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THE ODE/IM CORRESPONDENCE AND $\mathcal{PT}$-SYMMETRIC QUANTUM MECHANICS

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Abstract A connection between integrable quantum field theory and the spectral theory of ordinary differential equations is reviewed, with particular emphasis being given to its relevance to certain problems in $\mathcal{PT}$-symmetric quantum mechanics.
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Keywords: Ordinary differential equations; the Bethe ansatz; conformal field theory; spectral problems; quantum mechanics; PT symmetry

1. Introduction

This talk concerned a recently-discovered relationship between a particular class of integrable models and the spectral theory of ordinary differential equations in complex domain, a link which is sometimes referred to as the 'ODE/IM correspondence'. We recently reviewed many aspects of this in [1] (see also [2]), and a more extended version of [1] is currently in preparation. Therefore in this contribution we shall content ourselves with a quick sketch of some of the main features of the story, updating [1] as we go along with some references to more recent developments.

Two previously-distinct areas of investigation form the backdrop to this work. On the one hand, there is the theory of integrable lattice models and integrable quantum field theories, in particular as extensively developed by Baxter [3] and then Klümper, Pearce and collaborators [4, 5] on the lattice side, and by Bazhanov, Lukyanov and Zamolodchikov [6, 7, 8] on the integrable quantum field theory side. On the other, there is an approach to the theory of ordinary differential equations via functional relations, pioneered by Sibuya [9] and

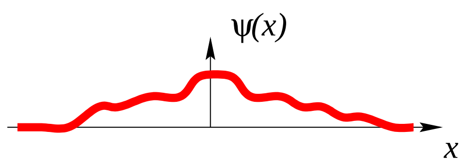
Voros [10]. The first observation of a concrete connection between these two subjects was made in [11]; related subsequent work includes [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Recently, the relevance of the ODE/IM correspondence to certain problems in $\mathcal{PT}$-symmetric quantum mechanics, first observed in [15], was re-emphasised [22, 23]; since this also formed the central theme of the talk given at the conference, the next section is devoted to an introduction to this topic.

2. $\mathcal{PT}$-symmetric quantum mechanics

Some years ago, considerations of the Yang-Lee edge singularity in two dimensions led Bessis and Zinn-Justin [25] to ask themselves about the spectrum of the following 'quantum-mechanical' Hamiltonian:

$$\mathcal{H} = p^2 + ix^3\,. \tag{1}$$

This is a cubic oscillator, with purely imaginary coupling i. To make the question precise, we shall say that the (possibly complex) number E is in the spectrum of (1) if and only if the equation $\mathcal{H}\psi(x) = E\psi(x)$ has a solution $\psi(x)$ lying in $L^2(\mathbb{R})$. For the Hamiltonian (1), this is equivalent to demanding that $\psi(x)$ should decay as $x \to \pm\infty$ along the real axis:



Even for real x and E , the wavefunction ψ cannot be everywhere real. Furthermore, the usual argument leading to reality of the eigenvalues does not apply here, as the Hamiltonian is not Hermitian. Nevertheless, perturbative and numerical studies led Bessis and Zinn-Justin to the following claim:

Conjecture 1 [25]: the spectrum of $\mathcal{H}$ is real and positive.

In 1997, Bender and Boettcher [26] suggested an interesting generalisation. Suppose N is a positive real number, and consider the spectrum of the Hamiltonian

$$\mathcal{H}_N = p^2 - (ix)^N \qquad (N \text{ real}, > 0) \tag{2}$$

where $\psi(x)$ is again required to lie in $L^2(\mathbb{R})$. For $N = 3$, this is the Bessis-Zinn-Justin problem (1), while for $N = 2$ it reduces to the much better-understood simple Harmonic oscillator. For non-integer values of N, the 'potential' $-(ix)^N$ is not single-valued, so a branch cut should be added running up the positive x-axis. Once this has been done, the problem as stated is well-defined (at least for $N < 4$) and can be studied numerically. This is what Bender and Boettcher did, with a surprising result: while the spectrum for $N \geq 2$ is real, as N decreases below 2, infinitely-many eigenvalues pair off and become complex, with only finitely-many remaining real. By the time N reaches 1.5, all but three have become complex, and as N tends to 1 the last real eigenvalue diverges to infinity. This curious behaviour is shown in figure 1, taken from [15]; it reproduces the results of [26].

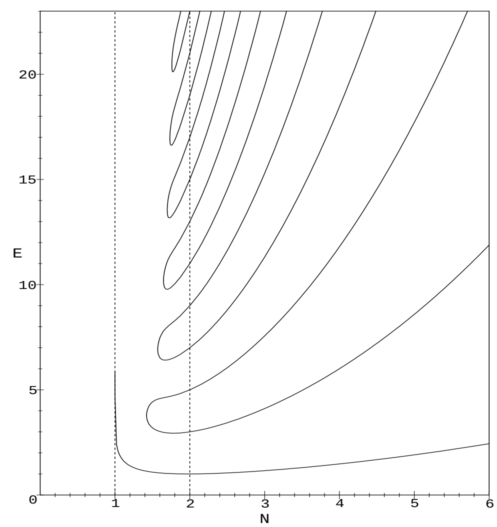


Figure 1. Eigenvalues of $\mathcal{H}_N \psi = E\psi$, plotted as a function of N.

For $N \geq 2$, the fact that their numerically-obtained spectrum was real and positive led Bender and Boettcher to generalise the Bessis-Zinn-Justin conjecture to

Conjecture 2 [26]: the spectrum of $\mathcal{H}_N$ is real and positive for $N \geq 2$.

At generic values of N, (2) is no more Hermitian than (1), but it does have a property, emphasised by Bender and Boettcher, known as $\mathcal{PT}$ symmetry. ('$\mathcal{P}$', or parity, acts by sending x to $-x^*$ and p to $-p$ while $\mathcal{T}$, time reversal, sends x to x, p to $-p$ and i to $-i$, so the combined effect of $\mathcal{PT}$ on a potential $V(x)$ is to send it to $V(-x^*)^*$.) This symmetry does not by itself guarantee reality of the spectrum, as is clearly demonstrated by figure 1, but it does imply that the eigenvalues are either real, or appear in complex-conjugate pairs. (This is analogous to the behaviour of the roots of a real polynomial.) In spite of this fact, reality proofs in $\mathcal{PT}$-symmetric quantum mechanics have been surprisingly elusive, and prior to the recent application of the ODE/IM correspondence to the problem [22], even the simple-to-state conjectures 1 and 2 were unproven: see [27, 28, 29, 30, 31, 32, 33] for some earlier discussions of the issues involved.

One further detail concerning figure 1 needs care: when N hits 4, the naive definition of the eigenvalue problem runs into difficulties. This can be traced to the fact that the asymptotic behaviour of the wavefunction along the real axis becomes oscillatory at $N = 4$. To avoid this difficulty, the contour along which $\psi(x)$ is defined should be deformed down from the real axis into the complex plane, so as to continue to avoid the so-called anti-Stokes lines for the problem. This point is explained at greater length in [26, 1], and so we shall not linger on it here. However, in figure 2 we show one possible contour for N just larger than 4.

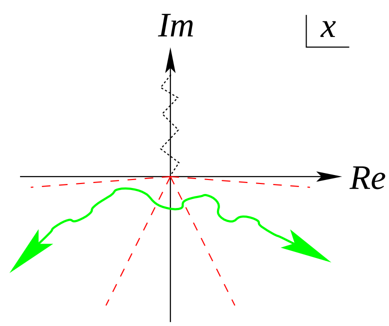


Figure 2. A possible wavefunction contour for $N > 4$.

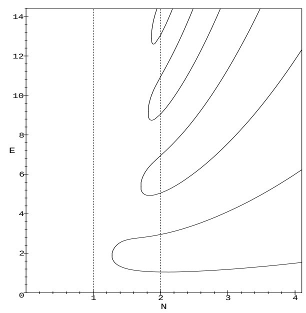
Finally, in [15] we suggested a further generalisation of the Bessis-Zinn-Justin conjecture, again $\mathcal{PT}$-symmetric, by including an angular-momentum term $l(l+1)x^{-2}$:

$$\mathcal{H}_{N,l} = p^2 - (ix)^N + l(l+1)/x^2. \tag{3}$$

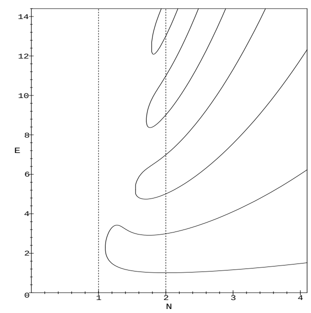
Numerical work gave us strong evidence for

Conjecture 3 [15]: the spectrum of $\mathcal{H}_{N,l}$ is real and positive for $N \geq 2$ and l small.

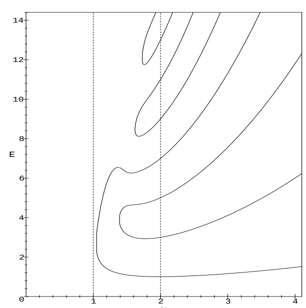
While the angular-momentum term does not affect the reality of the eigenvalues for $N \geq 2$, it can make a remarkable difference to the way in which they become complex. Figure 3 shows a sequence of spectral plots for l varying from -0.025 to 0.05. Notice that for $l = -0.025$, the connectivity of the real eigenvalues is completely reversed from that seen in figure 1, so that while for l=0 the first and second excited states pair off, at $l = -0.025$ the first excited state is instead paired with



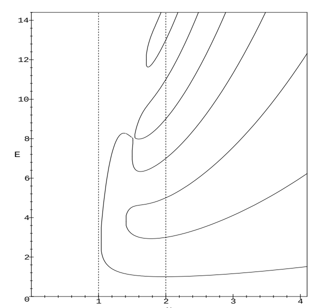
3a) $l = -0.025$



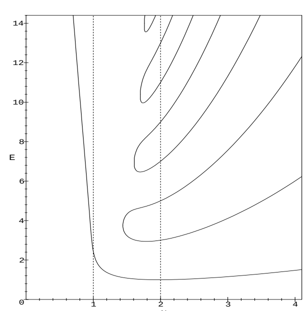
3b) $l = -0.005$



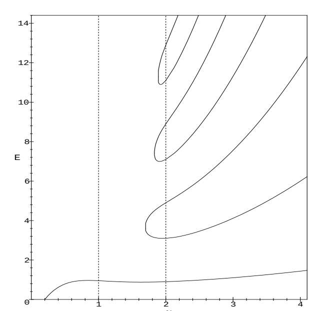
3c) $l = -0.0015$



3d) $l = -0.001$



3e) $l = 0.001$



3f) $l = 0.05$

Figure 3. Eigenvalues of $\mathcal{H}_{N,l}\psi = E\psi$, plotted as a function of N at various nonzero values of l.

the ground state, and so on up the spectrum. With this in mind it might be hard to see how it is possible to pass between the two sets of spectra simply by varying the continuous parameter l from -0.025 to zero. The intermediate plots shown in figures 3b–3d should help the reader to resolve this question, while figures 3e and 3f show how a real eigenvalue ventures into the region $N < 1$ once l becomes positive.

We hope that this discussion will have convinced the reader of the surprising richness of $\mathcal{PT}$-symmetric quantum mechanics. The field has now developed a considerable momentum of its own right, with many other examples being discussed in the literature – refs. [34, 35, 36, 37, 38, 39, 40, 41] offer a selection of this work. The conceptual basis of the subject is also a topic of interest, with recent discussions to be found in [42, 43, 44, 45, 46, 47]. The ODE/IM correspondence has little to say directly about this question, but it has yielded what to our knowledge is the first complete proof of spectral reality for a family of non-trivial $\mathcal{PT}$-symmetric problems, including conjectures 1–3 above as special cases [22].

Limitations of space preclude a complete explanation of how the proof goes, but in the remainder of this article we shall at least describe the basic elements of the correspondence which led to it.

3. Functional relations in integrable models

It was Baxter who first observed that an alternative to the Bethe ansatz method of solution for the six-vertex model was provided by a functional equation, now called a T-Q relation [3]. Related ideas were then used with much success in other lattice models (see for example [4, 5]), but it was not until the work of Bazhanov, Lukyanov and Zamolodchikov that they were systematically applied to integrable quantum field theories defined directly in the continuum [6, 7, 8] (elements of this structure had also been observed by Fendley, Lesage and Saleur [48]). A key feature of the work [6, 7] was the construction of two continuous families of commuting operators acting in the Hilbert space of the quantum field theory, defined on a circle: $\mathbf{T}(\lambda)$, sometimes called the quantum transfer matrix, and $\mathbf{Q}(\lambda)$. These were shown to satisfy the following relation, the field-theory analogue of Baxter's T-Q relation:

$$\mathbf{T}(\lambda)\mathbf{Q}(\lambda) = \mathbf{Q}(q^{-1}\lambda) + \mathbf{Q}(q\lambda)\,, \tag{4}$$

where $q = e^{i\pi\beta^2}$, and β is a parameter characterising the particular integrable field theory under consideration. Each theory has a set of ground states ('θ-vacua') $|p\rangle$. These are labelled by a 'momentum' p (*not* the same as the momentum operator of the last section) and are eigenstates of the $\mathbf{T}(\lambda)$ and $\mathbf{Q}(\lambda)$ operators. If we focus on the corresponding eigenvalues by setting $T(\lambda,p) = \langle p|\mathbf{T}(\lambda)|p\rangle$, $A(\lambda,p) = \lambda^{2p/\beta^2}\langle p|\mathbf{Q}(\lambda)|p\rangle$, then the T-Q relation for these vacuum eigenvalues can be written as

$$T(\lambda,p)A(\lambda,p) = e^{-2\pi ip}A(q^{-1}\lambda,p) + e^{+2\pi ip}A(q\lambda,p)\,, \tag{5}$$

an equation that was also obtained in [48]. The prefactor inserted into the definition of $A(\lambda,p)$ ensures that it, like $T(\lambda,p)$, is a single-valued function of λ^2. But not only are A and T single-valued, they are also *entire* functions of λ^2. This makes (5) a very powerful constraint: when combined with the leading asymptotics of T and A, it admits just a discrete, albeit infinite, set of solutions [3, 7]. The ground state eigenvalues are known to have further analyticity properties, connected with the distribution of their zeroes, which allow even this ambiguity to be removed. From the point of view of integrable models, all of this is important because the eigenvalues $T(\lambda)$ and $A(\lambda)$ encode the values of an infinite set of conserved quantities acting on the corresponding states; for the ODE/IM correspondence, the immediate relevance is rather that it allows a precise link to be established with certain quantities arising in the study of ordinary differential equations, as will now be described.

4. Functional relations in differential equations

Surprisingly, the T-Q relation introduced in the last section also governs the problems in $\mathcal{PT}$-symmetric quantum mechanics discussed in section 2. For convenience we shift x to x/i and E to $-E$ so that the ODE associated with (3) is

$$\left[-\frac{d^2}{dx^2} + x^N + \frac{l(l+1)}{x^2} - E\right]\psi(x) = 0 \ . \tag{6}$$

Considering this equation for general complex values of x, it is a relatively simple matter to establish the following Stokes relation:

$$C(E,l)y(x,E,l) = \omega^{-1/2}y(\omega x, \omega^{-2}E, l) + \omega^{1/2}y(\omega^{-1}x, \omega^2 E, l) \ , \tag{7}$$

where $\omega = e^{2\pi i/(N+2)}$, and $y(x,E,l)$ is a particular solution to (6) vanishing as $x \to \infty$ along the real axis and uniquely determined by its asymptotic behaviour there (see [15] for more details). From another perspective, any solution to (6) can be written as a linear combination of a solution, $\psi^+(x)$, behaving near $x = 0$ as x^{l+1}, and one, $\psi^-(x)$, behaving there as x^{-l}. Setting $y(x,E,l) = D(E,l)\psi^-(x) + D(E,-1-l)\psi^+(x)$, (7) implies that

$$C(E,l)D(E,l) = \omega^{-(l+1/2)}D(\omega^{-2}E,l) + \omega^{(l+1/2)}D(\omega^2 E,l) \ . \tag{8}$$

The functions $D(E,l)$ and $C(E,l)$ are entire in E, and have analogous analyticity properties to the ground-state eigenvalues $T(\lambda)$ and $A(\lambda)$ described in the last section. Together with the obvious similarity of form between (5) and (8), this permits a precise relationship to be established between the two sets of objects, and this was the approach to the ODE/IM correspondence that was adopted in [15]. (It is also possible to proceed via so-called 'quantum Wronskian' relations, as in [11, 12].)

From its definition, the zeroes of $D(E,l)$ are the values of E at which, for $l > -1/2$, the function $x^{-1/2}y(x)$ vanishes both at $x=\infty$ and $x=0$. This means that D can be interpreted as a spectral determinant, for a problem which, in contrast to the problems of section 2, *is* Hermitian. The $\mathcal{PT}$-symmetric problems, on the other hand, turn out to be encoded in the zeroes of C (or equivalently T): the Stokes multiplier $C(-E,l)$ vanishes if and only if, at that value of E, (3) has a nontrivial solution which decays to zero as x tends to infinity on the quantisation contour shown in figure 2. As a result, C is a spectral determinant for the Bessis-Zinn-Justin-Bender-Boettcher problem, and its generalisation [15] to non-zero angular momentum. Even better, the zeroes of C are constrained by the T-Q relation. By combining this with the positive-reality of the zeroes of D – reality being a consequence of the Hermiticity of the problem that D encodes – it is not too hard, for $N>2$, to prove that the zeroes of C must be real, thus settling conjectures 1–3 of section 2 [22]. In fact, in [22] we were able to prove a slightly more general result, but we refer the reader to that paper and to [23] for details.

5. Conclusions

A principal aim of this talk has been to give some hints of the link between two fascinating research fields: the spectral theory of ordinary differential equations, and the theory of integrable models. Whether results from ordinary differential equations are used in the study of integrable models, or vice versa, is largely a matter of taste and personal background; in either case the correspondence promises to be very fruitful. Besides the proof of the reality conjectures mentioned above, the non-linear integral equation technique, an established tool in integrable models, has been successfully introduced into the study of spectral problems [11, 15, 18] (figures 1 and 3 are concrete examples of this application), and Bethe ansatz equations have been used to derive spectral equivalences [22]. Surprisingly, the latter result links integrable models to the recent and exciting discovery of polynomial generalisations of supersymmetry in quantum mechanics [49, 50, 51, 52, 53, 54]. In the other direction, certain duality relations, important to the condensed matter physics applications of integrable models, can be now proved by simple variable changes in the relevant differential equations [12, 16, 19, 24].

Acknowledgments

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Note added

Ref. [55] appeared as we were preparing a version of this article for submission to the electronic archives. In this very interesting preprint, Shin observes that positive-reality of the zeroes of $D(E,l)$ is not a necessary condition for the reality proof discussed at the end of section 4 above to go through – it can be weakened considerably. This allows the argument to be generalised to cover a greatly expanded class of potentials, and shows that the relevance of ideas inspired by the ODE/IM correspondence to problems in $\mathcal{PT}$-symmetric quantum mechanics goes much further than had previously been suspected.

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