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Bergman–Bianchi identities in field theories*

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Abstract

We relate the generalized Bergman–Bianchi identities for Lagrangian field theories on gauge-natural bundles with the kernel of the associated gauge-natural Jacobi morphism.

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Key words: gauge-natural bundles, Bergman–Bianchi identities, Jacobi morphisms.

1 The Bergman–Bianchi morphism

Our general framework is the calculus of variations on finite order *gauge-natural bundles* [3, 8]. Such geometric structures have been widely recognized to suitably describe so-called gauge-natural field theories, *i.e.* physical theories in which right-invariant infinitesimal automorphisms of the structure bundle $\boldsymbol{P}$ uniquely define the transformation laws of the fields themselves (see *e.g.* [4] and references quoted therein). We shall in particular consider *finite order variational sequences on gauge-natural bundles*, whereby foundamental objects of calculus of variations such as Lagrangians, Euler–Lagrange and Jacobi morphisms are conveniently represented as quotient morphisms (see *e.g.* [9, 6]). For basic notions and fixing notation we refer to [1, 3, 4, 5, 6, 8, 11] and references therein.

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Recall that generalized Bergman–Bianchi identities for field theories are necessary and (locally) sufficient conditions for the Noether conserved current to be not only closed but also the divergence of a skew-symmetric (tensor) density along solutions of the Euler–Lagrange equations [1]. It was also stressed that in the general theory of relativity these identities coincide with the contracted Bianchi identities for the curvature tensor of the pseudo-Riemannian metric.

Let $\boldsymbol{Y}_\zeta$ be a gauge-natural bundle and let λ be a gauge-natural Lagrangian [4, 8] on the s-th order prolongation $J_s\boldsymbol{Y}_\zeta$. Let $\mathcal{A}^{(r,k)}$ be the vector bundle of right-invariant principal automorphisms of the underlying principal structure bundle $\boldsymbol{P}$. In the following we shall consider variation vector fields which are vertical parts of gauge-natural lifts of a given $\bar{\Xi} \in \mathcal{A}^{(r,k)}$. Let $\mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \simeq J_{2s+1}\mathcal{A}^{(r,k)} \times_{J_{2s}\mathcal{A}^{(r,k)}} VJ_{2s}\mathcal{A}^{(r,k)}$. By a slight abuse of notation, we denote by $\mathfrak{G}(\bar{\Xi})_V$ the vertical part – with respect to the contact structure induced by the projections $J_{s+1}\boldsymbol{Y}_\zeta \to J_s\boldsymbol{Y}_\zeta$ – of (jet prolongation of) the gauge-natural lift $\mathfrak{G}(\bar{\Xi})$ [3, 4, 5]. We set

$$\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) \equiv \pounds_{\bar{\Xi}} \rfloor \mathcal{E}_n(\lambda) : J_{2s}\boldsymbol{Y}_\zeta \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X}) \,, \quad (1)$$

where $\pounds_{\bar{\Xi}}$ is the Lie derivative operator acting on sections of the gauge-natural bundle [5], $\rfloor$ is the interior product and $\mathcal{E}_n(\lambda)$ is the generalized Euler-Lagrange morphism associated with λ [6]. The morphism $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ so defined is a generalized Lagrangian associated with the field equations of the original Lagrangian λ and it has been considered in applications *e.g.* in General Relativity. By the linearity of $\pounds$ we can regard $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ as the extended morphism defined on $J_{2s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{2s}\mathcal{A}^{(r,k)}$. We have $D_H\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$, where D_H is the exterior differential; thus, as a consequence of a global decomposition formula for vertical morphisms [7], we can state the following [11].

Lemma 1 *Let $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ be as above. On the domain of $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ we have (up to pull-backs):*

$$\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) = \beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) + F_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)} \,,$$

where

$$\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) \equiv E_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$$

and, locally, $F_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)} = D_H M_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$.

Definition 1 We call the global morphism $\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) := E_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$ the *generalized Bergman–Bianchi morphism* associated with the Lagrangian λ and the variation vector field $\bar{\Xi}$. □

Let $\mathfrak{K}$ be the *kernel* of $\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V)$. We have the following characterization of the Bergman-Bianchi identities for gauge-natural theories [11].

Theorem 1 *The generalized Bergman–Bianchi morphism is globally vanishing for the variation vector field* $\bar{\Xi}$ *if and only if* $\delta^2_{\mathfrak{G}}\lambda \equiv \mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$, i.e. *if and only if* $\mathfrak{G}(\bar{\Xi})_V \in \mathfrak{K}$.

From now on we shall write $\omega(\lambda, \mathfrak{K})$ to denote $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ when $\mathfrak{G}(\bar{\Xi})_V$ belongs to $\mathfrak{K}$. Analogously for β and other morphisms.

First of all let us make the following important consideration. Let $\mathcal{L}_{j_s\bar{\Xi}}$ be the variational Lie derivative operator [6] acting on gerneralized variational morphisms.

Proposition 1 *For each* $\bar{\Xi} \in \mathcal{A}^{(r,k)}$ *such that* $\bar{\Xi}_V \in \mathfrak{K}$, *we have*

$$\mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda, \mathfrak{K}) = -D_H(-j_s \pounds_{\bar{\Xi}} \rfloor p_{D_V\omega(\lambda,\mathfrak{K})})\,. \tag{2}$$

Proof. We have

$$\mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda, \mathfrak{K}) = \mathcal{L}_{j_s\bar{\Xi}_V}\mathcal{L}_{j_s\bar{\Xi}}\lambda = \mathcal{L}_{j_s[\bar{\Xi}_V,\bar{\Xi}_H]}\lambda\,.$$

On the other hand it is also easy to verify that

$$\mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda, \mathfrak{K}) = \mathcal{L}_{j_s[\bar{\Xi}_H,\bar{\Xi}_V]}\lambda = -\mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda, \mathfrak{K})\,.$$

Since

$$\begin{aligned}\mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda, \mathfrak{K}) &= -\pounds_{\bar{\Xi}}\rfloor\mathcal{E}_n(\omega(\lambda, \mathfrak{K})) + D_H(-j_s\pounds_{\bar{\Xi}}\rfloor p_{D_V\omega(\lambda,\mathfrak{K})}) = \\ &= \beta(\lambda, \mathfrak{K}) + D_H(-j_s\pounds_{\bar{\Xi}}\rfloor p_{D_V\omega(\lambda,\mathfrak{K})})\,,\end{aligned}$$

from the Theorem above we get the assertion. □

The new generalized Lagrangian $\omega(\lambda, \mathfrak{K})$ is gauge-natural invariant too, *i.e.* $\mathcal{L}_{j_s\bar{\Xi}}\,\omega(\lambda, \mathfrak{K}) = 0$.

Even more, we can state the following

Proposition 2 *Let* $\bar{\Xi}_V \in \mathfrak{K}$. *We have*

$$\mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda, \mathfrak{K}) = 0\,. \tag{3}$$

Corollary 1 *Let $\bar{\Xi}_V \in \mathfrak{K}$. We have the* covariant *conservation law*

$$D_H(-j_s \pounds_{\bar{\Xi}} \rfloor p_{D_V \omega(\lambda, \mathfrak{K})}) = 0\,. \tag{4}$$

Definition 2 We define the covariantly conserved current

$$\mathcal{H}(\lambda, \mathfrak{K}) = -j_s \pounds_{\bar{\Xi}} \rfloor p_{D_V \omega(\lambda, \mathfrak{K})}\,, \tag{5}$$

to be a Hamiltonian form for $\omega(\lambda, \mathfrak{K})$ (in the sense of [10]). □

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