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Towards a fully stringy computation of Yukawa couplings on non-factorized tori and non-abelian twist correlators (I): The classical solution and action

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Abstract

We consider the simplest possible setting of non-abelian twist fields which corresponds to $SU(2)$ monodromies. We first review the theory of hypergeometric function and of the solutions of the most general Fuchsian second order equation with three singularities. Then we solve the problem of writing the general solution with prescribed $U(2)$ monodromies. We use this result to compute the classical string solution corresponding to three $D2$ branes in $\mathbb{R}^4$. Despite the fact that the configuration is supersymmetric the classical string solution is not holomorphic. Using the equation of motion and not the KLT approach we give a very simple expression for the classical action of the string. We find that the classical action is not proportional to the area of the triangle determined by the branes intersection points since the solution is not holomorphic. Phenomenologically this means that the Yukawa couplings for these supersymmetric configurations on non-factorized tori are suppressed with respect to the factorized case.

1. Introduction and conclusions

Since the beginning, D-branes have been very important in the formal development of string theory as well as in attempts to apply string theory to particle phenomenology and cosmology. However, the requirement of chirality in any physically realistic model leads to a somewhat

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restricted number of possible D-brane set-ups. An important class of models are intersecting brane models where chiral fermions can arise at the intersection of two branes at angles. Most of these computable models are based on $D6$ branes at angles in T^6 or its orbifolds.

To ascertain the phenomenological viability of a model the computation of Yukawa couplings and flavor changing neutral currents plays an important role. This kind of computations involves the computations of (excited) twist fields correlators. Besides the previous computations many other computations often involve correlators of twist fields and excited twist fields. It is therefore important and interesting in its own right to be able to compute these correlators. The literature concerning orbifolds (see for example [1,2]) intersecting D-branes on factorized tori (see for example [3]), magnetic branes with commuting magnetic fluxes (see for example [4]) or involving "abelian" twist fields in various applications (see for example [5]) is very vast. These results are mainly based on the so-called stress–tensor method [1] and concerns mainly non-excited twists even if results for excited twists [6] were obtained. Some of the previous results were also obtained in the infinite charge formalism and boundary state formalism [7]. Within the Reggeon framework (see for example [8,9]) the generating functions for the three point correlators were also obtained in a somewhat complex way. Finally in [10] and [11] based on previous results [12] and a mixture of the path integral approach with the Reggeon approach the generating function of all the correlators with an arbitrary number of (excited) twist fields and usual vertices was given in the case of abelian twist fields. These computations boil down to the knowledge of the Green function in presence of twist fields and of the correlators of the plain twist fields. In this way the computations were made systematic differently from many previous papers where correlators with excited twisted fields have been computed on a case by case basis without a clear global picture. The same results were then recovered using the canonical quantization approach in [13].

Until now only the case of factorized tori has been considered at the stringy level. It is clear that the non-factorized case is more generic and technically by far more complex. It concerns the so-called non-abelian twists for which only a handful papers can be found in the literature of the last 30 years [14]. It is therefore interesting to try to understand how special the results from the factorized case are and to try to clarify the technical issues involved.

In this very technical paper we start the investigation of these configurations. We start considering the case of three $D6$ branes embedded in $\mathbb{R}^{10}$.[1] The relevant configuration can be effectively described by three euclidean $E2$ branes in $\mathbb{C}^2 = \mathbb{R}^4$. We can think of embedding the first $E2$ brane as $\Im Z^1 = \Im Z^2 = 0$. Then the second and third $E2$ branes are generically characterized by a $SO(4)$ matrix (or more precisely by an equivalence class, i.e. a point in the Grassmannian $SO(4)/SO(2) \times SO(2)$) which describes how they are embedded with respect to the first one. However we limit our analysis to the simplest case where these matrices are characterized by an equivalence class of $SU(2)$. If these two matrices commute then we are in the abelian case if not we deal with the by far more difficult non-abelian case. Even if we do not consider the most general case it is however interesting enough to start grasping the issues involved. Moreover this configuration is supersymmetric since there are spinors invariant under the other $SU(2)$ of the "internal" rotation $SO(4) \equiv SU(2) \times SU(2)$.

Due to the technicality of the computations involved we have preferred to write down the details therefore the paper has grown in dimension making necessary to split it into different parts.

[1] It is obviously possible to extend the analysis to configurations on $\mathbb{R}^6 \times T^4$ alone the lines of [15]. It would also be interesting to see in which way the results in this paper improve the conclusion of [16] as far as the moduli stabilization without fluxes.

In this part we recapitulate the mathematical tools necessary and we find the classical solution of the bosonic string. In a companion paper we deal with the Green function which is necessary to compute the correlators involving excited twist fields. We are nevertheless still not very close to determine (from first principles) the normalization of the three twist field correlator as it happens also for all the other papers on the subject [14]. The reason being the impossibility of writing explicitly the classical solution of the string with four branes with a non-abelian configuration or, equivalently, a basis to use for computing the Green functions.

We can nevertheless draw some interesting conclusions. In particular using the path integral approach the $\hat{N}_B$ point twist field correlator can be written roughly as

$$\langle \sigma_{M_1}(x_1)\dots\sigma_{M_{\hat{N}_B}}(x_{\hat{N}_B})\rangle = \mathcal{N}(\{x_t,\ M_t\}_{1\le t\le \hat{N}_B})e^{-S_{E,cl}(\{x_t,\ M_t\}_{1\le t\le \hat{N}_B})},$$

where M_t with $1 \le t \le \hat{N}_B$ are the monodromies. Therefore the knowledge of the classical solution gives the main contribution $e^{-S_{E,cl}(\{x_t,\ M_t\}_{1\le t\le \hat{N}_B})}$ even if the quantum contribution $\mathcal{N}(\{x_t,\ M_t\}_{1\le t\le \hat{N}_B})$ is necessary for the complete result. It then follows that given three $D6_t$ $(1 \le t \le 3)$ branes the leading order of the Yukawa coupling in a truly stringy computation is given by

$$Y_{123} \propto e^{-S_{E,cl}(\{x_t,\ M_t\}_{1\le t\le 3})}.$$

Naively one could think that $S_{E,cl}$ is simply the area of the triangle determined by the three interaction points but it is not so. These interaction points always define a 2 dimensional real plane in $\mathbb{R}^4$ but differently from the cases discussed before in the literature the embedding of the string worldsheet which follows from the equation of motion is not a flat triangle, i.e. a triangle which lies in the plane determined by the three interactions points. In fact Fig. 10 shows the actual line traced by the endpoint of the classical string while the naive path should be a segment. This implies that Yukawa couplings in non-factorized models are suppressed with respect to the factorized ones. The reason is that the classical string solution is not holomorphic (this must not be confused with the fact that the branes embeddings are holomorphic in the proper set of coordinates).

The paper is organized as follows. In section 2 we recapitulate the classical mathematics needed for the computation. In particular we consider the monodromies associated with the general solution of the second order Fuchsian equation with three singular points located at 0, 1 and ∞. Given the relation between the parameters of the equation and the monodromies we solve the inverse problem, i.e. given the monodromies in $U(2)$ find the properly normalized combination of solutions which has the desired monodromy set. This solution is obviously expressed using the hypergeometric function as in eqs. (25), (39), (41). Another not so commonly appreciated result of the discussion is that monodromies depend on whether the base point is the upper or lower half plane.

In section 3 we consider the string action and the boundary conditions we have to impose. The boundary conditions are better expressed as a local problem for the monodromies on the double string coordinates and a global problem.

In section 4 we then proceed to find the actual classical solution. The upshot of this is more general than the three D-branes case. It turns out that for the $U(2)$ case most of the information is contained in the local behavior of the solution, the indices of the Fuchsian equation, which are determined by the modulus of the vector $\vec{n}$ which parametrizes the $SU(2)$ monodromy as $M = \exp(i2\pi\vec{n}\cdot\vec{\sigma})$. The normalization of the solution depends also on the other parameters. Nevertheless the indices are not sufficient to completely fix the solution but in the simplest case we consider since there are the accessory parameters when the number of branes is bigger than or equal to four.

Finally, in section 5 we compute the classical action corresponding to the solution found. We do this in a more general way which allows us express the action as a linear function of some coefficients opposed to the usual way of getting an expression quadratic. Moreover we clearly show that in the holomorphic case the action has a geometrical meaning.

2. Monodromies of the hypergeometric function

To solve the problem of finding the string classical solution, we are interested in finding complex functions with a given set of singular points and monodromies. A good starting point to construct these functions is to consider the Fuchsian linear differential equations (which are reviewed in Appendix A) since the solutions come naturally in vectors with given monodromies.

Since we are interested in the case with three singular points and $U(2)$ monodromies we would now like to summarize some basics facts on the hypergeometric function which we need in the following.

Our main interest is the derivation of monodromies. In particular we discuss one point which seems to be overlooked or implicit in the literature, i.e. the monodromies do depend on the point we start the loop.[2] It is in fact well known that the homotopy group is defined starting from a base point and that all of these groups are isomorphic. This does not however mean that their representations in the vector space of the solutions of the hypergeometric equation are equal. And actually they are not. In Appendix B we show this point in a local setup and at the end of section 3.2 we explicitly show how this fact is needed to demonstrate that the string action is well defined when it is written using the string coordinates obtained by the doubling trick.

2.1. Paths

We consider the loops[3] $\gamma^{(+)}_{[\mathbf{0}]}, \gamma^{(+)}_{[\mathbf{1}]}, \gamma^{(+)}_{[\infty]}$ having a base point in the upper half plane $H^+ \equiv H$[4] and looping in counterclockwise direction around the marked points $z_0 = 0, 1, \infty$ respectively as shown in Fig. 1. We consider also the corresponding loops $\gamma^{(-)}_{[\mathbf{0}]}, \gamma^{(-)}_{[\mathbf{1}]}, \gamma^{(-)}_{[\infty]}$ with base point in lower half-plane H^- as shown in Fig. 2.

More explicitly we define

$$\gamma^{(\pm)} : t \in [0, 1] \to \mathbb{C}, \quad \gamma^{(\pm)}(0) = \gamma^{(\pm)}(1) = z_0 \in H^{\pm}. \tag{1}$$

We denote $\gamma_{[\mathbf{a}]} * \gamma_{[\mathbf{b}]}$ the loop formed by first going around $\gamma_{[\mathbf{a}]}$ and then $\gamma_{[\mathbf{b}]}$, i.e.

$$\gamma_{[\mathbf{a}]} * \gamma_{[\mathbf{b}]}(t) = \begin{cases} \gamma_{[\mathbf{a}]}(2t) & t \in [0, 1/2] \\ \gamma_{[\mathbf{b}]}(2t-1) & t \in [1/2, 1] \end{cases}. \tag{2}$$

We have then that

[2] Historically we notice that even the first paper on the monodromies for the hypergeometric function by Riemann in 1857 [17] seems not to consider the two cases.

[3] Here and in the following we denote the singular point by a subscript in square parenthesis, e.g. $\gamma_{[\mathbf{1}]}$ this is to avoid confusion with the index associated with the brane. Indices associated with the branes are put in round parenthesis, e.g. $f_{(1)}$. Moreover we use $\{\mathbf{1}\}$ as subscript to denote well adapted objects for the singular point $z = 1$, see for example eq. (14).

[4] We define $H = \{z \in \mathbb{C}; \Im z \geq 0\}$ and $H^- = \{z \in \mathbb{C}; \Im z \leq 0\}$.

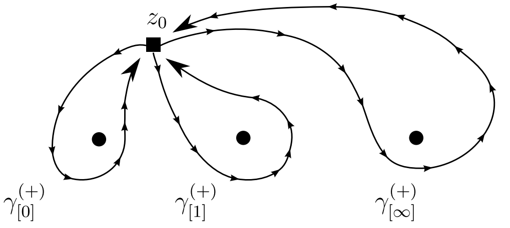

Fig. 1. The three different paths around the marked points 0, 1, ∞ starting in the upper half plane.

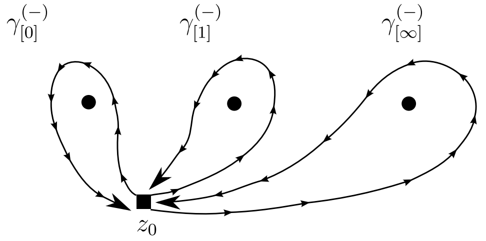

Fig. 2. The three different paths around the marked points 0, 1, ∞ starting in the lower half plane.

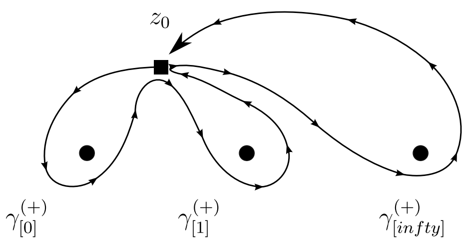

Fig. 3. The product $\gamma^{(+)}_{[\mathbf{0}]} * \gamma^{(+)}_{[\mathbf{1}]} * \gamma^{(+)}_{[\infty]} = 1$.

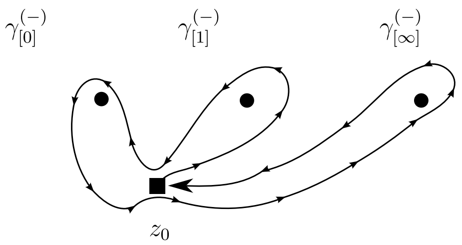

Fig. 4. The product $\gamma^{(-)}_{[\mathbf{1}]} * \gamma^{(-)}_{[\mathbf{0}]} * \gamma^{(-)}_{[\infty]} = 1$.

$$\gamma^{(+)}_{[\mathbf{0}]} * \gamma^{(+)}_{[\mathbf{1}]} * \gamma^{(+)}_{[\infty]} = 1, \quad \gamma^{(-)}_{[\infty]} * \gamma^{(-)}_{[\mathbf{1}]} * \gamma^{(-)}_{[\mathbf{0}]} = 1. \tag{3}$$

These equations can be generalized to N points with coordinates z_i such that $|z_i| < |z_{i-i}|$ as

$$\gamma^{+}_{[\mathbf{z_N}]} * \gamma^{+}_{[\mathbf{z_{N-1}}]} * \gamma^{+}_{[\mathbf{z_{N-2}}]} \cdots * \gamma^{+}_{[\mathbf{z_1}]} = 1, \quad \gamma^{-}_{[\mathbf{z_1}]} * \gamma^{-}_{[\mathbf{z_2}]} * \gamma^{-}_{[\mathbf{z_3}]} \cdots * \gamma^{+}_{[\mathbf{z_N}]} = 1. \tag{4}$$

The first equation in (3) is shown in Fig. 3 and equivalent version of the second one, i.e. $\gamma^{(-)}_{[\mathbf{1}]} * \gamma^{(-)}_{[\mathbf{0}]} * \gamma^{(-)}_{[\infty]} = 1$ in Fig. 4.

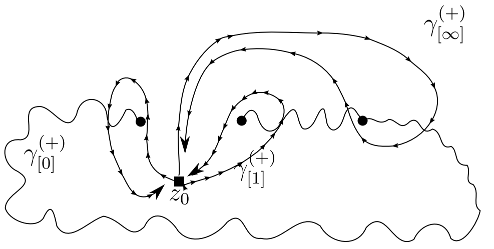

Fig. 5. Moving the base point between 0 and 1 does not map $\gamma^{(+)}_{[\infty]}$ into $\gamma^{(\cdot)}_{[\infty]}$.

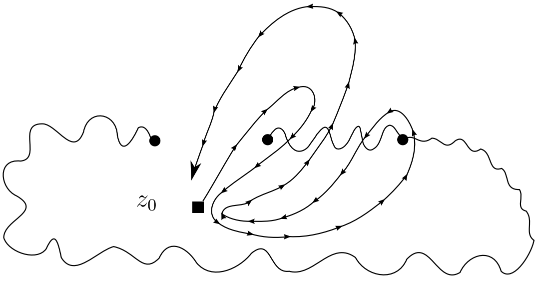

Fig. 6. $\gamma^{(+)}_{[\infty]}$ is transformed into $\gamma^{(-)\,-1}_{[\mathbf{1}]} * \gamma^{(-)}_{[\infty]} * \gamma^{(-)}_{[\mathbf{1}]}$.

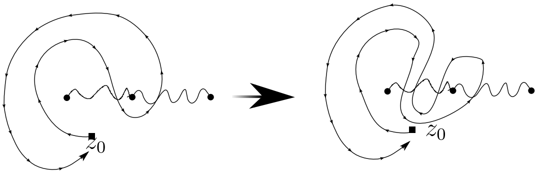

Fig. 7. $\gamma^{(+)}_{[\mathbf{1}]}$ is transformed into $\gamma^{(-)\,-1}_{[\mathbf{0}]} * \gamma^{(-)}_{[\mathbf{1}]} * \gamma^{(-)}_{[\mathbf{0}]} \equiv \gamma^{(-)}_{[\infty]} * \gamma^{(-)}_{[\mathbf{1}]} * \gamma^{(-)\,-1}_{[\infty]}$.

The reason why we find two different products in the upper and lower half plane is simple. Not all the paths $\gamma^{(+)}$ do transform into the $\gamma^{(-)}$ ones when we move the base point from the upper half plane to the lower one. Moreover there are three different ways of moving a point from the upper half plane to the lower one. These ways are characterized by the two marked points between which we move the base point. There are therefore three different possibilities.

In Fig. 5 we show what happens when we move the base point from the upper half plane to the lower half plane between 0 and 1. Both $\gamma^{(+)}_{[\mathbf{0}]}$ and $\gamma^{(+)}_{[\mathbf{1}]}$ are transformed into the corresponding paths $\gamma^{(-)}_{[\mathbf{0}]}$ and $\gamma^{(-)}_{[\mathbf{1}]}$ while $\gamma^{(+)}_{[\infty]}$ is transformed into $\gamma^{(-)\,-1}_{[\mathbf{1}]} * \gamma^{(-)}_{[\infty]} * \gamma^{(-)}_{[\mathbf{1}]}$ as it is shown in Fig. 6 or equivalently to $\gamma^{(-)}_{[\mathbf{0}]} * \gamma^{(-)}_{[\infty]} * \gamma^{(-)\,-1}_{[\mathbf{0}]}$ when we consider the sphere and we move the path on the sphere. Both these expressions are compatible with eqs. (3).

If we move the base point between 0 and ∞ then $\gamma^{(+)}_{[\mathbf{1}]}$ is transformed into $\gamma^{(-)\,-1}_{[\mathbf{0}]} * \gamma^{(-)}_{[\mathbf{1}]} * \gamma^{(-)}_{[\mathbf{0}]}$ when we move counterclockwise around the cuts as shown in Fig. 7 and $\gamma^{(-)}_{[\infty]} * \gamma^{(-)}_{[\mathbf{1}]} * \gamma^{(-)\,-1}_{[\infty]}$ when we move clockwise. The two expressions are nevertheless equal because of the second equation in (3).

Finally in Fig. 8 we show what happens when we move the base point between 1 and ∞.

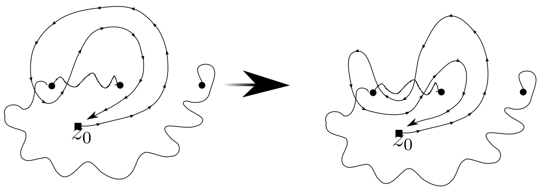

Fig. 8. $\gamma^{(+)}_{[\mathbf{0}]}$ is transformed into $\gamma^{(-)}_{[\mathbf{1}]} * \gamma^{(-)}_{[\mathbf{0}]} * \gamma^{(-)\,-1}_{[\mathbf{1}]}$.

2.2. Hypergeometric equation and its solutions

As discussed before in order to find a basis of solutions with $U(2)$ monodromies we start considering the most general Fuchsian differential equation of order $n = 2$ with $N = 3$ singularities. Specializing what described in Appendix A we can write it as

$$\frac{d^2 y}{dz^2} + \left[\frac{1-\rho_{11}-\rho_{12}}{z-z_1} + \frac{1-\rho_{21}-\rho_{22}}{z-z_2} + \frac{1-\rho_{31}-\rho_{32}}{z-z_3}\right]\frac{dy}{dz}$$
$$+\left[\frac{\rho_{11}\rho_{12}(z_1-z_2)(z_1-z_3)}{z-z_1} + \frac{\rho_{21}\rho_{22}(z_2-z_1)(z_2-z_3)}{z-z_2} + \frac{\rho_{31}\rho_{32}(z_3-z_1)(z_3-z_1)}{z-z_3}\right]$$
$$\times \frac{y}{\prod_{i=1}^{N=3}(z-z_i)} = 0, \tag{5}$$

where $\rho_{i\,a}$ ($i = 1, 2, N = 3$, $a = 1, n = 2$) are called the indices and give the possible behaviors of the solutions around the singular points, i.e. generically we have a mixture as $y \sim c_1(z - z_i)^{\rho_{i\,1}} + c_2(z - z_i)^{\rho_{i\,a2}}$. The indices are constrained as $\sum_{a=1}^{n=2}\sum_{i=1}^{N=3} \rho_{i\,a} = 1$. Its general solution can be formally written by using the Papperitz–Riemann P-symbol as

$$y = P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \rho_{1\,1} & \rho_{2\,1} & \rho_{3\,1} & z \\ \rho_{1\,2} & \rho_{2\,2} & \rho_{N\,n} & \end{Bmatrix}. \tag{6}$$

This symbol represents all the ∞^2 solutions of the Fuchsian equation obtained by the linear combination of two independent solutions.

Since it represents all the solutions and not one particular solution it has a number of remarkable properties:

- it is invariant under conformal transformations
$$y = P\begin{Bmatrix} z'_1 & z'_2 & z'_3 & \\ \rho_{1\,1} & \rho_{2\,1} & \rho_{3\,1} & z' \\ \rho_{1\,2} & \rho_{2\,2} & \rho_{N\,n} & \end{Bmatrix}, \tag{7}$$
with $z' = (az + b)/(cz + d)$ and $ad - bc = 1$;
- it is invariant under columns and lines permutations, for example
$$y = P\begin{Bmatrix} z'_1 & z'_2 & z'_3 & \\ \rho_{1\,2} & \rho_{2\,1} & \rho_{3\,1} & z' \\ \rho_{1\,1} & \rho_{2\,2} & \rho_{N\,n} & \end{Bmatrix}, \tag{8}$$
is one of the $3!\,2^2$ cases obtained permuting the singular points and exchanging their indexes wrt a fixed pair;

- it transforms as

$$y = \left(\frac{z_1 - z}{z_3 - z}\right)^{\delta} \left(\frac{z_2 - z}{z_3 - z}\right)^{\epsilon} P \left\{ \begin{matrix} z_1 & z_2 & z_3 & \\ \rho_{1\,1} - \delta & \rho_{2\,1} - \epsilon & \rho_{3\,1} + \delta + \epsilon & z \\ \rho_{1\,2} - \delta & \rho_{2\,2} - \epsilon & \rho_{N\,n} + \delta + \epsilon & \end{matrix} \right\}. \tag{9}$$

This is interpreted as the statement that for any solution associated with the P-symbol in eq. (6) there is one solution associated to the new P-symbol which is equal to the original one when multiplied by $\left(\frac{z_1-z}{z_3-z}\right)^{\delta} \left(\frac{z_2-z}{z_3-z}\right)^{\epsilon}$.

Using the last property we can limit ourselves to consider the P-symbol[5]

$$y = P \left\{ \begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{matrix} \right\}, \tag{10}$$

which is associated with the hypergeometric equation

$$z(1-z)\frac{d^2y}{dz^2} + [c - (a+b+1)z]\frac{dy}{dz} - aby = 0, \tag{11}$$

which has singular points $z = 0, 1$ and $z = \infty$ where the respective indices are $0, 1 - c$, $0, c - a - b$ and a, b.

This equation has the obvious perturbative solution around the $z = 0$ singular point given by[6]

$$y = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} z^n = F\begin{pmatrix} a \quad b \\ c \end{pmatrix}; z\Big) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} {}_2F_1(a, b; c; z). \tag{12}$$

Hence $F\left(\begin{matrix} a \quad b \\ c \end{matrix}; z\right)$ is among the solutions represented by the P-symbol (10). Notice also that $F\left(\begin{matrix} a \quad b \\ c \end{matrix}; z\right)$ is a function defined on the whole complex plane minus the cut and not only for $|z| < 1$ where the series converges.

The other independent solution around $z = 0$ can be found using the P-symbol properties which yield

$$y = P \left\{ \begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{matrix} \right\} = (-z)^{1-c} P \left\{ \begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a-c+1 & z \\ c-1 & c-a-b & b-c+1 & \end{matrix} \right\} \tag{13}$$

which implies that $(-z)^{1-c} F\left(\begin{matrix} a+1-c \quad b+1-c \\ 2-c \end{matrix}; z\right)$ is among the solutions. Since its behavior for $z \to 0$ is different it is independent of $F\left(\begin{matrix} a \quad b \\ c \end{matrix}; z\right)$.

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5 Notice that for $z \to \infty$ the P-symbol means $y \sim \left(\frac{1}{z}\right)^a, \left(\frac{1}{z}\right)^b$.

6 While we use the same notation used by NIST the normalization differs. The reason of this choice is to have simpler monodromy matrices as eq. (22) shows.

Now in order to derive the monodromy matrices we need to understand how the natural basis of the solutions at $z = 0$ to the hypergeometric equation behaves away from the singular point $z = 0$ and around the other singular points. In the fundamental sheet this basis (in which it has a diagonal monodromy matrix and has a simple power expansion around $\{\mathbf{0}\}$) is given by

$$\mathcal{B}_{\{\mathbf{0}\}}(z) = \begin{pmatrix} F\begin{pmatrix} a & b \\ & c \end{pmatrix}; z \\ (-z)^{1-c} F\begin{pmatrix} a+1-c & b+1-c \\ & 2-c \end{pmatrix}; z \end{pmatrix}. \tag{14}$$

Again these are the functions defined over all the complex plane minus cuts in the fundamental sheet. The cuts arise from both the $F\left(\begin{matrix} a & b \\ & c \end{matrix}; z\right)$ and the $(-z)^{1-c}$. While the cut from $F\left(\begin{matrix} a & b \\ & c \end{matrix}; z\right)$ is natural (but non-compulsory) between 1 and $+\infty$ the cut from $(-z)^{1-c}$ can be set either from 0 to $-\infty$ or from 0 to $+\infty$. The former situation is the one depicted in Figs. 5 and 6 while the latter is the one in Fig. 7. Our choice is to set both cuts along the real positive axis, i.e. what it is depicted in Fig. 7.

Using again the properties of the P-symbol we realize that the basis of solutions in $1/z$ (which has a diagonal monodromy matrix and have a simple power expansion around $\{\infty\}$) is given by

$$\mathcal{B}_{\{\infty\}}(z) = \begin{pmatrix} \left(-\frac{1}{z}\right)^a F\left(\begin{matrix} a & a+1-c \\ & a+1-b \end{matrix}; \frac{1}{z}\right) \\ \left(-\frac{1}{z}\right)^b F\left(\begin{matrix} b & b+1-c \\ & b+1-a \end{matrix}; \frac{1}{z}\right) \end{pmatrix}. \tag{15}$$

The starting point to connect the two basis is the Barnes integral representation of a solution of the hypergeometric equation given by

$$F\left(\begin{matrix} a & b \\ & c \end{matrix}; z\right) = \int_\Gamma \frac{ds}{2\pi i} \frac{\Gamma(s+a)\Gamma(s+b)\Gamma(-s)}{\Gamma(s+c)} (-z)^s \tag{16}$$

where $a, b, c \in \mathbb{C}$ and Γ is the path from $-i\infty$ to $+i\infty$ which has all the $s = -a - 1 - n$, $s = -b - 1 - n$ with $n \in \mathbb{N}$ poles to the left and $s = n$ ones to the right.

Notice that the integrand excluded the factor $(-z)^s$ is a function with only isolated singularities on the complex plane, i.e. the Riemann sphere but the infinite while the $F\left(\begin{matrix} a & b \\ & c \end{matrix}; z\right)$ has both isolated singularities and one cut which originates from the logarithm in $(-z)^s = \exp(\log(-z)s)$. The cut starts apparently at $z = 0$ but the expansion (12) shows it actually starts at $z = 1$.

Notice also that $F\left(\begin{matrix} a & b \\ & c \end{matrix}; z\right)$ is a function defined on the whole complex plane minus the cut and not only for $|z| < 1$.

The presence of the logarithmic cut means that we must specify the value of the logarithm in order to compute the integral and establish its existence. Moreover for seeing in a clear way the connection of the previous function with the usual hypergeometric ${}_2F_1$ and therefore the absence of the cut between $z = 0$ and $z = 1$ we can expand it for $|z| < 1$. To do so we close the path Γ on the left but this can be done only if $-\pi < arg(-z) < \pi$ because the modulus of the integrand behaves as $|z|^{Re(s)} e^{Im(s)\,[-\pi\, sign(arg(s) - arg(-z)]}$ for large $|s|$. When we close on the left we find the original perturbative solution (12)

$$F\begin{pmatrix} a & b \\ & c \end{pmatrix}; z\Big) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} z^n = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} {}_2F_1(a,b;c;z). \tag{17}$$

In a similar way for $|z| > 1$ and still with $-\pi < arg(-z) < \pi$ we can close the Γ path on the right and get for $b - a \notin \mathbb{Z}$

$$F\left(\begin{matrix} a & b \\ & c \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{\Gamma(b-a-n)\Gamma(a+n)}{\Gamma(c-a-n)\,n!} (-1)^n (-z)^{-a-n} + \sum_{n=0}^{\infty} \frac{\Gamma(a-b-n)\Gamma(b+n)}{\Gamma(c-b-n)\,n!} (-1)^n (-z)^{-b-n} \tag{18}$$

$$= \frac{\sin[\pi(c-a)]}{\sin[\pi(b-a)]} \left(-\frac{1}{z}\right)^{-a} F\left(\begin{matrix} a & a+1-c \\ & a+1-b \end{matrix}; \frac{1}{z}\right) - \frac{\sin[\pi(c-b)]}{\sin[\pi(b-a)]} \left(-\frac{1}{z}\right)^{-b} F\left(\begin{matrix} b & b+1-c \\ & b+1-a \end{matrix}; \frac{1}{z}\right), \tag{19}$$

where the second equality is obtained using the perturbative definition of $F\left(\begin{matrix} a & b \\ & c \end{matrix}; \frac{1}{z}\right)$ given in eq. (12) with argument $1/z$.

Once we have decided where the cuts are we can compute the series expansion of the basis of solutions (14) around $z = 0$ as in eq. (12) or around $z = \infty$ as in eq. (18) by expanding the Barnes integral for $|z| > 1$ and closing the path to the left.

We can then relate the basis of solutions (14) in z with the basis of solutions in $1/z$ as

$$\mathcal{B}_{\{\mathbf{0}\}}(z) = C\, \mathcal{B}_{\{\infty\}}(z) = \frac{1}{\sin[\pi(b-a)]} \begin{pmatrix} \sin[\pi(c-a)] & -\sin[\pi(c-b)] \\ -\sin[\pi a] & \sin[\pi b] \end{pmatrix} \mathcal{B}_{[\infty]}(z). \tag{20}$$

It is then immediate to compute the monodromies for the basis of solutions in z (14) as

$$M_{\{\mathbf{0}\}[\mathbf{0}]} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i2\pi c} \end{pmatrix} \tag{21}$$

$$M_{\{\mathbf{0}\}[\infty]} = \frac{i\, e^{i\pi(a+b)}}{\sin(\pi c)} \times \begin{pmatrix} -\cos[\pi(a+b-c)] + e^{-i\pi c}\cos[\pi(a-b)] & -2\sin[\pi(c-a)]\sin[\pi(c-b)] \\ +2\sin[\pi a]\sin[\pi b] & \cos[\pi(a+b-c)] - e^{+i\pi c}\cos[\pi(a-b)] \end{pmatrix}, \tag{22}$$

where the second expression comes from the obvious monodromy at $z = \infty$ for the basis of solutions in $1/z$ (15)

$$M_{\{\infty\}[\infty]} = \begin{pmatrix} e^{i2\pi a} & 0 \\ 0 & e^{i2\pi b} \end{pmatrix} \tag{23}$$

along with eq. (20).

These monodromy matrices are the same whether we start in the upper of lower half-plane because of our choice of cuts. This is not however true for the monodromy matrices $M^{(\pm)}_{\{\mathbf{0}\}[\mathbf{1}]}$ around $z = 1$ which do generically[7] depend on the base point. From the fact that monodromy matrices are a representation of the homotopy group (3) they can be derived as

[7] It can happen that the monodromy group is abelian and hence there is no difference.

$$M^{(+)}_{\{\mathbf{0}\}[\mathbf{1}]} = M^{-1}_{\{\mathbf{0}\}[\mathbf{0}]} M^{-1}_{\{\mathbf{0}\}[\infty]}, \quad M^{(-)}_{\{\mathbf{0}\}[\mathbf{1}]} = M^{-1}_{\{\mathbf{0}\}[\infty]} M^{-1}_{\{\mathbf{0}\}[\mathbf{0}]}. \tag{24}$$

A naive way of understanding why this happens is to look at Fig. 7 and realize that the neighborhood of $z = 1$ is cut into two disconnected pieces by the cuts.

Obviously the same relations hold also for the matrices $M_{\{\infty\}}$ which are obtained starting from the good basis at $z = \infty$ (20) since the two sets of matrices are connected by a conjugation by the C matrix.

2.3. $U(2)$ monodromies: constraints on the Papperitz equation

As sketched in the introduction and better explained in the following sections we want to use the doublet of solutions as the key element to build solutions to the string e.o.m with monodromies in $U(2)$. Therefore we are interested in finding a doublet of functions whose monodromies belong to $U(2)$. As it is clear from eqs. (21) that $M_{\{\mathbf{0}\}[\mathbf{0}]}$ is in $U(2)$ for real c, $M_{\{\mathbf{0}\}[\infty]}$ does not generically belong to $U(2)$. Therefore the doublet of solutions of the hypergeometric equation (11) given by the basis (14) is not what we are looking for but it is a close relative.

To fix the problem around $z = 0$ and $z = \infty$ we can consider

$$\begin{aligned}\mathcal{E}_{\{\mathbf{0}\}}(z) &= (-z)^d (1-z)^{f-d} D^{-1}_{\{\mathbf{0}\}} \mathcal{B}_{\{\mathbf{0}\}}(z) \\ &= \begin{pmatrix} d^{-1}_{\{\mathbf{0}\}1} (-z)^d (1-z)^{f-d} F\begin{pmatrix} a & b \\ & c \end{pmatrix}; z \end{pmatrix} \\ d^{-1}_{\{\mathbf{0}\}2} (-z)^{d+1-c} (1-z)^{f-d} F\begin{pmatrix} a+1-c & b+1-c \\ & 2-c \end{pmatrix}; z \end{pmatrix} \end{pmatrix}\end{aligned} \tag{25}$$

which is a solution of a Fuchsian equation with three singular points at $z = 0$, 1 and ∞ and where we have allowed for arbitrary complex rescaling

$$D_{\{\mathbf{0}\}} = \begin{pmatrix} d_{\{\mathbf{0}\}1} & \\ & d_{\{\mathbf{0}\}2} \end{pmatrix}. \tag{26}$$

Comparing the indices we see therefore that our solution is within the general solution represented by the Papperitz–Riemann symbol as

$$P \left\{ \begin{matrix} 0 & 1 & \infty & \\ d & f-d & a-f & z \\ 1-c+d & c+f-a-b-d & b-f & \end{matrix} \right\}. \tag{27}$$

Notice that given the previous symbol we can easily find two independent solutions but generically these solutions will not generate the desired monodromies. In order to get the desired we need to normalize and recombine the solutions to get the solution in eq. (25) back.

We parametrize a $U(2)$ matrix as

$$\begin{aligned} U &= \exp(i\, 2\pi\, N) \exp(i\, 2\pi\, n^i \sigma_i) \\ &= e^{i\, 2\pi\, N} \left(\cos(2\pi n) \mathbb{I}_2 + i \sin(2\pi n) \frac{\vec{n}}{n} \cdot \vec{\sigma} \right), \end{aligned} \tag{28}$$

with

$$(N, \vec{n}) \in \left\{ -\frac{1}{4} \le N < \frac{1}{4}, 0 \le n \le \frac{1}{2} \right\} / \sim \quad (N, \vec{n}) \sim (N', \vec{n}') \text{ iff } N = N', n = n' = \frac{1}{2}, \tag{29}$$

where $\vec{n} = (n^1, n^2, n^3)$ and $n = |\vec{n}|$. In Appendix C we report some useful formula such as the effect on the parameters $N, \vec{n}$ given by the product of two elements or the opposite of an element.

We are then interested in the relation among the parameters a, b, c, d, f, $d_{\{\mathbf{0}\}1}$ and $d_{\{\mathbf{0}\}2}$ and the parameters $N_{[\mathbf{0}]}, \vec{n}_{[\mathbf{0}]}$ and $N_{[\infty]}, \vec{n}_{[\infty]}$ which parametrize the $U(2)$ monodromy matrices $U_{[\mathbf{0}]} = U(N_{[\mathbf{0}]}, \vec{n}_{[\mathbf{0}]}) = M_{\{\mathbf{0}\}[\mathbf{0}]}$ around $z = 0$ and $U_{[\infty]} = U(N_{[\infty]}, \vec{n}_{[\infty]})$ (related to $M_{\{\mathbf{0}\}[\infty]}$ by a rescaling) around $z = \infty$ for the basis around $z = 0$ given by $\mathcal{E}_{[\mathbf{0}]}(z)$ in eq. (25).

As discussed in the Appendix D the monodromy around $z = 0$ can only be in the maximal torus of $U(2)$ and we get $U(2)$ monodromies when the parameters are real

$$a, b, c, d, f \in \mathbb{R}. \tag{30}$$

Moreover they must satisfy the constraint

$$-\sin(\pi a)\sin(\pi b)\sin[\pi(a-c)]\sin[\pi(b-c)] > 0 \tag{31}$$

which is necessary in order to be able to find a value for $\frac{d_{\{\mathbf{0}\}2}}{d_{\{\mathbf{0}\}1}}$. In fact the ratio of the moduli of parameters $d_{\{\mathbf{0}\}1}$ and $d_{\{\mathbf{0}\}2}$ is fixed in order to have $U(2)$ monodromies as

$$\left|\frac{d_{\{\mathbf{0}\}2}}{d_{\{\mathbf{0}\}1}}\right|^2 = -\frac{\sin(\pi a)\sin(\pi b)}{\sin[\pi(a-c)]\sin[\pi(b-c)]}. \tag{32}$$

Their relative phase $e^{i2\pi\delta_{[\mathbf{0}]}}$ defined by

$$\frac{d_{\{\mathbf{0}\}2}}{d_{\{\mathbf{0}\}1}} = \left|\frac{d_{\{\mathbf{0}\}2}}{d_{\{\mathbf{0}\}1}}\right| e^{i2\pi\delta_{[\mathbf{0}]}}, \qquad -\frac{1}{2} \le \delta_{[\mathbf{0}]} < \frac{1}{2} \tag{33}$$

is arbitrary. Nevertheless it fixes part of the information on the versor associated to $\vec{n}_{[\infty]}$ as it enters the last of eqs. (38).

It is then immediate to see that the monodromy around $z = 0$ has $U(2)$ parameters

$$\begin{aligned} 2N_{[\mathbf{0}]} &= 2d - c + k_{N_{[\mathbf{0}]}} \\ 2n^3_{[\mathbf{0}]} &= c + k_{n_{[\mathbf{0}]}} \\ n^1_{[\mathbf{0}]} &= n^2_{[\mathbf{0}]} = 0 \end{aligned} \tag{34}$$

where the integers $k_{N_{[\mathbf{0}]}} \in \mathbb{Z}$ and $k_{n_{[\mathbf{0}]}} \in \mathbb{Z}$ are uniquely fixed by the range in which $N_{[\mathbf{0}]}, n_{[\mathbf{0}]3}$ can vary and by

$$k_{N_{[\mathbf{0}]}} \equiv k_{n_{[\mathbf{0}]}} \quad mod\ 2. \tag{35}$$

Explicitly $0 \le N_{[\mathbf{0}]} < \frac{1}{2}$ fixes $k_{N_{[\mathbf{0}]}}$ then this fixes the parity of $k_{n_{[\mathbf{0}]}}$ and $-\frac{1}{2} \le n_{[\mathbf{0}]3} < \frac{1}{2}$ fixes it completely.

Similarly from the trace of the monodromy around $z = \infty$ we find that $U(2)$ has parameters

$$\begin{aligned} 2N_{[\infty]} &= a + b - 2f + k_{N_{[\infty]}} \\ 2n_{[\infty]} &= (-)^{s_{n[\infty]}}(a - b) + k_{n_{[\infty]}} \end{aligned} \tag{36}$$

where the integers $k_{N_{[\infty]}} \in \mathbb{Z}$ and $k_{n_{[\infty]}} \in \mathbb{Z}$ and the “sign” $s_{n[\infty]} \in \{0, 1\}$ are again uniquely fixed by the range in which $N_{[\infty]}, n_{[\infty]}$ can vary and by

$$k_{N_{[\infty]}} \equiv k_{n_{[\infty]}} \quad mod\ 2. \tag{37}$$

Moreover we have also

$$\frac{n^3_{[\infty]}}{n_{[\infty]}} = (-)^{s_{n[\infty]}+1} \frac{\cos[\pi(a+b-c)] - \cos(\pi c)\cos[\pi(a-b)]}{\sin(\pi c)\sin[\pi(a-b)]}$$
$$\frac{n^1_{[\infty]} + i\, n^2_{[\infty]}}{n_{[\infty]}} = e^{-i2\pi\delta_{[\mathbf{0}]}} (-)^{s_{n[\infty]}+1}\, sign(\sin(\pi a)\sin(\pi b))$$
$$\frac{\sqrt{-4\sin(\pi a)\sin(\pi b)\sin[\pi(a-c)]\sin[\pi(b-c)]}}{\sin(\pi c)\sin[\pi(a-b)]}. \tag{38}$$

2.4. From $U(2)$ monodromies to parameters of Papperitz equation

The previous equations can be also inverted. In this way we can find the parameters a, b, c, d, f and $d_{\{\mathbf{0}\}2}/d_{\{\mathbf{0}\}1}$ given the $U(2)$ monodromies at $z = 0$ and $z = \infty$. The former must be in the maximal torus of $U(2)$ generated by the Cartan subalgebra and is characterized by $N_{[\mathbf{0}]}$ and $n^3_{[\mathbf{0}]}$. The latter is a generic $U(2)$ matrix and is fixed by giving $N_{[\infty]}$ and $\vec{n}_{[\infty]}$.

With a simple algebra we find for $(-1)^{s_{n[\infty]}} = +1$[8]

$$\begin{aligned}
a &= n_{[\mathbf{0}]3} + n_{[\infty]} + (-)^{s_A} A + k_a \\
b &= n_{[\mathbf{0}]3} - n_{[\infty]} + (-)^{s_A} A + k_b \\
c &= 2n_{[\mathbf{0}]3} + k_c \\
d &= n_{[\mathbf{0}]3} + N_{[\mathbf{0}]} + k_d \\
f &= n_{[\mathbf{0}]3} - N_{[\infty]} + (-)^{s_A} A + k_f,
\end{aligned} \tag{39}$$

where all ks are arbitrary integers, $(-)^{s_A} \in \{\pm 1\}$ and the quantity A with $0 \le A < 1/2$ is defined as[9]

$$\cos(2\pi A) = \cos(2\pi n_{[\infty]})\cos(2\pi n_{[\mathbf{0}]3}) - \sin(2\pi n_{[\infty]})\sin(2\pi n_{[\mathbf{0}]3})\frac{n_{[\infty]3}}{n_{[\infty]}}. \tag{40}$$

In order to fix almost completely the solution we need also

$$\left|\frac{d_{\{\mathbf{0}\}2}}{d_{\{\mathbf{0}\}1}}\right|^2 = -\frac{\sin(\pi a)\sin(\pi b)}{\sin[\pi(a-c)]\sin[\pi(b-c)]}$$
$$\frac{d_{\{\mathbf{0}\}2}}{d_{\{\mathbf{0}\}1}} \Big/ \left|\frac{d_{\{\mathbf{0}\}2}}{d_{\{\mathbf{0}\}1}}\right| = e^{-i2\pi\delta_{\{\mathbf{0}\}}}$$
$$= -\frac{n_{[\infty]1} - i\, n_{[\infty]2}}{\sqrt{n^2_{[\infty]1} + n^2_{[\infty]2}}}\, sign\left(\sin(\pi a)\sin(\pi b)\sin(\pi c)\sin[\pi(a-b)]\right). \tag{41}$$

We then fix completely our definition of the solution by choosing

[8] For $(-1)^{s_{n[\infty]}} = -1$ simply exchange a with b.

[9] Because of the relation (24) among the monodromies we can expect that A is connected with $n_{[\mathbf{1}]}$. Using eq. (168) we can establish in a more precise way this relation.

When $-\frac{1}{4} \le -N_{[\mathbf{0}]} - N[\infty] < \frac{1}{4}$, i.e. $-N_{[\mathbf{0}]} - N[\infty]$ is in the proper definition range the quantity A is actually $n_{[\mathbf{1}]}$ the modulus of the vector $\vec{n}_{[\mathbf{1}]}$ which parametrizes the $SU(2)$ part of the upper half plane monodromy matrix $M^{(+)}_{[\mathbf{0}][\mathbf{1}]}$ in $z = 1$ as defined in eq. (24). Moreover $N_{[\mathbf{1}]} = -N_{[\mathbf{0}]} - N[\infty]$.

When $-\frac{1}{2} < -N_{[\mathbf{0}]} - N[\infty] < -\frac{1}{4}$, the quantity A is actually $\frac{1}{2} - n_{[\mathbf{1}]}$ and $N_{[\mathbf{1}]} = \frac{1}{2} - N_{[\mathbf{0}]} - N[\infty]$.

Finally, when $\frac{1}{4} \le -N_{[\mathbf{0}]} - N[\infty] \le \frac{1}{2}$, the quantity A is actually $\frac{1}{2} - n_{[\mathbf{1}]}$ and $N_{[\mathbf{1}]} = -\frac{1}{2} - N_{[\mathbf{0}]} - N[\infty]$.

$$D_{\{\mathbf{0}\}} = \begin{pmatrix} 1 & \\ & \left|\frac{d_{\{\mathbf{0}\}2}}{d_{\{\mathbf{0}\}1}}\right| e^{-i2\pi\delta_{\{\mathbf{0}\}}} \end{pmatrix}, \quad -\frac{1}{2} \le \delta_{\{\mathbf{0}\}} < \frac{1}{2}. \tag{42}$$

Using the explicit values of the parameters it is possible to verify that

$$\begin{aligned} &-\sin(\pi a)\sin(\pi b)\sin[\pi(a-c)]\sin[\pi(b-c)] \\ &\quad = \sin^2(2\pi n_{[\infty]})\sin^2(2\pi n_{[\mathbf{0}]3})\left[1-\left(\frac{n_{[\infty]3}}{n_{[\infty]}}\right)^2\right] \end{aligned}$$

and therefore that the previous expression for $\left|\frac{d_{\{\mathbf{0}\}2}}{d_{\{\mathbf{0}\}1}}\right|$ is always meaningful.

At first sight the presence of the sign ambiguity $(-)^{s_A} \in \{\pm 1\}$ would hint to the existence of two different families of solutions where each member of any class is labeled by the integers ks. It is not so. This can be seen using Euler relation

$$F\begin{pmatrix} a & & b \\ & c & \end{pmatrix}; z) = (1-z)^{c-a-b} F\begin{pmatrix} c-a & & c-b \\ & c & \end{pmatrix}; z). \tag{43}$$

Denoting by $\tilde{a}, \tilde{b}, \ldots$ the quantities for $(-)^{s_A} = -1$ it is easy to verify that

$$(-z)^{\tilde{d}}(1-z)^{\tilde{f}-\tilde{d}} F\begin{pmatrix} \tilde{a} & & \tilde{b} \\ & \tilde{c} & \end{pmatrix}; z) = (-z)^{d}(1-z)^{f-d} F\begin{pmatrix} a & & b \\ & c & \end{pmatrix}; z) \tag{44}$$

when $\tilde{k}_a = k_c - k_b$, $\tilde{k}_b = k_c - k_a$, $\tilde{k}_c = k_c$, $\tilde{k}_d = k_d$ and $\tilde{k}_f = k_f + k_c - k_a - k_b$. The analogous relation for the second component of $\mathcal{E}$ is then also satisfied.

This equivalence can be seen in a less precise way by comparing the Papperitz–Riemann symbols associated to the two solutions. Using eq. (27) we write for the $(-)^{s_A} = +1$ solution

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ n_{[\mathbf{0}]3} + N_{[\mathbf{0}]} + k_d & A - N_{[\mathbf{0}]} - N_{[\infty]} + k_f - k_d & n_{[\infty]} + N_{[\infty]} + k_a - k_f & z \\ -n_{[\mathbf{0}]3} + N_{[\mathbf{0}]} + k_d - k_c + 1 & -A - N_{[\mathbf{0}]} - N_{[\infty]} + k_f + k_c - k_a - k_b - k_d & -n_{[\infty]} + N_{[\infty]} + k_b - k_f & \end{matrix}\right\} \tag{45}$$

and for the $(-)^{s_A} = -1$ solution

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ n_{[\mathbf{0}]3} + N_{[\mathbf{0}]} + \tilde{k}_d & -A - N_{[\mathbf{0}]} - N_{[\infty]} + \tilde{k}_f - \tilde{k}_d & n_{[\infty]} + N_{[\infty]} + \tilde{k}_a - \tilde{k}_f & z \\ -n_{[\mathbf{0}]3} + N_{[\mathbf{0}]} + \tilde{k}_d - \tilde{k}_c + 1 & +A - N_{[\mathbf{0}]} - N_{[\infty]} + \tilde{k}_f + \tilde{k}_c - \tilde{k}_a - \tilde{k}_b - \tilde{k}_d & -n_{[\infty]} + N_{[\infty]} + \tilde{k}_b - \tilde{k}_f & \end{matrix}\right\}. \tag{46}$$

The two symbols coincide again when we use the previous identifications therefore the two families are actually the same.

As we have discussed above there are actually two possible monodromies at $z = 1$ (24) depending on whether our base point is in the upper or lower half plane, nevertheless since the $SU(2)$ in $z = 0$ is in the maximal torus, i.e. $\vec{n}_{[\mathbf{0}]} = n_{[\mathbf{0}]3}\vec{k}$ there is no difference in the modulus of $\vec{n}_{[\mathbf{1}]}$. Actually also $n_{[\mathbf{1}]3}$ is invariant and only $n_{[\mathbf{1}]1}$ and $n_{[\mathbf{1}]2}$ are different.

2.5. The complete abelian solution cannot be recovered

It is then interesting to take the abelian limit of the $U(2)$ monodromies, i.e. consider the case where all monodromies are in the maximal torus $U(1)^2$. This in order to make contact with the previous papers dealing with the factorized cases. Naively this seems to be possible since we have monodromies in $U(2)$ but this is not actually the case. One intuitive reason is that we are not considering the general monodromies in R^4. We can nevertheless obviously obtain a $U(1)$ monodromy at the price of setting to zero one of the two solutions of the $U(2)$ case.

More technically the reason why we cannot obtain solutions with $U(1)^2 \subset U(2)$ monodromy is the following. Suppose we want to find a second order equation which has as solutions the following two abelian solutions

$$w_1 = (-z)^{\epsilon_{[\mathbf{0}]1}}(1-z)^{\epsilon_{[\mathbf{1}]1}}$$
$$w_2 = (-z)^{\epsilon_{[\mathbf{0}]2}}(1-z)^{\epsilon_{[\mathbf{1}]2}}, \tag{47}$$

then the indices would be $\epsilon_{[\mathbf{0}]1}, \epsilon_{[\mathbf{1}]1}, \epsilon_{[\infty]1} = -\epsilon_{[\mathbf{0}]1} - \epsilon_{[\mathbf{0}]1}$ and $\epsilon_{[\mathbf{0}]2}, \epsilon_{[\mathbf{1}]2}, \epsilon_{[\infty]2} = -\epsilon_{[\mathbf{0}]2} - \epsilon_{[\mathbf{0}]2}$. It follows immediately that the sum of the indices is zero and therefore there cannot exist a Fuchsian second order equation with these solutions since for any Fuchsian second order equation with three singularities the sum of indices is one.

This can also be confirmed directly computing the associated second order differential equation for w as

$$\begin{vmatrix} w'' & w' & w \\ w_1'' & w_1' & w_1 \\ w_2'' & w_2' & w_2 \end{vmatrix} = 0, \tag{48}$$

and checking that the leading behavior for $z \to \infty$ for the coefficient of w is $O\left(\frac{1}{z^2}\right)$ while it should be $O\left(\frac{1}{z^4}\right)$ in order to be Fuchsian.

In any case it is interesting to consider how far we can go trying to recover the abelian solutions. We can actually recover one of the two abelian solutions while the other is set to zero because of the scaling coefficients $d_{\{\mathbf{0}\}1}$ and $d_{\{\mathbf{0}\}2}$.

Explicitly the abelian case corresponds to $\vec{n}_{[\mathbf{0}]} = n_{[\mathbf{0}]3}\vec{k}$ and $\vec{n}_{[\infty]} = n_{[\infty]3}\vec{k}$. Then from eq. (40) we read immediately that

$$A = \begin{cases} |n_{[\mathbf{0}]3} + n_{[\infty]3}| & |n_{[\mathbf{0}]3} + n_{[\infty]3}| < \frac{1}{2} \\ 1 - |n_{[\mathbf{0}]3} + n_{[\infty]3}| & |n_{[\mathbf{0}]3} + n_{[\infty]3}| > \frac{1}{2} \end{cases}. \tag{49}$$

In any of these cases one of the quantities a, b, $a-c$ and $b-c$ is an integer and therefore the ratio $\frac{d_{\{\mathbf{0}\}2}}{d_{\{\mathbf{0}\}1}}$ is either zero or infinity. This means that either $d_{\{\mathbf{0}\}1}$ or $d_{\{\mathbf{0}\}2}$ is zero and hence that one of the components of the vector of the basis solutions $\mathcal{E}_{[\mathbf{0}]}(z)$ is zero. Another way of explaining this it is to notice that the monodromy matrix at $z=\infty$ cannot ever become diagonal but at most triangular.

3. String action and branes configuration

Our aim is to describe the configuration of three $D2$ branes in R^4 with global monodromies in $U(2) \subset SO(4)$. Here we discuss the simplest possible setting.

The part of our interest of the Euclidean action for the string in conformal gauge is given by

$$S_E = \frac{1}{2\pi\alpha'} \int_H d^2u \, \frac{1}{2} \partial_u X^I \bar{\partial}_{\bar{u}} X^I$$
$$= \frac{1}{2\pi\alpha'} \int_H d^2u \, \frac{1}{2} \left(\partial_u Z^i \bar{\partial}_{\bar{u}} \bar{Z}^i + \bar{\partial}_{\bar{u}} Z^i \partial_u \bar{Z}^i \right) \tag{50}$$

where $u, v, \dots \in H$ belong to the upper half plane H ($\Im u \geq 0$), $I = 1, \dots 2N$ (for the case of our interest $N=2$) are the labels of flat coordinates and $Z^i = \frac{1}{\sqrt{2}}(X^i + iX^{N+i})$ are complex flat

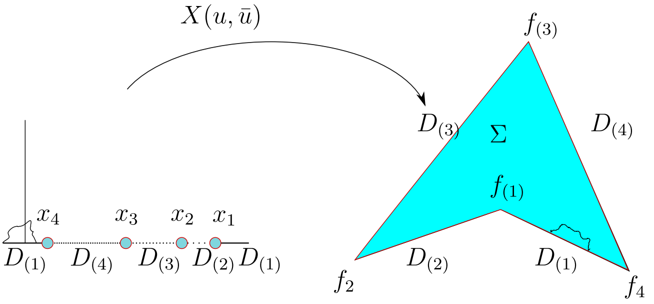

Fig. 9. Map from the upper half plane to the target polygon Σ with untwisted in and out strings. The map $X(u,\bar{u})$ folds the boundary of the upper half plane starting from $x = -\infty$ in a counterclockwise direction and preserves the orientation.

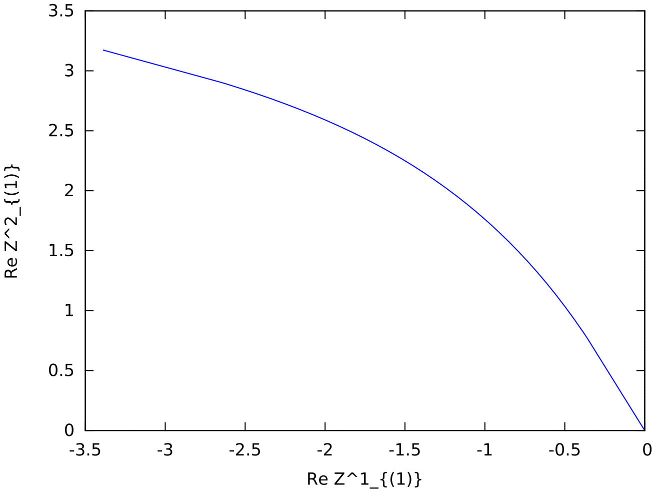

Fig. 10. The string endpoint on a brane. A flat surface would intersect a brane along a segment. Data are with numerical error $< 10^{-15}$.

coordinates with $i = 1, \dots N$. The complex string coordinate is a map from the upper half plane to a real surface with boundaries in $\mathbb{R}^4$.

In the cases considered previously the map was from the upper half plane to a polygon Σ in $\mathbb{C} \equiv \mathbb{R}^2$, i.e. $X : H \to \Sigma \subset \mathbb{C}$. For example in Fig. 9 we have pictured the interaction of $\hat{N}_B = 4$ branes at angles $D_{(t)}$ with $t = 1, \dots \hat{N}_B$. The interaction between brane $D_{(t)}$ and $D_{(t+1)}$ is in $f_{(t)} \in \mathbb{C}$. We use the conventions that index t is defined modulo $\hat{N}_B$ and that

$$x_t < x_{t-1}. \tag{51}$$

In the case we consider we have only three branes and therefore only three interaction points. These interaction points always define a 2 dimensional real plane in $\mathbb{R}^4$ but differently from the cases discussed before in the literature the embedding of the string worldsheet which follows from the equation of motion is not a flat triangle, i.e. a triangle which lies in the plane determined by the three interactions points. In fact Fig. 10 shows the actual line traced by the endpoint of the classical string. This line should be compared with the naive path which is a segment.

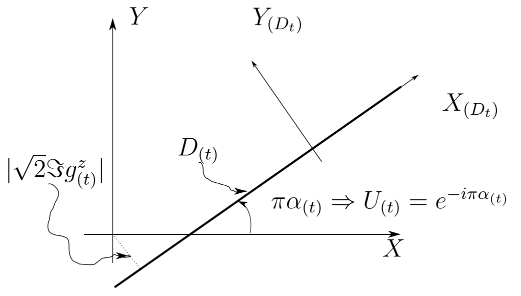

Fig. 11. The relation between local and global coordinates in the simplest case of $\mathbb{R}^2$. The bold line is the brane D_t with local coordinates $Z_{(D_t)} = (X_{(D_t)} + iY_{(D_t)})/\sqrt{2}$. It has a distance $|\sqrt{2}\Im g^z_{(t)}|$ from the origin.

Using the notation described in the next subsection we can describe more precisely the setup. The endpoint is shown in good local coordinates for brane $D2_{(1)} \subset \mathbb{R}^4$ with embedding $\Im Z^i_{(D_1)} = 0$. The embedding matrices are

$$U_{(1)} = e^{i2\pi\, 0.4\,\sigma_3},\ U_{(2)} = e^{i2\pi\,(0.3\,\sigma_1+0.4\,\sigma_2+0.5\,\sigma_3)},\ U_{(3)} = e^{i2\pi\,(0.5\,\sigma_1+0.6\,\sigma_2+0.7\,\sigma_3)}$$

$$M_{(1)} = \begin{pmatrix} 0.276341\,i - 0.00249408 & 0.914018 - 0.296982\,i \\ -0.296982\,i - 0.914018 & -0.276341\,i - 0.002494081 \end{pmatrix}$$

3.1. Local and global branes configuration

Any brane $D_{(t)}$ $t = 1, \ldots \hat{N}_B$ can be described in locally adapted real coordinates $X^I_{(D_t)}$ as

$$X^{N_{(t)}}_{(D_t)} = 0, \tag{52}$$

where $N_{(t)}$ runs over the normal directions. For a $D2$ in $\mathbb{C}^2 \equiv \mathbb{R}^4$ we have $I = 1, 2, 3, 4$ and the normal directions can be taken to be in two classes either $N_{(t)} = 3, 4$ (more generally $N_{(t)} = N + i$ with $i = 1 \ldots N$) or $N_{(t)} = 1, 3$ (more generally $N_{(t)} = i, N + i$ with $i = 1 \ldots N/2$). The former leads to an embedding in locally adapted complex coordinates as $Z^1_{(D_t)} = 0$ while the former leads to

$$\Im Z^1_{(D_t)} = \Im Z^2_{(D_t)} = 0. \tag{53}$$

We want to make contact with what usually done for $D1$ embedded into $\mathbb{C}$ where the embedding is described $\Im Z^1_{(D_t)} = 0$ therefore we use this former embedding given in eqs. (53) even if it is apparently less elegant. Because of this choice the tangent directions index T runs over the complementary coordinates which in the $D2$ in R^4 case are $T = 3, 4$.

The locally adapted complex coordinates are connected to the global complex coordinates used in defining the string action by a roto-translation as

$$Z^i_{(D_t)} = U^i_{(t)\,j} Z^j - g^i_{(t)}, \quad i = 1, \ldots N, \tag{54}$$

where the rotation is restricted to $U(N) \subset O(N)$. This is shown in Fig. 11 in the case of $\mathbb{R}^2$ where we have set $U_{(t)} = e^{-i\pi\alpha_t}$ $(0 \le \alpha_t < 1)$ to make contact with the notation used in previous papers [10].

Our choice is dictated by the fact that we want to find the simplest configurations which lead to $U(2)$ monodromies of our interest.

This means that the local embedding conditions for $D_{(t)}$ can be written in global coordinates as

$$\Im\left[U^i_{(t)\,j} Z^j - g^i_{(t)}\right] = 0, \quad i = 1, \dots N, \tag{55}$$

and hence only $\Im g_{(t)}$ matters for determining the embedding. Their real d.o.f.s are equal to the number of dimensions transverse to the brane as it should.

Let us now count the d.o.f.s needed to specify the configuration. This will teach us something about the kind of configurations we consider. We start considering the d.o.f.s in rotation matrices only and then we discuss the full problem with the shifts $\Im g_{(t)}$. Would we consider the generic configuration then we would consider the Grassmannian $SO(2N)/SO(N) \times SO(N)$ with N^2 real d.o.f.s. On the other side we do not consider its pure complex version $U(N)/U(N/2) \times U(N/2)$ with $N^2/2$ real d.o.f.s. To see what we are actually doing we start by rewriting eq. (54) in real coordinates, explicitly

$$X^I_{(D_t)} = R^I_{(t)\ J} X^J - g^I_{(t)}, \quad R_{(t)} = \begin{pmatrix} \Re U^i_{(t)\,j} & -\Im U^i_{(t)\,j} \\ \Im U^i_{(t)\,j} & \Re U^i_{(t)\,j} \end{pmatrix}, \quad g_{(t)} = \sqrt{2} \begin{pmatrix} \Re g^i_{(t)} \\ \Im g^i_{(t)} \end{pmatrix}, \tag{56}$$

then transformations $\begin{pmatrix} X^i_{(D_t)} \\ X^{N+i}_{(D_t)} \end{pmatrix} = \begin{pmatrix} O_\parallel & \\ & O_\perp \end{pmatrix} \begin{pmatrix} X^i_{(D_t)}{}' \\ X^{N+i}_{(D_t)}{}' \end{pmatrix}$ with $O_\parallel, O_\perp \in SO(N)$ keep the embedding equations invariant but destroy the relation between the good and global coordinates. This force us to consider $O_\parallel = O_\perp$ hence rotations we consider are in $U(N)/SO(N)$ and have $N(N+1)/2$ real d.o.f.s. To these d.o.f.s we need then to add other N real d.o.f.s associated with the shift $\Im g_{(t)}$ in order to completely specify the embedding.

We conclude therefore that our configuration with $\hat{N}_B$ branes require $\hat{N}_B N(N+3)/2$ real d.o.f.s to be specified. This number will be the same when we count it in a different way the next section.

3.2. *String boundary conditions*

It is immediate to write down the e.o.m. associated with the action (50) as

$$(\partial_x^2 + \partial_y^2) X^I = \partial_u \bar{\partial}_{\bar{u}} X^I = 0 \tag{57}$$

along with their general solution as

$$X^I(u, \bar{u}) = X^I_L(u) + X^I_R(\bar{u}), \tag{58}$$

with $u \in H$. On this solution we must impose the boundary conditions. In local real adapted coordinates they read

$$X^{N_{(t)}}_{(D_t)}|_{y=0} = \partial_y X^{P_{(t)}}_{(D_t)}|_{y=0} \quad x_t < x < x_{t-1}, \tag{59}$$

where $N_{(t)}$ runs over the normal directions and $P_{(t)}$ runs over the parallel directions to the brane D_t. The same conditions can be expressed using the well adapted complex coordinates as

$$\Im\left[Z^i_{(D_t)}\right]|_{y=0} = \Re\left[\partial_y Z^i_{(D_t)}\right]|_{y=0} = 0, \quad x_t < x < x_{t-1}, \tag{60}$$

where we have supposed that the string boundary lies on $D_{(t)}$ when $x_t < x < x_{t-1}$ and that the normal and tangent directions in local coordinates are always labeled by the same indexes. These boundary conditions imply (but are not equivalent because of the necessity of taking a derivative)

$$\begin{pmatrix} \bar\partial_{\bar u} Z_{(D_t)R}(x-i0^+) \\ \bar\partial_{\bar u} \bar Z_{(D_t)R}(x-i0^+) \end{pmatrix} = \mathcal{R}_{(D_t)(Z)} \begin{pmatrix} \partial_u Z_{(D_t)L}(x+i0^+) \\ \partial_u \bar Z_{(D_t)L}(x+i0^+) \end{pmatrix}, \tag{61}$$

where the reflection matrix $\mathcal{R}_{(D_t)(Z)(t)}$ in local adapted complex coordinates $Z_{(D_t)}$ is given by

$$\mathcal{R}_{(D_t)(Z)} = \begin{pmatrix} & \mathbb{I}_N \\ \mathbb{I}_N & \end{pmatrix}, \tag{62}$$

and is idempotent $\mathcal{R}^2_{(D_t)(Z)(t)} = \mathbb{I}_{2N}$.

In global coordinates the previous equations become

$$\Im\left[U^i_{(t)\,j}\, Z^j\right]|_{y=0} - \Im g^i_{(t)} = \Re\left[U^i_{(t)\,j}\, \partial_y Z^j\right]|_{y=0} = 0, \quad x_t < x < x_{t-1}. \tag{63}$$

Then the global boundary conditions are equivalent to the following conditions

$$\begin{pmatrix} \bar\partial_{\bar u} Z_R(x-i0^+) \\ \bar\partial_{\bar u} \bar Z_R(x-i0^+) \end{pmatrix} = \mathcal{R}_{(Z)(t)} \begin{pmatrix} \partial_u Z_L(x+i0^+) \\ \partial_u \bar Z_L(x+i0^+) \end{pmatrix}, \quad x_t < x < x_{t-1},$$
$$Z(x_t, x_t) = f_{(t)}, \tag{64}$$

where $\mathcal{R}_{(Z)(t)}$ is an idempotent matrix, i.e. $\mathcal{R}^2_{(Z)(t)} = \mathbb{I}_{2N}$ and $f_{(t)}$ is the intersection point between $D_{(t)}$ and $D_{(t+1)}$. The matrix $\mathcal{R}_{(Z)(t)}$ is idempotent since it is conjugated to $\mathcal{R}_{(D_t)(Z)(t)}$. Explicitly $\mathcal{R}_{(Z)(t)}$ is given by

$$\mathcal{R}_{(Z)\,(t)} = \begin{pmatrix} & \mathcal{U}^*_{(t)} \\ \mathcal{U}_{(t)} & \end{pmatrix} = \begin{pmatrix} U_{(t)} & \\ & U^*_{(t)} \end{pmatrix}^{-1} \mathcal{R}_{(D_t)(Z)} \begin{pmatrix} U_{(t)} & \\ & U^*_{(t)} \end{pmatrix}, \tag{65}$$

with

$$\mathcal{U}_{(t)} = \mathcal{U}^T_{(t)} = U^T_{(t)} U_{(t)}. \tag{66}$$

The intersection point[10] between $D_{(t)}$ and $D_{(t+1)}$ $f_{(t)}$ is given by

$$f_{(t)} = 2\, i\, \left[\mathcal{U}_{(t)} - \mathcal{U}_{(t+1)}\right]^{-1} (U^T_{(t)} \Im g_t - U^T_{(t+1)} \Im g_{t+1}). \tag{67}$$

The matrices $\mathcal{R}_{(Z)(t)}$ are somewhat trivial since they are conjugate to a reflection matrix and therefore we have $\mathcal{R}_{(Z)(t)} = \mathcal{R}^{-1}_{(Z)(t)}$ and $\det \mathcal{R}_{(Z)(t)} = (-1)^N$.

Nevertheless we need to specify $\hat N_B N(N+3)/2$ real d.o.f.s in order to fix the boundary conditions completely. $N(N+1)/2$ of these d.o.f.s come from the parameters of the unitary symmetric matrices $\mathcal{U}_{(t)}$ $(t = 1 \dots \hat N_B)$ as it can be easily seen by writing $\mathcal{U} = \exp(iH)$. The other $\hat N_B N$ real d.o.f.s come from the complex vectors $f_{(t)}$ $(t = 1 \dots \hat N_B)$. These are subject to the following constraints

$$\sum_{t=1}^{\hat N_B} \left[\mathcal{U}_{(t)} - \mathcal{U}_{(t+1)}\right] f_{(t)} = 0,$$
$$\left[f_{(t)} - f_{(t-1)}\right]^* = \mathcal{U}_{(t)} \left[f_{(t)} - f_{(t-1)}\right], \tag{68}$$

where the first one is actually a consequence of the second set of constraints. Actually the second set of constraints is simply stating the following geometrical fact: $\{f_{(t)}\} = D_{(t)} \cap D_{(t+1)}$, $\{f_{(t-1)}\} = D_{(t)} \cap D_{(t-1)}$ and therefore $f_{(t)} - f_{(t-1)} \in D_{(t)}$ hence $\Im\left[U_{(t)}(f_{(t)} - f_{(t-1)})\right] = 0$.

[10] It is a point because the co-dimension of the system of two branes is zero.

Therefore for our computation of the d.o.f.s we need to consider the second set of constraints only which halves the real d.o.f.s in the set $\{f_{(t)}\}$ from $2\hat{N}_B N$ real d.o.f.s to $\hat{N}_B N$. The meaning of these constraints is roughly to say that we need the "rotation" matrices $\mathcal{U}_{(t)}$ ($t = 1 \dots \hat{N}_B$), one corner $f_{(\bar{t})}$ for a fixed $\bar{t}$ and the "lengths" of $\hat{N}_B - 2$ sides to describe the string configuration.

3.3. String boundary conditions for double fields

Non-trivial rotation matrices arise when we use the doubling trick. In particular we can glue the upper and lower half planes along the segment $(x_{\bar{t}}, x_{\bar{t}-1})$ which corresponds to the $\bar{t}$ brane as

$$\partial \mathcal{Z}_{(\bar{t})}(z) = \begin{cases} \partial_u Z_L(u) & z = u \in \mathring{H} \cup (x_{\bar{t}}, x_{\bar{t}-1}) \\ \mathcal{U}_{(\bar{t})}^{-1} \bar{\partial}_{\bar{u}} \bar{Z}_R(\bar{u}) & z = \bar{u} \in \mathring{H}^- \cup (x_{\bar{t}}, x_{\bar{t}-1}) \end{cases},$$
$$\partial \tilde{\mathcal{Z}}_{(\bar{t})}(z) = \begin{cases} \partial_u \bar{Z}_L(u) & z = u \in \mathring{H} \cup (x_{\bar{t}}, x_{\bar{t}-1}) \\ \tilde{\mathcal{U}}_{(\bar{t})}^{-1} \bar{\partial}_{\bar{u}} Z_R(\bar{u}) & z = \bar{u} \in \mathring{H}^- \cup (x_{\bar{t}}, x_{\bar{t}-1}) \end{cases}, \tag{69}$$

with $\tilde{\mathcal{U}}_{(\bar{t})} = \mathcal{U}^*_{(\bar{t})}$ and where $\mathring{H}$ is the interior of the upper half plane.

Then the boundary conditions become the discontinuities

$$\partial \mathcal{Z}_{(\bar{t})}(x - i0^+) = \mathcal{U}_{(\bar{t})}^{-1} \mathcal{U}_{(t)}\, \partial \mathcal{Z}_{(\bar{t})}(x + i0^+),$$
$$\partial \tilde{\mathcal{Z}}_{(\bar{t})}(x - i0^+) = \tilde{\mathcal{U}}_{(\bar{t})}^{-1} \tilde{\mathcal{U}}_{(t)}\, \partial \tilde{\mathcal{Z}}_{(\bar{t})}(x + i0^+), \quad x_t < x < x_{t-1}, \tag{70}$$

and the boundary values

$$\mathcal{Z}_{(\bar{t})}(x_t, x_t) = f_{(t)}. \tag{71}$$

Notice that the two fields are independent and not connected by a complex conjugation even if their monodromies are complex conjugate. In fact we find

$$\left[\partial \mathcal{Z}_{(\bar{t})}(z)\right]^* = \mathcal{U}_{(\bar{t})} \partial \mathcal{Z}_{(\bar{t})}(w)|_{w=\bar{z}}$$
$$\left[\partial \tilde{\mathcal{Z}}_{(\bar{t})}(z)\right]^* = \tilde{\mathcal{U}}_{(\bar{t})} \partial \tilde{\mathcal{Z}}_{(\bar{t})}(w)|_{w=\bar{z}}, \tag{72}$$

with $\tilde{\mathcal{U}}_{(\bar{t})} = \mathcal{U}^*_{(\bar{t})}$. As discussed in Appendix B where we pay attention to the ordering of the generically non-commuting matrices $\mathcal{U}_{(u)}^{-1} \mathcal{U}_{(t)}$ the previous discontinuities can be rewritten as monodromies as ($\epsilon \in \mathbb{R}$, $0 < \epsilon$, $min(x_{t-1} - x_t, x_t - x_{t+1})$)

$$\partial \mathcal{Z}_{(\bar{t})}(x_t + e^{i2\pi}(\epsilon + i0^+)) = M_{(t)} \partial \mathcal{Z}_{(\bar{t})}(x_t + \epsilon - i0^+), \quad M_{(t)} = \mathcal{U}_{(t+1)}^{-1} \mathcal{U}_{(t)}$$
$$\partial \tilde{\mathcal{Z}}_{(\bar{t})}(x_t + e^{i2\pi}(\epsilon + i0^+)) = \widetilde{M_{(t)}} \partial \tilde{\mathcal{Z}}_{(\bar{t})}(x_t + \epsilon - i0^+), \quad \widetilde{M_{(t)}} = M^*_{(t)} \tag{73}$$

if we start in the upper half plane and

$$\partial \mathcal{Z}_{(\bar{t})}(x_t + e^{i2\pi}(\epsilon - i0^+)) = \hat{M}_{(\bar{t},t)} \partial \mathcal{Z}_{(\bar{t})}(x_t + \epsilon - i0^+), \quad \hat{M}_{(\bar{t},t)} = \mathcal{U}_{(\bar{t})}^{-1} \mathcal{U}_{(t)} \mathcal{U}_{(t+1)}^{-1} \mathcal{U}_{(\bar{t})}$$
$$\partial \tilde{\mathcal{Z}}_{(\bar{t})}(x_t + e^{i2\pi}(\epsilon - i0^+)) = \widetilde{\hat{M}_{(\bar{t},t)}} \partial \tilde{\mathcal{Z}}_{(\bar{t})}(x_t + \epsilon - i0^+), \quad \widetilde{\hat{M}_{(\bar{t},t)}} = \hat{M}^*_{(\bar{t},t)} \tag{74}$$

if we start in the lower half plane. The previous monodromy matrices are not completely arbitrary $U(N)$ matrices since for example $\hat{M}^T_{(\bar{t},t)} = \mathcal{U}_{(t+1)} \hat{M}^T_{(\bar{t},t)} \mathcal{U}^*_{(t+1)}$. Moreover they satisfy a constraint which follows from the fact they are a representation of the homotopy group given in eqs. (4), for example in the case $N_B = 3$ since $0 < x_3 < x_2 < x_1$ we have

$$M_{(3)} M_{(2)} M_{(1)} = \mathbb{I}. \tag{75}$$

While the existence of two sets of monodromy matrices may seem weird it is necessary to verify that the Euclidean action written using the double fields

$$S_E = \frac{1}{8\pi\alpha'} \int_{\mathbb{C}} d^2 z \, \left(\partial \mathcal{Z}^i_{(\bar{t})}(z)\, \mathcal{U}^i_{(\bar{t})\, j}\, \partial \mathcal{Z}^j_{(\bar{t})}(w)|_{w=\bar{z}} + \partial \tilde{\mathcal{Z}}^i_{(\bar{t})}(z)\, \mathcal{U}^{i\,*}_{(\bar{t})\, j}\, \partial \tilde{\mathcal{Z}}^j_{(\bar{t})}(w)|_{w=\bar{z}} \right) \tag{76}$$

has not any cut. In the previous expression $\mathcal{Z}^j_{(\bar{t})}(w)|_{w=\bar{z}}$ means that $\mathcal{Z}^j_{(\bar{t})}(w)$ is evaluated for $w \to \bar{z}$. Because of this when we go around x_t counterclockwise with $z \in H$ we have $\bar{z} \in H^-$ and we go round clockwise in $\bar{z}$ hence the first factor in the first addend contributes $[\mathcal{U}^{-1}_{(t+1)} \mathcal{U}_{(t)}]^T$ and the second factor in the first addend contributes $[\mathcal{U}^{-1}_{(\bar{t})} \mathcal{U}_{(t)} \mathcal{U}^{-1}_{(t+1)} \mathcal{U}_{(\bar{t})}]^{-1}$. Similarly for the tilded fields.

4. The classical solution

We are now ready to explicitly compute the classical solution. We try to be as general as possible as far as possible but then we apply the general procedure to the case of interest, i.e. the $SU(2)$ monodromies.

4.1. Summary of the previous steps

Let us summarize the what done up to now and see how we proceed further on.

- We start with the embedding data given by $U_{(t)}$ and $\Im g_{(t)}$ as follows from the embedding of the $D2_{(t)}$ is given in eq. (55).
- Using these data we can compute the intersection points $f_{(t)}$ between $D_{(t)}$ and $D_{(t+1)}$ as in eq. (67) and the symmetric matrices $\mathcal{U}_{(t)} = U^T_{(t)} U_{(t)}$.
- We choose a brane $D_{(\bar{t})}$ which we use for the doubling trick as in eqs. (69).
- We can compute the monodromy matrices $M_{(t)} = \mathcal{U}^{-1}_{(t+1)} \mathcal{U}_{(t)}$ as in eqs. (73). These data are nevertheless independent of the way we perform the doubling trick and are all what it is needed to compute the derivatives $\partial \mathcal{Z}$ and $\partial \tilde{\mathcal{Z}}$. However these matrices do depend on the order of the interaction points x_t and these can be changed by changing $\Im g_{(t)}$ as it is easy to see in the simplest abelian case with $\hat{N}_B = 3$.
- The world sheet interaction points are x_t $(t = 1, \dots \hat{N}_B)$ where the twists are inserted. They are singular points which need to be remapped to the singular points of any possible solution.[11] Moreover the brane $D_{(\bar{t})}$ for which $x_{\bar{t}} < x < x_{\bar{t}-1}$ where the doubled solution (69) has no cut must be remapped to the interval where solutions have no cut. In analogy with what happens for the hypergeometric function (and for the fixed singularities of Heun function) we can assume that the three canonical singularities are 0, 1 and ∞ and that the interval without cut is the real axes interval $(-\infty, 0)$ as discussed after eq. (14).
  This remapping can then be done with the $SL(2, \mathbb{R})$ transformation

$$\omega_{(\bar{t})}(u) = \frac{(u - x_{\bar{t}-1})(x_{\bar{t}+1} - x_{\bar{t}})}{(u - x_{\bar{t}})(x_{\bar{t}+1} - x_{\bar{t}-1})}. \tag{77}$$

[11] We write any solution and not the solution because there can be many solutions compatible with all constraints.

In particular we have called $\omega_{(\bar{t})}$ the complex variable on which any solution depends in order not to confuse it with the original doubled complex variable z.
In our case $\hat{N}_B = 3$ and the possible solutions are given by any $\mathcal{E}_{\{\mathbf{0}\}}$ compatible with all constraints given in eq. (39). If we choose $\bar{t} = 1$ then the $SL(2, \mathbb{R})$ transformation maps x_3, x_2 and $x_1^{\pm}$ to 0, 1 and $\mp\infty$ respectively. The general mapping is

$$(x_{\bar{t}+1}, x_{\bar{t}}^{\pm}, x_{\bar{t}-1}) \to (1, \pm\infty, 0). \tag{78}$$

4.2. A first naive look

After having summarized what done until now we can proceed further without paying attention to some subtleties which will be addressed in the next subsection.

- In general the monodromy matrix $M_{(\bar{t}-1)} = \mathcal{U}_{(\bar{t})}^{-1}\mathcal{U}_{(\bar{t}-1)}$ at $\omega_{(\bar{t})\bar{t}-1} = \omega_{(\bar{t})}(x_{\bar{t}-1}) = 0$ is not diagonal therefore we need a $U(N)$ ($N = 2$ in the case at hand) transformation $V_{(\bar{t})}$ to diagonalize it since we want to use the previous results where the monodromy matrix $U_{[\mathbf{0}]}$ at $\omega_{(\bar{t})} = 0$ is diagonal. In particular we want

$$U_{[\mathbf{0}]} = V_{(\bar{t})}\, M_{(\bar{t}-1)} V_{(\bar{t})}^{\dagger}\,. \tag{79}$$

  Naively we could think that given whichever solution of the previous equation would work. It is not so as we discuss in the next subsection.
- Given the matrix $V_{(\bar{t})}$ we can map the monodromy matrices $M_{(t)} = \mathcal{U}_{(t+1,t)}$ into the monodromy matrices of any solution $\mathcal{E}_{\{\mathbf{0}\}}$ given in eq. (25) as

$$\begin{aligned} U_{[\mathbf{0}]} &= U(N_{[\mathbf{0}]}, n_{[\mathbf{0}]}\vec{k}) = V_{(\bar{t})}\; M_{(\bar{t}-1)} V_{(\bar{t})}^{\dagger}\,, \\ U_{[\mathbf{1}]} &= U(-N_{[\mathbf{0}]} - N_{[\infty]}, \vec{n}_{[\mathbf{1}]}) = V_{(\bar{t})}\; M_{(\bar{t}+1)} V_{(\bar{t})}^{\dagger}\,, \\ U_{[\infty]} &= U(N_{[\infty]}, \vec{n}_{[\infty]}) = V_{(\bar{t})}\; M_{(\bar{t})} V_{(\bar{t})}^{\dagger}\,. \end{aligned} \tag{80}$$

  Correspondingly the monodromy matrices $\hat{M}_{(\bar{t},t)} = \mathcal{U}^*_{(t,t+1)}$ for $\partial\tilde{\mathcal{Z}}_{(\bar{t})}$ are mapped into

$$\begin{aligned} \tilde{U}_{[\mathbf{0}]} &= \tilde{V}_{(\bar{t})}\; \hat{M}_{(\bar{t},\bar{t}-1)} \tilde{V}_{(\bar{t})}^{\dagger}\,, \\ \tilde{U}_{[\mathbf{1}]} &= \tilde{V}_{(\bar{t})}\; \hat{M}_{(\bar{t},\bar{t}+1)} \tilde{V}_{(\bar{t})}^{\dagger}\,, \\ \tilde{U}_{[\infty]} &= \tilde{V}_{(\bar{t})}\; \hat{M}_{(\bar{t},\bar{t})} \tilde{V}_{(\bar{t})}^{\dagger}\,. \end{aligned} \tag{81}$$

  We can take $\tilde{V}_{()} = V_{()}^*$ so that $\tilde{U}_{[]} = U_{[]}^*$.
  When we specialize the previous generic discussion to our case with $N = 2$ we can parametrize the monodromies as

$$M_{(t)} = U(N_{(t)}, \vec{n}_{(t)}). \tag{82}$$

  Then given $V_{(t)}$ from the previous transformations we can compute the vectors $(N_{[]}, \vec{n}_{[]})$ which are used to canonically parametrize the matrices $U_{[]}$ according to eq. (29). Obviously their moduli $n_{[]}$ and $N_{[]}$ are the same of the corresponding quantities of the monodromies $M_{()}$ since they are conjugate, i.e. we have for example $n_{[\mathbf{0}]} = \tilde{n}_{[\mathbf{0}]} = n_{(\bar{t}-1)} = \tilde{n}_{(\bar{t}-1)}$. In this last expression we have also included the relations for the tilded quantities.

- Given the parameters $(N_{[]}, \vec{n}_{[]})$ we can finally compute the parameters associated to any solution $\mathcal{E}_{\{\mathbf{0}\}}$, in our case $a, \dots f$ up to integers. Similarly for the tilded quantities.
  Notice that the existence of solutions depends on the possibility of fixing these integers so that we get a finite Euclidean action.
- We can now look for the general solution for the derivative of the classical string e.o.m (which depend only on the monodromy matrices) among the linear combinations

$$\partial \mathcal{Z}_{(\bar{t})}(z) = \frac{\partial \omega_{(\bar{t})}}{\partial z} \sum_r a_{(\bar{t})r}\, \partial \mathcal{Z}_{(\bar{t})r}(z), \qquad \partial \mathcal{Z}_{(\bar{t})r}(z) = V^{\dagger}_{(\bar{t})\,r}\, \mathcal{E}_{\{\mathbf{0}\}r}(\omega_{(\bar{t})}),$$
$$\partial \tilde{\mathcal{Z}}_{(\bar{t})}(z) = \frac{\partial \omega_{(\bar{t})}}{\partial z} \sum_s b_{(\bar{t})s}\, \partial \tilde{\mathcal{Z}}_{(\bar{t})z}(z), \qquad \partial \tilde{\mathcal{Z}}_{(\bar{t})z}(z) = \tilde{V}^{\dagger}_{(\bar{t})\,s}\, \tilde{\mathcal{E}}_{\{\mathbf{0}\}s}(\omega_{(\bar{t})}), \tag{83}$$

  where r labels the possible independent solutions $\mathcal{E}_{\{\mathbf{0}\}r}$ which have the required monodromies and finite action as necessary for a classical solution. We have also allowed for a different $V_{(\bar{t})\,r}$ for any possible solution. Because of this we could in principle believe that also the corresponding monodromy matrices $U_{[]}$ depend on r. As we show in the next subsection for the case at hand it is not the case and that the only dependence of $V_{(\bar{t})\,r}$ on r is through a phase. Similarly for s with the tilded quantities.
- Finally we can determine the classical solution by fixing the constants as and bs from the global conditions. In fact we can write the classical solution as

$$\begin{aligned}
Z(u,\bar{u}) = f_{(\bar{t}-1)} \quad & + \sum_r a_{(\bar{t})r} \int_{0;\omega_{(\bar{t})}\in H}^{\omega_{(\bar{t})}(u)} d\omega_{(\bar{t})}\, V^{\dagger}_{(\bar{t})r} \mathcal{E}_{\{\mathbf{0}\}r}(\omega_{(\bar{t})}) \\
& + \sum_s b_{(\bar{t})s} \int_{0;\bar{\omega}_{(\bar{t})}\in H^-}^{\omega_{(\bar{t})}(\bar{u})} d\bar{\omega}_{(\bar{t})}\, \mathcal{U}^*_{(\bar{t})} \tilde{V}^{\dagger}_{(\bar{t})s} \tilde{\mathcal{E}}_{\{\mathbf{0}\}s}(\bar{\omega}_{(\bar{t})}), \\
\bar{Z}(u,\bar{u}) = \bar{f}_{(\bar{t}-1)} \quad & + \sum_r a_{(\bar{t})r} \int_{0;\bar{\omega}_{(\bar{t})}\in H^-}^{\omega_{(\bar{t})}(\bar{u})} d\bar{\omega}_{(\bar{t})}\, \mathcal{U}_{(\bar{t})} V^{\dagger}_{(\bar{t})r} \mathcal{E}_{\{\mathbf{0}\}r}(\bar{\omega}_{(\bar{t})}) \\
& + \sum_s b_{(\bar{t})s} \int_{0;\omega_{(\bar{t})}\in H}^{\omega_{(\bar{t})}(u)} d\omega_{(\bar{t})}\, \tilde{V}^{\dagger}_{(\bar{t})s} \tilde{\mathcal{E}}_{\{\mathbf{0}\}s}(\omega_{(\bar{t})}),
\end{aligned} \tag{84}$$

  and then impose the further global conditions (71)

$$Z(x_t, x_t) = f_{(t)}, \quad t \neq \bar{t} - 1 \tag{85}$$

  in order to fix the constants.

We can summarize what we have got until now by saying that twist conditions are local and therefore knowing the modulus of the twists is enough to determine up to integers the indices. The sentence "up to integer" is important since the sum of all indices for the Papperitz–Riemann equation, i.e. the general Fuchsian second order equation with three singularities must add to 1 for $\hat{N}_B = 3$ twists. In fact we can write the previous statement for the case of monodromies in $SU(2)$, $\hat{N}_B = 3$ twist fields in a sketchy way as

$$\mathcal{E}(\omega_{(\bar{t})}) \sim P \left\{ \begin{array}{cccc} 0 & 1 & \infty & \\ n_{(\bar{t}-1)} & n_{(\bar{t}+1)} & n_{(\bar{t})} & \omega_{(\bar{t})} \\ -n_{(\bar{t}-1)} & -n_{(\bar{t}+1)} & -n_{(\bar{t})} & \end{array} \right\} \tag{86}$$

where all indices sum to 0 while they should sum to 1. In a more precise way we can write[12]

$$\mathcal{E}(\omega_{(\bar{t})}) = P \left\{ \begin{array}{cccc} 0 & 1 & \infty & \\ n_{(\bar{t}-1)} + k_{(\bar{t}-1)+} & n_{(\bar{t}+1)} + k_{(\bar{t}+1)+} & n_{(\bar{t})} + k_{(\bar{t})+} & \omega_{(\bar{t})} \\ -n_{(\bar{t}-1)} + k_{(\bar{t}-1)-} & -n_{(\bar{t}+1)} + k_{(\bar{t}+1)-} & -n_{(\bar{t})} + k_{(\bar{t})-} & \end{array} \right\} \tag{87}$$

with $\sum_{t=1}^{3} \sum_{s\in\{\pm\}} k_{(t)s} = 1$. Now requiring that the action be finite implies that we must set

$$\mathcal{E}(\omega_{(\bar{t})}) = P \left\{ \begin{array}{cccc} 0 & 1 & \infty & \\ -\bar{n}_{(\bar{t}-1)} & -\bar{n}_{(\bar{t}+1)} & 2-\bar{n}_{(\bar{t})} & \omega_{(\bar{t})} \\ -n_{(\bar{t}-1)} & -n_{(\bar{t}+1)} & 2-n_{(\bar{t})} & \end{array} \right\} \tag{88}$$

where we have introduced for notation simplicity

$$\frac{1}{2} \le \bar{n} = 1 - n < 1, \tag{89}$$

which has however a different range w.r.t. n ($0 \le n < \frac{1}{2}$). Similarly we have

$$\tilde{\mathcal{E}}(\omega_{(\bar{t})}) = P \left\{ \begin{array}{cccc} 0 & 1 & \infty & \\ -\bar{n}_{(\bar{t}-1)} & -\bar{n}_{(\bar{t}+1)} & 2-\bar{n}_{(\bar{t})} & \omega_{(\bar{t})} \\ -n_{(\bar{t}-1)} & -n_{(\bar{t}+1)} & 2-n_{(\bar{t})} & \end{array} \right\}. \tag{90}$$

Notice that generically the behavior of the solution around any singular point is given by the sum of the two possible behaviors as for example in eq. (19) which shows that even if we start from a function having a unique index in a singularity we end up with a mixture of indices in other singularities. This means that we must require the all combinations of the two indices at any singular point must give a finite action. For example at $\omega_{(\bar{t})} = \infty$ we require $2(n_{(\bar{t})} + k_{(\bar{t})+}) > 2$, $2(-n_{(\bar{t})} + k_{(\bar{t})-}) > 2$ and $(n_{(\bar{t})} + k_{(\bar{t})+}) + (-n_{(\bar{t})} + k_{(\bar{t})-}) > 2$. In particular the last equation means that $k_{(\bar{t})+} + k_{(\bar{t})-} \ge 4$.

As it is obvious from the explicit expressions there is an asymmetry among points and this is disturbing since the P symbol is invariant under $SL(2,\mathbb{C})$ transformations. This asymmetry is however apparent since the P symbol is directly connected to $\partial_{\omega_{(\bar{t})}} \mathcal{Z}$ and not to $\partial_z \mathcal{Z}$. To see how this solve the asymmetry issue consider one of the simplest cases where we use $\hat{\omega}_{(\bar{t})} = 1/\omega_{(\bar{t})}$ as new variable then

$$\mathcal{E}(\hat{\omega}_{(\bar{t})}) = P \left\{ \begin{array}{cccc} \infty & 1 & 0 & \\ -\bar{n}_{(\bar{t}-1)} & -\bar{n}_{(\bar{t}+1)} & 2-\bar{n}_{(\bar{t})} & \hat{\omega}_{(\bar{t})} \\ -n_{(\bar{t}-1)} & -n_{(\bar{t}+1)} & 2-n_{(\bar{t})} & \end{array} \right\}. \tag{91}$$

But now $\partial_{\hat{\omega}_{(\bar{t})}} \mathcal{Z} = \hat{\omega}_{(\bar{t})}^{-2} \partial_{\omega_{(\bar{t})}} \mathcal{Z}$ and hence around $\hat{\omega}_{(\bar{t})} = 0$ we get $\partial_{\hat{\omega}_{(\bar{t})}} \mathcal{Z} \sim \hat{\omega}_{(\bar{t})}^{-\bar{n}_{(\bar{t})}}$ and $\partial_{\hat{\omega}_{(\bar{t})}} \mathcal{Z} \sim \hat{\omega}_{(\bar{t})}^{-n_{(\bar{t})}}$. Similarly for $\hat{\omega}_{(\bar{t})} = \infty$ where we get $\partial_{\hat{\omega}_{(\bar{t})}} \mathcal{Z} \sim 1/\hat{\omega}_{(\bar{t})}^{2-\bar{n}_{(\bar{t})}}$ and $\partial_{\hat{\omega}_{(\bar{t})}} \mathcal{Z} \sim 1/\hat{\omega}_{(\bar{t})}^{2-n_{(\bar{t})}}$. This restores completely the symmetry among the points.

Another point on which is worth noticing and commenting is the appearance of an antiholomorphic part in Z while in the corresponding case with abelian monodromies this does not

[12] Remember that the P symbol represents all the ∞^2 solutions therefore by writing $=$ it is meant that y is one of these solutions.

happen. In fact the previous discussion shows that there is one solution for $\partial \mathcal{Z}$ and one for $\partial \tilde{\mathcal{Z}}$. The reason why we get two solutions can be easily understood by noticing that we have more equations to fix the coefficients a_r and b_s than in the abelian case. In fact according to the discussion after eqs. (68) the configuration is determined by $\hat{N}_B N(N+3)/2 = 15$ real parameters and the proposed solution (84) depends on $f_{(\bar{t}-1)}$, $\mathcal{U}_{(\bar{t})}$ and $M_{(t)}$ ($t = 1, \dots \hat{N}_B$) for a total of $2N + \hat{N}_B N(N+1)/2$ real parameters. This happens since given $\mathcal{U}_{(\bar{t})}$ and $M_{(t)}$ we can compute all $\mathcal{U}_{(t)}$ and any symmetric unitary $\mathcal{U}_{(t)}$ is specified by $N(N+1)/2$ real d.o.f.s. Hence we still need $(\hat{N}_B - 2)N = 2$ real equations to fix the vector the remaining quantities $f_{(t)}$ ($t \neq \bar{t} - 1$), in particular we can simply determine $f_{(\bar{t})} - f_{(\bar{t}-1)}$. On the other side the reason why we get both a holomorphic and antiholomorphic contribution to Z is less obvious and may be traced back to the fact that minimal area surface is not anymore drawable on a plane despite we have only three interaction points which uniquely fix a plane in $\mathbb{R}^4$. This happens because the rotations of the $D2$ branes are not abelian.

This can give the impression that anything can be done easily also for more complex cases. Unfortunately, it is not so. Let us see why.

The same approach can be generalized to the $\hat{N}_B = 4$ case with $SU(2)$ global symmetry. So we can write using a generalized P-symbol in the case $\bar{t} = 4$ in a sketchy way

$$\mathcal{E}_{\hat{N}_B=4, SU(2)}(\omega_{(4)}) \sim P \left\{ \begin{array}{cccccc} 0 & a & 1 & \infty & & \\ k_{(1)+} - \bar{n}_{(1)} & k_{(2)+} - \bar{n}_{(2)} & k_{(3)+} - \bar{n}_{(3)} & k_{(4)+} + 2 - \bar{n}_{(4)} & q & \omega_{(4)} \\ k_{(1)-} - n_{(1)} & k_{(2)-} - n_{(2)} & k_{(3)-} - n_{(3)} & k_{(4)-} + 2 - n_{(4)} & & \end{array} \right\}, \quad (92)$$

where a is the location of the fourth singularity whence we have fixed the other three and with $\sum_{t=1}^{4} \sum_{s\in\{\pm\}} k_{(t)s} = S = 2$ and $k_{(t)s} \geq 0$ in order to have a finite action. Therefore we have $\frac{[S+(2\hat{N}_B-1)]!}{S!(2\hat{N}_B-1)!} = \frac{(2+7)!}{2!7!} = 36$ possible solutions $\mathcal{E}$ and 36 possible $\tilde{\mathcal{E}}$s even if we expect to need $(\hat{N}_B - 2)N = 4$ solutions only. It could be that many of these possible solutions are linearly dependent as it happens for the $U(1)$ case as discussed in section 4.4.

The question could be solved by a direct computation but there are unfortunately two issues. The first one is that the general solution of the general Fuchsian second order equation with four singularities is not uniquely determined by the indices as it happens for the hypergeometric function but it has an accessory parameter q. In fact, as reviewed in Appendix A, a Fuchsian equation of order N with $\hat{N}_B$ singular points has $\hat{N}_B N(N-1)/2 - N^2 + 1$ free accessory parameters. The fixing of the accessory parameters is therefore the first issue it is necessary to solve if we want to consider more complex cases than the actual one.

Another issue and more fundamental is that in order to write the actual solution we need to normalize the solutions in order to get the desired monodromies. This is what done in eq. (25) for the case $\hat{N}_B = 3$ and $N = 2$ which discuss in this paper. However as it is clear from the discussion in the present case this normalization does depend on the continuation formulas for the different basis of solutions around the different singular points. Unfortunately this problem is not solved in the general case even in the simplest case of Heun function whose P symbol is given in eq. (92) and which corresponds to the second order Fuchsian equation with four singularities.

At least a couple of possible ways forward can be imagined to try to solve these issues in this case:

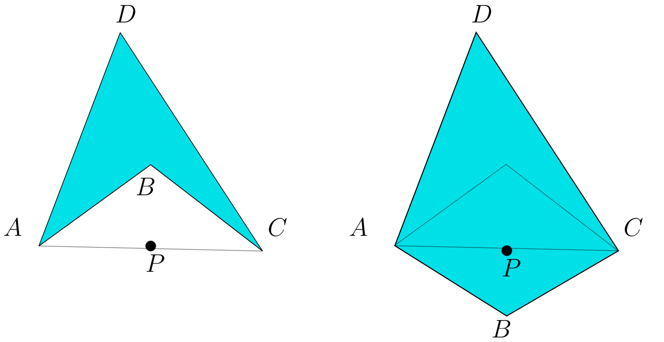

Fig. 12. On the left the case $\hat{N}_B = 4$ $M = 1$ and on the right the case $\hat{N}_B = 4$ $M = 2$ with the minimal areas shadowed.

- consider special $M_{(t)}$ values which correspond to algebraic solutions to the differential equation. This amounts to say that we are actually working a higher genus Riemann surface as done in the paper by Inoue [14];
- try to use CFT factorization.

Another point is worth discussing. In the usual factorized case where we consider only $\mathbb{R}^2$ there are two possible cases for $\hat{N}_B = 4$ as discussed in [10] and they are labeled by an integer M which in this case can be either 1 or 2. The situation is pictured in Fig. 12. As long as we limit ourselves to the $\mathbb{R}^2$ case we cannot move from one case to the other by moving the point B without going through the straight line, i.e. the case of no twist. This explains why the two cases are different in $\mathbb{R}^2$. At first sight this should be not true in $\mathbb{R}^3$ since we can rotate the curve ABC around the APC axis in a third dimension without going through the straight line. Hence we would expect to have only one case. Actually it is no. The reason is that while rotating the curve ABC in the third dimension in order to deform the left configuration to the right one the minimal area bounded by $ABCD$ starts increasing and a certain point a second configuration bounded by $APCD$ and $ABCP$ of equal area appears.[13]

The same issues are present also for the $\hat{N}_B = 3$ case with $SU(3)$ global monodromies where we can expect to write in a sketchy way

$$\mathcal{E}_{\hat{N}_B=3,SU(3)}(\omega_{(3)}) \sim P \left\{ \begin{array}{ccccc} 0 & 1 & \infty & & \\ n_{(1)} & n_{(2)} & n_{(3)} & q & \omega_{(3)} \\ m_{(1)} & m_{(2)} & m_{(3)} & & \\ -m_{(1)} - n_{(1)} & -m_{(1)} - n_{(2)} & -m_{(3)} - n_{(3)} & & \end{array} \right\} \quad (93)$$

where $n_{(i)}$ and $m_{(i)}$ are the independent local $SU(3)$ rotation parameters and again we have one accessory parameter q. Nevertheless we can consider special cases where the monodromies are known [18].

[13] An easy model to see what is going on is to approximate the minimal area configurations by a sum of triangles. We consider the simplest case with $A \equiv (0,0,0)$, $P \equiv (B,0,0)$, $C \equiv (2B,0,0)$, $D \equiv (B,H,0)$ and $B \equiv (B, h\cos\theta, h\sin\theta)$. This case corresponds to the case where both the triangles ACD and ACB are isosceles. Then the approximated area for the $M = 1$ case is twice the area of the triangle ABD and is $A_{M=1} = \sqrt{H^2h^2 + B^2H^2 + B^2h^2 - (H^2h^2\cos^2\theta + 2B^2\cos\theta)}$ and the approximated area for the $M = 2$ case is the sum of the two triangles ACD and ABC and is $A_{M=2} = B(H + h)$. It is the immediate to see that for the case $B^2 < Hh$ the maximum of $A_{M=1}$ is $\max A_{M=1} = B\sqrt{(H + h)^2 + (H - h)^2}$ which is greater than $A_{M=2}$.

4.3. A more detailed look

From the discussion of the previous subsection it seems that any $V_{(\bar{t})}$ would do the job since all of them fix the same indices. However it is not so.

The reason is that there are further constraints from the action of complex conjugation on $\partial\mathcal{Z}_{(\bar{t})}(z)$ and $\partial\tilde{\mathcal{Z}}_{(\bar{t})}(z)$. In fact eqs. (72) (or eqs. (84)) imply for any r and s

$$\mathcal{U}_{(\bar{t})} = \frac{a^*_{(\bar{t})r}}{a_{(\bar{t})r}} V^T_{(\bar{t})\,r}\, D^{-1*}_{\{\mathbf{0}\}} D_{\{\mathbf{0}\}} V_{(\bar{t})\,r}$$
$$\tilde{\mathcal{U}}_{(\bar{t})} = \frac{b^*_{(\bar{t})s}}{b_{(\bar{t})s}} \tilde{V}^T_{(\bar{t})\,s}\, \tilde{D}^{-1*}_{\{\mathbf{0}\}} \tilde{D}_{\{\mathbf{0}\}} \tilde{V}_{(\bar{t})\,s}, \tag{94}$$

since $\mathcal{E} \sim D^{-1}\mathcal{B}$ and $\mathcal{B}$ is a pair of hypergeometric functions with real parameters. These constraints are fundamental to get a solutions which satisfy the required boundary conditions, i.e. with the string boundaries on the branes.

They are however not satisfied by a random solution $V_{(\bar{t})\,r}$ of eq. (79) or the equivalent for $\tilde{V}_{(\bar{t})\,s}$. The problem in implementing them at this stage is that $D_{\{\mathbf{0}\}}$ does depend on $\vec{n}_{[\infty]}$ as in eqs. (41). On the other side $\vec{n}_{[\infty]}$ in turn depends on $V_{(\bar{t})}$ because $\vec{n}_{[\infty]}$ is the parameter associated with the conjugation of $\mathcal{U}_{(\bar{t}+1,\bar{t})}$ by $V_{(\bar{t})}$ as in the last of eqs. (80). Therefore we must solve eqs. (94) and (80) together.

Let us now solve the previous constraints. This is done by comparing the previous equations (94) from the behavior of the doubled solution under complex conjugation with the definition $\mathcal{U}_{(\bar{t})} = U^T_{(\bar{t})} U_{(\bar{t})}$ as given in eq. (66). This comparison suggests to choose the "gauge"[14]

$$n_{[\infty]2} = 0, \tag{95}$$

so that we have

$$D^{-1*}_{\{\mathbf{0}\}} D_{\{\mathbf{0}\}} = \mathbb{I}_2 \tag{96}$$

then we can make the ansatz $U_{(\bar{t})} = e^{-i2\pi\alpha_{(\bar{t})r}} R_{(\bar{t})}\ V_{(\bar{t})\,r}$[15] where the phase is defined from $a_{(\bar{t})r} = |a_{(\bar{t})r}| e^{i2\pi\alpha_{(\bar{t})r}}$ and

$$R^T_{(\bar{t})}\ R_{(\bar{t})} = \mathbb{I}_2, \tag{97}$$

moreover $R_{(\bar{t})} \in U(2)$ since both U and V are in $U(2)$. This implies $R_{(\bar{t})} = U(0, r_{(\bar{t})\,2}\vec{j})$.[16]

Then we get for any unknown $V_{(\bar{t})\,r}$

$$V_{(\bar{t})\,r} = e^{i2\pi\alpha_{(\bar{t})r}} R^\dagger_{(\bar{t})}\ U_{(\bar{t})}. \tag{98}$$

We are now left with the problem of computing $R_{(\bar{t})}$ and $U_{[\infty]}$.[17] These are the only unknowns since, as noticed before, $U_{[0]}$ is completely determined and independent of r. This happens because it is in the Cartan, i.e. $\vec{n}_{[0]} = n_{[0]3}\vec{k}$ and $M_{(\bar{t}-1)}$ and $U_{[0]}$ are conjugate, therefore if we write as before $M_{(\bar{t}-1)} = U(0, \vec{n}_{(\bar{t}-1)})$ we can always set $n_{[0]3} = n_{(\bar{t}-1)} > 0$.

---

14 Notice that we can always choose the gauge (95) by a "rotation" in the Cartan group of $SU(2)$ since this does not change $U_{[\mathbf{0}]}$.

15 In principle $R_{(\bar{t})}$ should be written as $R_{(\bar{t})\,r}$ but as we now show it is actually independent of r.

16 The parametrization given in the main text is such that $R_{(\bar{t})} \in SU(2)$ but we could for example have $R_{(\bar{t})} = U(0, r_{(\bar{t})\,2}\vec{j})\sigma_1 \in U(2)$. We can exploit this fact to choose $\vec{n}_{[0]} = n_{[0]}\vec{k}$, i.e. $n_{[0]3} = n_{[0]} > 0$.

17 Also $U_{[\infty]}$ turns out to be independent of r.

Because of the same reason $U_{[\infty]r}$ is partially determined and we must only fix the ratio $n_{[\infty]1}/n_{[\infty]3}$. We have therefore two unknowns $r_{(\bar{t})\,2}$ and $n_{[\infty]1}/n_{[\infty]3}$. They can be fixed using the first and second equation in the group of eqs. (80). The first one can be rewritten as

$$U^*_{(\bar{t})}\mathcal{U}_{(\bar{t}-1)}U^\dagger_{(\bar{t})} = (U_{(\bar{t}-1)}U^\dagger_{(\bar{t})})^T(U_{(\bar{t}-1)}U^\dagger_{(\bar{t})}) = R_{(\bar{t})}\,U_{[0]}R^T_{(\bar{t})}\,. \tag{99}$$

Since[18] both the rhs and the lhs are symmetric matrices we can write $U^*_{(\bar{t})}\mathcal{U}_{(\bar{t}-1)}U^\dagger_{(\bar{t})} = U(0, \hat{m}_{(\bar{t}-1)\,1}\vec{i} + \hat{m}_{(\bar{t}-1)\,3}\vec{k})$ (with $\hat{m}_{(\bar{t}-1)} = n_{(\bar{t}-1)}$) and then get easily

$$\cos(4\pi r_{(\bar{t})\,2}) = \frac{\hat{m}_{(\bar{t}-1)\,3}}{m_{(\bar{t}-1)}},$$
$$\sin(4\pi r_{(\bar{t})\,2}) = -\frac{\hat{m}_{(\bar{t}-1)\,1}}{m_{(\bar{t}-1)}}.$$

Since $-\frac{1}{2} \le r_{(\bar{t})\,2} < \frac{1}{2}$ the matrix $R_{(\bar{t})}$ is completely determined up to a sign. This explicit solution shows that $R_{(\bar{t})}$ is independent of r.

The second equation of eqs. (80) can be used to fix completely $U_{[\infty]}$ since it can be rewritten as

$$U_{[\infty]} = R^{-1}_{(\bar{t})}(U_{(\bar{t})}U^\dagger_{(\bar{t}-1)})(U_{(\bar{t})}U^\dagger_{(\bar{t}-1)})^T R_{(\bar{t})}\,. \tag{100}$$

All the previous equations can be solved since they are consistent with the various properties of the involved matrices, i.e. $\mathcal{U}^T = \mathcal{U}$, $U^T_{[0]} = U_{[0]}$ and $U^T_{[\infty]} = U_{[\infty]}$. In particular this last property is valid only because of the choice of "gauge" (95).

Let us now consider what happens to the tilded quantities associated with $\partial\tilde{\mathcal{Z}}$. All the previous equations remain unchanged with the substitution of the untilded quantities with the tilded ones. In particular the given quantities $M_{(t)}$ and $\mathcal{U}_{(t)}$ are connected with the tilded ones by complex conjugation. This means that $\tilde{V}$ is the same of V^* up to a phase. It also follows that $\tilde{R} = \pm R$ since R is a real matrix. Finally, $\tilde{U}_{[]}$ are the complex conjugate of the corresponding $U_{[]}$. The important consequence is then that $\vec{\tilde{n}}_{()}$ are the parameters associated with the complex conjugate of $U_{[]}$, i.e. $\vec{\tilde{n}} = (-n_1, +n_2, -n_3)$. However this does not mean that there exists one $\tilde{\mathcal{E}}_{\{\mathbf{0}\}s}$ for each $\mathcal{E}_{\{\mathbf{0}\}r}$ since the determination of the possible solutions requires fixing all integers ks and $\tilde{k}$s. In fact we can determine the parameters ks and $\tilde{k}$s of the solutions $\mathcal{E}_{\{\mathbf{0}\}r}$ and $\tilde{\mathcal{E}}_{\{\mathbf{0}\}s}$ by requiring a finite classical action.

Finally, the fact that the solution for R is unique up to a sign has as a consequence for the way we write the conditions which can be used to fix the coefficients a and b. In fact we write the global boundary conditions (71) as

$$Z(x_{t-1}, x_{t-1}) - Z(x_t, x_t) = f_{(t-1)} - f_{(t)}$$
$$= \int_{x_t;u\in H}^{x_{t-1}} du\,\partial_u Z_L(u) + \int_{x_t;\bar{u}\in H^-}^{x_{t-1}} d\bar{u}\,\bar{\partial}_{\bar{u}} Z_R(\bar{u})$$

[18] Since $U_{[0]}$ is diagonal we can easily compute $U^{\frac{1}{2}}_{[0]}$ and then naively get $R_{(\bar{t})} = \pm(U_{(\bar{t}-1)}U^\dagger_{(\bar{t})})^T U^{\frac{1}{2}}_{[0]}$. This is however wrong since this would-be solution does not generically satisfy $R^T_{(\bar{t})}R_{(\bar{t})} = \mathbb{I}$.

$$= \int_{x_t; u \in H}^{x_{t-1}} du\, \partial_z \mathcal{Z}_{(\bar{t})}(z)|_{z=u} + \mathcal{U}_{(\bar{t})} \int_{x_t; \bar{u} \in H^-}^{x_{t-1}} d\bar{u}\, \bar{\partial}_z \tilde{\mathcal{Z}}_{(\bar{t})}(z)|_{z=\bar{u}}. \tag{101}$$

Now $\mathcal{U}_{(\bar{t})}$ and $\tilde{V}_{(\bar{t})} = V^*_{(\bar{t})}$ conspire to allow to write

$$\begin{aligned} &Z(x_{t-1}, x_{t-1}) - Z(x_t, x_t) = f_{(t-1)} - f_{(t)} \\ &= U^\dagger_{(\bar{t})} R_{(\bar{t})} \Big[\sum_r a_r I_{L\,r\,(t)} + \sum_s b_s I_{R\,s\,(t)} \Big], \end{aligned} \tag{102}$$

where we have defined the coefficients

$$\begin{aligned} I^i_{L\,r\,(t)} &= \int_{\omega_{(\bar{t})t}; \omega \in H}^{\omega_{(\bar{t})t-1}} d\omega\, \mathcal{E}^i_{\{\mathbf{0}\}r}(\omega), \\ I^i_{R\,s\,(t)} &= \int_{\omega_{(\bar{t})t}; \bar{\omega} \in H^-}^{\omega_{(\bar{t})t-1}} d\bar{\omega}\, \tilde{\mathcal{E}}^i_{\{\mathbf{0}\}s}(\bar{\omega}). \end{aligned} \tag{103}$$

In the previous expression (102) we can take $a, b \in \mathbb{R}$ and we are not obliged to consider the previous expression with $a \to |a|$ and $b \to |b|$ as it would follow from the direct application of eq. (98) where the phase would cancel the corresponding phase of a (and similarly for b).

Finally notice that the previous equation is simply asserting that there exist well adapted coordinates where the computations are more straightforward. In facts when we have decided which branes $D_{(\bar{t})}$ to use for the doubling trick and then we have mapped in the proper way the original worldsheet coordinate u into $\omega_{(\bar{t})}$ as in eq. (77) then the space coordinates

$$Z_{\{F_{(\bar{t}-1)}\}} = R^T_{(\bar{t})} U_{(\bar{t})} Z = R^T_{(\bar{t})} Z_{(D_{\bar{t}})} \tag{104}$$

are good local coordinates at the interaction point $x_{\bar{t}-1}$, i.e. coordinates for which the monodromy at $\omega_{(\bar{t})\bar{t}-1} = 0$ are specially well suited to perform the explicit computations.

4.4. Warming up and recovering the $U(1)$ classical solution

Before using the previous discussion to find the desired $SU(2)$ solution we want to see how we can recover the usual abelian solution.

As noticed in section 3.1 and shown in Fig. 11 the rotation matrices in the $\mathbb{R}^2$ case are given by

$$U_{(t)} = e^{-i\pi\alpha_{(t)}}, \quad 0 \le \alpha_{(t)} < 1. \tag{105}$$

From this expression we can immediately compute the auxiliary object $\mathcal{U}_{(t)}$ defined in (66) and the space interaction point $f_{(t)} \in \mathbb{C}$ given in eq. (67)

$$\begin{aligned} &\mathcal{U}_{(t)} = U^T_{(t)} U_{(t)} = e^{-i2\pi\alpha_{(t)}}, \\ &f_{(t)} = 2\,i\, \left[\mathcal{U}_{(t)} - \mathcal{U}_{(t+1)}\right]^{-1} (U^T_{(t)} \Im g_t - U^T_{(t+1)} \Im g_{t+1}) = \frac{e^{-i\pi\alpha_{(t+1)}} \Im g_t - e^{-i\pi\alpha_{(t)}} \Im g_{t+1}}{\sin(\pi(-\alpha_{(t)} + \alpha_{(t+1)}))}. \end{aligned} \tag{106}$$

Following the discussion of the section 3.3 we can use the doubling trick to glue along the $\bar{t}$ brane and define the double fields

$$\partial \mathcal{Z}_{(\bar t)}(z) = \begin{cases} \partial_u Z_L(u) & z = u \in \mathring{H} \cup (x_{\bar t}, x_{\bar t -1}) \\ \mathcal{U}^{-1}_{(\bar t)} \bar\partial_{\bar u} \bar Z_R(\bar u) = e^{i\pi\alpha_{(\bar t)}} \bar\partial_{\bar u} \bar Z_R(\bar u) & z = \bar u \in \mathring{H}^- \cup (x_{\bar t}, x_{\bar t -1}) \end{cases},$$

$$\partial \tilde{\mathcal{Z}}_{(\bar t)}(z) = \begin{cases} \partial_u \bar Z_L(u) & z = u \in \mathring{H} \cup (x_{\bar t}, x_{\bar t -1}) \\ \tilde{\mathcal{U}}^{-1}_{(\bar t)} \bar\partial_{\bar u} Z_R(\bar u) = e^{-i\pi\alpha_{(\bar t)}} \bar\partial_{\bar u} Z_R(\bar u) & z = \bar u \in \mathring{H}^- \cup (x_{\bar t}, x_{\bar t -1}) \end{cases}, \tag{107}$$

which have monodromies

$$\begin{aligned} &\partial \mathcal{Z}_{(\bar t)}(x_t + e^{i2\pi}(\epsilon + i0^+)) = M_{(t)} \partial \mathcal{Z}_{(\bar t)}(x_t + \epsilon - i0^+) = e^{i2\pi\epsilon_{(t)}} \partial \mathcal{Z}_{(\bar t)}(x_t + \epsilon - i0^+), \\ &\partial \tilde{\mathcal{Z}}_{(\bar t)}(x_t + e^{i2\pi}(\epsilon + i0^+)) = \widetilde{M_{(t)}} \partial \tilde{\mathcal{Z}}_{(\bar t)}(x_t + \epsilon - i0^+) = e^{i2\pi\tilde\epsilon_{(t)}} \partial \tilde{\mathcal{Z}}_{(\bar t)}(x_t + \epsilon - i0^+). \end{aligned} \tag{108}$$

The monodromy matrices are computed as in eqs. (73)

$$\begin{aligned} M_{(t)} &= \mathcal{U}^{-1}_{(t+1)} \mathcal{U}_{(t)} = e^{i2\pi(\alpha_{(t+1)} - \alpha_{(t)})} = e^{i2\pi\epsilon_{(t)}}, \\ \widetilde{M_{(t)}} &= e^{-i2\pi(\alpha_{(t+1)} - \alpha_{(t)})} = e^{i2\pi\tilde\epsilon_{(t)}}. \end{aligned} \tag{109}$$

Notice that in this case there is no issue with the base point being in the upper or lower half plane and in fact $\hat M_{(\bar t, t)} = M_{(t)}$ since all matrices commute. From the previous expression for the monodromies we can read the mode frequency shifts as

$$\begin{aligned} \epsilon_{(t)} &= \begin{cases} \alpha_{(t+1)} - \alpha_{(t)} & \alpha_{(t+1)} > \alpha_{(t)} \\ 1 + \alpha_{(t+1)} - \alpha_{(t)} & \alpha_{(t+1)} < \alpha_{(t)} \end{cases} \\ \tilde\epsilon_{(t)} &= 1 - \epsilon_{(t)}, \end{aligned} \tag{110}$$

in such a way that $0 < \epsilon_{(t)} < 1$ since modes are integer valued.

We can now follow the steps of section 4 and map the interaction points x_t $(t = 1, \dots \hat N_B)$ to a canonical set with $(x_{\bar t+1}, x^{\pm}_{\bar t}, x_{\bar t-1}) \to (1, \pm\infty, 0)$ with the $SL(2, \mathbb{R})$ transformation $\omega_{(\bar t)}(u) = (u - x_{\bar t -1})(x_{\bar t+1} - x_{\bar t}) / (u - x_{\bar t})(x_{\bar t+1} - x_{\bar t-1})$ already given in eq. (77).

Then we can consider the solutions of a first order Fuchsian equation with P symbol given by

$$\mathcal{E}_r(\omega) = P \begin{Bmatrix} \Omega_{\bar t} = \infty & \Omega_{\bar t-1} = 0 & \dots & \Omega_t & \dots & \Omega_{\bar t+1} = 1 & \\ \epsilon_{(\bar t)} + k_{(\bar t)} & \epsilon_{(\bar t-1)} + k_{(\bar t-1)} & \dots & \epsilon_{(t)} + k_{(t)} & \dots & \epsilon_{(\bar t+1)} + k_{(\bar t+1)} & \omega \end{Bmatrix} \tag{111}$$

where $\Omega_t = \omega_{(\bar t)t} = \omega_{(\bar t)}(x_t)$, t is considered up to $\hat N_B$ as before, r labels the possible independent combinations of $k_{(t)}$ and the cut is along the real positive axis since the gluing happens along $D_{(\bar t)}$ which is mapped into the interval $(-\infty, 0)$. Notice that the requirement of having a Fuchsian equation implies $\sum_t (\epsilon_t + k_t) = 0$, hence

$$\sum_t k_t = -M, \quad \sum_t \epsilon_t = M \in \mathbb{N}, \tag{112}$$

where $\pi \sum_t \epsilon_t$ is, up to multiples of π, the sum of the internal angles of the polygon delimited by the branes and therefore an integer multiple of π. Again finiteness of the action requires $\epsilon_{(t)} + k_{(t)} > -1$ for $t \neq \bar t$ and $\epsilon_{(\bar t)} + k_{(\bar t)} > +1$ hence $k_{(t)} \geq -1$ for $t \neq \bar t$ and $k_{(\bar t)} \geq 1$. We get therefore a set of possible functions on which we can expand the solution as follows $\begin{pmatrix} N-1 \\ N-1 \end{pmatrix} =$ 1 function with all $k_{(t)} = -1$ and $k_{(\bar t)} = N - M - 1$, $\begin{pmatrix} N-1 \\ N-2 \end{pmatrix} = N - 1$ functions with all but one $k_{(t)} = -1$ and $k_{(\bar t)} = N - M - 2$ and so on until we have $k_{(\bar t)} = 1$ when $\sum_{t \neq \bar t} k_{(t)} = -M - 1$

which corresponds to $\begin{pmatrix} (M+1)+(N-2) \\ N-2 \end{pmatrix}$ possible functions. It would therefore seem that we find a huge number of functions on which we can expand the solution. It is not so since many of the previous functions are linearly dependent, in fact when we write explicitly the function corresponding to the previous P symbol we get

$$(-\omega)^{\epsilon_{(\bar{t}-1)}+k_{(\bar{t}-1)}}(1-\omega)^{\epsilon_{(\bar{t}+1)}+k_{(\bar{t}+1)}} \prod_{t\neq\bar{t},\bar{t}\pm1} (\Omega_t-\omega)^{\epsilon_{(t)}+k_{(t)}}. \tag{113}$$

We see therefore that the independent functions are simply

$$\mathcal{E}_r(\omega) = (-\omega)^{\epsilon_{(\bar{t}-1)}-1+r}(1-\omega)^{\epsilon_{(\bar{t}+1)}-1} \prod_{t\neq\bar{t},\bar{t}\pm1} (\Omega_t-\omega)^{\epsilon_{(t)}-1}, \quad 0\leq r\leq N-M-2. \tag{114}$$

Similarly for $\tilde{\mathcal{E}}_s(\omega)$ we get with the substitution $\epsilon \to \tilde{\epsilon} = 1-\epsilon$ (which implies $\sum \tilde{\epsilon}_{(t)} = N-M$)

$$\tilde{\mathcal{E}}_s(\omega) = (-\omega)^{-\epsilon_{(\bar{t}-1)}+s}(1-\omega)^{-\epsilon_{(\bar{t}+1)}} \prod_{t\neq\bar{t},\bar{t}\pm1} (\Omega_t-\omega)^{-\epsilon_{(t)}}, \quad 0\leq s\leq M-2. \tag{115}$$

Since all monodromies are abelian the matrices $V_{(\bar{t})r}$ could be taken to be the unity but it is more convenient to take $V_{(\bar{t})r} = U^{\dagger}_{(\bar{t})} = e^{i\pi\alpha_{(\bar{t})}}$ as suggested in the previous section 4.3. Then the solution can we written as in eqs. (84), explicitly

$$\begin{aligned}
Z(u,\bar{u}) = f_{(\bar{t}-1)} &+ \sum_{r=0}^{N-M-2} a_{(\bar{t})r} \int_{0;\omega\in H}^{\omega_{(\bar{t})}(u)} d\omega\, e^{+i\pi\alpha_{(\bar{t})}} \mathcal{E}_r(\omega) \\
&+ \sum_{s=0}^{M-2} b_{(\bar{t})s} \int_{0;\bar{\omega}\in H^-}^{\omega_{(\bar{t})}(\bar{u})} d\bar{\omega}\, e^{+i\pi\alpha_{(\bar{t})}} \tilde{\mathcal{E}}_{\{\mathbf{0}\}s}(\bar{\omega}), \\
\bar{Z}(u,\bar{u}) = \bar{f}_{(\bar{t}-1)} &+ \sum_{r=0}^{N-M-2} a_{(\bar{t})r} \int_{0;\bar{\omega}\in H^-}^{\omega_{(\bar{t})}(\bar{u})} d\bar{\omega}\, e^{-i\pi\alpha_{(\bar{t})}} \mathcal{E}_r(\bar{\omega}) \\
&+ \sum_{s=0}^{M-2} b_{(\bar{t})s} \int_{0;\omega\in H}^{\omega_{(\bar{t})}(u)} d\omega\, e^{-i\pi\alpha_{(\bar{t})}} \tilde{\mathcal{E}}_s(\omega).
\end{aligned} \tag{116}$$

Now the real coefficients a and b are determined by the global conditions (85) or in the more convenient way given in eq. (102), explicitly

$$\begin{aligned}
&Z(x_{t-1},x_{t-1}) - Z(x_t,x_t) = f_{(t-1)} - f_{(t)} \\
&= e^{i\pi\alpha_{(\bar{t})}} \Big[\sum_{r=0}^{N-M-2} a_{(\bar{t})r} I_{L\,r\,(t)} + \sum_{s=0}^{M-2} b_{(\bar{t})s} I_{R\,s\,(t)} \Big],
\end{aligned} \tag{117}$$

where we have defined the coefficients

$$I_{L\,r\,(t)}(\Omega_t,\epsilon_t) = \int_{\omega_{(\bar{t})t};\omega\in H}^{\omega_{(\bar{t})t-1}} d\omega\, \mathcal{E}_r(\omega),$$

$$I_{R\,s\,(t)}(\Omega_t,\epsilon_t)=\int_{\omega_{(\bar t)t};\bar\omega\in H^-}^{\omega_{(\bar t)t-1}} d\bar\omega\,\tilde{\mathcal{E}}_s(\bar\omega). \tag{118}$$

In this way we recover the previous results for the abelian case. Notice however that unless the solution is holomorphic or antiholomorphic there is a further constraint which seems to have gone unnoticed. It comes from the requirement of the vanishing of the world-sheet energy–momentum tensor, i.e. $T_{uu}=T_{\bar u\bar u}=0$. If this constraint is not satisfied the solution is not a solution of the Nambu–Goto action and its action evaluated on the solution is not the area of the string, i.e. the area of the polygon bounded by the branes in $\mathbb{R}^2$ as it should happen. This fact will emerge when we discuss how to write in a simple way the classical action in section 5. It is easy to see that the constraint $T_{uu}=0$ can be written as

$$\omega(1-\omega)\prod_{t\neq\bar t,\bar t\pm1}(\Omega_t-\omega)^{-1}\sum_{r=0}^{N-M-2}a_{(\bar t)r}\omega^r\sum_{s=0}^{M-2}b_{(\bar t)s}\omega^s=0 \tag{119}$$

which implies the $N-3$ quadratic constraints on the coefficients a and b

$$\sum_{r=0}^{N-M-2}\sum_{s=0}^{M-2}\delta_{r+s,n}\,a_{(\bar t)r}\,b_{(\bar t)s}=0,\quad 0\le n\le N-4. \tag{120}$$

These constraints are constraints on the possible world-sheet interaction points Ω_t $(t\neq\bar t,\bar t\pm1)$ and on the possible values of $f_{(t-1)}-f_{(t)}$ which are mediated by the coefficients a and b. For example for the $N=4$ $M=2$ case we get $a_{(\bar t)r=0}\,b_{(\bar t)s=0}=0$. A possible solution is $b_{(\bar t)s=0}=0$. Then using eqs. (117) we get

$$\frac{I_{L\,r\,(\bar t-1)}(\Omega_{\bar t-2},\epsilon_t)}{I_{L\,r\,(\bar t-2)}(\Omega_{\bar t-2},\epsilon_t)}=\frac{f_{(\bar t-2)}-f_{(\bar t-1)}}{f_{(\bar t-3)}-f_{(\bar t-2)}}, \tag{121}$$

and similarly for the $a_{(\bar t)r=0}=0$ case with $\epsilon\to\tilde\epsilon$.

This example could give the impression that all solutions are holomorphic or antiholomorphic, this is not the case as it can be seen for $N=5$ $M=3$.

Now the question is what is the interpretation of the previous facts and whether it invalidates the results in the literature. A possible interpretation is that when we fix the conformal gauge and give the polygon there is only one possible set of points Ω_t which give a solution of the classical string equation. This seems at odd with our intuitive understanding but we have to remember that in the case of the embedding of the world-sheet into a polygon in $\mathbb{R}^2$ the classical string action is simply the area and the equations of motion are essentially empty. As far as the computation of the Green functions and the normalizations of the twist fields amplitudes the previous observation is not a problem since the solutions found are used as a basis on which constraints are imposed and therefore the previous results in literature are still valid.

4.5. *The classical solution with $SU(2)$ monodromy*

We are now ready to compute the classical solution in the simplest of all non-abelian cases, i.e. when the global monodromy group is $SU(2)$. We leave for future publications more complex cases. As we have discussed before in section 4.2 there is only one solution of the hypergeometric which is needed for $\partial\mathcal{Z}_{(\bar t)}(z)$

$$\mathcal{E}_{\{\mathbf{0}\}r=1}(\omega_{(\bar t)}) = P \left\{ \begin{matrix} 0 & 1 & \infty & \\ -\bar n_{(\bar t-1)} & -\bar n_{(\bar t+1)} & 2-\bar n_{(\bar t)} & \omega_{(\bar t)} \\ -n_{(\bar t-1)} & -n_{(\bar t+1)} & 2-n_{(\bar t)} & \end{matrix} \right\} \tag{122}$$

and only one solution of the hypergeometric for $\partial \tilde{\mathcal{Z}}_{(\bar t)}(z)$

$$\tilde{\mathcal{E}}_{\{\mathbf{0}\}s=1}(\omega_{(\bar t)}) = P \left\{ \begin{matrix} 0 & 1 & \infty & \\ -\bar n_{(\bar t-1)} & -\bar n_{(\bar t+1)} & 2-\bar n_{(\bar t)} & \omega_{(\bar t)} \\ -n_{(\bar t-1)} & -n_{(\bar t+1)} & 2-n_{(\bar t)} & \end{matrix} \right\}. \tag{123}$$

For determining the values of the constants a, b, c, d and f we have to make use of the discussion done in section 2.4. Would we not we could make many different associations between the previous P symbols and the P symbol in eq. (27) because of the permutation property of the P symbol (8). Moreover we would miss the proper normalizations.

Now comparing the previous P symbols with the P symbol in eq. (27) and using eqs. (39) we get the following values

$$\begin{aligned} a &= n_{(\bar t-1)} + n_{(\bar t)} + n_{(\bar t+1)} - 1 \\ b &= n_{(\bar t-1)} - n_{(\bar t)} + n_{(\bar t+1)} \\ c &= 2n_{(\bar t-1)} \\ d &= n_{(\bar t-1)} - 1 \\ f &= n_{(\bar t-1)} + n_{(\bar t+1)} - 2, \end{aligned} \tag{124}$$

since we can identify

$$n_{[\mathbf{0}]} = n_{(\bar t-1)}, \quad n_{[\mathbf{1}]} = n_{(\bar t+1)}, \quad n_{[\infty]} = n_{(\bar t)}. \tag{125}$$

For the $\tilde{\mathcal{E}}$ solution we have to remember that $\tilde n_{(\bar t-1)3} = -n_{(\bar t-1)3}$ and $\tilde n_{(\bar t)1} = -n_{(\bar t)1}$ which stems from $\tilde M = M^*$. Therefore we get

$$\begin{aligned} \tilde a &= -n_{(\bar t-1)} + n_{(\bar t)} + n_{(\bar t+1)} && = a + 1 - c \\ \tilde b &= -n_{(\bar t-1)} - n_{(\bar t)} + n_{(\bar t+1)} + 1 && = b + 1 - c \\ \tilde c &= 2 - 2n_{(\bar t-1)} && = 2 - c \\ \tilde d &= -n_{(\bar t-1)} \\ \tilde f &= -n_{(\bar t-1)} + n_{(\bar t+1)} - 1. \end{aligned} \tag{126}$$

We are now ready to write the $\mathcal{E}_{\{\mathbf{0}\}}$ and $\tilde{\mathcal{E}}_{\{\mathbf{0}\}}$ as

$$\begin{aligned} &\mathcal{E}_{\{\mathbf{0}\}r=1}(\omega) \\ &= (-\omega)^{n_{(\bar t-1)}-1}(1-\omega)^{n_{(\bar t+1)}-1} \times \\ &\begin{pmatrix} F\left(\begin{matrix} n_{(\bar t-1)} + n_{(\bar t)} + n_{(\bar t+1)} - 1 \quad n_{(\bar t-1)} - n_{(\bar t)} + n_{(\bar t+1)} \\ 2n_{(\bar t-1)} \end{matrix} ; \omega \right) \\ \mathcal{N}(-\omega)^{1-2n_{(\bar t-1)}} F\left(\begin{matrix} -n_{(\bar t-1)} + n_{(\bar t)} + n_{(\bar t+1)} \quad -n_{(\bar t-1)} - n_{(\bar t)} + n_{(\bar t+1)} + 1 \\ 2 - 2n_{(\bar t-1)} \end{matrix} ; \omega \right) \end{pmatrix}, \end{aligned} \tag{127}$$

and

$$\tilde{\mathcal{E}}_{\{\mathbf{0}\}s=1}(\omega) = \mathcal{N}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{E}_{\{\mathbf{0}\}r=1}(\omega) \tag{128}$$

with[19]

$$\mathcal{N} = -sign(n_{(\bar{t})1}) \sqrt{\frac{\sin[\pi(n_{(\bar{t}-1)} + n_{(\bar{t})} - n_{(\bar{t}+1)})]}{\sin[\pi(n_{(\bar{t}-1)} + n_{(\bar{t})} + n_{(\bar{t}+1)})]} \frac{\sin[\pi(-(n_{(\bar{t}-1)} - n_{(\bar{t})}) + n_{(\bar{t}+1)})]}{\sin[\pi((n_{(\bar{t}-1)} - n_{(\bar{t})}) + n_{(\bar{t}+1)})]}}. \tag{129}$$

Notice that $\mathcal{E}_{\{\mathbf{0}\}r=1}(\omega)$ and $\tilde{\mathcal{E}}_{\{\mathbf{0}\}s=1}(\omega)$ differ in nuce by the σ_2 matrix because for any $SU(2)$ matrix we have $U^* = \sigma_2 U \sigma_2$.

Because of this we reabsorb $\mathcal{N}$ in the definition of $\mathcal{E}_{\{\mathbf{0}\}r=1}(\omega)$ and from now on we use

$$\tilde{\mathcal{E}}_{\{\mathbf{0}\}s=1}(\omega) = i\sigma_2 \, \mathcal{E}_{\{\mathbf{0}\}r=1}(\omega). \tag{130}$$

Then we can write

$$\begin{aligned} \partial \mathcal{Z}_{(\bar{t})}(z) &= a_{(\bar{t})r=1} \frac{\partial \omega_z}{\partial z} U^{\dagger}_{(\bar{t})} \, R_{(\bar{t})} \, \mathcal{E}_{\{\mathbf{0}\}r=1}(\omega) \\ \partial \tilde{\mathcal{Z}}_{(\bar{t})}(z) &= b_{(\bar{t})s=1} \frac{\partial \omega_z}{\partial z} U^{T}_{(\bar{t})} \, R_{(\bar{t})} \, \tilde{\mathcal{E}}_{\{\mathbf{0}\}s=1}(\omega), \end{aligned} \tag{131}$$

with $a_{(\bar{t})r=1}, b_{(\bar{t})s=1} \in \mathbb{R}$.

Notice that because of the previous equations and eq. (130) the solution satisfies in a non-trivial way the energy momentum constraint

$$T_{uu} = (T_{\bar{u}\bar{u}})^* \propto \partial_u Z^i \, \partial_u \bar{Z}^{\bar{i}} = \partial \tilde{\mathcal{Z}}_{(\bar{t})}(z)^T \, \partial \bar{\mathcal{Z}}_{(\bar{t})}(z)|_{z=u} = 0, \tag{132}$$

despite the fact that $Z^i(u, \bar{u})$ is not a holomorphic function.

We are left with the task of fixing the two real coefficients $a_{(\bar{t})r=1}$ and $b_{(\bar{t})s=1}$. This can be done using any interval (x_t, x_{t-1}) in the equation (102). All of them are equivalent and must be consistent because the counting of the d.o.f.s. For example we can write

$$R^T_{(\bar{t})} \, U_{(\bar{t})}(f_{(\bar{t}+1)} - f_{(\bar{t}-1)}) = a_{(\bar{t})r=1} \int_{0;\omega\in H}^{1} d\omega \, \mathcal{E}_{\{\mathbf{0}\}r=1}(\omega) + b_{(\bar{t})s=1} \int_{0;\bar{\omega}\in H^-}^{1} d\bar{\omega} \, \mathcal{E}_{\{\mathbf{0}\}r=1}(\bar{\omega}). \tag{133}$$

One can in principle doubt that this system is solvable since we have two real unknowns a and b while we have to match four real numbers $f_{(t)} - f_{(t-1)} \in \mathbb{C}^2$. In order to see whether it is consistent and solvable we can however consider the following case

$$R^T_{(\bar{t})} \, U_{(\bar{t})}(f_{(\bar{t}-1)} - f_{(\bar{t})}) = a_{(\bar{t})r=1} \int_{-\infty}^{0} d\omega \, \mathcal{E}_{\{\mathbf{0}\}r=1}(\omega) + b_{(\bar{t})s=1} \int_{-\infty}^{0} d\omega \, \tilde{\mathcal{E}}_{\{\mathbf{0}\}s=1}(\omega), \tag{134}$$

where we do not need to distinguish whether we are integrating in the upper or lower half plane since we are working in the principal sheet in a region where the solution has not any cut which

[19] From the fact that the quantity under square root is positive along with $|n_{(\bar{t}-1)} - n_{(\bar{t})}| < n_{(\bar{t}+1)} < n_{(\bar{t}-1)} + n_{(\bar{t})}$ which follows from $U(2)$ multiplication law we deduce that $\sin[\pi(n_{(\bar{t}-1)} + n_{(\bar{t})} - n_{(\bar{t}+1)})] \sin[\pi(n_{(\bar{t}-1)} + n_{(\bar{t})} + n_{(\bar{t}+1)})] > 0$ and hence the $sign(\dots)$ in eq. (41) is positive.

are those depicted in Fig. 7. In this case the rhs is real. Therefore we need to verify that also the lhs is real. This is however true because of eq. (68).

The integrals in the previous equation can be expressed with the help of generalized hypergeometric functions as

$$\int_{-\infty}^{0} d\omega\,(-\omega)^{\alpha-1}(1-\omega)^{\beta-1} F\left(\begin{matrix} a & & b \\ & c & \end{matrix};\omega\right)$$
$$= \frac{\Gamma(c-a)\,\Gamma(b)\,\Gamma(\alpha)}{\Gamma(c)\,\Gamma(b-\beta+1)}\Gamma(b-\beta-\alpha+1){}_3F_2(c-a,b,\alpha\,;\,c,b-\beta+1\,;\,1). \tag{135}$$

Explicitly we find

$$\begin{pmatrix} a_{(\bar t)r=1}\, I^1_{L\,1\,(\bar t)} + b_{(\bar t)s=1}\, I^2_{L\,1\,(\bar t)} \\ a_{(\bar t)r=1}\, I^2_{L\,1\,(\bar t)} + b_{(\bar t)s=1}\, I^1_{L\,1\,(\bar t)} \end{pmatrix} = R^T_{(\bar t)}\, U_{(\bar t)}(f_{(\bar t-1)} - f_{(\bar t)}), \tag{136}$$

where in accordance with eq. (103) we have defined

$$\begin{aligned}
I^1_{L\,1\,(\bar t)} = {} & \frac{\Gamma(n_{(\bar t-1)}-n_{(\bar t+1)}-n_{(\bar t)}+1)\,\Gamma(n_{(\bar t-1)}+n_{(\bar t+1)}-n_{(\bar t)})\,\Gamma(n_{(\bar t-1)})}{\Gamma(2n_{(\bar t-1)})\,\Gamma(n_{(\bar t-1)}-n_{(\bar t)}+1)}\,\Gamma(-n_{(\bar t)}+1) \\
& {}_3F_2(n_{(\bar t-1)}-n_{(\bar t+1)}-n_{(\bar t)}+1,\ n_{(\bar t-1)}+n_{(\bar t+1)}-n_{(\bar t)},\ n_{(\bar t-1)}\,; \\
& 2n_{(\bar t-1)},\ n_{(\bar t-1)}-n_{(\bar t)}+1\,;\ 1) \\
I^2_{L\,1\,(\bar t)} = {} & \mathcal{N}\frac{\Gamma(-n_{(\bar t-1)}-n_{(\bar t+1)}-n_{(\bar t)}+2)\,\Gamma(-n_{(\bar t-1)}+n_{(\bar t+1)}-n_{(\bar t)}+1)\,\Gamma(-n_{(\bar t-1)}+1)}{\Gamma(-2n_{(\bar t-1)}+2)\,\Gamma(-n_{(\bar t-1)}-n_{(\bar t)}+2)} \\
& \Gamma(-n_{(\bar t)}+1) \\
& {}_3F_2(-n_{(\bar t-1)}-n_{(\bar t+1)}-n_{(\bar t)}+2,\ -n_{(\bar t-1)}+n_{(\bar t+1)}-n_{(\bar t)}+1,\ -n_{(\bar t-1)}+1\,; \\
& -2n_{(\bar t-1)}+2,\ -n_{(\bar t-1)}-n_{(\bar t)}+2\,;\ 1) \\
= {} & \mathcal{N}\frac{\Gamma(\bar n_{(\bar t-1)}-n_{(\bar t+1)}-n_{(\bar t)}+1)\,\Gamma(\bar n_{(\bar t-1)}+n_{(\bar t+1)}-n_{(\bar t)})\,\Gamma(\bar n_{(\bar t-1)})}{\Gamma(2\bar n_{(\bar t-1)})\,\Gamma(\bar n_{(\bar t-1)}-n_{(\bar t)}+1)}\,\Gamma(-n_{(\bar t)}+1) \\
& {}_3F_2(\bar n_{(\bar t-1)}-n_{(\bar t+1)}-n_{(\bar t)}+1,\bar n_{(\bar t-1)}+n_{(\bar t+1)}-n_{(\bar t)},\bar n_{(\bar t-1)}\,; \\
& 2\bar n_{(\bar t-1)},\bar n_{(\bar t-1)}-n_{(\bar t)}+1\,;\ 1)
\end{aligned} \tag{137}$$

5. The classical action

The next task is to compute the classical action of the classical configuration we have found. Many of the recent papers use to express the classical action using the KLT formalism, i.e. they express the double integral on the complex plane as a sum of products of two line integrals. In the abelian case win $\hat N_B = 3$ the final answer is then simply the area of a triangle. In view of the fact that we expect the classical action must have something to do with an area we should suspect that an easier approach should be available. In facts it turns out that using the e.o.m. in the case of holomorphic solutions, which is unfortunately not our case, we show that the classical action can be easily expressed using the embedding data.

We start from the classical action (50) and we use the e.o.m. along with the finiteness of the action to immediately write

$$4\pi\alpha' S_E^{(classical)} = \int_H dx\, dy \left((\partial_x X^I)^2 + (\partial_y X^I)^2\right)$$
$$= -\int_{-\infty}^{\infty} dx\, X^I \partial_y X^I|_{y=i0^+} \tag{138}$$

then we use the existence of different boundary conditions and the boundary conditions (59) along with (56) to write

$$= -\sum_{t=1}^{\hat{N}_B} \int_{x_t}^{x_{t-1}} dx\, X^I \partial_y X^I|_{y=i0^+}$$
$$= -\sum_{t=1}^{\hat{N}_B} g_{(t)}^{N_{(t)}} \int_{x_t}^{x_{t-1}} dx\, \partial_y X_{(D_t)}^{N_{(t)}}|_{y=i0^+}$$
$$= -i \sum_{t=1}^{\hat{N}_B} g_{(t)}^{N_{(t)}} R_{(t)\ J}^{N_{(t)}} \int_{x_t}^{x_{t-1}} dx\, (X_L^{J'}(x+i0^+) - X_R^{J'}(x-i0^+))$$
$$= 2\sum_{t=1}^{\hat{N}_B} g_{(t)}^{N_{(t)}} R_{(t)\ J}^{N_{(t)}} \Im(X_L^J(x_{t-1}+i0^+) - X_L^J(x_t+i0^+)). \tag{139}$$

This expression can be written using complex coordinates as

$$= -2\sum_{t=1}^{\hat{N}_B} \Im\left[g_{(t)}^i\right] \Re\left[U_{(t)\ j}^i Z_L^j(x+i0^+) - U_{(t)\ j}^i Z_R^j(x-i0^+)\right]\Big|_{x=x_t}^{x=x_{t-1}} \tag{140}$$

5.1. The holomorphic case

It is now immediate to finish the computation when X^i is holomorphic (or antiholomorphic) since in this case $X^j(x_t, \bar{x}_t) = X_L^j(x_t) = f_{(t)}^j$. We therefore get

$$4\pi\alpha' S_E^{(classical)} = -2\sum_{t=1}^{\hat{N}_B} \Im\left[g_{(t)}^j\right] \Re\left[U_{(t)\ j}^i (f_{(t-1)}^j - f_{(t)}^j)\right]. \tag{141}$$

In the case of $\mathbb{R}^2$ it is not difficult to see that the previous expression is actually the area of the polygon bounding the string. Few observations are needed to show this. The quantity $\sqrt{2}\,|\Im\left[g_{(t)}^z\right]|$ is the distance of the side from the origin and $\sqrt{2}\,|\Re\left[U_{(t)\ z}^z(f_{(t-1)}^z - f_{(t)}^z)\right]|$ is the length of the side. Given the equation of the line through the side $Y = aX + b$ the sign of $\Im\left[g_{(t)}^z\right]$ is the same of the sign of the product ab. The sign of $\Re\left[U_{(t)\ z}^z(f_{(t-1)}^z - f_{(t)}^z)\right]$ is the same of the component of the vector $f_{(t-1)}^z - f_{(t)}^z$ on the Y axis. Then using the previous rules it is easy to see that each term of the sum computes the (signed) area of a sub-triangle of the original polygon and that the sum is the area of the polygon. In Fig. 13 we give a very simple example of how this works and in Fig. 14 a less trivial one.

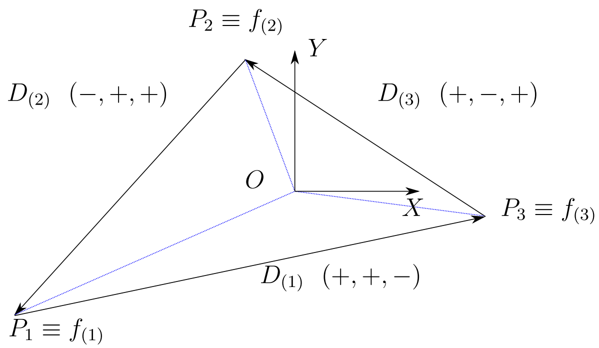

Fig. 13. It is shown as the area of the triangle $P_1 P_2 P_3$ is decomposed into three sub-triangles $P_1 P_3 O$, $P_3 P_2 O$ and $P_2 P_1 O$. Along each side the three signs are the signs of $\Re\left[U^z_{(t)\ z}(f^z_{(t-1)} - f^z_{(t)})\right]$, $a_{(t)}$ and $b_{(t)}$ where $Y = a_{(t)} X + b_{(t)}$ is the equation of the line through the side.

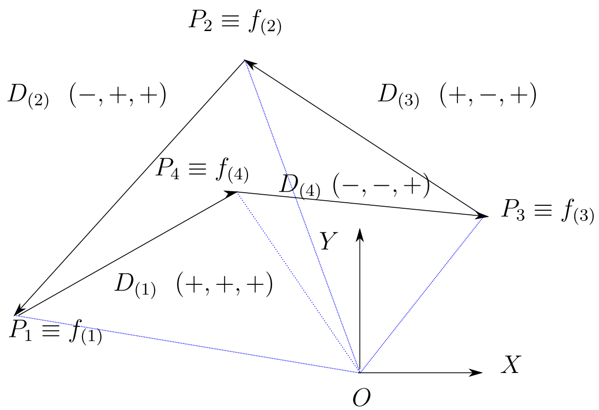

Fig. 14. It is shown as the area of the polygon $P_1 P_2 P_3 P_4$ is decomposed into four sub-triangles $P_1 P_4 O$, $P_4 P_3 O$ and $P_3 P_2 O$, $P_2 P_1 O$. The contributions to $S_E^{classical}$ of the first two triangles are positive while the ones from the last two is negative thus yielding minus the area of the interior of the polygon. Along each side the three signs are the signs of $\Re\left[U^z_{(t)\ z}(f^z_{(t-1)} - f^z_{(t)})\right]$, $a_{(t)}$ and $b_{(t)}$ where $Y = a_{(t)} X + b_{(t)}$ is the equation of the line through the side.

5.2. The general case

In the case where the solution is neither holomorphic nor antiholomorphic the computation is more complex and requires the explicit knowledge of the solution. To simplify the expression (140) we start noticing that eqs. (63) imply that

$$\Im\left[U^i_{(t)\ j}\,\partial_x Z^j_L(x)\right] = \Im\left[U^i_{(t)\ j}\,\partial_x Z^j_R(x)\right] = 0, \quad x \in \mathbb{R} - \{x_t\}_{t=1,\dots\hat{N}_B}, \tag{142}$$

i.e. both $\Im\left[U_{(t)}\, Z_L(x)\right]$ and $\Im\left[U_{(t)}\, Z_R(x)\right]$ are step functions with discontinuities in the set of the interaction points $\{x_t\}_{t=1,\dots\hat{N}_B}$. In particular, from eqs. (142) it follows that $U^i_{(t)\ j}\,\partial_x Z^j_L(x)$ and $U^i_{(t)\ j}\,\partial_x Z^j_R(x)$ are real vectors hence eq. (140) can be written without the real projection as

$$4\pi\alpha' S_E^{(classical)} = -2\sum_{t=1}^{\hat{N}_B} \Im\left[g^i_{(t)}\right] U^i_{(t)\ j}\left[Z^j_L(x+i0^+) - Z^j_R(x-i0^+)\right]\Big|^{x=x_{t-1}}_{x=x_t}. \quad (143)$$

The advantage of this expression is that we can directly compare with the global boundary conditions which can be written as

$$U^i_{(t)\ j}\left[Z^j_L(x+i0^+) + Z^j_R(x-i0^+)\right]\Big|_{x=x_t} = f^i_{(t)}, \quad (144)$$

to see that once we have solved for the global condition we need not to do more efforts to compute the classical action.

The classical action can be written also as

$$4\pi\alpha' S_E^{(classical)} = -2\sum_{t=1}^{\hat{N}_B} \Im\left[g^i_{(t)}\right] U^i_{(t)\ j}\left[\left(f^i_{(t-1)} - f^i_{(t)}\right) - 2Z^j_R(x-i0^+)\Big|^{x=x_{t-1}}_{x=x_t}\right], \quad (145)$$

in such a way to show the deviation from the holomorphic case and that the real projection in eq. (141) is not really necessary. We can also make contact with the more explicit expression in eq. (102) by writing

$$4\pi\alpha' S_E^{(classical)} = -2\sum_{t=1}^{\hat{N}_B} \Im\left[g^i_{(t)}\right]\left[U^i_{(t)\ j}\left(f^i_{(t-1)} - f^i_{(t)}\right) - 2\left(U_{(t)}U^\dagger_{(\bar{t})}R_{(\bar{t})}\right)^i_{\ j}\sum_s b_{(\bar{t})s} I^j_{R\,s\,(t)}\right]. \quad (146)$$

This expression shows the problem pointed out in section 4.4, i.e. that for the planar case where the world-sheet in embedded into a polygon in $\mathbb{R}^2$ the classical action generically is not the area of the polygon while this should be so because the classical action is nothing but the area. On the other side for the non-planar case we consider in this paper this is exactly what can be expected even if three interaction points always define a triangle. This happens because the string is not anymore constrained to be on the planar surface given by the triangle as Fig. 10 shows.

Acknowledgements

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Appendix A. Fuchsian differential equations

Fuchsian differential equations are linear differential equations in the complex plane P^1 where all solutions near the singular points of the coefficients are regular, i.e. their growth is bounded (in any small sector) by an algebraic function.

We can therefore consider the linear differential equation of order n with meromorphic coefficients $p_i(z)$ $(i = 1 \dots n)$ as

$$\frac{d^n y}{dz^n} + p_1(z)\frac{d^{n-1}y}{dz^{n-1}} + p_2(z)\frac{d^{n-2}y}{dz^{n-2}} + p_n(z)y = 0. \quad (147)$$

If we suppose that $z = z_i$ at finite is a singular point of the meromorphic coefficients and we require to have n solutions at this point with behavior like $y \sim (z - z_i)^{\rho_i}$ we deduce immediately that the previous equation can be written as

$$\frac{d^n y}{dz^n} + \frac{R_1(z-z_i)}{(z-z_i)} \frac{d^{n-1} y}{dz^{n-1}} + \frac{R_2(z-z_i)}{(z-z_i)^2} \frac{d^{n-2} y}{dz^{n-2}} + \cdots + \frac{R_n(z-z_i)}{(z-z_i)^n} y = 0, \tag{148}$$

where R_i are regular functions at $z = z_i$. The possible values of ρ_i are the indices at $z = z_i$.

If we consider $N-1$ singular points at finite we get therefore

$$\frac{d^n y}{dz^n} + \frac{\hat{P}_1(z)}{Q_{N-1}(z)} \frac{d^{n-1} y}{dz^{n-1}} + \frac{\hat{P}_2(z)}{Q^2_{N-1}(z)} \frac{d^{n-2} y}{dz^{n-2}} + \cdots + \frac{\hat{P}_n(z)}{Q^n_{N-1}(z)} y = 0, \tag{149}$$

where $Q_{N-1}(z) = \prod_{j=1}^{N-1}(z - z_j)$ and $\hat{P}_i$ are polynomials.

If we now want the infinity to be a regular singular point, i.e. we require to have n solutions with behavior like $y \sim \left(\frac{1}{z}\right)^{\rho_\infty}$ we deduce that the most general Fuchsian equation with $N-1$ regular singular points at finite and one at the infinite is given by

$$\frac{d^n y}{dz^n} + \frac{P_{N-2}(z)}{Q_{N-1}(z)} \frac{d^{n-1} y}{dz^{n-1}} + \frac{P_{2(N-2)}(z)}{Q^2_{N-1}(z)} \frac{d^{n-2} y}{dz^{n-2}} + \cdots + \frac{P_{n(N-2)}(z)}{Q^n_{N-1}(z)} y = 0 \tag{150}$$

where $Q_{N-1}(z) = \prod_{j=1}^{N-1}(z - z_j)$ when $P_k(z)$ are arbitrary polynomials of order k. We consider the case where the infinity is a singular point since it is not immediate to write down the conditions for the regularity a infinity.

The previous equation for the special case $n = 2$ can be written in a more explicit form as

$$\frac{d^2 y}{dz^2} + \sum_{i=1}^{N-1} \frac{1 - \rho_{i\,1}\rho_{i\,2}}{z - z_i} \frac{dy}{dz} + \sum_{i=1}^{N-1} \frac{\rho_{i\,1}\rho_{i\,2} + \gamma_i (z - z_i)}{(z - z_i)^2} y = 0, \tag{151}$$

with $\sum_i \gamma_i = 0$.

It is also easy to prove Fuchs' result according to which the sum of all indices must be equal to $(N-2)n(n-1)/2$, i.e.

$$\sum_{a=1}^{n} \left[\sum_{i=1}^{N} \rho_{i\,a} + \rho_{\infty\,a} \right] = (N-2)n(n-1)/2. \tag{152}$$

Given the behavior $y \sim (z - z_i)^{\rho_i}$ at the singular point $z = z_i$ at finite it is easy to show that the sum of all indices is given by $\sum_{a=1}^{n} \rho_{i\,a} = n(n-1)/2 - P_{N-2}(z_i)/[\prod_{k \neq i}(z_i - z_k)]$. In fact we get near $z = z_i$

$$\rho_i(\rho_i - 1)\ldots(\rho_i - n + 1)(z - z_i)^{\rho_i} + \rho_i(\rho_i - 1)\ldots(\rho_i - n + 2) \frac{P_{N-2}(z_i)}{\prod_{j=1, j\neq i}^{N-1}(z_i - z_j)} (z - z_i)^{\rho_i} + \cdots = 0. \tag{153}$$

Similarly assuming $y \sim \left(\frac{1}{z}\right)^{\rho_\infty}$ near $z = \infty$ we get $\sum_{a=1}^{n} \rho_{\infty\,a} = -n(n-1)/2 + A$ with $P_{N-2}(z) \sim A z^{N-2}$ since

$$z^{-\rho_\infty + n(N-2)} [-\rho_\infty(-\rho_\infty - 1)\ldots(-\rho_\infty - n + 1) - \rho_\infty(-\rho_\infty - 1)\ldots(-\rho_\infty - n + 2)A + \ldots] = 0. \tag{154}$$

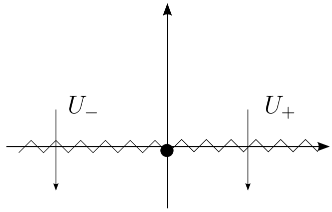

Fig. 15. The array of functions $f(z)$ with their discontinuities.

Using the residue theorem we get

$$\oint_{|z|>max\{|z_i|\}} \frac{P_{N-2}(z)}{Q_{N-1}(z)} \frac{dz}{2\pi i} = \sum_{i=1}^{N-2} \frac{P_{N-2}(z_i)}{\prod_{k\neq i}(z_i - z_k)} = A \tag{155}$$

from which the theorem follows.

Now a simple parameters counting gives $\sum_{i=1}^{n}[i(N-2)+1]$ parameters in the polynomials P and nN indices of which only $nN-1$ are arbitrary by virtue of Fuchs' result. Therefore we have $Nn(n-1)/2 - n^2 + 1$ free accessory parameters. Only for $n=1$ and $n=2,\ N=3$ there are no accessory parameters.

The general solution of the Fuchsian differential equation of order n with N singularities can be represented by a generalized P-symbol as

$$y = P\left\{\begin{array}{cccccc} x_1 & x_2 & \dots & x_N & & \\ \rho_{1\,1} & \rho_{2\,1} & \dots & \rho_{N\,1} & \vec{q} & z \\ \vdots & \vdots & \dots & \vdots & & \\ \rho_{1\,n} & \rho_{2\,n} & \dots & \rho_{N\,n} & & \end{array}\right\}, \tag{156}$$

where $\vec{q} \in \mathbb{C}^{Nn(n-1)/2-n^2+1}$. This symbol represents the space of all ∞^n solutions.

Since the equation is defined on P^1 the symbol is invariant under a $SL(2,\mathbb{C})$ transformation $\hat{z} = (az+b)/(cz+d)$, i.e.

$$y = P\left\{\begin{array}{cccccc} x_1 & x_2 & \dots & x_N & & \\ \rho_{1\,1} & \rho_{2\,1} & \dots & \rho_{N\,1} & \vec{q} & z \\ \vdots & \vdots & \dots & \vdots & & \\ \rho_{1\,n} & \rho_{2\,n} & \dots & \rho_{N\,n} & & \end{array}\right\} = P\left\{\begin{array}{cccccc} \hat{x}_1 & \hat{x}_2 & \dots & \hat{x}_N & & \\ \rho_{1\,1} & \rho_{2\,1} & \dots & \rho_{N\,1} & \vec{q} & \hat{z} \\ \vdots & \vdots & \dots & \vdots & & \\ \rho_{1\,n} & \rho_{2\,n} & \dots & \rho_{N\,n} & & \end{array}\right\}. \tag{157}$$

Appendix B. From discontinuities to monodromies

In this appendix we would like to show explicitly in a local setup that the monodromies depend on the base point. This discussion is useful for the derivation of the monodromies of the non-abelian twists.

Suppose we are given a vector of analytic functions $f(u)$ with discontinuities

$$\begin{aligned} f(x - i0^+) &= U_+ f(x + i0^+) \\ f(-x - i0^+) &= U_- f(-x + i0^+), \end{aligned} \tag{158}$$

with $x \in R$ and $x > 0$ as shown in Fig. 15.

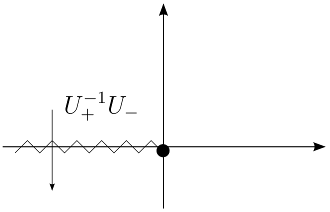

Fig. 16. The array of functions $F_+(z)$ with their discontinuity.

We want to compute the monodromies associated with $f(e^{i2\pi}(x+i0^+))$ and $f(e^{i2\pi}(x-i0^+))$. From Fig. 15 it is clear that when we compute $f(e^{i2\pi}(x+i0^+))$ we first cross the U_- discontinuity and then the U_+ discontinuity in the opposite direction with respect to the definition and therefore we have a contribution U_+^{-1}. What it is not obvious is whether the final contribution is $U_-U_+^{-1}$ or $U_+^{-1}U_-$. To solve this issue we define a new function $F_+(z)$ by the doubling trick as

$$F_+(z) = \begin{cases} f(u) & z=u\in \mathring{H}\cup(0,+\infty) \\ U_+^{-1} f(\bar u) & z=\bar u \in \mathring{H}^-\cup(0,+\infty) \end{cases} . \tag{159}$$

This new function has only one discontinuity

$$F_+(-x-i0^+) = U_+^{-1}U_- F_+(-x+i0^+), \tag{160}$$

as shown in Fig. 16. It follows then that the monodromy is simply

$$F_+\left(e^{i2\pi}(x\pm i0^+)\right) = U_-^{-1}U_+F_+(x\pm i0^+). \tag{161}$$

In particular using the definition it follows that

$$\begin{aligned} f\left(e^{i2\pi}(x+i0^+)\right) &= U_-^{-1}U_+ f(x+i0^+). \\ f\left(e^{i2\pi}(x-i0^+)\right) &= U_+U_-^{-1} f(x-i0^+). \end{aligned} \tag{162}$$

This last results show again that the monodromies depend on the base point and it is in accordance with the fact that to compute $f(e^{i2\pi}(x-i0^+))$ we first cross the U_+^{-1} discontinuity in the opposite direction and then the U_- discontinuity.

Finally, notice that the same result can be obtained using a different gluing, i.e.

$$F_-(z) = \begin{cases} f(u) & z=u\in \mathring{H}\cup(-\infty,0) \\ U_-^{-1} f(\bar u) & z=\bar u \in \mathring{H}^-\cup(-\infty,0) \end{cases} . \tag{163}$$

Appendix C. Useful formula for $U(2)$

We parametrize a $U(2)$ matrix as

$$\begin{aligned} U &= \exp(i\,2\pi\,N)\exp(i\,2\pi\,n^i\sigma_i) \\ &= e^{i\,2\pi\,N}\left(\cos(2\pi n) + i\sin(2\pi n)\frac{\vec n}{n}\cdot\vec\sigma\right), \end{aligned} \tag{164}$$

with

$$(N,\vec{n}) \in \left\{-\frac{1}{4} \le N < \frac{1}{4}, 0 \le n \le \frac{1}{2}\right\} / \sim \quad (N,\vec{n}) \sim (N',\vec{n}') \text{ iff } N = N', n = n' = \frac{1}{2}, \tag{165}$$

where $\vec{n} = (n^1, n^2, n^3)$ and $n = |\vec{n}|$.

This parametrization has the following two properties:

$$-U(N,\vec{n}) = U\left(N, -(\frac{1}{2} - |\vec{n}|)\frac{\vec{n}}{n}\right), \tag{166}$$

and

$$[U(N,\vec{n})]^* = U\left(-N, \vec{\tilde{n}}\right) = \sigma_2 U(-N,\vec{n})\sigma_2, \tag{167}$$

for all N with $\vec{\tilde{n}} = (-n^1, +n^2, -n^3)$.

It is also useful to record the product of two $U(2)$ elements. We have in fact $U(M*N, \vec{m}*\vec{n}) = U(M,\vec{m})U(N,\vec{n})$ with

$$U(N*M, \vec{m}*\vec{n}) = \begin{cases} U(N+M+\frac{1}{2}, -(\frac{1}{2}-A)\frac{\vec{A}}{A}) & -\frac{1}{2} \le N+M < -\frac{1}{4} \\ U(N+M, \vec{A}) & -\frac{1}{4} \le N+M < \frac{1}{4} \\ U(N+M-\frac{1}{2}, -(\frac{1}{2}-A)\frac{\vec{A}}{A}) & -\frac{1}{4} \le N+M < -\frac{1}{2} \end{cases}, \tag{168}$$

where the vector $\vec{A}$ with $0 \le A \le \frac{1}{2}$ is defined

$$\begin{aligned} \cos(2\pi A) &= \cos(2\pi m)\cos(2\pi n) - \sin(2\pi m)\sin(2\pi n)\frac{\vec{m}\cdot\vec{n}}{mn} \\ \sin(2\pi A)\frac{\vec{A}}{A} &= \cos(2\pi m)\sin(2\pi n)\frac{\vec{n}}{n} + \sin(2\pi m)\cos(2\pi n)\frac{\vec{m}}{m} \\ &\quad - \sin(2\pi m)\sin(2\pi n)\frac{\vec{m}\times\vec{n}}{mn}. \end{aligned} \tag{169}$$

Notice that it is also possible to choose a different range for N $0 \le N < \frac{1}{2}$ and all the formulas in this section remain unchanged in form. With this range the mapping between the parameters of a complex conjugate $U(2)$ element and the corresponding ones of the usual element is more complex, explicitly

$$[U(N,\vec{n})]^* = U\left(\frac{1}{2} - N, -(\frac{1}{2} - |\vec{n}|)\frac{\vec{\tilde{n}}}{\tilde{n}}\right) = \sigma_2 U\left(\frac{1}{2} - N, -(\frac{1}{2} - |\vec{n}|)\frac{\vec{n}}{n}\right)\sigma_2 \tag{170}$$

for $N \ne 0$ and

$$[U(N=0,\vec{n})]^* = U(N=0,\vec{\tilde{n}}) = \sigma_2 U(N=0,\vec{n})\sigma_2 \tag{171}$$

with $\frac{\vec{\tilde{n}}}{\tilde{n}} = \frac{(-n^1, +n^2, -n^3)}{n}$.

In Appendix E we exam the exact mapping.

Appendix D. Details on $U(2)$ monodromies

In this appendix we would like to give the details of the derivation of the relation between the indices and $\frac{d_{\{\mathbf{0}\}2}}{d_{\{\mathbf{0}\}1}}$ and the parameters of the $U(2)$ monodromies.

Since the monodromy at $z = 1$ is fixed once we have the monodromies at $z = 0$ and $z = \infty$ we will not deal with it.

Our strategy is first to find the constraints on the solution parameters so that the monodromy matrices are in $U(2)$ and then find the relation between them and the $U(2)$ parameters.

D.1. Constraints from the monodromy at $z = 0$

If we start from the basis of solutions $\mathcal{E}_{\{\mathbf{0}\}}(z)$ in eq. (25) the monodromy at $z = 0$

$$\mathcal{E}_{\{\mathbf{0}\}}(z e^{i2\pi}) = U_{\{\mathbf{0}\}[\mathbf{0}]}\, \mathcal{E}_{\{\mathbf{0}\}}(z), \quad |z| < 1 \tag{172}$$

becomes

$$U_{\{\mathbf{0}\}[\mathbf{0}]} = D_{\{\mathbf{0}\}}^{-1} M_{\{\mathbf{0}\}[\mathbf{0}]} D_{\{\mathbf{0}\}}, \tag{173}$$

where

$$M_{\{\mathbf{0}\}[\mathbf{0}]} = \begin{pmatrix} e^{i2\pi d} & \\ & e^{i2\pi(d-c)} \end{pmatrix} = e^{i2\pi d} M_{(B)\{\mathbf{0}\}[\mathbf{0}]} \tag{174}$$

is the monodromy of the Barnes basis at $z = 0$ (14) times $(-z)^d\,(1-z)^{f-d}$ around $z = 0$ and

$$d_{\{\mathbf{0}\}} = \begin{pmatrix} d_{\{\mathbf{0}\}1} & \\ & d_{\{\mathbf{0}\}2} \end{pmatrix} \tag{175}$$

is the matrix used to rescale the two independent solutions of the Barnes basis at $z = 0$ (14) independently of each other. Now requiring that $U_{\{\mathbf{0}\}[\mathbf{0}]} U_{\{\mathbf{0}\}[\mathbf{0}]}^\dagger = \mathbb{I}$ since $D_{\{\mathbf{0}\}}$ commutes with $M_{\{\mathbf{0}\}[\mathbf{0}]}$ implies immediately that

$$d, c \in \mathbb{R}. \tag{176}$$

D.2. Constraints from the monodromy at $z = \infty$

In an analogous way the monodromy at $z = \infty$

$$\mathcal{E}_{\{\mathbf{0}\}}(z e^{-i2\pi}) = U_{\{\mathbf{0}\}[\infty]}\, \mathcal{E}_{\{\mathbf{0}\}}(z), \quad |z| > 1 \tag{177}$$

reads

$$U_{\{\mathbf{0}\}[\infty]} = e^{-i2\pi f} D_{\{\mathbf{0}\}}^{-1} M_{\{\mathbf{0}\}[\infty]} D_{\{\mathbf{0}\}} = e^{-i2\pi f} D_{\{\mathbf{0}\}}^{-1} C M_{\{\infty\}[\infty]} C^{-1} D_{\{\mathbf{0}\}}, \tag{178}$$

where $M_{\{\mathbf{0}\}[\infty]}$ is the monodromy of the Barnes basis at $z = 0$ (14) around $z = \infty$, $M_{\{\infty\}[\infty]}$ is the monodromy of the Barnes basis at $z = \infty$ (15) around $z = \infty$ given in eq. (23), C is the matrix which connects the Barnes basis at $z = 0$ and $z = \infty$ given in eq. (20), i.e.

$$C = \frac{1}{\sin[\pi(b-a)]} \begin{pmatrix} \sin[\pi(c-a)] & -\sin[\pi(c-b)] \\ -\sin[\pi a] & \sin[\pi b] \end{pmatrix}. \tag{179}$$

Because of the previous expression we check whether the matrix $U_{\{\mathbf{0}\}[\infty]}$ is unitary by checking $U_{\{\mathbf{0}\}[\infty]}^\dagger = U_{\{\mathbf{0}\}[\infty]}^{-1}$. This amounts to impose

$$M_{\{\infty\}[\mathbf{0}]}^\dagger\, P = e^{-4\pi \Im(f)}\, P\, M_{\{\infty\}[\mathbf{0}]}^{-1}, \tag{180}$$

with $P = C^\dagger (D_{\{\mathbf{0}\}} D_{\{\mathbf{0}\}}^\dagger)^{-1} C$. Since P must be invertible we have $P_{11}, P_{22} > 0$ therefore we get

$$\Im(a) = \Im(b) = \Im(f) \tag{181}$$

while the constraints involving P_{12}, P_{21} imply

$$P_{12} = 0 \to \left|\frac{d_{\{\mathbf{0}\}2}}{d_{\{\mathbf{0}\}1}}\right|^2 = -\frac{C^*_{21} C_{22}}{C^*_{11} C_{12}} = -\frac{\sin(\pi a^*)\sin(\pi b)}{\sin[\pi(a^* - c)]\sin[\pi(b - c)]}. \tag{182}$$

This constraint can be implemented as

$$-4\sin(\pi a^*)\sin(\pi b)\sin[\pi(a^* - c)]\sin[\pi(b - c)] \in \mathbb{R}^+. \tag{183}$$

The imaginary part of the previous product must vanish and can be written as

$$\sin(i4\pi\Im(a))\ \sin(\pi c)\ (sin(\pi(2\Re(a) - c)) - sin(\pi(2\Re(b) - c))) \tag{184}$$

therefore we get three cases either $\Im(a) = \Im(b) = \Im(f)$ or $\Re(a) = \Re(b)\ mod\ 1$ or $c = \Re(a) + \Re(b)\ mod\ 1$. The latter case is not admissible since $\left|\frac{d_{\{\mathbf{0}\}2}}{d_{\{\mathbf{0}\}1}}\right|^2$ would be negative. While the second gives an abelian monodromy.

D.3. The monodromy at $z = \infty$

We start by taking the trace of eq. (178) from which we immediately get

$$e^{i\,2\pi\,N_{[\infty]}}\cos(2\pi n_{[\infty]}) = e^{i\pi(a+b-f)}\cos(\pi(a - b)) \tag{185}$$

which implies eqs. (36).

Now taking the imaginary part of $U_{\{\mathbf{0}\}[\infty]\,11}$ and using eqs. (36) we find immediately the first of eqs. (38), i.e.

$$\frac{n^3_{[\infty]}}{n_{[\infty]}} = (-)^{s_{n[\infty]}+1}\frac{\cos[\pi(a + b - c)] - \cos(\pi c)\cos[\pi(a - b)]}{\sin(\pi c)\sin[\pi(a - b)]}. \tag{186}$$

From the product $U_{\{\mathbf{0}\}[\infty]\,12}U_{\{\mathbf{0}\}[\infty]\,21} = |U_{\{\mathbf{0}\}[\infty]\,12}|^2$ it follows the constraint in eq. (31)

$$-\sin(\pi a)\sin(\pi b)\sin[\pi(a - c)]\sin[\pi(b - c)] > 0. \tag{187}$$

From the ratio $U_{\{\mathbf{0}\}[\infty]\,12}/U_{\{\mathbf{0}\}[\infty]\,21}$ and from the previous equation it follows eq. (32)

$$\left|\frac{d_{\{\mathbf{0}\}2}}{d_{\{\mathbf{0}\}1}}\right|^2 = -\frac{\sin(\pi a)\sin(\pi b)}{\sin[\pi(a - c)]\sin[\pi(b - c)]}. \tag{188}$$

Finally from $U_{\{\mathbf{0}\}[\infty]\,12}$ we get the second equation in eqs. (38)

$$\frac{n^1_{[\infty]} + i\,n^2_{[\infty]}}{n_{[\infty]}} = e^{-i2\pi\delta_{[\mathbf{0}]}}(-)^{s_{n[\infty]}+1}\,sign(\sin(\pi a)\sin(\pi b))\ \frac{\sqrt{-4\sin(\pi a)\sin(\pi b)\sin[\pi(a - c)]\sin[\pi(b - c)]}}{\sin(\pi c)\sin[\pi(a - b)]}. \tag{189}$$

Appendix E. Details on the mapping of the Papperitz–Riemann parameters for the two $U(2)$ parametrizations

An element of $U(2)$ can be represented as

$$U = \exp(i\, 2\pi\, N) \exp(i\, 2\pi\, n^i \sigma_i) = e^{i\, 2\pi\, N} \left(\cos(2\pi n) + i \sin(2\pi n) \frac{\vec{n}}{n} \cdot \vec{\sigma} \right). \tag{190}$$

There are at least two obvious parametrizations, the one chosen in the main text

$$\left\{ -\frac{1}{4} \le N < \frac{1}{4},\ 0 \le n \le \frac{1}{2} \mid n = n' = \frac{1}{2} \Rightarrow \vec{n} \equiv \vec{n}' \right\}, \tag{191}$$

and a different one given by

$$\left\{ 0 \le \hat{N} < \frac{1}{2},\ 0 \le \hat{n} \le \frac{1}{2} \mid \hat{n} = \hat{n}' = \frac{1}{2} \Rightarrow \vec{\hat{n}} \equiv \vec{\hat{n}}' \right\}. \tag{192}$$

These two parametrization are connected by

$$(\hat{N}, \vec{\hat{n}}) = \left\{ \begin{array}{cc} (N,\ \vec{n}) & 0 \le N < \frac{1}{4} \\ (N + \frac{1}{2},\ (n - \frac{1}{2}) \frac{\vec{n}}{n}) & -\frac{1}{4} \le N < 0 \end{array} \right. . \tag{193}$$

As a check of eqs. (39) we can verify that the parameters of the Papperitz–Riemann symbols are the same in the two parametrizations up to the Papperitz–Riemann symbols symmetries.

For example for $\frac{1}{4} < N_{[\mathbf{0}]} < \frac{1}{2}$ and $0 \le N_{[\infty]} < \frac{1}{4}$ we get

$$\hat{a} = a,\ \hat{b} = b,\ \hat{c} = c,\ \hat{d} = d,\ \hat{f} = f \tag{194}$$

when we map

$$\begin{aligned} \hat{A} &= \frac{1}{2} - A \\ (-)^{\hat{s}_A} &= -(-)^{s_A} \\ \hat{k}_a &= k_a + \frac{1}{2}(1 + (-)^{s_A}) \\ \hat{k}_b &= k_b + \frac{1}{2}(1 + (-)^{s_A}) \\ \hat{k}_c &= k_c + 1 \\ \hat{k}_d &= k_d \\ \hat{k}_f &= k_f + \frac{1}{2}(1 - (-)^{s_A}). \end{aligned} \tag{195}$$

Notice that $\hat{A} = \frac{1}{2} - A$ is enforced by eq. (40).

Another example is given by $0 \le N_{[\mathbf{0}]} < \frac{1}{4}$ and $\frac{1}{4} < N_{[\infty]} < \frac{1}{2}$ for which we have

$$\hat{a} = b,\ \hat{b} = a,\ \hat{c} = c,\ \hat{d} = d,\ \hat{f} = f \tag{196}$$

when we map

$$\begin{aligned}
\hat{A} &= \frac{1}{2} - A \\
(-)^{\hat{s}_A} &= -(-)^{s_A} \\
\hat{k}_a &= k_b - \frac{1}{2}(1-(-)^{s_A}) \\
\hat{k}_b &= k_a + \frac{1}{2}(1-(-)^{s_A}) \\
\hat{k}_c &= k_c \\
\hat{k}_d &= k_d \\
\hat{k}_f &= k_f + \frac{1}{2}(1-(-)^{s_A}).
\end{aligned} \tag{197}$$

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