable truncation of its Hilbert space, using the basis provided by the (exactly known) space of states of the unperturbed conformal field theory. Figures 3 and 4 reproduce some of the results found by Yurov and Zamolodchikov in 1990. Figure 3 shows the spectrum of the Yang–Lee model at fixed $(h - h_c) > 0$, as the circumference R of the cylinder is increased. One can see that the spectrum is completely real. The bottom line represents the ground-state energy E_0 and above this line lies the single one-particle state E_1, the energy gap tending asymptotically to the mass m of a single neutral particle at rest. Above this second line, asymptotically again at $E_2 - E_1 = m$ there is the lowest two-particle excited state, and above it the continuum.

The situation for $(h - h_c) < 0$ is very different. For small values of R all the energy levels are still real, but the ground and the first excited states meet at a critical value R_c, beyond which they form a complex-conjugate pair. At the special point where this happens, there is a square-root branch point of the ground-state energy.

Figure 4 shows this and other mergings, and illustrates the contrast between the situations for the two signs of $(h - h_c)$. The plots show the so-called scaling functions $F(R) \equiv RE(R)/\pi$. Conformal field theory gives the (finite) limits of these functions as $R \to 0$. For $R \neq 0$ the scaling functions move away from these limits (in fact, as analytic functions of $(h - h_c)R^{12/5}$), and for $(h - h_c) < 0$ certain pairs meet at square-root branch points, in a way that will see a striking echo in a quantum mechanical context in the next section.

As a more technical remark, note that the explicit factor of i appearing in the action (5.1) is a result of the particular normalization of the perturbing field $\phi(x)$ used in [4], which ensured that the corresponding primary state in the conformal field theory had positive norm-squared. However there is no compelling reason to make such a choice: if $L_n^\dagger = L_{-n}$, then it follows from the Virasoro algebra (4.4)

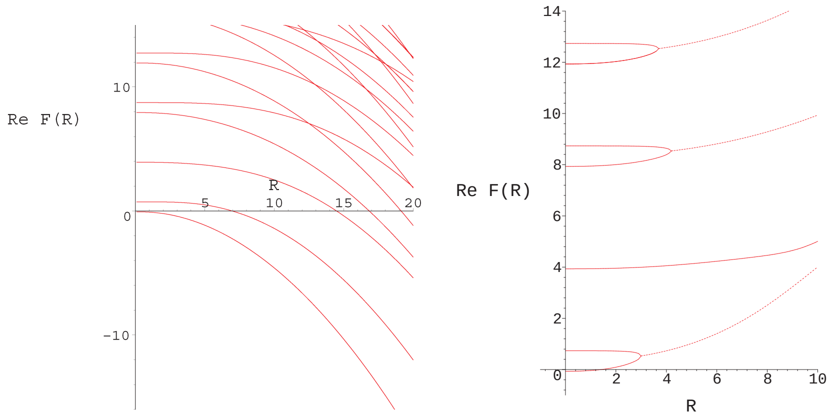

Figure 4. Yang–Lee scaling functions $F(R) = RE(R)/\pi$ for $(h - h_c) > 0$ (left-hand plot) and $(h - h_c) < 0$ (right-hand plot).

that, if $|\psi\rangle$ is the state corresponding to a primary field $\psi(x)$, then the squared norm of, for example, $L_{-n}|\psi\rangle$ is $2n(h + \frac{c}{24}(n^2-1))$ times that of $|\phi\rangle$. In a non-unitary conformal field theory such as the Yang–Lee model, where both c and some of the conformal dimensions h are negative, there is therefore no natural way for all (primary and descendant) states to have positive squared norms, and from many points of view it is preferable to modify the normalizations chosen in [4], so that all coefficients in operator product expansions are real (see, for example, [5,27,28]). This removes the factor of i from (5.1), but of course the theory remains non-Hermitian, since the norm in the underlying Hilbert space is not positive-definite. A similarly non-positive-definite norm will appear in a quantum mechanical context in the next section.

Finally for this section, we should mention that $\mathcal{PT}$ symmetry can also be found in integrable lattice models, where the continuous dimensions of space and time are replaced by a discrete lattice. An initial set of examples was discussed in [29], but it is clear that many further cases remain to be studied.

6. $\mathcal{PT}$-symmetric quantum mechanics

6.1 *Spectral reality*

In the early 1990s, motivated by the Yang–Lee edge singularity, Bessis and Zinn-Justin considered the spectrum of the non-Hermitian Hamiltonian

$$H_{\mathrm{BZJ}} = p^2 + ix^3 \tag{6.1}$$

defined on the full real line. The corresponding Schrödinger equation is

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + ix^3\psi(x) = E_n\psi(x), \quad \psi(x) \in L^2(\mathbb{R}). \tag{6.2}$$

Perturbative and numerical work led them to conjecture [30]

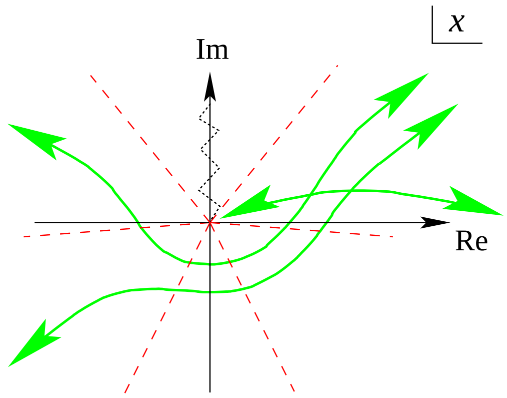

Figure 5. Sample quantization contours. The dashed lines separate Stokes sectors.

- The spectrum of H_{BZJ} is real, and positive [31].

In 1997, Bender and Boettcher [15] proposed an apparently simple generalization:

$$H_M = p^2 - (ix)^{2M}, \quad M \in \mathbb{R}, M > 0. \tag{6.3}$$

To render the 'potential' $-(ix)^{2M}$ single-valued for all values of M, a branch cut should be placed along the positive imaginary axis of the complex x plane. For $M < 2$, the analytic continuation of the spectrum at $M = 3/2$ (Bessis and Zinn-Justin's original problem) can be recovered by imposing the boundary condition $\psi(x) \in L^2(\mathbb{R})$, or equivalently demanding that $\psi(x)$ should tend exponentially to zero as $|x| \to \infty$ along the positive and negative real axes. However, when M reaches 2, the potential becomes $-x^4$ and the eigenvalue problem as just stated changes its nature dramatically. To see why, consider the WKB approximation to the wave function as x tends to infinity along a general ray in the complex plane:

$$\psi_\pm \sim \rho^{-M/2} \exp\left[\pm \frac{1}{M+1} \mathrm{e}^{i\theta(1+M)} \rho^{1+M}\right], \quad x = \rho \mathrm{e}^{i\theta}/i. \tag{6.4}$$

For most values of θ, one of these solutions will be exponentially growing as $\rho \to \infty$ (a dominant solution) and the other exponentially decaying (or subdominant). But when θ is such that

$$\Re e[\mathrm{e}^{i\theta(1+M)}] = 0, \tag{6.5}$$

neither solution is dominant: both decay algebraically. The angles

$$\theta = \pm \frac{\pi}{2M+2}, \quad \pm \frac{3\pi}{2M+2}, \quad \pm \frac{5\pi}{2M+2}, \dots \tag{6.6}$$

define anti-Stokes lines, which divide the complex plane into Stokes sectors. Across an anti-Stokes line, the dominant and subdominant solutions swap roles.

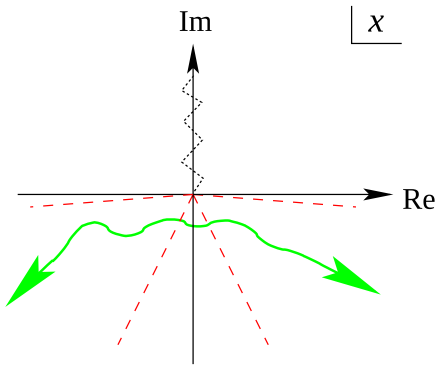

Figure 6. The Bender–Boettcher contour, for M just larger than 2.

At $M = 2$, the positive and negative real axes coincide with anti-Stokes lines, and the original eigenproblem ceases to have a discrete spectrum. The issue can be resolved if the perspective is widened by allowing the wave function to be defined along general contours in the complex plane, rather than just the real axis. Two types of eigenvalue problems are natural. Lateral problems correspond to boundary conditions which require wave functions to decay at large $|x|$ along a contour that begins and ends in a pair of Stokes sectors. By contrast, radial problems correspond to a quantization contour which starts at $x = 0$, and runs out to infinity in a given Stokes sector. Figure 5 depicts some sample quantization contours. Some eigenproblems defined in this way will be related via simple changes of variables, but those that are not, have completely different spectra.

For $M < 2$, the Bender–Boettcher problem (6.3) is an instance of the lateral problem with the (next-nearest-neighbour) Stokes sectors that include the real axis. For $M > 2$, these sectors rotate down in the complex plane as shown in figure 6, and no longer include the real axis. Demanding that the quantization contour begins and ends inside this pair of Stokes sectors for all positive values of M gives the analytic continuation of the Bender–Boettcher problems as M increases past 2.

Numerical and analytical evidence led Bender and Boettcher to conjecture that, with these boundary conditions understood, the spectrum of (6.3) is real and positive for all $M > 1$, despite the non-Hermiticity of the problem. The real eigenvalues as a function of M are shown in figure 7. As soon as M decreases below 1, infinitely-many eigenvalues pair off and become complex, while the spectrum at $M = 1/2$ is empty.

For $M = 2$, the upside-down quartic potential along the complex contour described above, the reality of the spectrum can be understood via a spectrally-equivalent Hermitian Hamiltonian [35–37]. Jones and Mateo [37] found a particularly transparent way to derive this result using the simple variable change $x = -2i\sqrt{1 + iw}$, which maps the complex contour onto the full real line. This variable change followed by a Fourier transform turns the $\mathcal{PT}$-symmetric Hamiltonian H_2 defined on the contour in figure 6 into

$$\widetilde{H}_2 = p^2 + 4w^4 - 2w \tag{6.7}$$

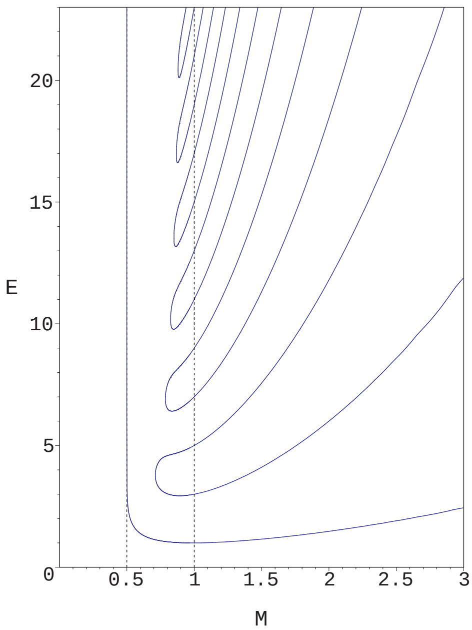

Figure 7. Real eigenvalues of $p^2 - (ix)^{2M}$ as a function of M.

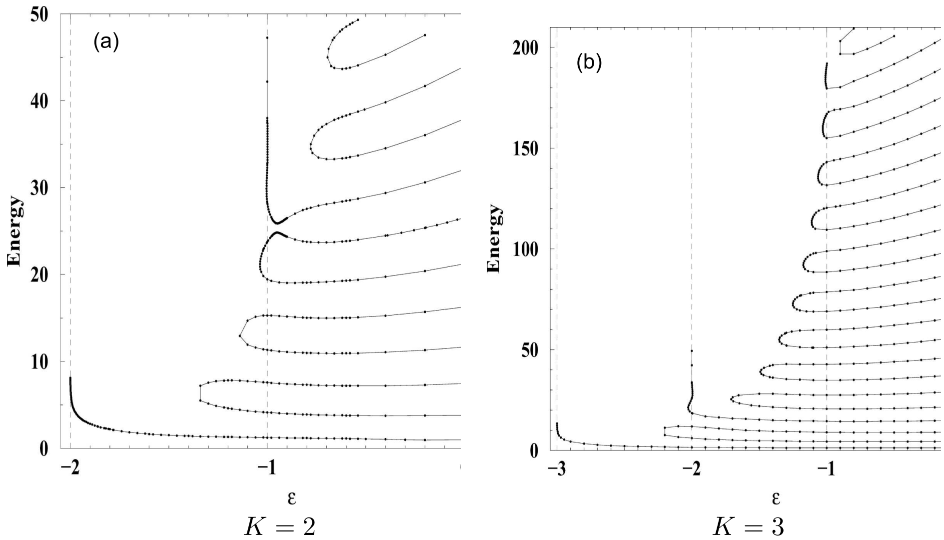

Figure 8. Real eigenvalues of $H_\varepsilon^{(K)} = p^2 + x^{2K}(ix)^\varepsilon$ as a function of ε (plots taken from [16]).

defined on the full real line, which is explicitly Hermitian. Apart from $M = 2$ and $M = 3$ [34], mappings to Hermitian Hamiltonians have not been found for the Bender–Boettcher problems H_M for other values of $M > 1$.

Bender and Boettcher interpreted the phase transition from entirely real eigenvalues to infinitely-many complex eigenvalues as a spontaneous breaking of $\mathcal{PT}$ symmetry. The parity reflection $\mathcal{P}$ and time-reversal $\mathcal{T}$ operators act as follows:

$$\mathcal{P}\colon x \to -x, \quad p \to -p$$
$$\mathcal{T}\colon x \to x, \quad p \to -p, \quad i \to -i.$$

The combined operator $\mathcal{PT}$ commutes with the Hamiltonian H_M for all values of M. The eigenvalues of $\mathcal{PT}$ are pure phases and with a multiplicative rescaling of the eigenfunctions they can be set equal to 1. If the eigenfunction ψ_n of H_M is also an eigenfunction of $\mathcal{PT}$, then we say that the $\mathcal{PT}$ symmetry is unbroken. In this case, using commutativity we have

$$\mathcal{PT}H_M\psi_n = \mathcal{PT}E_n\psi_n = E_n^*\mathcal{PT}\psi_n = E_n^*\psi_n \tag{6.8}$$

$$H_M\mathcal{PT}\psi_n = H_M\psi_n = E_n\psi_n \tag{6.9}$$

and so E_n is real. Unfortunately, this does not seem to offer a quick route to a proof of spectral reality, since it is a non-trivial task to prove that $\mathcal{PT}$ symmetry is unbroken for all wave functions. The only proof currently known of the reality of the spectrum of H_M for general $M > 1$ [34] was obtained by a rather different route, using ideas from the ODE/IM correspondence [23].

Spectra for problems involving next-next-nearest-neighbour Stokes sectors and next-next-next-nearest-neighbour Stokes sectors are shown in figure 8. These plots, taken from [16], show the real eigenvalues as a function of ε for the $K = 2$ and $K = 3$ cases of

$$H_\varepsilon^{(K)} = p^2 + x^{2K}(ix)^\varepsilon, \quad (K \in \mathbb{N}, \quad \varepsilon \text{ real}), \tag{6.10}$$

with the implicit understanding that in each case the quantization contour at $\varepsilon = 0$ is the real line. The spectrum at $K = 2$, shown in figure 8a, is conjectured to be real for all $\varepsilon \geq 0$, and undergoes an extra 'phase transition' at $\varepsilon = -1$. The real eigenvalues for next-next-next-neighbour Stokes sectors (the $K = 3$ case of (6.10)) are shown in figure 8b. This time there are phase transitions at $\varepsilon = -2, -1$ and 0, and the spectrum is conjectured to be real for all $\varepsilon \geq 0$. For integer values of $\varepsilon \geq 1 - K$, the reality of the spectrum of (6.10) was proved in [38], using techniques related to those of [34]. (The negative values of ε covered by the proof correspond to the extra phase transitions just mentioned.)

An alternative to looking at eigenproblems in varying pairs of Stokes sectors is to stay with the same Stokes sectors as for the original Bender–Boettcher problem and explore other possibilities for the potential. In [39] the effect of the addition of an angular momentum-like term $l(l+1)/x^2$ was investigated, the Hamiltonian becoming

$$H_{M,l} = p^2 - (ix)^{2M} + l(l+1)/x^2. \tag{6.11}$$

The results are shown in figure 9: note the completely reversed connectivity of the real levels in figure 9a, compared to that of figure 7. The remaining figures

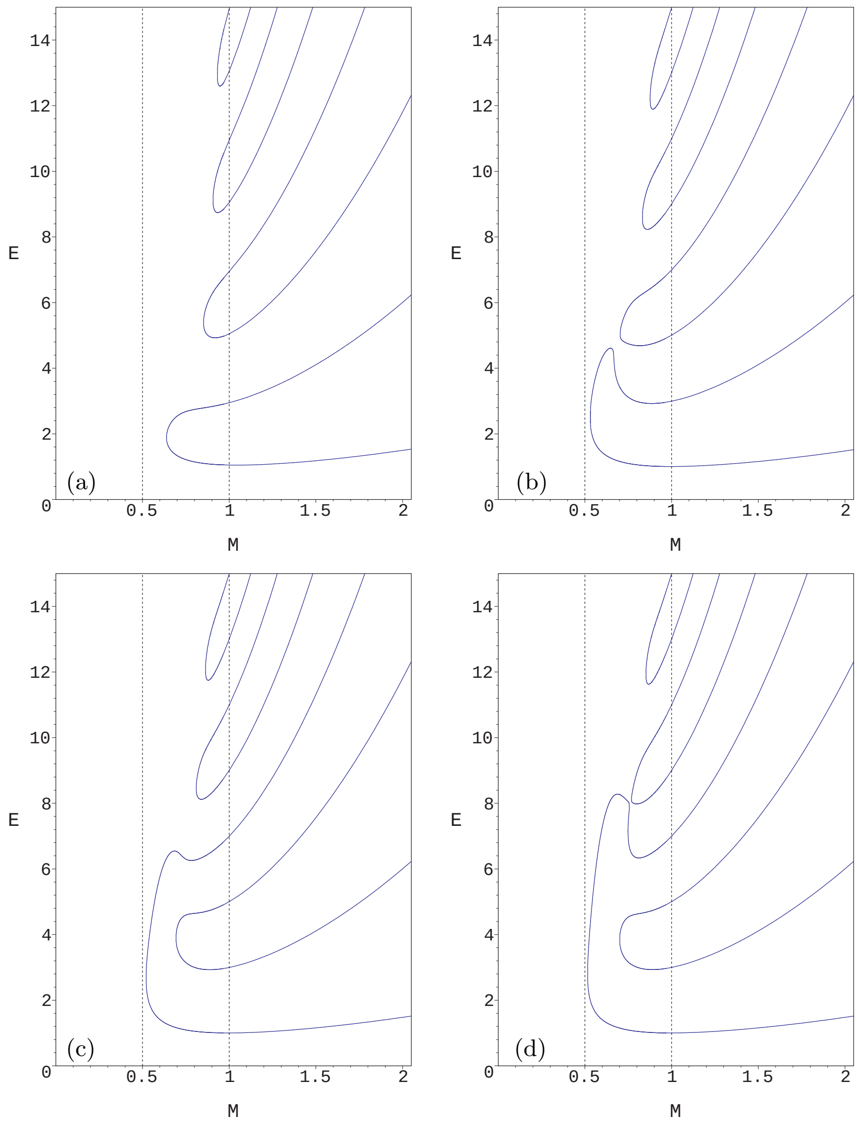

Figure 9. Real eigenvalues of $p^2 - (ix)^{2M} + l(l+1)/x^2$ as functions of M. (**a**) $l = -0.025$, (**b**) $l = -0.0025$, (**c**) $l = -0.0015$ and (**d**) $l = -0.001$.

show how this connectivity changes as l increases to zero, driven by the lowest eigenvalue. This behaviour was subsequently understood analytically, again using tools borrowed from the world of integrable models, in [40].

Finally, we mention one further generalization, first studied in detail in [34,41]: the 3-parameter family of $\mathcal{PT}$-symmetric Hamiltonians

$$H_{M,l,\alpha} = p^2 - (ix)^{2M} - \alpha(ix)^{2M-1} + l(l+1)/x^2 \tag{6.12}$$

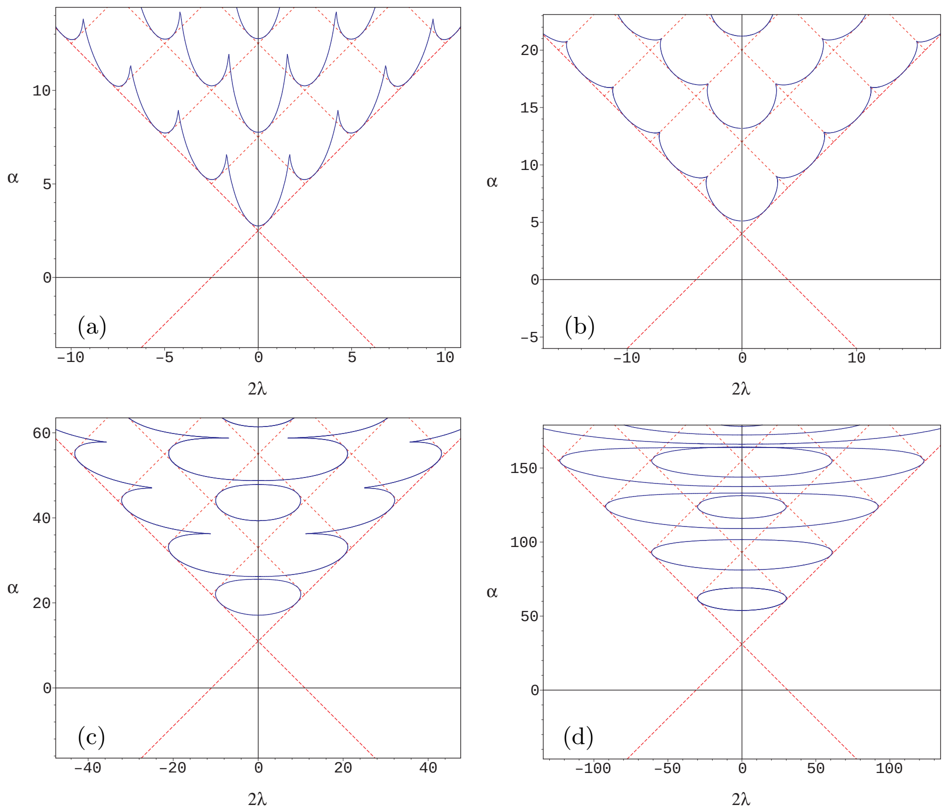

Figure 10. Lines in the $(2\lambda, \alpha)$ plane across which complex pairs of eigenvalues appear, for various values of M. In each case, the spectrum is entirely real in the lower portion of the plot, and acquires complex eigenvalues as the curved lines are crossed. (**a**) $M = 1.5$, (**b**) $M = 3$, (**c**) $M = 10$ and (**d**) $M = 30$.

with boundary conditions on the Schrödinger-picture wave functions imposed on the same next-nearest-neighbour pair of Stokes sectors as before. The reality proof of [34] shows that the spectrum of $\mathcal{H}_{M,\alpha,l}$ is

- real for $M > 1$ and $\alpha < M + 1 + |2l + 1|$;
- positive for $M > 1$ and $\alpha < M + 1 - |2l + 1|$.

This translates into the spectrum being entirely real in the regions below the dark dashed lines (red in colour) $\alpha = M + 1 \pm 2\lambda$ on the plots in figure 10, where $2\lambda = 2l + 1$. In fact, the full region of spectral reality is considerably more complicated than this initial result would suggest: the curved cusped lines on the plots (blue in colour) show the lines in the $(2\lambda, \alpha)$ plane across which the number of complex eigenvalues changes, for various values of M. Further details will appear in [42].

6.2 *Physical consistency*

Given these surprising reality properties, a natural question arises: are $\mathcal{PT}$-symmetric Schrödinger problems with entirely real spectra such as (6.1) fully consistent quantum mechanical systems? Many recent papers [43–47] (see also the review articles [48,49]) have addressed this question. Given a $\mathcal{PT}$-symmetric Hamiltonian H in its unbroken $\mathcal{PT}$ phase, and denoting by $\{\psi_n\}$ its set of eigenfunctions, up to a multiplicative rescaling we have

$$\mathcal{PT}\psi_n(x) = \psi_n(x). \tag{6.13}$$

We can then define a $\mathcal{PT}$ inner product as

$$(f, g)_{\mathcal{PT}} = \int_\gamma \mathrm{d}x[\mathcal{PT}f(x)]g(x), \tag{6.14}$$

where γ is the contour in the complex plane along which the wave function is defined. However, the inner product $(\cdot, \cdot)_{\mathcal{PT}}$ is not positive definite and it is easy to check numerically that the orthogonality condition for the wave functions is

$$(\psi_n, \psi_m)_{\mathcal{PT}} = (-1)^n\delta_{n,m}, \quad n, m = 0, 1, 2, \ldots. \tag{6.15}$$

Similarly, indirect numerical support [50–53] leads to

$$\sum_{n=0}^{\infty} \psi_n(x)\psi_n(y) = (-1)^n\delta(x - y). \tag{6.16}$$

Given the results (6.15) and (6.16) it is natural to introduce a new operator $\mathcal{C}$ such that

$$\psi_n^{\mathcal{CPT}} = \mathcal{CPT}\psi_n = (-1)^n\psi_n, \tag{6.17}$$

and define a $\mathcal{CPT}$ inner product

$$(f, g)_{\mathcal{CPT}} = \int_\gamma \mathrm{d}x[\mathcal{CPT}f(x)]g(x). \tag{6.18}$$

One can check that a $\mathcal{PT}$-symmetric Hamiltonian H commutes with the $\mathcal{CPT}$ operator. Moreover, H is Hermitian with respect to the $\mathcal{CPT}$ inner product and it defines a unitary time evolution. The observables correspond to operators satisfying the following condition:

$$O^{\mathrm{T}} = \mathcal{CPT}O\,\mathcal{CPT}, \tag{6.19}$$

where O^{T} stands for the transpose of O. A major difference between conventional quantum mechanics and $\mathcal{PT}$-symmetric quantum mechanics is that the inner product (6.18) is generated dynamically by the Hamiltonian: in position space the linear operator $\mathcal{C}$ is given by

$$\mathcal{C}(x,y)=\sum_{n=0}^{\infty}\psi_n(x)\psi_n(y). \tag{6.20}$$

The parity operator $\mathcal{P}$ in position space is

$$\mathcal{P}(x,y)=\sum_{n=0}^{\infty}(-1)^n\psi_n(x)\psi_n(-y)=\delta(x+y), \tag{6.21}$$

and the time reversal operator $\mathcal{T}$ is just complex conjugation

$$\mathcal{T}(x,y)=\sum_{n=0}^{\infty}(-1)^n\psi_n^*(x)\psi_n(y). \tag{6.22}$$

Finally we note that further generalizations of $\mathcal{PT}$ symmetry have been considered – see for example the $\mathcal{CPT}$ quantum mechanical models discussed in [54,55].

7. The ODE/IM correspondence for the minimal models

The ODE/IM correspondence [23] connects the spectral properties of ordinary differential equations, including some of the $\mathcal{PT}$-symmetric problems discussed above, with objects which arise in the study of integrable models. The simplest example [23,56] links the following 1d Schrödinger equation:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+(x^{2M}-E)+\frac{l(l+1)}{x^2}\right)\psi(x)=0, \tag{7.1}$$

where $M>0$ and l is real, to the conformal field theories with $c\le 1$ in the framework proposed by Bazhanov *et al* [57,58].

More precisely, eq. (7.1) encodes properties of the primary operator with conformal dimension

$$h(M,l)=\frac{(2l+1)^2-4M^2}{16(M+1)} \tag{7.2}$$

belonging to a conformal field theory with central charge

$$c(M)=1-\frac{6M^2}{M+1}. \tag{7.3}$$

Give a pair of coprime integers $\mathsf{p}<\mathsf{q}$, the ground state of the minimal model $\mathcal{M}_{\mathsf{p},\mathsf{q}}$ is selected by setting

$$M+1=\frac{\mathsf{q}}{\mathsf{p}},\quad l+\frac{1}{2}=\frac{1}{\mathsf{p}} \tag{7.4}$$

in (7.1). This corresponds to the central charge $c(M)=c_{\mathsf{pq}}=1-\frac{6}{\mathsf{pq}}(\mathsf{q}-\mathsf{p})^2$, and lowest possible conformal dimension

$$h(M,l) = \frac{4}{\mathsf{pq}}(1 - (\mathsf{q}{-}\mathsf{p})^2). \tag{7.5}$$

For the Ising model $\mathcal{M}_{3,4}$, eq. (7.5) gives $h(\frac{1}{3}, -\frac{1}{6}) = 0$ and the correspondence is with the identity operator, $\mathbb{1}$. For the Yang–Lee model $\mathcal{M}_{2,5}$, we have $h(\frac{3}{2}, -\frac{1}{2}) = -\frac{1}{5}$ and the correspondence is with the relevant operator ϕ. Now we observe that the singular term $l(l{+}1)/x^2$ in (7.1) with $l = \frac{1}{\mathsf{p}} - \frac{1}{2}$ can be eliminated by a change of variables

$$x = z^{\mathsf{p}/2}, \quad \psi(x,E) = z^{\mathsf{p}/4-1/2} y(z,E). \tag{7.6}$$

After rescaling $z \to (2/\mathsf{p})^{2/\mathsf{q}} z$, eq. (7.1) finally becomes

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + z^{\mathsf{p}-2}(z^{\mathsf{q}-\mathsf{p}} - \tilde{E})\right) y(z,\tilde{E}) = 0, \tag{7.7}$$

where

$$\tilde{E} = \left(\frac{\mathsf{p}}{2}\right)^{2-2\mathsf{p}/\mathsf{q}} E. \tag{7.8}$$

The change of variable (7.6) has replaced a singular potential defined on a multi-sheeted Riemann surface by a simple polynomial. Therefore, any solution to the transformed eq. (7.7) is automatically single-valued around $z = 0$.

Zero monodromy conditions have already played a central role in the ODE/IM correspondence [59,60], and therefore it is natural to explore the constraints that this property imposes on the constant l in (7.1) and (7.2) [61].

7.1 *Monodromy properties and the Kac table*

To see which other primary states have similar trivial monodromy, start from (7.1) with generic $l > -1/2$ and perform the transformation (7.6). The result is

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{\tilde{l}(\tilde{l}+1)}{z^2} + z^{\mathsf{p}-2}(z^{\mathsf{q}-\mathsf{p}} - \tilde{E})\right) y(z,\tilde{E},\tilde{l}) = 0, \tag{7.9}$$

where $2(\tilde{l} + \frac{1}{2}) = \mathsf{p}(l + \frac{1}{2})$. The presence of the Fuchsian singularity at $z = 0$, compared to (7.7), means that the zero monodromy condition of a generic solution is no longer guaranteed. In general, eq. (7.9) admits a pair of solutions

$$\chi_1(z) = z^{\lambda_1} \sum_{n=0}^{\infty} c_n z^n; \quad \chi_2(z) = z^{\lambda_2} \sum_{n=0}^{\infty} d_n z^n, \tag{7.10}$$

where $\lambda_1 = \tilde{l} + 1 > \lambda_2 = -\tilde{l}$ are the two roots of the indicial equation $\lambda(\lambda - 1) - \tilde{l}(\tilde{l} + 1) = 0$ and

$$\chi_j(\mathrm{e}^{2\pi i} z) = \mathrm{e}^{2\pi i \lambda_j} \chi_j(z), \quad j = 1, 2. \tag{7.11}$$

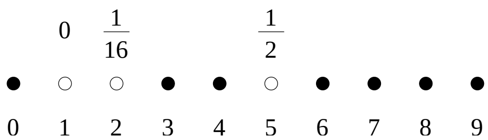

Figure 11. The holes in the infinite sequence of integers for the critical Ising model $\mathcal{M}_{3,4}$. The holes are at 1, 2 and 5.

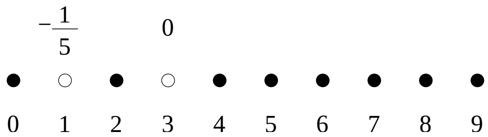

Figure 12. Holes for the Yang–Lee model $\mathcal{M}_{2,5}$, at 1 and 3.

Therefore, the general solution is a linear combination

$$y(z,\tilde{E},\tilde{l}) = \sigma\chi_1(z) + \tau\chi_2(z) \tag{7.12}$$

and we shall demand that the monodromy of $y(z)$ around $z = 0$ is projectively trivial: $y(\mathrm{e}^{2\pi i}z) \propto y(z)$. We shall see that this condition imposes

(i) $2\tilde{l}+1$ is a positive integer;
(ii) The allowed values of $2\tilde{l}+1$ form the set of holes of the infinite sequence

$$\mathsf{p}\,r + \mathsf{q}\,s, \quad r,s = 0,1,2,3\ldots. \tag{7.13}$$

We call the set of integers (7.13) 'representable', and denote them by $\mathbb{R}_{\mathsf{pq}}$.

As a consequence of (i) and (ii), it turns out that the allowed conformal dimensions

$$h(M,l) = \frac{(2\tilde{l}+1)^2 - (\mathsf{p}-\mathsf{q})^2}{4\mathsf{pq}} \tag{7.14}$$

exactly reproduce the set of conformal weights of the primary operators in the Kac table of $\mathcal{M}_{\mathsf{p},\mathsf{q}}$. Figures 11 and 12 illustrate the situation for the Ising and the Yang–Lee models.

To establish these claims, we first notice that the requirement that the general solution $y(z)$ be projectively trivial means that $\chi_1(z)$ and $\chi_2(z)$ must have the same monodromy. This implies

$$\lambda_1 - \lambda_2 = 2\tilde{l}+1 \in \mathbb{N}, \tag{7.15}$$

and therefore restricts $\tilde{l}$ to be an integer or half integer. However, in such a circumstance, while $\chi_1(z)$ keeps its power series expansion, $\chi_2(z)$ generally acquires a logarithmic contribution

$$\chi_2(z) = D\chi_1(z)\log(z) + \frac{1}{z^{\tilde{l}}}\sum_{n=0}^{\infty} d_n z^n, \tag{7.16}$$

and unless $D = 0$, this will spoil the projectively trivial monodromy of $y(z)$. The log term is absent if and only if the recursion relation for the d_n's with $D = 0$

$$n\,(n - 2\tilde{l} - 1)\, d_n = d_{n-\mathsf{q}} - \tilde{E}\, d_{n-\mathsf{p}} \tag{7.17}$$

with the initial conditions $d_0 = 1$, $d_{m<0} = 0$ admits a solution.

Consider first the case $2\tilde{l} + 1 \notin \mathbb{R}_{\mathsf{pq}}$. Starting from the given initial conditions, the recursion relation generates a solution of the form

$$\chi_2(z) = \frac{1}{z^{\tilde{l}}} \sum_{n=0}^{\infty} d_n z^n, \tag{7.18}$$

where the only non-zero d_n's are those for which the label n lies in the set $\mathbb{R}_{\mathsf{pq}}$. Given that $2\tilde{l} + 1 \notin \mathbb{R}_{\mathsf{pq}}$, for these values of n the factor $n(n - 2\tilde{l} - 1)$ on the LHS of the recursion relation is never zero, and the procedure is well-defined. If instead $2\tilde{l} + 1 \in \mathbb{R}_{\mathsf{pq}}$, then the recursion equation taken at $n = 2\tilde{l} + 1$ yields the additional condition

$$\tilde{E}\, d_{2\tilde{l}+1-\mathsf{p}} - d_{2\tilde{l}+1-\mathsf{q}} = 0, \tag{7.19}$$

which is inconsistent for generic $\tilde{E}$, and so the log term is required. If the coprime integers p and q are larger than 1 then the number of 'non-representable' integers $\mathbb{N}_{\mathsf{pq}}$ with $\mathbb{Z}^+ = \mathbb{R}_{\mathsf{pq}} \cup \mathbb{N}_{\mathsf{pq}}$ is in fact $|\mathbb{N}_{\mathsf{pq}}| = \frac{1}{2}(\mathsf{p}{-}1)(\mathsf{q}{-}1)$, a result which goes back to Sylvester [62] and matches the number of primary fields in the Kac table.

8. Applications to condensed matter physics

There are many interesting potential physical applications of $\mathcal{PT}$ symmetry, ranging from optics to the more speculative proposals related to conformal gravity [63], and Higgs phenomena in the Standard Model of particle physics [64]. A partial list of examples can be found in the review [49] on pseudo-Hermitian quantum mechanics. An 'experimental' means of distinguishing between a (Dirac) Hermitian and a $\mathcal{PT}$-symmetric Hamiltonian through the quantum brachistochrone is an interesting recent development [65,66]. In §3, a general relationship between the Yang–Lee model and non-intersecting branched polymers was discussed. Here we would like to report on a simple application of $\mathcal{PT}$-symmetric quantum mechanics to the ferromagnetic Ising system and briefly mention a surprising correspondence between eq. (7.1) and experiments in cold atom physics.

Recently, Lamacraft and Fendley [67] were able to connect the study of universal amplitude ratios (see also [68]) in the quantum spin-chain version of the Ising model to a simple $\mathcal{PT}$-symmetric Schrödinger problem. The Hamiltonian of the quantum spin-chain Ising model is

$$\widehat{\mathcal{H}}[\sigma, h] = -\sum_{i=1}^{L} [h\sigma_i^x + \sigma_i^z \sigma_{i+1}^z], \quad \sigma_{L+1}^z = \sigma_1^z, \tag{8.1}$$

where σ^x and σ^z are Pauli matrices. The model (8.1) is critical at $h = h_c = 1$ and in its ordered and disordered phases for $h < 1$ and $h > 1$ respectively. The magnetization operator $\widehat{M} = \sum_i \sigma_i^z/2$ does not commute with $\widehat{\mathcal{H}}$. Setting $h = h_c$ and considering the continuum limit versions of the operators $\widehat{\mathcal{H}}$ and $\widehat{M} = \int_0^L \mathrm{d}x\, \sigma(x)$, we can define the magnetization distribution in the ground state $|0\rangle$ as

$$P(m) = \langle 0|\delta(m - \widehat{M})|0\rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{-i\lambda m} \chi(\lambda), \tag{8.2}$$

where

$$\chi(\lambda) = \langle 0|\mathrm{e}^{i\lambda\widehat{M}}|0\rangle \tag{8.3}$$

is the generating function of the moments of the distribution $P(m)$

$$\langle 0|\widehat{M}^{2n}|0\rangle = \langle m^{2n}\rangle = \int_{-\infty}^{\infty} m^{2n} P(m)\, \mathrm{d}m = (-1)^n \left[\frac{\mathrm{d}}{\mathrm{d}\lambda^{2n}} \chi(\lambda)\right]_{\lambda=0}. \tag{8.4}$$

It is a direct consequence of the Yang and Lee results [17,18] that the function $\chi(\lambda)$ is an entire function of λ, and a consequence of the Hadamard factorization theorem that

$$\chi(\lambda) = \prod_{n=0}^{\infty} \left(1 - \frac{\lambda^2}{E_n}\right). \tag{8.5}$$

Finally, χ can be related to the spectral determinant T [39] of the following $\mathcal{PT}$-symmetric Hamiltonian (see (6.11)):

$$H_{7,-\frac{1}{2}} = p^2 - (ix)^{14} - \frac{1}{4x^2}. \tag{8.6}$$

The relation is simply

$$\chi(\lambda) = \frac{1}{2} T(E), \quad \lambda^2 \propto E. \tag{8.7}$$

The second application, proposed by Gritsev *et al* in [69], is related to interference experiments in cold atom physics. In a typical interference experiment [69–71], a pair of parallel 1d condensates of length L is created using standard cooling techniques and a highly anisotropic double well radio frequency-induced microtrap. After the condensates are released from the trap, interference fringes are observed using a laser beam. If the condensate has the form of two coaxial rings [71], the associated 'full distribution function' in the zero temperature limit can be mapped, through the ODE/IM correspondence, to the spectral determinant of the Schrödinger equation (7.1).

9. Conclusions

In this short review we have discussed non-Hermitian quantum mechanics, integrable quantum field theory, links between them via an ODE/IM correspondence and some applications to condensed matter physics. The material is by no means exhaustive but rather reflects the authors' experience and interests. Apart from a small number of exceptions, most of the papers on $\mathcal{PT}$ symmetry concern one-dimensional Schrödinger problems in the complex domain; many fundamental issues related to their full consistency as quantum mechanical models have been successfully addressed. Since the early 1990s, two-dimensional relativistic non-unitary quantum field theory, particularly the 2d scaling Yang–Lee model, has been extensively studied by the integrable model community. Considering the relatively restricted physical relevance usually given to these models, a large quantity of exact results have been produced over the years. However, none of the fundamental interpretation issues, of great concern in the $\mathcal{PT}$ symmetry community, have been completely clarified. We think this in an interesting open problem which deserves attention.

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References

[1] J L Cardy and G Mussardo, *Phys. Lett.* **B225**, 275 (1989)
[2] F A Smirnov, *Int. J. Mod. Phys.* **A4**, 4213 (1989)
[3] Al B Zamolodchikov, *Nucl. Phys.* **B342**, 695 (1990)
[4] V P Yurov and Al B Zamolodchikov, *Int. J. Mod. Phys.* **A5**, 3221 (1990)
[5] P Dorey, A Pocklington, R Tateo and G Watts, *Nucl. Phys.* **B525**, 641 (1998), arXiv:hep-th/9712197
[6] Al B Zamolodchikov, *Nucl. Phys.* **B348**, 619 (1991)
[7] P Dorey, M Pillin, R Tateo and G M T Watts, *Nucl. Phys.* **B594**, 625 (2001), arXiv:hep-th/0007077
[8] A LeClair, G Mussardo, H Saleur and S Skorik, *Nucl. Phys.* **B453**, 581 (1995), arXiv:hep-th/9503227
[9] P Dorey, D Fioravanti, C Rim and R Tateo, *Nucl. Phys.* **B696**, 445 (2004), arXiv:hep-th/0404014
[10] V V Bazhanov, S L Lukyanov and A B Zamolodchikov, *Nucl. Phys.* **B489**, 487 (1997), arXiv:hep-th/9607099
[11] P Dorey and R Tateo, *Nucl. Phys.* **B482**, 639 (1996), arXiv:hep-th/9607167

[12] J L Cardy, *Phys. Rev. Lett.* **54**, 1354 (1985)
[13] P Di Francesco, P Mathieu and D Senechal, *Conformal field theory* (Springer, New York, USA, 1997)
[14] G Parisi and N Sourlas, *Phys. Rev. Lett.* **46**, 871 (1981)
[15] C M Bender and S Boettcher, *Phys. Rev. Lett.* **80**, 5243 (1998), arXiv:physics/9712001
[16] C M Bender, S Boettcher and P Meisinger, *J. Math. Phys.* **40**, 2201 (1999), arXiv:quant-ph/9809072
[17] C N Yang and T D Lee, *Phys. Rev.* **87**, 404 (1952)
[18] T D Lee and C N Yang, *Phys. Rev.* **87**, 410 (1952)
[19] M E Fisher, *Phys. Rev. Lett.* **40**, 1610 (1978)
[20] J L Cardy, *J. Phys.* **A34**, L665 (2001), arXiv:cond-mat/0107223
[21] J L Cardy, *J. Stat. Phys.* **110**, 519 (2003)
[22] C Richard, A J Guttmann and I Jensen, *J. Phys.* **A34**, L495 (2001), arXiv:cond-mat/0107329
[23] P Dorey and R Tateo, *J. Phys.* **A32**, L419 (1999), arXiv:hep-th/9812211
[24] In the following we are only interested in spin-zero operators with $h = \bar{h}$.
[25] A A Belavin, A M Polyakov and A B Zamolodchikov, *Nucl. Phys.* **B241**, 333 (1984)
[26] A B Zamolodchikov, *Adv. Stud. Pure Math.* **19**, 641 (1989)
[27] H Kausch, G Takacs and G Watts, *Nucl. Phys.* **B489**, 557 (1997), arXiv:hep-th/9605104
[28] I Runkel, *Nucl. Phys.* **B549**, 563 (1999), arXiv:hep-th/9811178
[29] C Korff and R A Weston, *J. Phys.* **A40**, 8845 (2007), arXiv:math-ph/0703085
[30] D Bessis and J Zinn-Justin, (around 1992) unpublished
[31] While the Bessis–Zinn-Justin conjecture initiated the line of work we want to describe here, similar observations had in fact been made before. For example, in 1980 the oscillators $p^2 + x^2 + igx^3$ were studied [32], and the reality of individual eigenvalues proved for g real and $|g|$ sufficiently small. However, it is important to note that this does not suffice to prove that the whole spectrum is real for any non-zero value of g. The reason is that the 'sufficiently small' value of $|g|$ may depend on the particular eigenvalue under consideration. Without a global bound away from zero for these values, reality can still be lost sufficiently high in the spectrum for all non-zero values of g, even if each individual eigenvalue ultimately becomes real as $|g| \to 0$. (This is exactly the behaviour seen in Bender and Boettcher's generalization of the Bessis–Zinn-Justin problem as $M \to 1^-$, illustrated in figure 7, and it is not ruled out by the arguments of [32]). A full proof of spectral reality for $p^2 + x^2 + igx^3$ – in fact valid for any real value of the parameter g – was finally provided by Shin in 2002 [33], following the approach of [34].
[32] E Caliceti, S Graffi and M Maioli, *Commun. Math. Phys.* **75**, 51 (1980)
[33] K C Shin, *Commun. Math. Phys.* **229**, 543 (2002)
[34] P Dorey, C Dunning and R Tateo, *J. Phys.* **A34**, 5679 (2001), arXiv:hep-th/0103051
[35] A A Andrianov, *Ann. Phys. (N.Y.)* **140**, 82 (1982)
[36] V Buslaev and V Grecchi, *J. Phys.* **A26**, 5541 (1993)
[37] H F Jones and J Mateo, *Phys. Rev.* **D73**, 085002 (2006), arXiv:quant-ph/0601188
[38] K C Shin, *J. Math. Phys.* **46**, 082110 (2005), arXiv:hep-ph/0207251
[39] P Dorey and R Tateo, *Nucl. Phys.* **B563**, 573 (1999), Erratum, *ibid.* **B603**, 581 (2001), arXiv:hep-th/9906219
[40] P Dorey, A Millican-Slater and R Tateo, *J. Phys.* **A38**, 1305 (2005), arXiv:hep-th/0410013
[41] P Dorey, C Dunning and R Tateo, *J. Phys.* **A34**, L391 (2001), arXiv:hep-th/0104119

[42] P Dorey, C Dunning, A Lishman and R Tateo, Durham preprint DCPT-08/37, submitted to *J. Phys. A*, arXiv:0907.3673/hep-th
[43] A Mostafazadeh, *J. Math. Phys.* **43**, 205 (2002)
[44] C M Bender, D C Brody and H F Jones, *Phys. Rev. Lett.* **89**, 270401 (2002); Erratum, *ibid.* **92**, 119902 (2004), arXiv:quant-ph/0208076
[45] A Mostafazadeh, *J. Math. Phys.* **43**, 3944 (2002), arXiv:math-ph/0203005
[46] C M Bender, S F Brandt, J H Chen and Q H Wang, *Phys. Rev.* **D71**, 065010 (2005), arXiv:hep-th/0412316
[47] T Curtright and L Mezincescu, *J. Math. Phys.* **48**, 092106 (2007), arXiv:quant-ph/0507015
[48] C M Bender, *Contemp. Phys.* **46**, 277 (2005), arXiv:quant-ph/0501052
[49] A Mostafazadeh, arXiv:0810.5643 [quant-ph]
[50] C M Bender, S Boettcher and V M Savage, *J. Math. Phys.* **41**, 6381 (2000), arXiv:math-ph/0005012
[51] C M Bender, S Boettcher, P N Meisinger and Q H Wang, *Phys. Lett.* **A302**, 286 (2002), arXiv:hep-th/0208136
[52] G A Mezincescu, *J. Phys.* **A33**, 4911 (2000)
[53] C M Bender and Q Wang, *J. Phys.* **A34**, 3325 (2001), arXiv:math-ph/0005012
[54] B Bagchi, A Banerjee, E Caliceti, F Cannata, H B Geyer, C Quesne and M Znojil, *Int. J. Mod. Phys.* **A20**, 7107 (2005), arXiv:hep-th/0412211
[55] E Caliceti, F Cannata, M Znojil and A Ventura, *Phys. Lett.* **A335**, 26 (2005), arXiv:math-ph/0406031
[56] V V Bazhanov, S L Lukyanov and A B Zamolodchikov, *J. Stat. Phys.* **102**, 567 (2001), arXiv:hep-th/9812247
[57] V V Bazhanov, S L Lukyanov and A B Zamolodchikov, *Commun. Math. Phys.* **177**, 381 (1996), arXiv:hep-th/9412229
[58] V V Bazhanov, S L Lukyanov and A B Zamolodchikov, *Commun. Math. Phys.* **190**, 247 (1997), arXiv:hep-th/9604044
[59] V V Bazhanov, S L Lukyanov and A B Zamolodchikov, *Adv. Theor. Math. Phys.* **7**, 711 (2004), arXiv:hep-th/0307108
[60] B Feigin and E Frenkel, arXiv:0705.2486 [math.QA]
[61] P Dorey, C Dunning, F Gliozzi and R Tateo, *J. Phys.* **A41**, 132001 (2008), arXiv:0712.2010 [hep-th]
[62] J J Sylvester, *Mathematical questions from the educational times* **41**, 21 (1884)
[63] C M Bender and P D Mannheim, *J. Phys.* **A41**, 304018 (2008), arXiv:0807.2607 [hep-th]
[64] C M Bender, K A Milton and V M Savage, *Phys. Rev.* **D62**, 085001 (2000), arXiv:hep-th/9907045
[65] C M Bender, D C Brody, H F Jones and B K Meister, *Phys. Rev. Lett.* **98**, 040403 (2007), arXiv:quant-ph/0609032
[66] P E G Assis and A Fring, *J. Phys.* **A41**, 244002 (2008), arXiv:quant-ph/0703254
[67] A Lamacraft and P Fendley, *Phys. Rev. Lett.* **100**, 165706 (2008), arXiv:0802.1246 [cond-mat.stat-mech]
[68] J Salas and A D Sokal, *J. Stat. Phys.* **98**, 551 (2000), arXiv:cond-mat/9904038
[69] V Gritsev, E Altman, E Demler and A Polkovnikov, *Nature Phys.* **2**, 705 (2006), arXiv:cond-mat/0602475
[70] A Imambekov, V Gritsev and E Demler, arXiv:cond-mat/0703766
[71] S Gupta, K W Murch, K L Moore, T P Purdy and D M Stamper-Kurn, *Phys. Rev. Lett.* **95**, 143201 (2005)