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The History of Differential Equations, 1670–1950

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Introduction by the Organisers

Differential equations have been a major branch of pure and applied mathematics since their inauguration in the mid 17th century. While their history has been well studied, it remains a vital field of on-going investigation, with the emergence of new connections with other parts of mathematics, fertile interplay with applied subjects, interesting reformulation of basic problems and theory in various periods, new vistas in the 20th century, and so on. In this meeting we considered some of the principal parts of this story, from the launch with Newton and Leibniz up to around 1950.

'Differential equations' began with Leibniz, the Bernoulli brothers and others from the 1680s, not long after Newton's 'fluxional equations' in the 1670s. Applications were made largely to geometry and mechanics; isoperimetrical problems were exercises in optimisation.

Most 18th-century developments consolidated the Leibnizian tradition, extending its multi-variate form, thus leading to partial differential equations. Generalisation of isoperimetrical problems led to the calculus of variations. New figures appeared, especially Euler, Daniel Bernoulli, Lagrange and Laplace. Development of the general theory of solutions included singular ones, functional solutions and those by infinite series. Many applications were made to mechanics, especially to astronomy and continuous media.

In the 19th century: general theory was enriched by development of the understanding of general and particular solutions, and of existence theorems. More

(3) In the period around 1905 — just after the publication of the papers by Fredholm, Hilbert, and E. Schmidt on the theory of integral equations — many Italian mathematicians changed their approach and started studying the equations of mathematical physics by using Fredholm's new theory;
(4) Levi-Civita believed that Fredholm's approach was fruitful and easy to apply to mathematical physics, and encouraged mathematicians to use the new theory, as his private correspondence shows.

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G. Peano and M. Gramegna on ordinary differential equations

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In this preliminary research I will assess the historical and mathematical value of Peano and Gramegna's studies on linear differential equations, focusing on the symbolic and vectorial approach, which I believe make these works interesting.

Giuseppe Peano (1858-1932) taught in the University of Turin for over fifty years, creating a famous School of mathematicians, teachers and engineers. He became Professor of *Calcolo Infinitesimale* in 1890 and he was appointed to the course of *Analisi Superiore* in the academic years 1908-1910. His very large production includes over three hundred writings, dealing with analysis, geometry, logic, foundational studies, history of mathematics, actuarial mathematics, glottology and linguistics.

Maria Paola Gramegna (1887-1915) was a student of Peano in his courses (*Calcolo Infinitesimale* and *Analisi Superiore*) and, under his supervision, she wrote the note *Serie di equazioni differenziali lineari ed equazioni integro-differenziali*, submitted by Peano at the Academy of Sciences in Turin, in the session of the 13 March 1910. This article would be discussed by Gramegna, with the same title, as her graduation thesis in mathematics, on the 7 July of the same year. In 1911 Gramegna became a teacher in Avezzano, holding a secondary school appointment at the Royal Normal School. Four years later, on the 13 January 1915, she died, a victim of the earthquake which destroyed that town.

The study of the articles by Peano and by Gramegna on systems of ordinary linear differential equations presents interesting implications. In the winter of 1887, Peano was able to deal with these systems for the first time in a rigorous way, and

he submitted an article entitled *Integrazione per serie delle equazioni differenziali lineari* to the Academy of Sciences of Turin. A slightly modified version of this note, in French, would be published the following year in the *Mathematische Annalen.* Here he applied the method of "successive approximations" or "successive integrations" — as Peano preferred to call it — based on the theory of linear substitutions.

The purpose of Peano's article is to prove the following theorem: let there be n homogeneous linear differential equations in n functions $x_1, x_2, \dots, x_n$ of the variable t, in which the coefficients α_{ij} are functions of t, continuous on a closed and bounded interval $[p, q]$:

$$\frac{dx_1}{dt} = \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n,$$
$$\frac{dx_2}{dt} = \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n,$$
$$\dots$$
$$\frac{dx_n}{dt} = \alpha_{n1}x_1 + \alpha_{n2}x_2 + \cdots + \alpha_{nn}x_n.$$

Substitute in the second members of the equations, n arbitrary constants $a_1, a_2, \dots, a_n$, in place of $x_1, x_2, \dots, x_n$, and integrate from t_0 to t, where $t_0, t \in (p, q)$. We obtain n functions of t, which will be denoted by $a'_1, a'_2, \dots, a'_n$. Now substitute in the second members of the proposed differential equations, $a'_1, a'_2, \dots, a'_n$ in place of $x_1, x_2, \dots, x_n$. With the same treatment we obtain n new functions of t, which will be denoted by $a''_1, a''_2, \dots, a''_n$. Repeating this process, we obtain

$$a_1 + a'_1 + a''_1 + \dots,$$
$$a_2 + a'_2 + a''_2 + \dots,$$
$$\dots$$
$$a_n + a'_n + a''_n + \dots.$$

The series are convergent throughout the interval (p, q). Their sums, which we shall denote by $x_1, x_2, \dots, x_n$ are functions of t and satisfy the given system. Moreover, for $t = t_0$, they assume the arbitrarily chosen values $a_1, a_2, \dots, a_n$.

In order to prove the preceding theorem, Peano introduces vectorial and matrix notations and some sketches of functional analysis on linear operators.

In 1910, Maria Gramegna again took up the method of successive integrations, in order to generalize the previous theorem to systems of infinite differential equations and to integro-differential equations. The original results exposed by Gramegna in the above-mentioned note *Serie di equazioni differenziali lineari ed equazioni integro-differenziali*, are an important example of the modern use of matrix notation, which will be central in the development of functional analysis. Moreover, the widespread application of Peano's symbolic language gives her work a modern slant. Gramegna proves the following extension of Peano's theorem: we consider an infinite system of differential linear equations with an infinite number

of unknowns:

$$\frac{dx_1}{dt} = u_{11}x_1 + u_{12}x_2 + \cdots + u_{1n}x_n + \ldots$$
$$\frac{dx_2}{dt} = u_{21}x_1 + u_{22}x_2 + \cdots + u_{2n}x_n + \ldots$$
$$\ldots$$

where the u_{ij} are constant with respect to time. Let us denote by A the substitution represented by the matrix of the u's. Let x be the sequence $(x_1, x_2, \ldots)$ and x_0 its initial value. We may write the given differential equations as $Dx = Ax$, and the integral is given by $x_t = e^{tA}x_0$, where the substitution e^{tA} has this representation:

$$1 + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \ldots.$$

In the last section of the article Gramegna applies the new analytic tools she has introduced (the concept of *mole*, the exponential of a substitution, etc.) in order to solve integro-differential equations, already studied by I. Fredholm, V. Volterra and E. H. Moore.

Unfortunately, the note by Gramegna did not have a large circulation. This was probably due to the difficulty, for many mathematicians, of understanding research in advanced analysis presented with Peano's logic symbolism. Besides, this is the last work in *Analisi Superiore* realised under the supervision of Peano, who was dismissed from the course in 1910, some days after the submission of Gramegna's article to the Academy, thereby losing the possibility of training other researchers.

This historical study will continue with the aim of investigating the possible influences of Gramegna's article on research in functional analysis in Italy and abroad.

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