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On the gauge chosen by the bosonic open string

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Abstract

String theory gives S matrix elements from which is not possible to read any gauge information. Using factorization we go off shell in the simplest and most naive way and we read which are the vertices suggested by string. To compare with the associated Effective Field Theory it is natural to use color ordered vertices. The $\alpha' = 0$ color ordered vertices suggested by string theory are more efficient than the usual ones since the three gluon color ordered vertex has three terms instead of six and the four gluon one has one term instead of three. They are written in the so called Gervais–Neveu gauge. The full Effective Field Theory is in a generalization of the Gervais–Neveu gauge with α' corrections. Moreover a field redefinition is required to be mapped to the field used by string theory.

We also give an intuitive way of understanding why string choose this gauge in terms of the minimal number of couplings necessary to reproduce the non-abelian amplitudes starting from color ordered ones.

1. Introduction and conclusions

String theory is a good candidate for describing all the interactions in Nature, gravity included. This happens because in its spectrum there are both massless spin 1 and spin 2 particles. Nevertheless the presence of these particles does not mean that they can be identified with gauge bosons and the graviton. This can only be established when interactions are considered. There-

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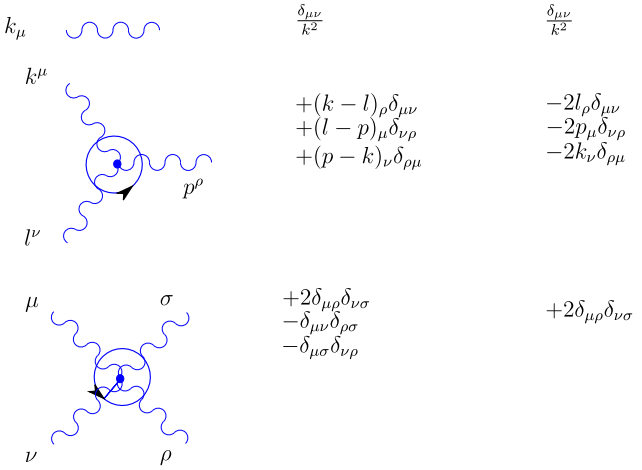

Fig. 1. The usual Euclidean color ordered vertices vs the ones suggested by string theory. Each tree diagram must be then multiplied by $g^{N-2}\frac{1}{\kappa}Tr(T_{a_1}\dots T_{a_N})$ with g the Yang–Mills coupling constant, N the number of legs and T_a the unitary algebra matrix normalized as in Appendix B.

fore the study and the derivation of effective field theory (EFT) actions (to be understood as 1PI actions) from string theory is a very well studied subject starting already at the beginning of 70s ([1–5]) and improved in the 80s (see for example [6]) but we want to approach it from a different point of view.

Usually the aim is to determine the gauge invariant effective field theory (see [7] for a nice approach).

Our main focus is slightly different since we are not mainly interested in the derivation of EFT action for gluons but we want to explore in agnostic way which is the gauge fixed EFT suggested by string theory and the connection between the fields used by string theory and the canonical ones usually used in defining EFT. Essentially we will derive and extend the gauge proposed in [8]. Our approach differs from the one used in [9] within the Witten open string field theory [10] since we use plain old string theory and we are interested in finding the gauge fixing suggested by it. It differs also from [11] since we try to read the gauge and the fields suggested by string theory rather than try to verify that the gauge suggested in [8] (or more precisely an extension to the background field method in the case of [11] as first proposed in [12]) works.

Since all choices made by string theory are clever it is worth trying to read in the most direct way what it suggests. Actually it turns out that the suggested color ordered vertices by string theory [8] are more efficient than the usual ones since the three gluon color ordered vertex has three terms instead of the usual six and the four gluon one has one term instead of the usual three as shown in Fig. 1. The reason is that the usual color ordered vertices are obtained starting from the Feynman rules in Feynman gauge and then mimicking string by performing a color decomposition (see [13,14] and references therein), here we adopt a more radical point of view and we try to mimick string in all.

Our starting point is to notice that while computing the EFT one is actually using a gauge fixed EFT action. The gauge fixing is necessary in order to have a well defined propagator and a well defined propagator is needed in order to compute the S matrix elements which are then compared with the ones from string theory. This happens because the EFT is a 1-PI action and the S matrix elements are computed from truncated on shell Green functions.

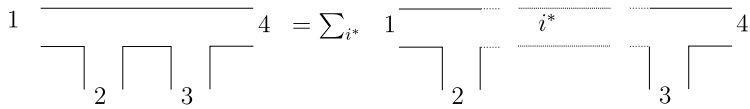

Fig. 2. The 4 state string S matrix is given by the sum of the product of two 3 state string amplitudes and a propagator where the intermediate state i^* is not required to be physical.

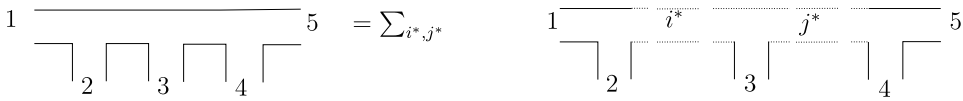

Fig. 3. The 5 state string S matrix is given by the sum of the product of three 3 state string amplitudes and two propagators where the intermediate 3 state string amplitude has two states i^*, j^* which are not required to be physical.

The S matrix elements of gauge invariant operators are obviously independent on the gauge fixing but the intermediate steps are not. So one could wonder how it is possible to extract any information on gauge fixing and fields comparing S matrices. In fact it is not possible. Nevertheless factorization of string amplitudes allows to have a glimpse on how string theory fixes the gauge since it yields amplitudes with off shell and unphysical states (see [15] for previous work on how to extend off shell the string amplitudes). For example Fig. 2 shows how it is possible to obtain a 3 state amplitude with one possibly unphysical state from a factorization of a 4 state physical one. In the same spirit it is possible to start with a 5 state amplitude and get an amplitude with one physical state and two possibly not physical ones as shown in Fig. 3.

Using these unphysical amplitudes we can try to understand which gauge fixing is suggested by string theory in the EFT computations. To find which gauge is chosen by string we have actually to introduce some other requirements. The reason is the following. The amplitude with two possibly non-physical states is Fig. 3 is not the full 3 point truncated Green function, i.e. the 3 vertex. It is only a part of it since Green functions are totally symmetric on the external legs and the amplitude we get is not. This means that either we compute the full 3 vertex or we compare with a color ordered vertex.

The first approach is not readily available since the off shell 3 point string partial amplitude treats in asymmetric way the off shell gluons.[1] This means that if we want to construct a 3 vertex, that is required to be totally symmetric in the exchange of gluons, we should sum over all the permutations of the external states. This would require to reexam the way we are used to do string computations and it would lead too far away.

We are therefore left with to the latter approach, also for ease of computation. In doing this partial identification then we introduce an element of arbitrarily. Since each on shell string diagram is cyclically invariant it is the natural to compare with a cyclically invariant color ordered vertex. It turns out that the 3 string truncated Green function we are dealing with is not cyclically invariant but it is up to gauge conditions. It follows then that we cannot identify the 3 state string truncated Green function with the cyclically invariant color ordered vertex but there is a left over, see eq. (21). This means that we cannot exactly match the naive off shell string amplitudes with an EFT but we can try to mimic them as close as possible. We are therefore left with the choice of how to choose the left over. Then the result on the gauge then depends on the assumption on

[1] This issue can be probably avoided using the twisted propagator at the price of having a non-canonical propagator (see [20] and references therein). The issue is under investigation.

what it means to mimic as close as possible the string truncated Green functions with the EFT vertices. Obviously we are not obliged to use the suggested gauge and use whichever gauge we want but trying to mimic as close as possible the string can give useful ideas. Our way of defining as close as possible is to try to minimize the number of left over terms in the 3 state vertex and then check that this implies that the number of terms in 4 point color ordered vertex, i.e. the contact terms, is also minimized. This is what done in this paper.

Using this approach we find that the gauge chosen is an α' corrected version of the Gervais–Neveu gauge [8] and that the field chosen by string theory to describe the gluon is not the gauge field used naturally in EFT but it is connected to it by a field redefinition. This kind of field redefinition is natural and expected in string field theory but it is a kind of surprise in the plain old string theory. Since at the end we are comparing partial color ordered S matrix elements we can also use a gauge fixed EFT expressed using the usual gauge field and the usual Feynman gauge at the price of having a bigger difference between the vertices suggested by the string and the ones computed from EFT.

It would also be interesting to consider the color ordered vertices suggested by string theory in a magnetic background using the techniques developed in [21,22] and compare with the ones used in [11]. It is very likely that the string suggestion is of a non-commutative nature. Also considering the superstring could be interesting in order to see whether a field redefinition is necessary.

The rest of this article is organized as follows. In section 2 we describe in more details the idea on how to read the vertices and color ordered vertices from string theory and we compare with the usual approach in determining the EFT. We introduce the color ordered vertices in a slightly different way as usual (see [13,14] and references therein) since they are introduced as a tool to mimic string diagrams as close as possible. In section 3 we perform the actual computation of the 3 color ordered vertex. We discuss how it compares to the most general 3 vector Lagrangian and the field redefinition which is needed to map the string field to the usual one used in EFT. We also discuss the string color ordered vertex as result of the minimal information which is needed to reconstruct the gauge invariant EFT. Finally in section 4 we recover the 4 point color ordered vertex up to two derivatives and we show that choice performed for the 3 vertex is the one which minimize the number of terms in this 4 point vertex.

2. The basic idea

In this section we would like to summarize some well known facts and then explain in more detail the basic idea behind this paper. The first point to quickly review is how factorization works in the simplest setting and allows to extract string amplitudes where some states are not required to be physical. Then we review the connection between Lagrangian interactions and Feynman vertices and we discuss the color ordered vertices (see [13,14] for a different way of introducing them) which are then used in the rest of the paper for extracting the gauge fixing. Finally we exemplify the approach with the simplest computation, i.e. the derivation of the propagator or that is the same the kinetic term.

2.1. Simple factorization

In the old days of string theory the tree amplitude of N open string physical states ϕ_i ($i = 1, \dots N$) was computed as (see Appendix A for conventions)

$$A(\phi_1,\dots\phi_N)=\langle\langle\phi_1|V(1;\phi_2)\frac{1}{L_0^{(X)}-1}V(1;\phi_3)\dots\frac{1}{L_0^{(X)}-1}V(1;\phi_{N-1})|\phi_N\rangle. \tag{1}$$

This amplitude is cyclically symmetric, i.e. $A(\phi_1,\dots\phi_N)=A(\phi_N,\dots\phi_1)$. In the previous expression $V(x;\phi)$ is the vertex operator associated to the physical state ϕ of conformal dimension 1, $|\phi\rangle=V(x=0,\phi)|0\rangle_{SL(2,\mathbb{R})}$. This expression roughly corresponds to a truncated Feynman diagram associated with a cubic theory and propagator $1/(L_0^{(X)}-1)$. Truncated diagram because the states are on shell and because of this there is not propagator immediately after (before) the bra(ket) state.

The previous expression gives part of the S matrix and the full S matrix is obtained by summing over all non-cyclically inequivalent permutations after having multiplied the previous expression for the Chan Paton contribution and having given a color a to all the physical states $\phi\to\phi_a$, explicitly

$$S(\phi_{1,a_1},\dots\phi_{N,a_N})=\imath\ \mathcal{A}(\phi_{1,a_1},\dots\phi_{N,a_N}), \tag{2}$$

where $\mathcal{A}$ is the connected truncated Green function

$$\begin{aligned}&\mathcal{A}(\phi_{1,a_1},\dots\phi_{N,a_N})\\&\quad=\frac{\alpha'^{N-3}}{\kappa}\mathcal{C}_0\mathcal{N}_0^N\sum_{\text{non-cyclical perm.s }\sigma}A(\phi_{\sigma(1),a_{\sigma(1)}},\dots\phi_{\sigma(N),a_{\sigma(N)}})\ tr(T_{a_{\sigma(1)}}\dots T_{a_{\sigma(N)}}),\end{aligned} \tag{3}$$

where the factor α'^{N-3} can be reabsorbed into the definition of the tree amplitude normalization $\mathcal{C}_0$ [17] and the vertex normalization $\mathcal{N}_0$ but we prefer to make it clear since it makes the propagator canonical.[2]

The previous amplitude (1) can be recast in a more modern form by writing the propagator in an integral form $1/(L_0^{(X)}-1)=\int_0^1 dy\ y^{L_0^{(X)}-2}$ and then moving all the terms involving $L_0^{(X)}$ to the right and changing integration variables to get a correlator integrated over the moduli space as

$$\begin{aligned}&A(\phi_1,\dots\phi_N)\\&\quad=\int_0^1 dx_3\int_0^{x_3}dx_4\dots\int_0^{x_{N-2}}dx_{N-1}\ \langle\langle\phi_1|V(1;\phi_2)V(x_3;\phi_3)\dots V(x_{N-1};\phi_{N-1})|\phi_N\rangle.\end{aligned} \tag{4}$$

For our purposes we need only the 3 point amplitude with two non-physical states and one physical which can be readily obtained by factorizing the $N=5$ amplitude in the old form

$$A(\phi_1,\dots\phi_5)=\langle\langle\phi_1|V(1;\phi_2)\frac{1}{L_0^{(X)}-1}V(1;\phi_3)\frac{1}{L_0^{(X)}-1}V(1;\phi_4)|\phi_4\rangle. \tag{5}$$

We can now insert four times the partition of unity

$$\begin{aligned}\mathbb{I}&=\int\frac{d^D\hat{k}}{(2\pi)^D}\Big[|\hat{k}\rangle\ \langle\langle\hat{k}|+\alpha_{-1}^\mu|\hat{k}\rangle\ \langle\langle\hat{k}|\alpha_1^\mu+\frac{\alpha_{-2}^\mu}{\sqrt{2}}|\hat{k}\rangle\ \langle\langle\hat{k}|\frac{\alpha_2^\mu}{\sqrt{2}}+\frac{\alpha_{-1}^\mu\alpha_{-1}^\nu}{\sqrt{2!}}|\hat{k}\rangle\ \langle\langle\hat{k}|\frac{\alpha_1^\mu\alpha_1^\nu}{\sqrt{2!}}+\dots\Big]\\&=\sum_\alpha|\alpha\rangle\ \langle\langle\alpha|,\end{aligned} \tag{6}$$

[2] Explicitly we have with respect to [17] $\mathcal{C}_0^{\text{here}}=\mathcal{C}_0^{\text{there}}\kappa\alpha'^3$ and $\mathcal{N}_0^{\text{here}}=\mathcal{N}_0^{\text{there}}/\alpha'$ when we consider the different trace normalizations $tr^{\text{there}}(T_aT_b)=\frac{1}{2}\delta_{ab}$ while we use the normalization given in eq. (69) which implies $\kappa=\frac{1}{2}$.

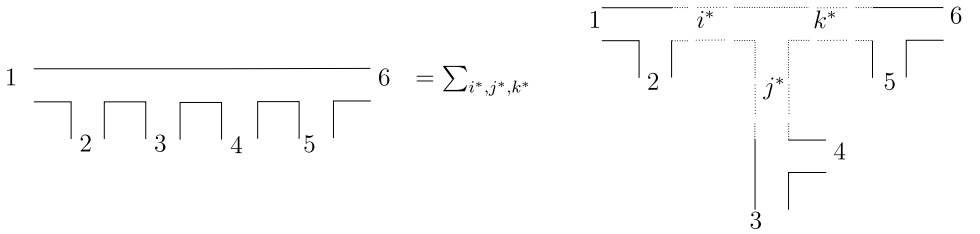

Fig. 4. The 6 states string S matrix is given by the sum of the product of four 3 states string amplitudes and three propagators where one of the 3 states vertices involves three i^*, j^*, k^* states which are not required to be physical.

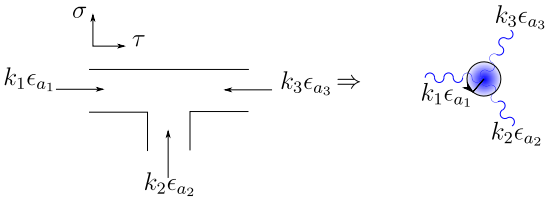

Fig. 5. The intuitive reason why we label the states in a counterclockwise fashion.

where $\hat{k}$ is the dimensionless momentum and $|\alpha\rangle$ is a generic basis element of the string Fock space which is eigenstate of $L_0^{(X)}$ with eigenvalue $l_0(\alpha)$. These states are normalized as $\langle\langle\beta|\alpha\rangle = \delta_{\alpha,\beta}$. We then immediately get the mathematical expression corresponding to Fig. 3

$$A(\phi_1,\dots\phi_5) = \sum_{\alpha,\beta}\langle\phi_1|V(1;\phi_2)|\alpha\rangle\frac{1}{l_0(\alpha)-1}\langle\langle\alpha|V(1;\phi_3)|\beta\rangle\frac{1}{l_0(\beta)-1}\langle\langle\beta|V(1;\phi_4)|\phi_4\rangle. \tag{7}$$

In this expression the sub-amplitude with two states which are not necessarily physical is $\langle\langle\alpha|V(1;\phi_3)|\beta\rangle$ and corresponds to the part of the Fig. 3 with dotted lines. Notice however that this amplitude is not cyclically symmetric as the corresponding amplitude with physical states (it is however actually sufficient to have off shell but transverse states to get cyclicity).

Even more generally starting from a 6 state amplitude is possible to find a 3 state amplitude where all states are possible unphysical as shown in Fig. 4 and first derived in the seminal paper [16]. In the rest of the paper we are not going to use this more general vertex and therefore we do not write its expression.

In the following we depict the string amplitudes mostly as interactions on a disc. On a disc the states are labeled counterclockwise because this is the natural way of labeling starting from the intuitive strip picture as shown in Fig. 5 for the three gluon amplitude.

In the case of 3 gluons the string S matrix element can then be depicted as in Fig. 6.

2.2. *Usual way of computing the EFT*

To compute the gauge invariant EFT we proceed order by order in the number of fields A^N, in power of derivatives ∂^n and in the YM coupling constant g^k. The Lagrangian of order N can be written schematically as $\mathcal{L}_{[N]} = \sqrt{2\alpha'}^{-D} \times[\sqrt{2\alpha'}^{2-\frac{1}{2}D} g]^k \times[\sqrt{2\alpha'}\partial]^m \times[\sqrt{2\alpha'}^{\frac{1}{2}D-1} A]^N$. Taking in consideration that gs originates efficaciously from cubic vertices in string we have the

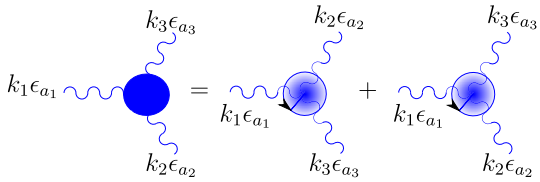

Fig. 6. The 3 gluon string S matrix is given by the sum of the two cyclically inequivalent orderings.

usual relations $3k = 2I + N$ and $I = L + k - 1$ where I is the number of internal lines. We can then write $\mathcal{L}_{[N]} = \sqrt{2\alpha'}^{N+m-L(D-4)} g^{N-2+2L} \partial^m A^N$ where L is the number of loops. Since the Lagrangian is a scalar and we want it to be expressed using gauge invariant field strength (we do not consider Chern–Simons theories) we need an even number of Lorentz indices and we get finally $\mathcal{L}_{[N]} = \sqrt{2\alpha'}^{N+n-4-L(D-4)} g^{N-2+2L} \partial^n F^N$.

In the following we are interested in the tree EFT, i.e. $L = 0$. If we are also only interested to up $(2\alpha')^n$ then only a finite number of terms are needed since $N \leq \frac{1}{2}n + 2$.

The usual procedure for computing the EFT is roughly as follows. Suppose we have computed the EFT to order $N - 1$ in the fields and order $(2\alpha')^n$. In order to do so we have fixed a gauge since in order to compute the S matrix elements with k particles we need the k point Green functions and they are obtained from 1PI vertices also by joining some of them with inverse propagators. To compute the next order involving N fields then [6]:

- write down the most general gauge invariant Lagrangian with at least N fields;
- check that all terms are independent;
- consider all the field redefinitions with at most N fields which do not change the S matrix (see [6] for a discussion for the open string theory) and how these field redefinitions change the coefficients of the independent terms of the Lagrangian;
- determine which combinations of the coefficients are left invariant by field redefinitions;
- compute a number of S matrix elements with N fields sufficient to determine the independent combinations of the coefficients
- compare the previous S matrix elements with the corresponding ones from string theory in order to fix explicitly the independent combinations.

Consider the Euclidean Lagrangian up to $N = 4$ and $(2\alpha')^2$ orders we have order by order in N[3]

$$S_{E\,[2]} = \int d^D x \, \frac{1}{\kappa} tr\Big[\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + (2\alpha')\Big(+v_{[2]1} D_\rho F_{\rho\mu} \; D_\sigma F_{\sigma\mu}\Big) + (2\alpha')^2\Big(v_{[2]2} D_\rho D_\sigma F_{\mu\nu} \; D_\rho D_\sigma F_{\mu\nu}\Big)\Big] \tag{8}$$

[3] Note that due to Bianchi identity we have [6] $tr(D_\rho F_{\mu\nu} \; D_\rho F_{\mu\nu}) \equiv 2tr(D_\rho F_{\rho\mu} \; D_\sigma F_{\sigma\mu} - 2F_{\rho\sigma} \; F_{\rho\lambda} \; F_{\sigma\lambda})$ up to total derivatives.

$$
\begin{aligned}
S_{E\,[3]} = \int d^D x\, \frac{1}{\kappa} tr\Big[& (2\alpha')\upsilon_{[3]0} F_{\mu\nu} F_{\nu\lambda} F_{\lambda\mu} \\
& + (2\alpha')^2 \Big(\upsilon_{[3]1} F_{\mu\nu}\ D_\rho F_{\mu\nu}\ D_\sigma F_{\sigma\rho} + \upsilon_{[3]2} F_{\mu\nu}\ D_\sigma F_{\sigma\rho}\ D_\rho F_{\mu\nu} \\
& + \upsilon_{[3]1} F_{\mu\nu}\ D_\rho F_{\rho\mu}\ D_\sigma F_{\sigma\nu} \Big) \Big]
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
S_{E\,[4]} = \int d^D x\, \frac{1}{\kappa} tr\Big[& (2\alpha')^2 \Big(\upsilon_{[4]0} F_{\mu\nu} F_{\nu\lambda} F_{\lambda\kappa} F_{\kappa\mu} + \upsilon_{[4]1} F_{\mu\rho} F_{\mu\sigma} F_{\lambda\rho} F_{\lambda\sigma} \\
& + \upsilon_{[4]2} F_{\mu\nu} F_{\mu\nu} F_{\rho\sigma} F_{\rho\sigma} \Big) \Big]
\end{aligned} \tag{10}
$$

As usual there is an ambiguity on how to write the derivative terms since $[D_\mu, D_\nu] \sim F_{\mu\nu}$. Then we can also write the gauge fixing Lagrangian

$$
\begin{aligned}
S_{E\,gf} = \int d^D x\, \frac{\xi}{\kappa} tr\Big(& \partial_\mu A_\mu + (2\alpha')\frac{g_0}{\xi}\partial^2\partial_\mu A_\mu + (2\alpha')^{2-D/2}\frac{g_1}{\xi} A_\mu A_\mu \\
& + (2\alpha')^{D/2}\Big[\frac{g_2}{\xi}\partial_\mu A_\mu \partial_\nu A_\nu + \frac{g_3}{\xi}\partial_\nu A_\mu \partial_\mu A_\nu + \frac{g_4}{\xi}\partial_\mu A_\nu \partial_\mu A_\nu + \frac{g_5}{\xi}\partial^2 A_\mu A_\mu \\
& + \frac{g_6}{\xi} A_\mu \partial^2 A_\mu \Big] + \dots \Big)^2 .
\end{aligned} \tag{11}
$$

Finally we can consider the field redefinitions. We can consider field redefinitions which do no change the gauge transformations like

$$
A_\mu = A'_\mu + r D'_\rho F'_{\rho\mu} + \dots \tag{12}
$$

or we can consider field redefinitions which do change the gauge transformations. If we are willing to change the gauge transformation then the only constraints are that all terms belong to the original algebra and that they do no change the S matrix elements. We will see that we need such more drastic field redefinitions in order to accomplish our program. They are like

$$
\begin{aligned}
A_\mu = & A'_\mu + (2\alpha')\left(r_1 \partial_\mu \partial_\rho A'_\rho + r_2 \partial_\rho \partial_\rho A'_\mu \right) \\
& + (2\alpha')^{D/2}\left(r_3 [A'_\mu, \partial_\rho A'_\rho] + r_4 [A'_\rho, \partial_\rho A'_\mu] + r_5 [A'_\rho, \partial_\mu A'_\rho] \right) + \dots
\end{aligned} \tag{13}
$$

The usual approach would then continue by finding the coefficients υs which are left unchanged by field redefinitions and then fix them by comparing the S matrix elements. This comparison is obviously independent on the gauge fixing.

2.3. The approach and the propagator

Differently from the usual approach the idea we want to implement is first to write blindly the EFT vertices mimicking the amplitudes with off shell/unphysical states computed from the string. Then to map these vertices to a gauge fixed EFT and determine the necessary field redefinitions at the same time.

To see how this work let us consider the propagator, i.e. the case $N = 2$. From the previous discussion we know that the propagator is given by

$$
\langle\langle \hat{k}_1, \mu_1 | \frac{\alpha'}{L_0^{(X)} - 1} | \hat{k}_2, \mu_2 \rangle = \frac{\delta^{\mu_1\mu_2}}{\hat{k}_1^2} \delta_{\hat{k}_1 + \hat{k}_2} . \tag{14}
$$

It follows then that the $N = 2$ part of the EFT is

$$S_{E\,[2]} = \int \prod_{i=1}^{2} \frac{d^D \hat{k}_i}{(2\pi)^D} \frac{1}{2!} \epsilon^a_{\mu_1}(\hat{k}_1) \left(\delta^{\mu_1\mu_2} k_1^2 \delta_{\hat{k}_1+\hat{k}_2} \right) \epsilon^a_{\mu_2}(\hat{k}_2). \tag{15}$$

Comparing the previous expression with the EFT expressed using the canonical fields we get at this order in the number of fields A

$$v_{[2]i} = 0, \quad i = 0, 1, 2, \quad \xi = -\frac{1}{2}, \quad g_0 = 0, \quad r_1 = r_2 = 0, \tag{16}$$

and the gauge fixing action, always up to A^2

$$S_{E\,[2],g.f.} = \int d^D x \left[-\frac{1}{2} (\partial^\mu A^a_\mu)^2 \right], \tag{17}$$

and no field redefinition is needed. In order to describe how we proceed with interaction terms we have to discuss what happens with Feynman vertices.

2.4. Vertices and color ordered vertices

When we start with a field theory we can compute the Feynman vertices and then compute Green functions by summing all the corresponding Feynman diagrams. Using these Green functions we can then compute the S matrix elements by using the LSZ reduction formula which amounts to put on shell the external legs after having truncated the legs.

In general given the part of the EFT action with N fields $\mathcal{L}_{[N]}$ the corresponding Feynman vertex can have up to $N!$ terms since it is built to be totally symmetric with respect the permutations of equal fields. For example in the case of the simplest $\phi^N(x)$ colorless scalar theory there is actually only 1 term in the vertex, while in the case of Yang–Mills for $N = 3$ we have $3! = 6$ terms but for $N = 4$ we have only $\frac{1}{2}4! = 12$ terms.

Consider a generic field $\Phi_A(x)$ with M components $A = 1, \dots M$ where A stands for both color and space time indices. Its polarization is then $\Phi_A(k)$. The totally symmetric Euclidean vertex $V_{[N]} \equiv V_{A_1 \dots A_N}(k_1 \dots, k_N)$ may have $N!$ terms and it is defined by

$$-S_{E\,[N]} = \int \prod_{i=1}^{N} \frac{d^D k_i}{(2\pi)^D} \frac{1}{N!} V_{A_1 \dots A_N}(k_1 \dots, k_N)\, \Phi_{A_1}(k_1) \dots \Phi_{A_N}(k_N), \tag{18}$$

where the momentum conservation $(2\pi)^D \delta^D(\sum_i k_i) \equiv \delta_{\sum_i k_i}$ is included into the definition of the vertex.

Because of the way we build the vertices a S matrix element with N fields may have $N!$ terms only from the vertex $V_{[N]}$. To these terms we must then add all the others coming from connecting vertices with fewer legs.

Nevertheless the comparison between open string theory and its EFT can be made easier if we split the Feynman vertices into cyclically invariant color ordered vertices. This split is shown in Fig. 7 where the 3 gluon Feynman vertex is written as the sum of two cyclically invariant color ordered vertices which are pictured with a circle with a direction. Then we can compare one (out of $(N-1)!$) string diagram with the corresponding color ordered Feynman diagram built using the color ordered color ordered vertices. In the case of the previous example with $N = 3$ this means comparing the first string diagram on the rhs in Fig. 6 with the first Feynman sub-diagram on the rhs in Fig. 7 (or that is the same the second ones in the same figures).

The same result applies when we compare Feynman diagrams involving more than one vertices. In general to a Feynman diagram build with N_3 3 vertices corresponds 2^{N_3} ordered

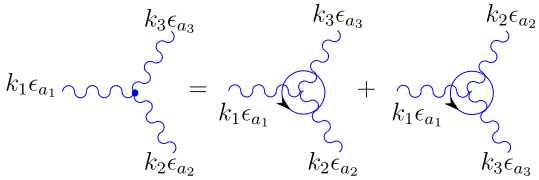

Fig. 7. The 3 point totally symmetric vertex $V_{[3]}$ is given as a sum of two cyclically symmetric ones $V_{[3]}^{(123)}$ and $V_{[3]}^{(132)}$.

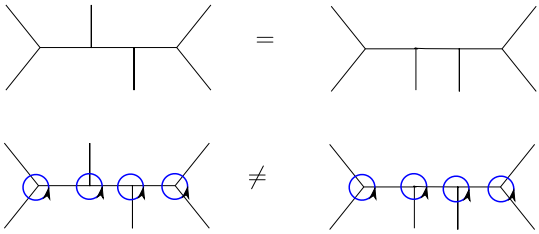

Fig. 8. The two Feynman diagrams of the first line are equal because vertices are totally symmetric under permutations while the 2 (out of $2^4 = 16$) cyclically symmetric (color ordered) Feynman diagrams in the second line differ.

Feynman diagrams. For example in Fig. 8 we show how a Feynman graph built using the usual 3 vertex can be drawn in many different ways because of the permutation symmetry of the vertex. Nevertheless using the cyclically symmetric vertex there is only one way of drawing a graph. When we write all vertices in a Feynman diagram as sum of cyclically symmetric color ordered vertices and we expand this "product" we get a 1–1 correspondence between these color ordered Feynman diagrams built using the ordered color ordered vertices and the string color ordered diagrams.

Fig. 10 shows what happens when we compare the string diagrams of $N = 4$ gluons which have a pole in the s channel with the corresponding Feynman diagram with a pole in the s channel.

2.5. Dealing with interaction terms

Since we can compare color ordered string amplitudes with EFT color ordered Feynman diagrams built using the cyclically symmetric color ordered vertices, it is natural to try to read the cyclically invariant Feynman vertices directly from string amplitudes with off shell/non-physical states which can be obtained from factorization. This can be described in a more precise way. In the case of $N = 3$ we can read directly the $V_{[3]}^{(123)}$ while for $N = 4$ and greater N we need first to subtract the poles and then read the contact interactions.

However already for the $N = 3$ gluons case this does not work exactly. It turns out to be possible to identify the $N = 3$ gluon string amplitude with the cyclically symmetric color ordered vertex up to gauge conditions, i.e. up to terms proportional to $\epsilon \cdot k$ as shown in eq. (21).

This difference between the off shell string vertex and the EFT cyclically invariant color ordered vertex is then at the origin of some contact terms in the quartic (and higher) coupling because of the Ward identity. Moreover this difference causes a more annoying fact that it is not

possible to compare off shell color ordered amplitudes but only on shell ones, i.e. pieces of an S matrix element.[4] We will discuss this point in section 4.1.

We read therefore the Feynman color ordered vertices as suggested by string theory by mimicking it as close as possible with a color ordered vertex. Then we can compute the totally symmetric Feynman vertices and compare these with the most general gauge fixed action. It turns out that they cannot be derived directly from a gauge fixed EFT written in terms of the canonical fields. In fact the resulting vertices are written using fields which are not the ones used to write the EFT but they are connected by to them by a field redefinition. Obviously one can use the canonical fields in the EFT but then the EFT vertices differ by more terms with respect to the string amplitudes.

3. String amplitudes: 3 points

We would now implement in practice what we have discussed in the previous section. In particular we would like to determine the 3 vertex suggested by string theory and then find the gauge fixing and the field redefinition necessary to map it to the EFT written the standard field.

3.1. Three gluons amplitude

It is standard matter (see for example [19]) to compute the three photons partial amplitude once we have given the photon vertex operator

$$V(x;\hat{k},\hat{\epsilon}) = +\imath\hat{\epsilon}\cdot\partial\hat{X}(x,x)e^{\imath\hat{k}\cdot\hat{X}(x,x)}, \tag{19}$$

where the hatted quantities are adimensional, for example $\hat{k} = \sqrt{2\alpha'}k$ is the adimensional momentum. We compute the partial amplitude not requiring that the in and out state be on shell or transverse. The reason is that this is what we see by factoring the 5 point amplitude. The basic contribution to the truncated Euclidean Green function is then

$$\begin{aligned}
A_{1^*23^*} &= A(\hat{k}_1^*,\hat{\epsilon}_1^*;\hat{k}_2,\hat{\epsilon}_2;\hat{k}_3^*,\hat{\epsilon}_3^*) = \langle\langle\hat{k}_1^*,\hat{\epsilon}_1^*|\ V(x=1;\hat{k}_2,\hat{\epsilon}_2)\ |\hat{k}_3^*,\hat{\epsilon}_3^*\rangle \\
&= \langle\langle\hat{k}_1^*,0|\hat{\epsilon}_1^*\cdot\alpha_1\ V(x=1;\hat{k}_2,\hat{\epsilon}_2)\ \hat{\epsilon}_3^*\cdot\alpha_{-1}|\hat{k}_3^*,0\rangle \\
&= \langle\langle\hat{k}_1^*,0|\hat{\epsilon}_1^*\cdot\alpha_1\ :\hat{\epsilon}_2\cdot(\alpha_{-1}+\alpha_0+\alpha_1)\,e^{i\hat{k}_2x_0}e^{\hat{k}_2\cdot\alpha_{-1}}e^{-\hat{k}_2\cdot\alpha_1}:\ \hat{\epsilon}_3^*\cdot\alpha_{-1}^*|\hat{k}_3,0\rangle \\
&= [-\hat{\epsilon}_1^*\cdot\hat{\epsilon}_2\,\hat{k}_2\cdot\hat{\epsilon}_3^* + \hat{\epsilon}_3^*\cdot\hat{\epsilon}_1^*\,\hat{\epsilon}_2\cdot\hat{k}_3^* - \hat{\epsilon}_1^*\cdot\hat{k}_2\,\hat{\epsilon}_2\cdot\hat{k}_3^*\,\hat{\epsilon}_3^*\cdot\hat{k}_2 + \hat{\epsilon}_2\cdot\hat{\epsilon}_3^*\,\hat{k}_2\cdot\hat{\epsilon}_1^*]\delta_{\hat{k}_1+\hat{k}_2+\hat{k}_3}
\end{aligned} \tag{20}$$

$$\begin{aligned}
&= [-\hat{\epsilon}_1^*\cdot\hat{\epsilon}_2\,\hat{k}_2\cdot\hat{\epsilon}_3^* - \hat{\epsilon}_2\cdot\hat{\epsilon}_3^*\,\hat{k}_3^*\cdot\hat{\epsilon}_1^* - \hat{\epsilon}_3^*\cdot\hat{\epsilon}_1^*\,\hat{k}_1^*\cdot\hat{\epsilon}_2 \\
&\quad + \hat{\epsilon}_1^*\cdot\hat{k}_2\,\hat{\epsilon}_2\cdot\hat{k}_3^*\,\hat{\epsilon}_3^*\cdot\hat{k}_1^* \\
&\quad + \hat{\epsilon}_1^*\cdot\hat{\epsilon}_2\,\hat{k}_3^*\cdot\hat{\epsilon}_3^* + \hat{\epsilon}_1^*\cdot\hat{k}_2\,\hat{\epsilon}_2\cdot\hat{k}_3^*\,\hat{\epsilon}_3^*\cdot\hat{k}_3^*]\delta_{\hat{k}_1+\hat{k}_2+\hat{k}_3},
\end{aligned} \tag{21}$$

where the * means that the corresponding starred quantity may not satisfy the physical conditions. It is the previous expression properly normalized, i.e. $\mathcal{C}_0\mathcal{N}_0^3 A_{1^*23^*}$ that we want to mimic with the color ordered vertex of the EFT. Few things are worth noticing. First eq. (20) is antisymmetric in the exchange of the two non-physical gluons 1 and 3. This makes impossible to interpret it

[4] All these problems may perhaps be avoided using the twisted propagator which allows for cyclically invariant vertices as the CSV vertex [18].

as a piece of a usual EFT since the Feynman vertices are totally symmetric in the exchange of gluons. Secondly the last way of writing the partial amplitude $A_{1^*23^*}$ in eq. (21) shows that the amplitude is cyclically invariant when we use the gauge condition $\hat{\epsilon}_3^* \cdot k_3 = 0$ for the third state, i.e. $A_{1^*23^*}$ is cyclically invariant when $\hat{\epsilon}_3^*$ is transverse but eventually off shell, since then the last line vanishes. Obviously it is possible to write an analogous expression where we require the transversality for the first state $\hat{\epsilon}_1 \cdot \hat{k}_1 = 0$.

Only when all states are physical, i.e. on shell and transverse the amplitude has on shell gauge invariance, i.e. it is invariant under $\hat{\epsilon} \to \hat{\epsilon} + \hat{k}$ with $\hat{k}^2 = 0$.

The S matrix element from string theory for non-abelian gluons can then be obtained from the amplitude as

$$\mathcal{A}_{123}(\hat{k}_1, \hat{\epsilon}_{\mu_1 a_1}; \hat{k}_2, \hat{\epsilon}_{\mu_2 a_2}; \hat{k}_3, \hat{\epsilon}_{\mu_3 a_3}) = \mathcal{C}_0 \mathcal{N}_0^3 \left[A_{123}\, tr(T_{a_1} T_{a_2} T_{a_3}) + A_{132}\, tr(T_{a_1} T_{a_3} T_{a_2}) \right] \tag{22}$$

and it is obtained by taking all states physical, substituting the abelian polarizations $\hat{\epsilon}_i$ with their non-abelian ones $\hat{\epsilon}_{a_i}$ and multiplying by the Chan–Paton factors, explicitly in the previous expression we have

$$A_{123} = A(\hat{k}_1, \hat{\epsilon}_{a_1}; \hat{k}_2, \hat{\epsilon}_{a_2}; \hat{k}_3, \hat{\epsilon}_{a_3}), \tag{23}$$

and there is no summation over the color indices. The full amplitude is depicted in Fig. 6.

Now because of the on shell condition $\hat{k}_i^2 = 0$ it follows that all the momenta $\hat{k}_i$ are parallel as can be easily seen since on shell $\hat{k}_i \cdot \hat{k}_j = 0$ and we can choose any $\hat{k}$ in the light cone direction. Therefore both the amplitude and the S matrix vanish

$$S_{123}(\hat{k}_1, \hat{\epsilon}_{\mu_1 a_1}; \hat{k}_2, \hat{\epsilon}_{\mu_2 a_2}; \hat{k}_3, \hat{\epsilon}_{\mu_3 a_3}) = 0. \tag{24}$$

3.2. The general three gluons up to three derivatives Lagrangian

In order to reconstruct the gauge fixed EFT from the previous S matrix we write down the most general Lagrangian with 3 gluons and up to 3 derivatives. From the Lagrangian we deduce the 3 Feynman vertex and then we require that it yields a 3 point S matrix element vanishing on shell. Besides this constraints we have nevertheless to respect the pole structure of the 4 and higher point S matrix amplitudes, i.e. given the 4 point S matrix amplitude the result of subtracting the contribution from the reducible Feynman diagrams obtained by joining two 3 point vertices must be pole free.[5]

Nevertheless as discussed in the previous section 2.5 our main idea is to proceed in a different way and we use the off shell extension $\mathcal{C}_0 \mathcal{N}_0^3 A_{1^*23^*}$ to read the 3 vertex suggested by string theory for a EFT. However we consider the most general Lagrangian in order to discuss how the string choice minimizes the number of terms in the 3 and 4 point vertices.

The general cubic effective action with up to three derivatives reads[6]

[5] This requirement is not true when dealing with Green functions as we show in section 4.1 since the stringy off shell amplitude cannot be interpreted as a piece of a usual Feynman vertex.

[6] The easiest way to obtain it is to work in momentum space. The terms with one momentum are immediate to find. The terms with three momenta fall into two categories either $(\epsilon \cdot k)^3$ or $(\epsilon \cdot \epsilon)(\epsilon \cdot k)(k \cdot k)$.

Let us consider the first class. Using cyclicity we have 3^3 terms $\epsilon_1 \cdot k_i\ \epsilon_2 \cdot k_j\ \epsilon_3 \cdot k_l$ since $i, j, k \in \{1, 2, 3\}$. Using momentum conservation we can consider only 2^3 terms, i.e. those with $i \neq 1$, $j \neq 2$, $l \neq 3$. Then using again cyclicity

$$
\begin{aligned}
S_{E\,[3]} = \int d^D x\, \frac{1}{\kappa} tr[&+ c_1 \partial_\mu A_\nu\, A_\mu\, A_\nu && + c_2 \partial_\mu A_\nu\, A_\nu\, A_\mu \\
&+ c_3 \partial_\mu A_\nu\, \partial_\nu A_\lambda\, \partial_\lambda A_\mu && + c_4 \partial_\lambda \partial_\nu A_\mu\, \partial_\mu A_\nu\, A_\lambda \\
&+ c_5 \partial_\lambda \partial_\nu A_\mu\, \partial_\mu A_\nu\, A_\lambda && + c_6 \partial_\nu A_\mu\, \partial_\lambda \partial_\mu A_\nu\, A_\lambda \\
&+ c_7 \partial^2 \partial_\lambda A_\mu\, A_\mu\, A_\lambda && + c_8 \partial_\lambda A_\mu\, \partial^2 A_\nu\, A_\lambda \\
&+ c_9 \partial_\rho \partial_\lambda A_\mu\, \partial_\rho A_\nu\, A_\lambda && \\
&+ c_{10} \partial^2 A_\mu\, \partial_\lambda A_\mu\, A_\lambda && + c_{11} A_\mu\, \partial^2 \partial_\lambda A_\mu\, A_\lambda \\
&+ c_{12} \partial_\rho A_\mu\, \partial_\rho \partial_\lambda A_\nu\, A_\lambda &&].
\end{aligned} \tag{25}
$$

Notice that all these terms give a vanishing 3 point S matrix. This can be more easily looking at the corresponding Feynman vertex in eqs. (28), (29). In particular it is necessary to remember that all k_i are parallel on shell and hence $\epsilon_i \cdot k_j = 0$.

Interpreting this cubic interaction as coming from a gauge invariant action with a non-linear gauge fixing as eq. (11) and a field redefinition as in eq. (13) (assuming a canonical kinetic term which implies $g_0 = v_{[2]1} = v_{[2]2} = 0$) requires[7]

$$
\begin{aligned}
&c_1 = -ig - 2g_1, \quad c_2 = +ig - 2g_1, \\
&c_3 = +v_{[3]0} - 2g_2, \quad c_4 = -v_{[3]0} - 2g_2, \\
&c_5 = -2g_3 - 6g_2, \quad c_6 = -2g_3 - 6g_2, \\
&c_7 = -2g_5 - r_3 - r_5, \quad c_{10} = -3v_{[3]0} - 2g_5 - r_3 - r_4 + r_5, \\
&c_8 = 3v_{[3]0} - 2g_6 + r_3 + r_4 - r_5, \quad c_{11} = -2g_6 + r_3 + r_5, \\
&c_9 = 3v_{[3]0} - 2g_4 - 2r_5, \quad c_{12} = -3v_{[3]0} - 2g_4 + 2r_5.
\end{aligned} \tag{26}
$$

In particular the previous vertex can **not** become the usual three vertex in the linear Lorentz gauge unless $c_2 = -c_1$, $3c_3 = -3c_4 = c_8 = c_9 = -c_{10} = -c_{12}$ and $c_{5,6,7,11} = 0$. This happens because the usual three vertex involves the commutator of the algebra elements $tr(T_a[T_b, T_c])$ which is totally antisymmetric in the exchange of a, b, c. When these conditions are not satisfied the cubic interaction does not originate from a gauge invariant action with linear gauge fixing and we must interpret it as originating from a gauge fixed action with non-linear gauge fixing and a field redefinition.

The previous cubic interaction gives raise to the Euclidean Feynman cubic vertex defined by

$$
-S_{E\,[3]} = \int \prod_{i=1}^{3} \frac{d^D k_i}{(2\pi)^D} \frac{1}{3!} V_{\mu_1 a_1, \mu_2 a_2, \mu_3 a_3}(k_1, k_2, k_3)\, \epsilon_{a_1}^{\mu_1}(k_1) \epsilon_{a_2}^{\mu_2}(k_2) \epsilon_{a_3}^{\mu_3}(k_3). \tag{27}
$$

As discussed in section 2.4 it is convenient to write this cubic vertex as the sum of two cyclically invariant color ordered vertices as shown in Fig. 7 as

---

we are left with 4 terms, those with coefficients $c_3, \ldots c_6$ in eq. (29). An example of the use of cyclicity is the fact that the term with $(i, j, k) = (2, 3, 2)$ is equivalent to $(i, j, k) = (2, 1, 1)$.

Now consider the second class. Using cyclicity we have 3^3 terms like $\epsilon_1 \cdot \epsilon_2\ \epsilon_3 \cdot k_l\ k_i \cdot k_j$. Again momentum conservation allows us to consider the cases $i, j, l \neq 3$. They are 6 and are the terms with coefficients $c_7, \ldots c_{12}$ in eq. (29).

[7] The dependence of coefficients $c_1 \ldots c_4$ on g and $v_{[3]0}$ can be immediately read by expanding the Lagrangian, the other requires a little more work.

$$V_{\mu_1 a_1,\mu_2 a_2,\mu_3 a_3}(k_1,k_2,k_3) = \frac{1}{\kappa}\Big[V^{(123)}_{\mu_1;\ \mu_2;\ \mu_3}(k_1,k_2,k_3)\ tr(T_{a_1}T_{a_2}T_{a_3}) + V^{(123)}_{\mu_1;\ \mu_3;\ \mu_2}(k_1,k_3,k_2)\ tr(T_{a_1}T_{a_3}T_{a_2})\ \Big], \tag{28}$$

where[8]

$$\begin{aligned} V^{(123)}_{\mu_1;\ \mu_2;\ \mu_3}\ \epsilon^{\mu_1}_{a_1}\epsilon^{\mu_2}_{a_2}\epsilon^{\mu_3}_{a_3} = (+\imath)\Big[& -c_1(\epsilon_{a_1}\cdot\epsilon_{a_2}\ \epsilon_{a_3}\cdot k_2 + \text{cycl}) - c_2(\epsilon_{a_1}\cdot\epsilon_{a_2}\ \epsilon_{a_3}\cdot k_1 + \text{cycl}) \\ & +3c_3\ \epsilon_{a_1}\cdot k_2\ \epsilon_{a_2}\cdot k_3\ \epsilon_{a_3}\cdot k_1 + 3c_4\ \epsilon_{a_1}\cdot k_3\ \epsilon_{a_2}\cdot k_1\ \epsilon_{a_2}\cdot k_2 \\ & +c_5(\epsilon_{a_1}\cdot k_2\ \epsilon_{a_2}\cdot k_1\ \epsilon_{a_3}\cdot k_1 + \text{cycl}) + c_6(\epsilon_{a_1}\cdot k_2\ \epsilon_{a_2}\cdot k_1\ \epsilon_{a_2}\cdot k_2 + \text{cycl}) \\ & +c_7(\epsilon_{a_1}\cdot\epsilon_{a_2}\ \epsilon_{a_3}\cdot k_1\ k_1^2 + \text{cycl}) + c_8(\epsilon_{a_1}\cdot\epsilon_{a_2}\ \epsilon_{a_3}\cdot k_1\ k_2^2 + \text{cycl}) \\ & +c_9(\epsilon_{a_1}\cdot\epsilon_{a_2}\ \epsilon_{a_3}\cdot k_1\ k_1\cdot k_2 + \text{cycl}) \\ & +c_{10}(\epsilon_{a_1}\cdot\epsilon_{a_2}\ \epsilon_{a_3}\cdot k_2\ k_1^2 + \text{cycl}) + c_{11}(\epsilon_{a_1}\cdot\epsilon_{a_2}\ \epsilon_{a_3}\cdot k_2\ k_2^2 + \text{cycl}) \\ & +c_{12}(\epsilon_{a_1}\cdot\epsilon_{a_2}\ \epsilon_{a_3}\cdot k_2\ k_1\cdot k_2 + \text{cycl})\Big]\delta_{k_1+k_2+k_3}. \end{aligned} \tag{29}$$

Matching the structure of cyclical color ordered vertex as close as possible to the off shell amplitude (21) gives

$$\begin{aligned} c_1 &= -2\imath g, \quad c_2 = 0, \\ c_3 &= 2v_{[3]0}, \quad c_4 = 0, \\ c_{5,6,7,8,9,10,11,12} &= 0 \\ g_1 &= \frac{1}{2}\imath g, \\ g_2 &= -\frac{1}{2}v_{[3]0}, \quad g_3 = -3g_2 \\ g_4 &= \frac{3}{2}v_{[3]0}, \quad g_5 = -g_6 = r_3 + r_5 \\ r_4 &= 0, \quad r_5 = \frac{3}{2}v_{[3]0}. \end{aligned} \tag{30}$$

A rapid look to eqs. (26) reveals that these coefficients cannot be reproduced simply using a gauge fixing and that we therefore need a field redefinition. We find the gauge fixed Lagrangian

$$S_{E\,[3]gauge\ fixed} = \int d^D x\ \frac{1}{\kappa} tr\Big(-2ig\partial_\mu A_\nu\ A_\mu\ A_\nu + 2v_{[3]0}\partial_\mu A_\nu\ \partial_\nu A_\lambda\ \partial_\lambda A_\mu\Big), \tag{31}$$

the gauge fixing Lagrangian

$$\begin{aligned} S_{E\,gf} = \int d^D x\ \frac{\xi}{\kappa} tr\Big(&\partial_\mu A_\mu + \imath\frac{g}{2\xi}A_\mu A_\mu \\ & - \frac{v_{[3]0}}{2\xi}\partial_\mu A_\mu \partial_\nu A_\nu + \frac{3v_{[3]0}}{2\xi}\partial_\nu A_\mu \partial_\mu A_\nu - \frac{v_{[3]0}+2r_3}{2\xi}[\partial^2 A_\mu, A_\mu] + \dots\Big)^2, \end{aligned} \tag{32}$$

[8] The coefficients $3c_3$ and $3c_4$ come from the fact that the corresponding structures are cyclically symmetric. The different signs from the different momentum powers ik vs $(ik)^3$.

with $\xi = -\frac{1}{2}$ as from eq. (17) and the field redefinition

$$A_\mu = A'_\mu + r_3[A'_\mu, \partial_\rho A'_\rho] + \frac{3v_{[3]0}}{2\xi}[A'_\rho, \partial_\mu A'_\rho] + \dots . \tag{33}$$

If we want to match also the coefficient we need to match the previous color ordered vertex with $\mathcal{C}_0\mathcal{N}_0^3 A_{1*23*}$ and set

$$c_1 = -2\imath g = -\imath\mathcal{C}_0\mathcal{N}_0^3(2\alpha')^{2-\frac{1}{2}D}, \quad c_3 = 2v_{[3]0} = -\frac{1}{3}\imath\mathcal{C}_0\mathcal{N}_0^3(2\alpha')^{3-\frac{1}{2}D}, \tag{34}$$

thus finding the usual result

$$v_{[3]0} = -\frac{1}{3}\imath(2\alpha')g. \tag{35}$$

If we do not want to use field redefinition we have more possibilities on the closest possible vertex has gauge fixed Lagrangian. One possibility is given by the gauge fixed Lagrangian

$$\begin{aligned} S_{E\,[3]gauge\,fixed} = \int d^Dx\,\frac{1}{\kappa}tr\Big(&-2ig\partial_\mu A_\nu\,A_\mu\,A_\nu + 2v_{[3]0}\partial_\mu A_\nu\,\partial_\nu A_\lambda\,\partial_\lambda A_\mu \\ &+ 3v_{[3]0}\partial_\mu A_\nu\,[\partial_\lambda A_\mu,\,\partial_\lambda A_\nu]\Big), \end{aligned} \tag{36}$$

and the gauge fixing Lagrangian

$$S_{E\,gf} = \int d^Dx\,\frac{\xi}{\kappa}tr\Big(\partial_\mu A_\mu + \imath\frac{g}{2\xi}A_\mu A_\mu - \frac{v_{[3]0}}{2\xi}\partial_\mu A_\mu\partial_\nu A_\nu + \frac{3v_{[3]0}}{2\xi}\partial_\nu A_\mu\partial_\mu A_\nu + \dots\Big)^2. \tag{37}$$

Another possibility is given by the gauge fixed Lagrangian

$$S_{E\,[3]gauge\,fixed} = \int d^Dx\,\frac{1}{\kappa}tr\Big(-2ig\partial_\mu A_\nu\,A_\mu\,A_\nu + v_{[3]0}F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}\Big), \tag{38}$$

and the gauge fixing Lagrangian

$$S_{E\,gf} = \int d^Dx\,\frac{\xi}{\kappa}tr\Big(\partial_\mu A_\mu + \imath\frac{g}{2\xi}A_\mu A_\mu + \dots\Big)^2. \tag{39}$$

3.3. The abelian limit and intuitive explanation of the Gervais–Neveu gauge

Looking to the possible terms in the color ordered vertex $V^{(123)}$ it is clear that some om them become exchanged under non-cyclical permutations. For example c_1 and c_2 are exchanged when $1 \leftrightarrow 2$. In more formal way $c_2^{(123)} = c_1^{(213)}$. This means that we can know $c_2^{(123)}$ if we know $c_1^{(123)}$ since by exchanging $1 \leftrightarrow 2$ we can compute $c_1^{(213)}$. Therefore $c_2^{(123)}$ is redundant and can be likely set to zero by choosing a gauge. In facts string theory chooses $c_2 = 0$ (or equivalently $c_1 = 0$). It seems that string theory be choosing the minimal number of terms from which we can reconstruct both the abelian and non-abelian theory. Because of this also the abelian theory has non-vanishing 3 vertex.

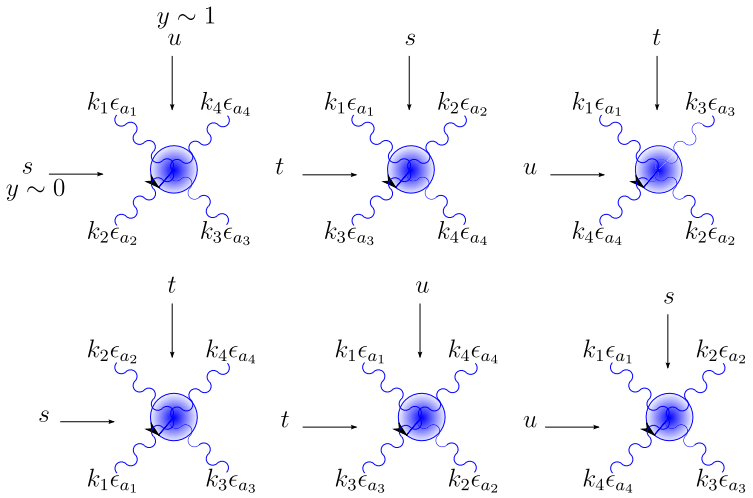

Fig. 9. The six diagrams contributing to the $N = 4$ amplitude with the indication of the channels to which each diagram contributes. The horizontal channel is the obvious one from the old way of writing the amplitude. The vertical channel is the obvious one when using the cyclicity of the amplitude.

4. Four gluons amplitude, propagator and contact terms

The basic partial amplitude (and not correlator since this is already the integrated correlator) is[9]

$$\begin{aligned} A_{1234} &= A(\hat{k}_1, \hat{\epsilon}_1; \dots \hat{k}_4, \hat{\epsilon}_4) \\ &= \langle\langle \hat{k}_1, \hat{\epsilon}_1 |\, V(x=1; \hat{k}_2, \hat{\epsilon}_2)\, \frac{1}{L_0^{(X)} - 1}\, V(x=1; \hat{k}_3, \hat{\epsilon}_4)\, | k_4, \hat{\epsilon}_4 \rangle. \end{aligned} \tag{40}$$

The full S matrix is then obtained from (see Fig. 9)

$$\begin{aligned} \mathcal{A}_{1234} = \frac{\alpha'}{\kappa} \mathcal{C}_0 \mathcal{N}_0^4 \Big\{ & [A_{1234}\, tr(T_{a_1} T_{a_2} T_{a_3} T_{a_4}) + A_{1243}\, tr(T_{a_1} T_{a_2} T_{a_4} T_{a_3})] \\ & + [A_{1342}\, tr(T_{a_1} T_{a_3} T_{a_4} T_{a_2}) + A_{1324}\, tr(T_{a_1} T_{a_3} T_{a_2} T_{a_4})] \\ & + [A_{1423}\, tr(T_{a_1} T_{a_4} T_{a_2} T_{a_3}) + A_{1432}\, tr(T_{a_1} T_{a_4} T_{a_3} T_{a_2})] \Big\}, \end{aligned} \tag{41}$$

where we substitute the abelian polarizations $\hat{\epsilon}_i$ with their non-abelian ones $\hat{\epsilon}_{i a_i}$. In the previous equation the first line gives poles in the s and u channels, the second to the s and t ones and the last to the t and u ones where we defined

[9] Notice that this expression is naive since it is divergent as it stands because of the sum over infinite intermediate states (this divergence seemed to be well known in 1971, see [20] after eq. (4.40)). This is easily seen in the four tachyons amplitude $\int_0^1 dx\, x^{\hat{k}_3 \cdot \hat{k}_4} (1-x)^{\hat{k}_2 \cdot \hat{k}_3}$ where the term $(1-x)^{\hat{k}_2 \cdot \hat{k}_3}$ can be expanded around $x = 0$ inside the integral and this gives the s channel poles Nevertheless the infinite summation cannot be exchanged with the integral because the series is not uniformly convergent.

To give a proper meaning we need to use a regularized propagator as $\Delta_r(\epsilon) = e^{-\epsilon N}/(L_0^{(X)} - 1)$ as well as consider a contribution from the A_{2341} amplitude like what happens in string field theory where the infinite sum is naturally performed. For the time being we do not consider this and take the previous expression as the integral of a correlator which is well defined.

$$s = -(k_1 + k_2)^2, \quad t = -(k_1 + k_3)^2, \quad u = -(k_1 + k_4)^2. \tag{42}$$

4.1. Factorizing the $N = 4$ amplitude on the gluons and constraints on the c_i coefficients

In order to discuss how the string minimize the number of terms in the vertices we would now find the constraints on the constants $c_{1,\dots 12}$ which arise in order to cancel the physical poles. In the following subsection we use these constraints to show that the string solution is minimal in ensuing that the 4 vertex has the minimal number of terms.

The cancellation of poles can be checked by comparing the ordered string diagrams with a pole in the s channel (all the other channels would do the same) with the Feynman diagram from EFT which has a pole in the same s channel. In order to do so we must see which of the six terms has a pole in the s channel. It is obvious that A_{1234} and A_{1243} have such a pole but because of the cyclicity also $A_{1342} \equiv A_{2134}$ and $A_{1432} \equiv A_{2143}$ have therefore

$$\begin{aligned}\mathcal{A}_{1234} \sim_{s\to 0} \frac{\alpha'}{\kappa}\mathcal{C}_0\mathcal{N}_0^4 \Big\{ &[A_{1234}\, tr(T_{a_1}T_{a_2}T_{a_3}T_{a_4}) + A_{1243}\, tr(T_{a_1}T_{a_2}T_{a_4}T_{a_3})] \\ &+[A_{1342}\, tr(T_{a_1}T_{a_3}T_{a_4}T_{a_2})] + [A_{1432}\, tr(T_{a_1}T_{a_4}T_{a_3}T_{a_2})] + O(1)\Big\}.\end{aligned} \tag{43}$$

To these ordered diagrams corresponds the Feynman diagram

$$\begin{aligned}&\epsilon^{\mu_1}_{a_1}\epsilon^{\mu_2}_{a_2}\frac{1}{\kappa}\Big[V^{(123)}_{\mu_1;\,\mu_2;\,\mu}(k_1,k_2,q^*)\, tr(T_{a_1}T_{a_2}T_b) + V^{(132)}_{\mu_1;\,\mu;\,\mu_2}(k_1,q^*,k_2)\, tr(T_{a_1}T_bT_{a_2})\Big] \\ &\quad\times\frac{\delta^{bc}P(q^*)^{\mu\nu}}{q^{*2}} \\ &\quad\times\frac{1}{\kappa}\Big[V^{(123)}_{\nu;\,\mu_3;\,\mu_4}(-q^*,k_3,k_4)\, tr(T_cT_{a_3}T_{a_4}) \\ &\qquad + V^{(132)}_{\nu;\,\mu_4;\,\mu_3}(-q^*,k_4,k_3)\, tr(T_cT_{a_4}T_{a_3})\Big]\,\epsilon^{\mu_4}_{a_4}\epsilon^{\mu_3}_{a_3}\delta_{\sum k_i},\end{aligned} \tag{44}$$

with $k_1+k_2+q^* = -q^*+k_3+k_4 = 0$. The request is then that the expressions (43) and (44) have the same pole. As shown in Fig. 10 and discussed above in section 2.5 the computation can be simplified since to any ordered string diagram corresponds a piece of the Feynman diagram built using the cyclically symmetric color ordered vertices. Because of this we only need to compute the expression graphically depicted in Fig. 11. Then the expression which corresponds to figure this is given by

$$\begin{aligned}&\frac{\alpha'}{\kappa}\mathcal{C}_0\mathcal{N}_0^4 A_{1234}\, tr(T_{a_1}T_{a_2}T_{a_3}T_{a_4}) \\ &\quad -\frac{1}{\kappa}\epsilon^{\mu_1}_{a_1}\epsilon^{\mu_2}_{a_2}V^{(123)}_{\mu_1;\,\mu_2;\,\mu}(k_1,k_2,q^*)\, tr(T_{a_1}T_{a_2}T_b)\frac{\delta^{bc}\delta^{\mu\nu}}{q^{*2}} \\ &\quad\times\frac{1}{\kappa}V^{(123)}_{\nu;\,\mu_3;\,\mu_4}(-q^*,k_3,k_4)\,\epsilon^{\mu_3}_{a_3}\epsilon^{\mu_4}_{a_4}\, tr(T_cT_{a_3}T_{a_4})\delta_{\sum k_i}\end{aligned} \tag{45}$$

$$\begin{aligned}&= \alpha'\mathcal{C}_0\mathcal{N}_0^4 A_{1234} - \Big[\epsilon^{\mu_1}_{a_1}\epsilon^{\mu_2}_{a_2}V^{(123)}_{\mu_1;\,\mu_2;\,\mu}(k_1,k_2,q^*)\frac{\delta^{\mu\nu}}{q^{*2}}V^{(123)}_{\nu;\,\mu_3;\,\mu_4}(-q^*,k_3,k_4)\,\epsilon^{\mu_3}_{a_3}\epsilon^{\mu_4}_{a_4}\Big]\delta_{\sum k_i} \\ &\quad\frac{1}{\kappa}tr(T_{a_1}T_{a_2}T_{a_3}T_{a_4}),\end{aligned} \tag{46}$$

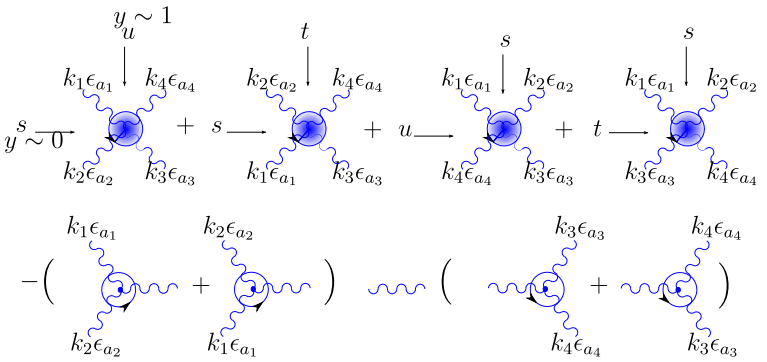

Fig. 10. The ordered string diagrams with poles in the s channel and the Feynman diagram with a pole in the same channel. To any ordered string diagram corresponds a part of the Feynman diagram computed with the ordered Feynman vertices.

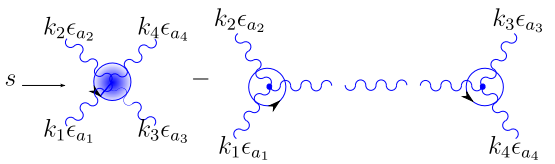

Fig. 11. Single diagram subtraction.

where we have already used the first suggestion which comes from string, i.e. to use the propagator in Feynman gauge.[10] We have also used

$$tr(X\,T_a)\,\delta^{ab}\,tr(T_b\,Y) = \kappa\,tr(X\,Y) \quad X, Y \in u(N). \tag{47}$$

The pole in the s channel of the string partial amplitude can be exposed by simply inserting twice the unity at level $N = 1$ in the string amplitude A_{1234} and get

$$A_{1234} \sim_{s\to 0} \int_{q^*} \langle\langle \hat{k}_1, \hat{\epsilon}_1 |\, V(x=1; \hat{k}_2, \hat{\epsilon}_2)\alpha^{\mu}_{-1}|q^*\rangle\, \frac{\delta_{\mu\nu}}{\alpha' q^{*2}} \langle\langle q^*|\alpha^{\nu}_1\, V(x=1; \hat{k}_3, \hat{\epsilon}_4)\, |\hat{k}_4, \hat{\epsilon}_4\rangle. \tag{48}$$

Comparing this expression with the EFT one in eq. (45) suggests to set[11]

$$\begin{aligned} \sqrt{-\mathcal{C}_0\mathcal{N}_0^4}\,\langle\langle \hat{k}_1, \hat{\epsilon}_1 |\, V(x=1; \hat{k}_2, \hat{\epsilon}_2)\alpha^{\mu}_{-1}|\hat{q}^*\rangle &= \epsilon^{\mu_1}_{a_1}\epsilon^{\mu_2}_{a_2} V^{(123)}_{\mu_1;\ \mu_2;\ \mu}(k_1, k_2, q^*) \\ \sqrt{-\mathcal{C}_0\mathcal{N}_0^4}\,\langle\langle \hat{q}^*|\alpha^{\nu}_1\, V(x=1; \hat{k}_3, \hat{\epsilon}_4)\, |\hat{k}_4, \hat{\epsilon}_4\rangle &= V^{(123)}_{\nu;\ \mu_3;\ \mu_4}(-q^*, k_3, k_4)\, \epsilon^{\mu_3}_{a_3}\epsilon^{\mu_4}_{a_4}. \end{aligned} \tag{49}$$

As discussed in the previous section this is not possible since the string truncated Green function is not cyclically invariant while the color ordered vertex is, the proper expressions are

[10] This does not mean that the gauge fixing is the usual Lorentz gauge but only that the linear part of the gauge fixing is the usual Lorentz gauge.

[11] At first sight the choice of $\sqrt{-\mathcal{C}_0\mathcal{N}_0^4}$ seems quite odd and the choice $\sqrt{+\mathcal{C}_0\mathcal{N}_0^4}$ would seem more natural but it is the proper one when considering the results of the comparison of the 3 vertex 34.

$$\sqrt{-\mathcal{C}_0\mathcal{N}_0^4}\langle\langle\hat{k}_1,\hat{\epsilon}_1|\ V(x=1;\hat{k}_2,\hat{\epsilon}_2)\alpha_{\mu\,-1}|\hat{q}^*\rangle$$
$$= \epsilon_{a_1}^{\mu_1}\epsilon_{a_2}^{\mu_2}V^{(123)}_{\mu_1;\ \mu_2;\ \mu}(k_1,k_2,q^*) + O(q^*{}_\mu) + O(q^{*2})$$
$$\sqrt{-\mathcal{C}_0\mathcal{N}_0^4}\langle\langle\hat{q}^*|\alpha_{\nu 1}\ V(x=1;k_3,\epsilon_4)\ |k_4,\epsilon_4\rangle$$
$$= V^{(123)}_{\nu;\ \mu_3;\ \mu_4}(-q,k_3,k_4)\ \epsilon_{a_3}^{\mu_3}\epsilon_{a_4}^{\mu_4} + O(q^*{}_\mu) + O(q^{*2}), \tag{50}$$

where the terms in $V^{(123)}$ proportional to $k_1\cdot k_2$ contribute as q^{*2} because of momentum conservation. It is possible to use the previous less restrictive identification since for example $q^{*\mu}V^{(123)}_{\nu;\ \mu_3;\ \mu_4}(-q^*,k_3,k_4)\ \epsilon_{a_3}^{\mu_3}\epsilon_{a_4}^{\mu_4}\propto q^{*2}$ so that the propagator pole is canceled. This happens because $q^{*\mu}V^{(123)}_{\nu;\ \mu_3;\ \mu_4}=0$ when we take $\epsilon=q^*$ and the gluon physical, i.e. $q^2=0$ because of gauge invariance.

Finally we get the constraints[12]

$$c_1-c_2=\frac{3c_3-3c_4+3c_5-3c_6}{(2\alpha')}=-\imath\mathcal{C}_0\mathcal{N}_0^3(2\alpha')^{2-\frac{1}{2}D}=\sqrt{-\mathcal{C}_0\mathcal{N}_0^4}(2\alpha')^{2-\frac{1}{2}D}. \tag{51}$$

No constraints are obtained on the other coefficients since all of them contribute terms proportional to q^{*2}.

It is also interesting and consistent with the previous line of thought to consider what happens when the gluons 1 and 4 are not physical. In this case the difference in eq. (46) must be a sum of terms proportional to one of the following factors $\hat{k}_1^2$, $\hat{k}_4^2$, $\hat{\epsilon}_1\cdot\hat{k}_1$ or $\hat{\epsilon}_4\cdot\hat{k}_4$ since these are vanishing when the particles are physical. A direct computation reveals that all of these terms are actually present. This means that the string truncated partially off shell $N=4$ Green function when subtracted the Feynman diagrams still has poles. This seems wrong but it is not so. The reason is that using the naive factorization we cannot compare directly the truncated Green functions since the $N=3$ truncated Green functions do not match perfectly between string theory and the usual EFT. Nevertheless the S matrix elements must match and not only the full S matrix but also the color ordered sub-pieces.

4.2. Computing the contact terms up to k^2 order

In order to compute the $N=4$ color ordered vertices we need to compute the usual string amplitude and then expand in momentum powers. We write the basic amplitude as

$$A_{1234}=\int_0^1 dy\ \langle\langle\hat{k}_1,\hat{\epsilon}_1|\ V(x=1;\hat{k}_2,\hat{\epsilon}_2)\ y^{L_0-2}\ V(x=1;\hat{k}_3,\hat{\epsilon}_4)\ |\hat{k}_4,\hat{\epsilon}_4\rangle$$
$$=\int_0^1 dy\ \langle\langle\hat{k}_1,\hat{\epsilon}_1|\ V(x=1;\hat{k}_2,\hat{\epsilon}_2)\ y^{L_0-2}\ V(x=1;\hat{k}_3,\hat{\epsilon}_4)\ |\hat{k}_4,\hat{\epsilon}_4\rangle$$
$$=\int_0^1 dy\ \langle\langle\hat{k}_1,\hat{\epsilon}_1|\ V(1;\hat{k}_2,\hat{\epsilon}_2)\ y^{(\hat{k}_3^2+1)-2}\ V(y;\hat{k}_3,\hat{\epsilon}_4)\ y^{L_0}|\hat{k}_4,\hat{\epsilon}_4\rangle \tag{52}$$

[12] From these equations and eq. (34) it follows that $\mathcal{C}_0\mathcal{N}_0^2=(2\alpha')^{\frac{1}{2}D-1}$ and then $\mathcal{N}_0=\frac{g}{\alpha'}$ and $\mathcal{C}_0=1/(2g)^2(2\alpha')^{\frac{1}{2}D+1}$.

The explicit expression for this contribution to the amplitude is given

$$
\begin{aligned}
A_{1234} = &+ \Big[\left(1 - \frac{1}{2}\hat{s} - \frac{1}{2}\hat{u}\right) C_{(0,0)} + \frac{1}{-\hat{s}/2} C_{(1,0)} + \frac{1}{-\hat{u}/2} C_{(0,1)} \\
&- \left(1 - \frac{\hat{u}}{\hat{s}}\right) \frac{1}{1-\hat{s}/2} C_{(2,0)} - \left(1 - \frac{\hat{s}}{\hat{u}}\right) \frac{1}{1-\hat{u}/2} C_{(0,2)} \Big] \\
&\times \frac{\Gamma\left(1 - \frac{1}{2}\hat{s}\right) \Gamma\left(1 - \frac{1}{2}\hat{u}\right)}{\Gamma\left(1 - \frac{1}{2}\hat{s} - \frac{1}{2}\hat{u}\right)} \delta_{\sum \hat{k}},
\end{aligned} \tag{53}
$$

where the coefficients $C_{(\cdot,\cdot)}$ are given in eqs. (88), (84), (80), (82), (86) in Appendix C. In order to compare with the EFT we need to expand the previous expression in momentum powers, explicitly we get

$$
\begin{aligned}
A_{1234} = + \Big\{ \quad &+ \Big[- C_{(2,0)}|_{k^0} \frac{u}{s} - C_{(0,2)}|_{k^0} \frac{s}{u} + C_{(1,0)}|_{k^2} \frac{1}{-s/2} + C_{(0,1)}|_{k^2} \frac{1}{-u/2} \\
&\quad - C_{(2,0)}|_{k^0} - C_{(0,2)}|_{k^0} + C_{(0,0)}|_{k^0} \Big] \\
&+ \Big[- C_{(2,0)}|_{k^2} \frac{u}{s} - C_{(0,2)}|_{k^2} \frac{s}{u} + C_{(1,0)}|_{k^4} \frac{1}{-s/2} + C_{(0,1)}|_{k^4} \frac{1}{-u/2} \\
&\quad + \left(-C_{(2,0)}|_{k^0} - C_{(0,2)}|_{k^0} + C_{(0,0)}|_{k^0}\right) \left(-\frac{1}{2}s - \frac{1}{2}u\right) \\
&\quad - C_{(2,0)}|_{k^2} - C_{(0,2)}|_{k^2} + C_{(0,0)}|_{k^2} \Big] \\
&+ \Big[- C_{(2,0)}|_{k^4} \frac{u}{s} - C_{(0,2)}|_{k^4} \frac{s}{u} \\
&\quad - C_{(2,0)}|_{k^4} - C_{(0,2)}|_{k^4} + C_{(0,0)}|_{k^4} \\
&\quad + \Big(-C_{(2,0)}|_{k^2} - C_{(0,2)}|_{k^2} + C_{(0,0)}|_{k^2} \\
&\quad + (\Gamma'(1)^2 - \Gamma''(1)) C_{(0,1)}|_{k^2}\Big) \frac{-s}{2} \\
&\quad + \Big(-C_{(2,0)}|_{k^2} - C_{(0,2)}|_{k^2} + C_{(0,0)}|_{k^2} \\
&\quad + (\Gamma'(1)^2 - \Gamma''(1)) C_{(1,0)}|_{k^2}\Big) \frac{-u}{2} \\
&\quad - \left(C_{(2,0)}|_{k^0} + (\Gamma'(1)^2 - \Gamma''(1)) C_{(0,2)}|_{k^0}\right) \left(\frac{-s}{2}\right)^2 \\
&\quad - \left(C_{(0,2)}|_{k^0} + (\Gamma'(1)^2 - \Gamma''(1)) C_{(2,0)}|_{k^0}\right) \left(\frac{-u}{2}\right)^2 \\
&\quad - \Big((\Gamma'(1)^2 - \Gamma''(1) + 1)\left(-C_{(2,0)}|_{k^0} - C_{(0,2)}|_{k^0} + C_{(0,0)}|_{k^0}\right) \\
&\quad - C_{(0,0)}\Big) \frac{-s}{2} \frac{-u}{2} \Big] \\
&+ O(k^6) \Big\} \delta_{\sum \hat{k}},
\end{aligned} \tag{54}
$$

where $C_{(\cdot,\cdot)}|_{k^n}$ stands for the part with n momentum powers in the coefficient $C_{(\cdot,\cdot)}$.

Since now we are considering the string amplitude for all possible values of the momenta we must subtract all the Feynman diagrams built with color ordered vertices which have the

proper color ordering and poles in the same channels of the string amplitude, both s and u for the amplitude A_{1234}. When canceling the poles we get again eqs. (51). The explicit computation gives at k^0 order

$$\begin{aligned}
&\alpha' \mathcal{C}_0 \mathcal{N}_0^4 A_{1234}|_{k^0} \\
&-\epsilon_{a_1}^{\mu_1}\epsilon_{a_2}^{\mu_2} V^{(123)}_{\mu_1;\ \mu_2;\ \mu}(k_1,k_2,q^*)|_{k^1}\frac{\delta^{\mu\nu}}{q^{*2}} V^{(123)}_{\nu;\ \mu_3;\ \mu_4}(-q^*,k_3,k_4)|_{k^1}\ \epsilon_{a_3}^{\mu_3}\epsilon_{a_4}^{\mu_4}\delta_{\sum k_i} \\
&-\ \epsilon_{a_4}^{\mu_4}\epsilon_{a_1}^{\mu_1} V^{(123)}_{\mu_4;\ \mu_1;\ \mu}(k_4,k_1,q^*)|_{k^1}\frac{\delta^{\mu\nu}}{q^{*2}} V^{(123)}_{\nu;\ \mu_2;\ \mu_3}(-q^*,k_2,k_3)|_{k^1}\ \epsilon_{a_2}^{\mu_2}\epsilon_{a_3}^{\mu_3}\delta_{\sum k_i} \\
&=\Big[-c_1\, c_2\ (\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot\epsilon_4+\epsilon_1\cdot\epsilon_4\,\epsilon_2\cdot\epsilon_3) \\
&\quad -\frac{1}{2}(c_1-c_2)^2\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot\epsilon_4\Big]\delta_{\sum k_i},
\end{aligned} \tag{55}$$

along with the constraint from pole cancellation

$$(c_1-c_2)^2 = -\mathcal{C}_0\mathcal{N}_0^4(2\alpha')^{3-\frac{1}{2}D}. \tag{56}$$

Notice that the previous expression is cyclically invariant therefore we can interpret it as the quartic color ordered vertex at order k^0.

From this expression it is then clear that the choice $c_2=0$ (or $c_1=0$) is the most economical. This is exactly the choice suggested by the string.

The previous color ordered vertex at order k^0 then becomes in the gauge suggested by the string

$$V^{(1234)}_{1234}(k_1,k_2,k_3,k_4)|_{k^0} = +2g^2\Big\{\epsilon_1\cdot\epsilon_3\,\epsilon_2\epsilon_4\Big\}\delta_{\sum k_i}, \tag{57}$$

which is the color ordered vertex depicted in Fig. 1.

The quartic vertex at k^0 order reads in general

$$\begin{aligned}
&V_{[4]}(k_1,k_2,k_3,k_4)|_{k^0} = \\
&\quad\Big\{+\epsilon_{a_1}\cdot\epsilon_{a_2}\,\epsilon_{a_3}\cdot\epsilon_{a_4}\Big[-c_1\,c_2\,tr\left(\{T_{a_1},T_{a_2}\}\{T_{a_3},T_{a_4}\}\right) \\
&\qquad\qquad -\frac{1}{2}(c_1-c_2)^2 tr\left(T_{a_1}T_{a_4}T_{a_2},T_{a_3}+T_{a_1}T_{a_3}T_{a_2},T_{a_4}\right)\Big] \\
&\quad+\epsilon_{a_1}\cdot\epsilon_{a_3}\,\epsilon_{a_2}\cdot\epsilon_{a_4}\Big[-c_1\,c_2\,tr\left(\{T_{a_1},T_{a_3}\}\{T_{a_2},T_{a_4}\}\right) \\
&\qquad\qquad -\frac{1}{2}(c_1-c_2)^2 tr\left(T_{a_1}T_{a_4}T_{a_3},T_{a_2}+T_{a_1}T_{a_2}T_{a_2},T_{a_4}\right)\Big] \\
&\quad+\epsilon_{a_1}\cdot\epsilon_{a_4}\,\epsilon_{a_2}\cdot\epsilon_{a_3}\Big[-c_1\,c_2\,tr\left(\{T_{a_1},T_{a_4}\}\{T_{a_2},T_{a_3}\}\right) \\
&\quad-\qquad\qquad\qquad\frac{1}{2}(c_1-c_2)^2 tr\left(T_{a_1}T_{a_3}T_{a_4},T_{a_2}+T_{a_1}T_{a_2}T_{a_4},T_{a_3}\right)\Big]\Big\}\delta_{\sum k_i}.
\end{aligned} \tag{58}$$

The explicit computation at k^2 order requires

$$(c_1-c_2)(-3c_3+3c_4+3c_5-3c_6) = \mathcal{C}_0\mathcal{N}_0^4(2\alpha')^{4-\frac{1}{2}D} \tag{59}$$

because of pole cancellation and gives

$$
\begin{aligned}
\alpha' \mathcal{C}_0 \mathcal{N}_0^4 A_{1234}|_{k^2} &- \frac{\delta^{\mu\nu}}{q^{*2}} \Big[\epsilon_{a_1}^{\mu_1} \epsilon_{a_2}^{\mu_2} V^{(123)}_{\mu_1;\ \mu_2;\ \mu}(k_1, k_2, q^*) \\
&\qquad \times V^{(123)}_{\nu;\ \mu_3;\ \mu_4}(-q^*, k_3, k_4)\, \epsilon_{a_3}^{\mu_3} \epsilon_{a_4}^{\mu_4} \Big]\Big|_{k^4} \delta_{\sum k_i} \\
&- \frac{\kappa\, \delta^{\mu\nu}}{q^{*2}} \Big[\epsilon_{a_4}^{\mu_4} \epsilon_{a_1}^{\mu_1} V^{(123)}_{\mu_4;\ \mu_1;\ \mu}(k_4, k_1, q^*) \\
&\qquad \times V^{(123)}_{\nu;\ \mu_2;\ \mu_3}(-q^*, k_2, k_3)\, \epsilon_{a_2}^{\mu_2} \epsilon_{a_3}^{\mu_3} \Big]\Big|_{k^4} \delta_{\sum k_i} \\
= \Big\{ &- \frac{1}{2} [c_1(-3c_4 - c_5 + 2c_6) + c_2(-3c_3 + 2c_5 - c_6)] \\
&\qquad \times [\epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot k_4\, \epsilon_4 \cdot k_3 + 3 \text{ terms obtained by cycling (1234) in the} \\
&\qquad\quad \text{previous term}] \\
&+ \Big\{ + \frac{1}{2} \epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot \epsilon_4 \Big[\Big(\frac{1}{2} (2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 \Big) s \\
&\qquad\qquad + \Big(\frac{1}{2} (2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 + c_1 c_9 + c_2 c_{12} \Big) u \\
&\qquad\qquad + \Big(\frac{1}{2} (2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 + c_1 c_{12} + c_2 c_9 \Big) t \Big] \\
&\qquad + 1 \text{ term obtained by cycling (1234) in the previous term} \Big\} \\
&- \frac{1}{2} \epsilon_3 \cdot \epsilon_1\, \epsilon_2 \cdot \epsilon_4 \Big[\Big(\frac{1}{2} (2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 \Big) u + \Big(\frac{1}{2} (2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 \Big) s \Big] \\
&+ \Big\{ + \epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot k_1\, \epsilon_4 \cdot k_1 \frac{1}{2} \Big[(2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 + c_1(-3c_4 - c_5 + 2c_6) \\
&\qquad\qquad + c_2(-3c_3 + 2c_5 - c_6 - 2c_7 + 2c_8 + 2c_{10} - 2c_{11}) \Big] \\
&\quad + \epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot k_1\, \epsilon_4 \cdot k_2 \frac{1}{2} \Big[(2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 + c_1(-3c_4 - c_5 + 2c_6 \\
&\qquad\qquad + 2c_8 - 2c_{11}) + c_2(-3c_3 + 2c_5 - c_6 - 2c_7 + 2c_{10}) \Big] \\
&\quad + \epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot k_1\, \epsilon_4 \cdot k_2 \frac{1}{2} \Big[(2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 \\
&\qquad\qquad + c_1(-3c_4 - c_5 + 2c_6 - 4c_7 - 2c_8 + 4c_9 + 2c_{10} \\
&\qquad\qquad + 2c_{11} - 2c_{12}) + c_2(-3c_3 + 2c_5 - c_6 + 2c_7 \\
&\qquad\qquad + 4c_8 - 2c_9 - 2c_{10} - 4c_{11} + 4c_{12}) \Big] \\
&\quad + \epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot k_2\, \epsilon_4 \cdot k_2 \frac{1}{2} \Big[(2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 \\
&\qquad\qquad + c_1(-3c_4 - c_5 + 2c_6 - 2c_7 + 2c_8) \\
&\qquad\qquad + c_2(-3c_3 + 2c_5 - c_6) \Big]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 3{*}4 \text{ terms obtained by cycling (1234) in the previous 4 terms}\Big\} \\
&+\Big\{+\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot k_1\,\epsilon_4\cdot k_1\Big[-\frac{1}{2}(2\alpha')^{4-\frac{1}{2}D}\mathcal{C}_0\mathcal{N}_0^4\Big] \\
&\quad +\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot k_1\,\epsilon_4\cdot k_3\Big[-\frac{1}{2}(2\alpha')^{4-\frac{1}{2}D}\mathcal{C}_0\mathcal{N}_0^4 \\
&\qquad\qquad -(c_1-c_2)(-2c_7+c_9+2c_{10}-c_{12})\Big] \\
&\quad +\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot k_3\,\epsilon_4\cdot k_1\Big[-\frac{1}{2}(2\alpha')^{4-\frac{1}{2}D}\mathcal{C}_0\mathcal{N}_0^4 \\
&\qquad\qquad +(c_1-c_2)(+2c_7-c_9-2c_{11}+c_{12})]\Big] \\
&\quad +\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot k_3\,\epsilon_4\cdot k_3\Big[-\frac{1}{2}(2\alpha')^{4-\frac{1}{2}D}\mathcal{C}_0\mathcal{N}_0^4\Big] \\
&\quad + 4 \text{ terms obtained by cycling (1234) in the previous 4 terms}\Big\}\Big\}\delta_{\sum k_i}.
\end{aligned}
\tag{60}
$$

Again the previous result can be interpreted as the $N=4$ gluon color ordered vertex since it is cyclically invariant.

Moreover the suggestion of string theory is the more economical since all terms coming from color ordered vertices vanish when only c_1 and c_3 are different from zero. This can also be understood by the fact that the 3 gluon color ordered vertex suggested by the string has the minimal contain to cancel the poles in the 4 gluons amplitude.

The previous color ordered vertex at order k^2 becomes in the gauge suggested by the string

$$
\begin{aligned}
&V_{1234}^{(1234)}(k_1,k_2,k_3,k_4)|_{k^2} = -4\alpha' g^2\Big\{ \\
&\quad +\Big[\epsilon_1\cdot\epsilon_2\,\epsilon_3\epsilon_4 + 1 \text{ term from cycling (1234)}\Big]\frac{t}{2} \\
&\quad +\Big[\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot\epsilon_4\Big]\frac{t}{2} \\
&\quad -\Big[+\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_3 + 4 \text{ terms from cycling (1234)}\Big] \\
&\quad +\Big[+\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot k_4\,\epsilon_4\cdot k_2 + 1 \text{ term from cycling (1234)}\Big]\Big\}\delta_{\sum k_i}.
\end{aligned}
\tag{61}
$$

Finally the full the quartic vertex at k^2 order in string gauge reads

$$
\begin{aligned}
&V_{[4]}|_{k^2} = \\
&\quad\Big\{+\ V_{a_1a_2a_3a_4}^{(1234)}\,tr\left(T_{a_1}\,T_{a_4}\,T_{a_2},T_{a_3}+T_{a_1}\,T_{a_3}\,T_{a_2},T_{a_4}\right) \\
&\quad + V_{a_1a_4a_2a_3}^{(1234)}\,tr\left(T_{a_1}\,T_{a_4}\,T_{a_3},T_{a_2}+T_{a_1}\,T_{a_2}\,T_{a_2},T_{a_4}\right) \\
&\quad + V_{a_1a_3a_4a_2}^{(1234)}\,tr\left(T_{a_1}\,T_{a_3}\,T_{a_4},T_{a_2}+T_{a_1}\,T_{a_2}\,T_{a_4},T_{a_3}\right)\Big\}\delta_{\sum k_i}
\end{aligned}
\tag{62}
$$

because $V_{1234}^{(1234)} = V_{1432}^{(1234)}$. We have also substituted $\epsilon_i \to \epsilon_{a_i}$. This expression is by far simpler than the one obtained in the usual Feynman gauge where a lot of c.s are different from zero in eq. (60).

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A. Conventions

We write the open string expansion for the dimensionless field $\hat{X} = \sqrt{2\alpha'}X$ as

$$\hat{X}^\mu(u,\bar{u}) = \frac{1}{2}(\hat{X}_L^\mu(u) + \hat{X}_R^\mu(\bar{u})) \tag{63}$$

with

$$u = e^{\tau_E + \imath\sigma} \in \mathbb{H}$$

and

$$\begin{aligned}\hat{X}_L(u) &= \hat{x}_0 + \hat{y}_0 - \imath\alpha_0 \ \ln(u) + \imath \sum_{n\neq 0} \frac{\alpha_n}{n} u^{-n} \\ \hat{X}_R(\bar{u}) &= \hat{x}_0 - \hat{y}_0 - \imath\alpha_0 \ \ln(\bar{u}) + \imath \sum_{n\neq 0} \frac{\alpha_n}{n} \bar{u}^{-n}\end{aligned} \tag{64}$$

The commutation relations read

$$[\alpha_n^\mu,\ \alpha_m^\nu] = n\ \delta^{\mu\nu}\ \delta_{m+n,0}. \tag{65}$$

The mass shell condition reads

$$L_0^{(X)}|phys\rangle = (\frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n)|phys\rangle = |phys\rangle. \tag{66}$$

The momentum states are defined as

$$e^{i\hat{k}\cdot\hat{x}_0}|0\rangle = |k\rangle, \quad \langle 0|e^{-i\hat{k}\cdot\hat{x}_0} = \langle\langle k| = \langle -k|. \tag{67}$$

B. YM conventions

The Euclidean YM Lagrangian reads

$$\mathcal{L}_E = +\frac{1}{4\kappa} tr(F_{\mu\nu}F_{\mu\nu}) = +\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a, \tag{68}$$

since we normalize the generators $T^a = T^{a\dagger}$ as

$$tr(T_a T_b) = \kappa\delta_{ab}, \quad [T_a, T_b] = i f_{abc} T^c. \tag{69}$$

It then follows that $tr(T_a[T_b, T_c]) = i\kappa f_{abc}$. We define the field strength of the gauge field $A = A_\mu dx^\mu = A_\mu^a T^a dx^\mu$ as

$$
\begin{aligned}
F &= dA - igA \wedge A = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu .
\end{aligned} \tag{70}
$$

The Lagrangian then becomes

$$
\begin{aligned}
\mathcal{L}_E =& \frac{1}{2\kappa} tr\Big(\partial_\mu A_\nu (\partial_\mu A_\nu - \partial_\nu A_\mu)\Big) \\
&+ \frac{-ig}{\kappa} tr\Big(\partial_\mu A_\nu [A_\mu, A_\nu]\Big) \\
&+ \frac{(-ig)^2}{4\kappa} tr\Big([A_\mu, A_\nu][A_\mu, A_\nu]\Big).
\end{aligned} \tag{71}
$$

Let us rewrite the cubic interaction term in momentum space

$$
\begin{aligned}
-S_{E\,[3]} =& -\int \prod_{i=1}^{3} \frac{d^D k_i}{(2\pi)^D} (2\pi)^D \delta^D(k_1 + k_2 + k_3) \times \frac{ig}{\kappa} tr(T_{a_1}[T_{a_2}, T_{a_3}])(ik_{1\mu_2})\delta_{\mu_2\mu_3} \\
&\times \epsilon_{a_1\mu_1}(k_1)\epsilon_{a_2\mu_2}(k_2)\epsilon_{a_3\mu_3}(k_3) \\
=& \int \prod_{i=1}^{3} \frac{d^D k_i}{(2\pi)^D} \delta_{\sum k} \times \frac{1}{3!} \frac{-g}{\kappa} tr(T_{a_1}[T_{a_2}, T_{a_3}]) \\
&\times [(k_{1\mu_2} - k_{3\mu_2})\delta_{\mu_3\mu_1} + (k_{3\mu_1} - k_{2\mu_1})\delta_{\mu_2\mu_3} + (k_{2\mu_3} - k_{1\mu_3})\delta_{\mu_1\mu_2}] \\
&\times \epsilon_{a_1\mu_1}(k_1)\epsilon_{a_2\mu_2}(k_2)\epsilon_{a_3\mu_3}(k_3),
\end{aligned} \tag{72}
$$

then it follows that

$$
\begin{aligned}
V_{a_1\mu_1;a_2\mu_2;a_3\mu_3}(k_1, k_2, k_3) =& V^{(123)}_{a_1\mu_1;a_2\mu_2;a_3\mu_3}(k_1, k_2, k_3) + V^{(123)}_{a_1\mu_1;a_3\mu_3;a_2\mu_2}(k_1, k_3, k_2) \\
=& \frac{-g}{\kappa} tr(T_{a_1}[T_{a_2}, T_{a_3}]) \\
&\times [(k_{1\mu_2} - k_{3\mu_2})\delta_{\mu_3\mu_1} + (k_{3\mu_1} - k_{2\mu_1})\delta_{\mu_2\mu_3} \\
&+ (k_{2\mu_3} - k_{1\mu_3})\delta_{\mu_1\mu_2}]\delta_{\sum k},
\end{aligned} \tag{73}
$$

and

$$
\begin{aligned}
V^{(123)}_{a_1\mu_1;a_2\mu_2;a_3\mu_3}(k_1, k_2, k_3) =& \frac{-g}{\kappa} tr(T_{a_1} T_{a_2} T_{a_3}) \\
&\times [(k_{1\mu_2} - k_{3\mu_2})\delta_{\mu_3\mu_1} + (k_{3\mu_1} - k_{2\mu_1})\delta_{\mu_2\mu_3} \\
&+ (k_{2\mu_3} - k_{1\mu_3})\delta_{\mu_1\mu_2}]\delta_{\sum k}.
\end{aligned} \tag{74}
$$

Similarly we write the quartic action as

$$
\begin{aligned}
-S_{E\,[4]} =& -\int \prod_{i=1}^{4} \frac{d^D k_i}{(2\pi)^D} (2\pi)^D \delta^D(k_1 + k_2 + k_3 + k_3) \\
&\times \frac{(-ig)^2}{4\kappa} tr([T_{a_1}, T_{a_2}][T_{a_3}, T_{a_4}])\delta_{\mu_1[\mu_3}\delta_{\mu_4]\mu_2}
\end{aligned}
$$

$$\times\, \epsilon_{a_1\mu_1}(k_1)\epsilon_{a_2\mu_2}(k_2)\epsilon_{a_3\mu_3}(k_3)\epsilon_{a_4\mu_4}(k_4), \tag{75}$$

the using the symmetries $1 \leftrightarrow 2$, $3 \leftrightarrow 4$ and $(1,2) \leftrightarrow (3,4)$ we sum over the remaining $4!/2^3 = 3$. Using the previous symmetries we can always set 1 in the first place of the permutation and then we are left with 1234, 1342 and 1423. So we get

$$\begin{aligned}
&V_{a_1\mu_1;a_2\mu_2;a_3\mu_3;a_4\mu_4}(k_1,k_2,k_3,k_4) = \\
&= V^{(1234)}_{a_1\mu_1;a_2\mu_2;a_3\mu_3;a_4\mu_4}(k_1,k_2,k_3,k_4) + V^{(1234)}_{a_1\mu_1;a_3\mu_3;a_4\mu_4;a_2\mu_2}(k_1,k_3,k_4,k_2) \\
&\qquad + V^{(1234)}_{a_1\mu_1;a_4\mu_4;a_2\mu_2;a_3\mu_3}(k_1,k_4,k_2,k_3) \\
&= \frac{2g^2}{\kappa}\Big[tr([T_{a_1},T_{a_2}][T_{a_3},T_{a_4}])\delta_{\mu_1[\mu_3}\delta_{\mu_4]\mu_2} + tr([T_{a_1},T_{a_3}][T_{a_4},T_{a_2}])\delta_{\mu_1[\mu_4}\delta_{\mu_2]\mu_3} \\
&\qquad + tr([T_{a_1},T_{a_4}][T_{a_2},T_{a_3}])\delta_{\mu_1[\mu_2}\delta_{\mu_3]\mu_4}\Big]\delta_{\sum k},
\end{aligned} \tag{76}$$

from which it follows

$$\begin{aligned}
&V^{(1234)}_{a_1\mu_1;a_2\mu_2;a_3\mu_3;a_4\mu_4}(k_1,k_2,k_3,k_4) = \\
&\quad = \frac{g^2}{\kappa} tr(T_{a_1}T_{a_2}T_{a_3}T_{a_4})\Big[2\delta_{\mu_1\mu_3}\delta_{\mu_4\mu_2} - \delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4} - \delta_{\mu_1\mu_4}\delta_{\mu_2\mu_3}\Big]\delta_{\sum k}.
\end{aligned} \tag{77}$$

C. Details on $N = 4$ gluons correlator

In this section we do not write the hat explicitly in order to make the notation lighter, i.e. $\hat{k}$ is simply written as k.

The direct computation of the correlator gives the following result

$$\begin{aligned}
A_{1234} &= + \int_0^1 dy\; C\delta_{k_1+k_2+k_3+k_4} \\
C &= C_{(0,0)} + \frac{1}{y}C_{(1,0)} + \frac{1}{y^2}C_{(2,0)} + \frac{1}{1-y}C_{(0,1)} + \frac{1}{(1-y)^2}C_{(0,2)},
\end{aligned} \tag{78}$$

after we write the rational expressions involving y as sum of simple factors, e.g. $y/(1-y) = 1 - 1/(1-y)$. The different contributions are given as follows.

Terms proportional to y^{-1}:

$$\begin{aligned}
C_{(1,0)} = &- \epsilon_1\cdot k_3\,\epsilon_2\cdot k_4\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_3 \\
&- \epsilon_1\cdot k_3\,\epsilon_2\cdot k_3\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_3 \\
&- \epsilon_1\cdot k_2\,\epsilon_2\cdot k_3\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_3 \\
&+ \epsilon_1\cdot k_2\,\epsilon_2\cdot k_4\,\epsilon_3\cdot k_2\,\epsilon_4\cdot k_3 \\
&+ \epsilon_1\cdot k_2\,\epsilon_2\cdot k_3\,\epsilon_3\cdot k_2\,\epsilon_4\cdot k_3 \\
&- \epsilon_1\cdot k_2\,\epsilon_2\cdot k_4\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_2 \\
&- \epsilon_1\cdot k_2\,\epsilon_2\cdot k_3\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_2 \\
&- \epsilon_1\cdot \epsilon_2\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_2
\end{aligned}$$

$$
\begin{aligned}
&+\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot k_2\,\epsilon_4\cdot k_3\\
&-\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot k_4\,\epsilon_4\cdot k_3\\
&-\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot k_3\,\epsilon_4\cdot k_3\\
&+\epsilon_1\cdot\epsilon_4\,\epsilon_2\cdot k_3\,\epsilon_3\cdot k_4\\
&+\epsilon_1\cdot\epsilon_4\,\epsilon_2\cdot k_4\,\epsilon_3\cdot k_4\\
&-\epsilon_1\cdot k_2\,\epsilon_2\cdot\epsilon_3\,\epsilon_4\cdot k_3\\
&+\epsilon_1\cdot k_2\,\epsilon_2\cdot\epsilon_4\,\epsilon_3\cdot k_4\\
&+\epsilon_1\cdot k_3\,\epsilon_2\cdot k_4\,\epsilon_3\cdot\epsilon_4\\
&+\epsilon_1\cdot k_3\,\epsilon_2\cdot k_3\,\epsilon_3\cdot\epsilon_4\\
&+\epsilon_1\cdot k_2\,\epsilon_2\cdot k_3\,\epsilon_3\cdot\epsilon_4
\end{aligned}
\tag{79}
$$

This can be simplified to[13]

$$
\begin{aligned}
C_{(1,0)} =&+\epsilon_1\cdot k_3\wedge k_4\cdot\epsilon_2\,\epsilon_3\cdot k_3\wedge k_4\cdot\epsilon_4\\
&+\epsilon_1\cdot k_1\wedge k_2\cdot\epsilon_2\,\epsilon_3\cdot k_1\wedge k_2\cdot\epsilon_4\\
&+\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot k_1\wedge k_2\cdot\epsilon_4\\
&+\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot k_1\,\epsilon_4\cdot k_3\\
&-\epsilon_1\cdot\epsilon_4\,\epsilon_2\cdot k_1\,\epsilon_3\cdot k_4\\
&-\epsilon_2\cdot\epsilon_3\,\epsilon_1\cdot k_2\,\epsilon_4\cdot k_3\\
&+\epsilon_2\cdot\epsilon_4\,\epsilon_1\cdot k_2\,\epsilon_3\cdot k_4\\
&+\epsilon_3\cdot\epsilon_4\,\epsilon_1\cdot k_3\wedge k_4\cdot\epsilon_2
\end{aligned}
\tag{80}
$$

Terms proportional to y^{-2}:

$$
\begin{aligned}
C_{(2,0)} =&-\epsilon_1\cdot k_2\,\epsilon_2\cdot k_4\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_3\\
&-\epsilon_1\cdot k_2\,\epsilon_2\cdot k_3\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_3\\
&-\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_3\\
&+\epsilon_1\cdot k_2\,\epsilon_2\cdot k_4\,\epsilon_3\cdot\epsilon_4\\
&+\epsilon_1\cdot k_2\,\epsilon_2\cdot k_3\,\epsilon_3\cdot\epsilon_4\\
&+\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot\epsilon_4
\end{aligned}
\tag{81}
$$

This can be simplified to

$$
\begin{aligned}
C_{(2,0)} =&+\epsilon_1\cdot k_2\,\epsilon_2\cdot k_1\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_3\\
&-\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_3\\
&-\epsilon_3\cdot\epsilon_4\,\epsilon_1\cdot k_2\,\epsilon_2\cdot k_1\\
&+\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot\epsilon_4
\end{aligned}
\tag{82}
$$

[13] We use $\epsilon_1\cdot k_3\wedge k_4\cdot\epsilon_2=\epsilon_1\cdot k_3\,k_4\cdot\epsilon_2-\epsilon_1\cdot k_4\,k_3\cdot\epsilon_2$.

Terms proportional to $(1-y)^{-1}$:

$$
\begin{aligned}
C_{(0,1)} = & -\epsilon_1\cdot k_3\,\epsilon_2\cdot k_3\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_3 \\
& -\epsilon_1\cdot k_2\,\epsilon_2\cdot k_3\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_3 \\
& +\epsilon_1\cdot k_3\,\epsilon_2\cdot k_4\,\epsilon_3\cdot k_2\,\epsilon_4\cdot k_3 \\
& +\epsilon_1\cdot k_2\,\epsilon_2\cdot k_4\,\epsilon_3\cdot k_2\,\epsilon_4\cdot k_3 \\
& +\epsilon_1\cdot k_2\,\epsilon_2\cdot k_3\,\epsilon_3\cdot k_2\,\epsilon_4\cdot k_3 \\
& -\epsilon_1\cdot k_3\,\epsilon_2\cdot k_3\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_2 \\
& -\epsilon_1\cdot k_2\,\epsilon_2\cdot k_3\,\epsilon_3\cdot k_4\,\epsilon_4\cdot k_2 \\
& +\epsilon_1\cdot k_3\,\epsilon_2\cdot k_4\,\epsilon_3\cdot k_2\,\epsilon_4\cdot k_2 \\
& +\epsilon_1\cdot k_2\,\epsilon_2\cdot k_4\,\epsilon_3\cdot k_2\,\epsilon_4\cdot k_2 \\
& -\epsilon_1\cdot k_3\,\epsilon_2\cdot k_3\,\epsilon_3\cdot k_2\,\epsilon_4\cdot k_2 \\
& +\epsilon_1\cdot \epsilon_2\,\epsilon_3\cdot k_2\,\epsilon_4\cdot k_3 \\
& +\epsilon_1\cdot \epsilon_2\,\epsilon_3\cdot k_2\,\epsilon_4\cdot k_2 \\
& -\epsilon_1\cdot \epsilon_3\,\epsilon_2\cdot k_3\,\epsilon_4\cdot k_2 \\
& -\epsilon_1\cdot \epsilon_3\,\epsilon_2\cdot k_3\,\epsilon_4\cdot k_3 \\
& +\epsilon_1\cdot \epsilon_4\,\epsilon_2\cdot k_3\,\epsilon_3\cdot k_4 \\
& -\epsilon_1\cdot \epsilon_4\,\epsilon_2\cdot k_4\,\epsilon_3\cdot k_2 \\
& -\epsilon_1\cdot k_2\,\epsilon_2\cdot \epsilon_3\,\epsilon_4\cdot k_3 \\
& +\epsilon_1\cdot k_3\,\epsilon_2\cdot \epsilon_3\,\epsilon_4\cdot k_2 \\
& -\epsilon_1\cdot k_3\,\epsilon_2\cdot \epsilon_4\,\epsilon_3\cdot k_2 \\
& -\epsilon_1\cdot k_2\,\epsilon_2\cdot \epsilon_4\,\epsilon_3\cdot k_2 \\
& +\epsilon_1\cdot k_3\,\epsilon_2\cdot k_3\,\epsilon_3\cdot \epsilon_4 \\
& +\epsilon_1\cdot k_2\,\epsilon_2\cdot k_3\,\epsilon_3\cdot \epsilon_4
\end{aligned}
\tag{83}
$$

This can be simplified to

$$
\begin{aligned}
C_{(0,1)} = & -\epsilon_1\cdot k_4\,\epsilon_4\cdot k_1\,\epsilon_2\cdot [k_4\wedge k_1]\cdot \epsilon_3 \\
& -\epsilon_2\cdot k_3\,\epsilon_3\cdot k_2\,\epsilon_1\cdot [k_3\wedge k_2]\cdot \epsilon_4 \\
& -\epsilon_1\cdot \epsilon_2\,\epsilon_3\cdot k_2\,\epsilon_4\cdot k_1 \\
& +\epsilon_1\cdot \epsilon_3\,\epsilon_2\cdot k_3\,\epsilon_4\cdot k_1 \\
& -\epsilon_1\cdot \epsilon_4\,\epsilon_2\cdot [k_1\wedge k_4]\cdot \epsilon_3 \\
& -\epsilon_2\cdot \epsilon_3\,\epsilon_1\cdot [k_2\wedge k_3]\cdot \epsilon_3 \\
& +\epsilon_2\cdot \epsilon_4\,\epsilon_1\cdot k_4\,\epsilon_3\cdot k_2 \\
& -\epsilon_3\cdot \epsilon_4\,\epsilon_1\cdot k_4\,\epsilon_2\cdot k_3
\end{aligned}
\tag{84}
$$

Terms proportional to $(1-y)^{-2}$:

$$
\begin{aligned}
C_{(0,2)} = &+ \epsilon_1 \cdot k_3\, \epsilon_2 \cdot k_3\, \epsilon_3 \cdot k_2\, \epsilon_4 \cdot k_3 \\
&+ \epsilon_1 \cdot k_2\, \epsilon_2 \cdot k_3\, \epsilon_3 \cdot k_2\, \epsilon_4 \cdot k_3 \\
&+ \epsilon_1 \cdot k_3\, \epsilon_2 \cdot k_3\, \epsilon_3 \cdot k_2\, \epsilon_4 \cdot k_2 \\
&+ \epsilon_1 \cdot k_2\, \epsilon_2 \cdot k_3\, \epsilon_3 \cdot k_2\, \epsilon_4 \cdot k_2 \\
&- \epsilon_1 \cdot \epsilon_4\, \epsilon_2 \cdot k_3\, \epsilon_3 \cdot k_2 \\
&- \epsilon_1 \cdot k_3\, \epsilon_2 \cdot \epsilon_3\, \epsilon_4 \cdot k_2 \\
&- \epsilon_1 \cdot k_3\, \epsilon_2 \cdot \epsilon_3\, \epsilon_4 \cdot k_3 \\
&- \epsilon_1 \cdot k_2\, \epsilon_2 \cdot \epsilon_3\, \epsilon_4 \cdot k_3 \\
&- \epsilon_1 \cdot k_2\, \epsilon_2 \cdot \epsilon_3\, \epsilon_4 \cdot k_2 \\
&+ \epsilon_1 \cdot \epsilon_4\, \epsilon_2 \cdot \epsilon_3
\end{aligned} \tag{85}
$$

This can be simplified to

$$
\begin{aligned}
C_{(0,2)} = &+ \epsilon_1 \cdot k_4\, \epsilon_2 \cdot k_3\, \epsilon_3 \cdot k_2\, \epsilon_4 \cdot k_1 \\
&- \epsilon_1 \cdot \epsilon_4\, \epsilon_2 \cdot k_3\, \epsilon_3 \cdot k_2 \\
&- \epsilon_2 \cdot \epsilon_3\, \epsilon_1 \cdot k_4\, \epsilon_4 \cdot k_1 \\
&+ \epsilon_1 \cdot \epsilon_4\, \epsilon_2 \cdot \epsilon_3
\end{aligned} \tag{86}
$$

Terms proportional to 1:

$$
\begin{aligned}
C_{(0,0)} = &- \epsilon_1 \cdot k_3\, \epsilon_2 \cdot k_4\, \epsilon_3 \cdot k_4\, \epsilon_4 \cdot k_2 \\
&- \epsilon_1 \cdot k_3\, \epsilon_2 \cdot k_4\, \epsilon_3 \cdot k_2\, \epsilon_4 \cdot k_2 \\
&- \epsilon_1 \cdot \epsilon_3\, \epsilon_2 \cdot k_4\, \epsilon_4 \cdot k_2 \\
&+ \epsilon_1 \cdot k_3\, \epsilon_2 \cdot \epsilon_4\, \epsilon_3 \cdot k_4 \\
&+ \epsilon_1 \cdot k_3\, \epsilon_2 \cdot \epsilon_4\, \epsilon_3 \cdot k_2 \\
&+ \epsilon_1 \cdot \epsilon_3\, \epsilon_2 \cdot \epsilon_4
\end{aligned} \tag{87}
$$

This can be simplified to

$$
\begin{aligned}
C_{(0,0)} = &+ \epsilon_1 \cdot k_3\, \epsilon_2 \cdot k_4\, \epsilon_3 \cdot k_1\, \epsilon_4 \cdot k_2 \\
&- \epsilon_1 \cdot \epsilon_3\, \epsilon_2 \cdot k_4\, \epsilon_4 \cdot k_2 \\
&- \epsilon_2 \cdot \epsilon_4\, \epsilon_1 \cdot k_3\, \epsilon_3 \cdot k_1 \\
&+ \epsilon_1 \cdot \epsilon_3\, \epsilon_2 \cdot \epsilon_4
\end{aligned} \tag{88}
$$

D. Details on the computation of $V_{[4]}$

The first step is to compute the color ordered vertices with two on shell legs

$$
\begin{aligned}
-\epsilon_1^{\mu_1} \epsilon_2^{\mu_2} V^{(123)}_{\mu_1;\ \mu_2;\ \mu}(k_1, k_2, q^*) = &\imath \Big\{ +[c_2 \epsilon_1 \cdot \epsilon_2] k_{1\mu} + [c_1 \epsilon_1 \cdot \epsilon_2] k_{2\mu} \\
&+ [(c_1 - c_2)\epsilon_2 \cdot k_1]\epsilon_{1\mu} + [-(c_1 - c_2)\epsilon_1 \cdot k_2]\epsilon_{2\mu} \Big\} \\
&+ \imath^3 \Big\{ +[(-3c_3 + 2c_5 - c_6)\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 + c_9 \epsilon_1 \cdot \epsilon_2 k_1 \cdot k_2] k_{1\mu}
\end{aligned}
$$

$$
\begin{aligned}
&+[(-3c_4-c_5+2c_6)\epsilon_1\cdot k_2\epsilon_2\cdot k_1+c_{12}\epsilon_1\cdot\epsilon_2 k_1\cdot k_2]k_{1\mu}\\
&+[(-2c_7+c_9+2c_{10}-c_{12})\epsilon_2\cdot k_1 k_1\cdot k_2]\epsilon_{1\mu}\\
&+[(+2c_8-c_9-2c_{11}+c_{12})\epsilon_1\cdot k_2 k_1\cdot k_2]\epsilon_{2\mu}\Big\}\delta_{\sum k_i}\, q^{*\mu},
\end{aligned}
\tag{89}
$$

and

$$
\begin{aligned}
&-V^{(123)}_{\mu;\ \mu_3;\ \mu_4}(-q^*,k_3,k_4)\epsilon_3^{\mu_3}\epsilon_4^{\mu_4}\\
&=\imath\Big\{+[c_2\epsilon_3\cdot\epsilon_4]k_{3\mu}+[c_1\epsilon_3\cdot\epsilon_4]k_{4\mu}\\
&\quad+[(c_1-c_2)\epsilon_4\cdot k_3]\epsilon_{3\mu}+[-(c_1-c_2)\epsilon_3\cdot k_4]\epsilon_{4\mu}\Big\}\\
&+\imath^3\Big\{+[(-3c_3+2c_5-c_6)\epsilon_3\cdot k_4\epsilon_4\cdot k_3+c_9\epsilon_3\cdot\epsilon_4 k_3\cdot k_4]k_{3\mu}\\
&\quad+[(-3c_4-c_5+2c_6)\epsilon_3\cdot k_4\epsilon_4\cdot k_3+c_{12}\epsilon_3\cdot\epsilon_4 k_3\cdot k_4]k_{1\mu}\\
&\quad+[(-2c_7+c_9+2c_{10}-c_{12})\epsilon_4\cdot k_3 k_3\cdot k_4]\epsilon_{3\mu}\\
&\quad+[(+2c_8-c_9-2c_{11}+c_{12})\epsilon_3\cdot k_4 k_3\cdot k_4]\epsilon_{4\mu}\Big\}\delta_{\sum k_i}\,(-q^*)^\mu.
\end{aligned}
\tag{90}
$$

The computation of the color ordered vertex at k^0 order is

$$
\begin{aligned}
&\alpha'\mathcal{C}_0\mathcal{N}_0^4 A_{1234}|_{k^0}\\
&-\epsilon_1^{\mu_1}\epsilon_2^{\mu_2}V^{(123)}_{\mu_1;\ \mu_2;\ \mu}(k_1,k_2,q^*)|_{k^1}\frac{\delta^{\mu\nu}}{q^{*2}}V^{(123)}_{\nu;\ \mu_3;\ \mu_4}(-q^*,k_3,k_4)|_{k^1}\,\epsilon_3^{\mu_3}\epsilon_4^{\mu_4}\delta_{\sum k_i}\\
&-\epsilon_4^{\mu_4}\epsilon_1^{\mu_1}V^{(123)}_{\mu_4;\ \mu_1;\ \mu}(k_4,k_1,q^*)|_{k^1}\frac{\delta^{\mu\nu}}{q^{*2}}V^{(123)}_{\nu;\ \mu_2;\ \mu_3}(-q^*,k_2,k_3)|_{k^1}\,\epsilon_2^{\mu_2}\epsilon_3^{\mu_3}\delta_{\sum k_i},
\end{aligned}
\tag{91}
$$

and it can be split into three pieces. The first piece contains the pole in the s channel

$$
\begin{aligned}
&-\alpha'\mathcal{C}_0\mathcal{N}_0^4\frac{1}{s}\left[C_{(2,0)}|_{k^0}u+2C_{(1,0)}|_{k^2}\right]\delta_{\sum\hat{k}_i}\\
&-\epsilon_1^{\mu_1}\epsilon_2^{\mu_2}V^{(123)}_{\mu_1;\ \mu_2;\ \mu}(k_1,k_2,q^*)|_{k^1}\frac{\delta^{\mu\nu}}{q^{*2}}V^{(123)}_{\nu;\ \mu_3;\ \mu_4}(-q^*,k_3,k_4)|_{k^1}\,\epsilon_3^{\mu_3}\epsilon_4^{\mu_4}\delta_{\sum k_i}\\
&=-\frac{1}{2}(c_1^2+c_2^2)\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot\epsilon_4,
\end{aligned}
\tag{92}
$$

when the constraint (56) [14]

$$
(c_1-c_2)^2=-\mathcal{C}_0\mathcal{N}_0^4(2\alpha')^{3-\frac{1}{2}D}
\tag{93}
$$

is satisfied. The second piece contains the pole in the u channel and can be obtained from the first one by a cyclic permutation (1234). Finally the third piece come from the string amplitude without poles, i.e.

$$
\begin{aligned}
\alpha'\mathcal{C}_0\mathcal{N}_0^4 A_{1234}|_{k^0}&=\alpha'\mathcal{C}_0\mathcal{N}_0^4\left[-C_{(2,0)}|_{k^0}-C_{(0,2)}|_{k^0}+C_{(0,0)}|_{k^0}\right]\delta_{\sum\hat{k}_i}\\
&=-\frac{(2\alpha')^{3-\frac{1}{2}D}}{2}\mathcal{C}_0\mathcal{N}_0^4\left[\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot\epsilon_4+\epsilon_4\cdot\epsilon_1\,\epsilon_2\cdot\epsilon_3-\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot\epsilon_4\right]\delta_{\sum k_i}.
\end{aligned}
\tag{94}
$$

[14] The powers of $(2\alpha')$ come from the fact that $\hat{k}=\sqrt{2\alpha'}k$ and $\hat{\epsilon}=\sqrt{2\alpha'}\epsilon$.

Assembling all pieces gives the result (55).

The computation of the color ordered vertex at k^2 order proceeds in the same way. The first term is given by the terms with a pole in the s channel

$$\begin{aligned}
&\alpha' \mathcal{C}_0 \mathcal{N}_0^4 \frac{1}{s} \left[C_{(2,0)}|_{k^2} u - 2C_{(1,0)}|_{k^4} \right] \delta_{\sum \hat{k}_i} \\
&- \left[V^{(123)}_{\mu_1;\ \mu_2;\ \mu} (k_1, k_2, q^*) V^{(123)}_{\nu;\ \mu_3;\ \mu_4} (-q^*, k_3, k_4) \right] |_{k^4} \epsilon_1^{\mu_1} \epsilon_2^{\mu_2} \frac{\delta^{\mu\nu}}{q^{*2}} \epsilon_3^{\mu_3} \epsilon_4^{\mu_4} \delta_{\sum k_i} \\
&= -\frac{1}{2} [c_1(-3c_4 - c_5 + 2c_6) + c_2(-3c_3 + 2c_5 - c_6)] \\
&\qquad \times [\epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot k_4\, \epsilon_4 \cdot k_3 + \epsilon_3 \cdot \epsilon_4\, \epsilon_1 \cdot k_2\, \epsilon_2 \cdot k_1]
\end{aligned} \tag{95}$$

when the constraint

$$(c_1 - c_2)(-3c_3 + 3c_4 + 3c_5 - 3c_6) = \mathcal{C}_0 \mathcal{N}_0^4 (2\alpha')^{4-\frac{1}{2}D} \tag{96}$$

is satisfied. The second piece contains the pole in the u channel and can be obtained from the first one by a cyclic permutation (1234). The third piece come from the string amplitude without poles and from the Feynman diagram obtained from color ordered vertices without poles. Explicitly we have that the terms of the form $(\epsilon \cdot \epsilon)^2 k \cdot k$ are

$$\begin{aligned}
&\Bigg\{ + \frac{1}{2} \epsilon_1 \cdot \epsilon_2\, \epsilon_3 \epsilon_4 \Bigg[\left(\frac{1}{2} (2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 \right) s + \left(\frac{1}{2} (2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 + c_1 c_9 + c_2 c_{12} \right) u \\
&\qquad\qquad + \left(\frac{1}{2} (2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 + c_1 c_{12} + c_2 c_9 \right) t \Bigg] \\
&\quad + 1 \text{ term obtained by cycling (1234) in the previous term} \Bigg\} \\
&\quad - \frac{1}{2} \epsilon_3 \cdot \epsilon_1\, \epsilon_2 \epsilon_4 \left[\left(\frac{1}{2} (2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 \right) u + \left(\frac{1}{2} (2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 \right) s \right]
\end{aligned} \tag{97}$$

The terms of the form $(\epsilon \cdot \epsilon)(\epsilon \cdot k)^2$ in a canonical form where the indices of the momenta are taken from the indices of $(\epsilon \cdot \epsilon)$ are

$$\begin{aligned}
+ \Bigg\{ &+ \epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot k_1\, \epsilon_4 \cdot k_1 \frac{1}{2} \Big[(2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 + c_1(-3c_4 - c_5 + 2c_6) \\
&\qquad + c_2(-3c_3 + 2c_5 - c_6 - 2c_7 + 2c_8 + 2c_{10} - 2c_{11}) \Big] \\
&+ \epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot k_1\, \epsilon_4 \cdot k_2 \frac{1}{2} \Big[(2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 + c_1(-3c_4 - c_5 + 2c_6 + 2c_8 - 2c_{11}) \\
&\qquad + c_2(-3c_3 + 2c_5 - c_6 - 2c_7 + 2c_{10}) \Big] \\
&+ \epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot k_1\, \epsilon_4 \cdot k_2 \frac{1}{2} \Big[(2\alpha')^{4-\frac{1}{2}D} \mathcal{C}_0 \mathcal{N}_0^4 + c_1(-3c_4 - c_5 + 2c_6 - 4c_7 - 2c_8 \\
&\qquad + 4c_9 + 2c_{10} + 2c_{11} - 2c_{12}) \\
&\qquad + c_2(-3c_3 + 2c_5 - c_6 + 2c_7 + 4c_8 - 2c_9 \\
&\qquad - 2c_{10} - 4c_{11} + 4c_{12}) \Big]
\end{aligned}$$

$$
\begin{aligned}
&+\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot k_2\,\epsilon_4\cdot k_2\frac{1}{2}\Big[(2\alpha')^{4-\frac{1}{2}D}\mathcal{C}_0\mathcal{N}_0^4+c_1(-3c_4-c_5+2c_6-2c_7+2c_8)\\
&\qquad\qquad+c_2(-3c_3+2c_5-c_6)\Big]\\
&\quad+3{*}4\text{ terms obtained by cycling (1234) in the previous 4 terms}\Big\}\\
&+\Big\{+\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot k_1\,\epsilon_4\cdot k_1\Big[-\frac{1}{2}(2\alpha')^{4-\frac{1}{2}D}\mathcal{C}_0\mathcal{N}_0^4\Big]\\
&\quad+\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot k_1\,\epsilon_4\cdot k_3\Big[-\frac{1}{2}(2\alpha')^{4-\frac{1}{2}D}\mathcal{C}_0\mathcal{N}_0^4-(c_1-c_2)(-2c_7+c_9+2c_{10}-c_{12})\Big]\\
&\quad+\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot k_3\,\epsilon_4\cdot k_1\Big[-\frac{1}{2}(2\alpha')^{4-\frac{1}{2}D}\mathcal{C}_0\mathcal{N}_0^4+(c_1-c_2)(+2c_7-c_9-2c_{11}+c_{12})]\Big]\\
&\quad+\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot k_3\,\epsilon_4\cdot k_3\Big[-\frac{1}{2}(2\alpha')^{4-\frac{1}{2}D}\mathcal{C}_0\mathcal{N}_0^4\Big]\\
&\quad+4\text{ terms obtained by cycling (1234) in the previous 4 terms}\Big\}
\end{aligned}
\tag{98}
$$

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