Voting as a lottery

Published version:

DOI:10.1016/j.jpubeco.2016.11.016

(Article begins on next page)

Voting as a Lottery - Online Appendix

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November 2016

Appendix A
Proofs

Proof. Equation 5. f^α and f^β are two normal densities: $f^k \sim N(\mu_k, \sigma_k)$, with $\sigma_k = \sigma$, $(k = \alpha, \beta)$. Thus,

$$f^k(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu_k)^2}{2\sigma^2}\right) \tag{A.1}$$

Inserting the above expressions for f^α and f^β into (4) in the main text, we obtain

$$\exp\left(-\frac{(q_j^0 - w_j - \mu_\alpha)^2}{2\sigma^2}\right) \cdot [u_j(\alpha) - u_j(\varsigma)] = \exp\left(-\frac{(q_j^0 - \mu_\beta)^2}{2\sigma^2}\right) \cdot [u_j(\varsigma) - u_j(\beta)]$$

Rearranging,

$$\exp\left(\frac{(q_j^0 - \mu_\beta)^2 - (q_j^0 - w_j - \mu_\alpha)^2}{2\sigma^2}\right) = \frac{u_j(\varsigma) - u_j(\beta)}{u_j(\alpha) - u_j(\varsigma)}$$

Taking the natural logarithm of both sides of the above expression and using the definition of $RASQ_j$ in (6),

$$(q_j^0 - \mu_\beta)^2 - (q_j^0 - w_j - \mu_\alpha)^2 = 2\sigma^2 \ln RASQ_j$$

After some algebraic manipulation,

$$2q_j^0 - (w_j + \mu_\alpha + \mu_\beta) = \frac{2\sigma^2 \ln RASQ_j}{w_j + \mu_\alpha - \mu_\beta}$$

Equation (5) follows from the above expression using the fact that $w_j + \mu_\alpha + \mu_\beta = m$. ■

Proof. Lemma 1. Recall that $f^\alpha(\cdot)$ and $f^\beta(\cdot)$ are normal density functions with specific parameters. Thus, we can write:

$$\frac{\partial^2 EU_j(L_j(q))}{\partial q^2} = \frac{(q-A) \cdot e^{-\frac{(q-A)^2}{2\sigma^2}} - RASQ_j \cdot (q-B) \cdot e^{-\frac{(q-B)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma^3 \left[u_j(\alpha) - u_j(\varsigma)\right]}$$

where $A = \mu_\alpha + w_j$ and $B = \mu_\beta$. This second-order derivative is positive if and only if:

$$(q - A) \cdot e^{-\frac{(q-A)^2}{2\sigma^2}} > RASQ_j \cdot (q - B) \cdot e^{-\frac{(q-B)^2}{2\sigma^2}} \tag{A.2}$$

If $A > B$, inequality (A.2) never holds for any $q \in [B, A]$: EU_j is always concave in $[B, A]$. Moreover, if $A < B$, (A.2) holds for any $q \in [A, B]$: EU_j is always convex in $[A, B]$.
If $q < \min(A, B)$, (A.2) becomes

$$\frac{(A - q) \cdot e^{-\frac{(q-A)^2}{2\sigma^2}}}{(B - q) \cdot e^{-\frac{(q-B)^2}{2\sigma^2}}} < RASQ_j$$

and if $q > \max(A, B)$, (A.2) becomes

$$\frac{(q - A) \cdot e^{-\frac{(q-A)^2}{2\sigma^2}}}{(q - B) \cdot e^{-\frac{(q-B)^2}{2\sigma^2}}} > RASQ_j$$

If the agent is confident, then $A > B$. In this case, both inequalities above hold if $A - B$ is sufficiently small. Thus, EU_j is concave for any $q \notin [B, A]$ if the level of confidence, as measured by $A - B$, is large.
If the agent is non-confident, then $A < B$, and both inequalities hold if $B - A$ is sufficiently large. Thus, EU_j is convex for any $q \notin [A, B]$ if the level of non-confidence is large. ■

Proof. Proposition 1.

To prove this Proposition first we prove the following Lemma.

Lemma A.1

i) The optimal threshold is q_j^0 in (5) if and only if j is confident and $q_j^0 \in [q^s, m]$.

ii) The optimal threshold is either simple majority or unanimity if j is non-confident or if $q_j^0 \notin [q^s, m]$.

Proof of Lemma A.1

The SOC at an interior stationary point, $\left.\frac{\partial^2 EU_j(L_j)}{\partial q^2}\right|_{q_j^0}$, is:

$$-f_q^\alpha(q_j^0 - w_j) \cdot [u_j(\alpha) - u_j(\varsigma)] + f_q^\beta(q_j^0) \cdot [u_j(\varsigma) - u_j(\beta)] < 0 \tag{A.3}$$

with $f_q^k(.) \equiv \frac{\partial f^\alpha(.)}{\partial q}$, $(k = \alpha, \beta)$. Rearranging expression (4) yields

$$\frac{f^\alpha(q_j^0 - w_j)}{f^\beta(q_j^0)} = \frac{[u_j(\varsigma) - u_j(\beta)]}{[u_j(\alpha) - u_j(\varsigma)]} \tag{A.4}$$

Using (A.4), the SOC in (A.3) can be written as

$$f_q^\beta(q_j^0) \cdot \frac{f^\alpha(q_j^0 - w_j)}{f^\beta(q_j^0)} < f_q^\alpha(q_j^0 - w_j)$$

from which

$$\frac{f_q^\beta(q_j^0)}{f^\beta(q_j^0)} < \frac{f_q^\alpha(q_j^0 - w_j)}{f^\alpha(q_j^0 - w_j)} \tag{A.5}$$

Recall that $f^k(\cdot)$ is a normal density. Therefore, the partial derivatives have precise analytical expressions:

$$\begin{aligned} f_q^\alpha(q_j^0 - w_j) &= & -\tfrac{q_j^0 - w_j - \mu_\alpha}{\sigma^3\sqrt{2\pi}} \exp\left(-\tfrac{(q_j^0 - w_j - \mu_\alpha)^2}{2\sigma^2}\right) \\ f_q^\beta(q_j^0) &= & -\tfrac{q_j^0 - \mu_\beta}{\sigma^3\sqrt{2\pi}} \exp\left(-\tfrac{(q_j^0 - w_j - \mu_\beta)^2}{2\sigma^2}\right) \end{aligned} \tag{A.6}$$

Using (A.1) and (A.6) and simplifying, we can rewrite (A.5) as

$$-\frac{1}{\sigma^2}(q_j^0 - \mu_\beta) < -\frac{1}{\sigma^2}(q_j^0 - w_j - \mu_\alpha)$$

or,

$$\mu_\alpha + w_j > \mu_\beta$$

Thus, an unconstrained maximum at the point q_j^0 implies that j is confident (the reverse is trivial). This proves part *i)* of the Lemma.

Moving to part *ii)*, first consider the case that j is confident and $q_j^0 \notin [q^s, m]$. Since q_j^0 is unique, if $q_j^0 < q^s$ ($q_j^0 > m$) then $EU_j(L_j(q))$ is decreasing (increasing) in $[q^s, m]$; thus the optimal threshold is simple majority (unanimity). Now consider the case j is non-confident. $EU_j(L_j(q))$ is convex in q. The sign of inequality (A.5) is reversed. Repeating the same steps,

$$\mu_\alpha + w_j < \mu_\beta$$

This proves that if q_j^0 is a minimum then j is non-confident (the reverse is trivial).

We can now proceed with the proof of Proposition 1.

i.1) By confidence and Lemma A.1, it follows that q_j^0 is a maximum. By (5), $q_j^0 \leq q^s$ if $\frac{\sigma^2 \ln RASQ_j}{w_j + \mu_\alpha - \mu_\beta} \leq 0$. By confidence, the denominator in this inequality is positive. This implies that $RASQ_j$ must not be higher than one. Therefore, the preferred threshold $q_j^* = q^s$.

i.2) Following the same argument, $q_j^0 > q^s$ if $RASQ_j > 1$. Therefore, $q_j^* \in (q^s, m]$.

ii.1) By Lemma A.1, q_j^0 is a minimum. Moreover, by (5), non-confidence and $RASQ_j < 1$, it follows that $\frac{\sigma^2 \ln RASQ_j}{w_j+\mu_\alpha-\mu_\beta} > 0$, thus $q_j^0 > q^s$. Agent j has to decide whether she is better off under simple majority or under unanimity. Under simple majority, outcomes α and β occur with probability $y_\alpha = \Pr\{\alpha|_{q=q^s}\}$ and $1-y_\alpha$, respectively. By non-confidence it follows that $y_\alpha < 0.5$. Recall that with simple majority the status quo is impossible, while under unanimity the status quo is (almost) certain. Thus, the expected utilities of voting under simple majority and under unanimity are $u_j(\alpha)\cdot y_\alpha + u_j(\beta)\cdot(1-y_\alpha)$ and $u_j(\varsigma)$, respectively. Agent j prefers simple majority to unanimity if

$$u_j(\alpha)\cdot y_\alpha + u_j(\beta)\cdot(1-y_\alpha) > u_j(\varsigma)$$

By rearranging,

$$[u_j(\alpha)-u_j(\varsigma)]\cdot y_\alpha > [u_j(\varsigma)-u_j(\beta)]\cdot(1-y_\alpha)$$

then,

$$y_\alpha > RASQ_j \cdot (1-y_\alpha)$$

which completes the proof.

ii.2) By Lemma A.1, q_j^0 is a minimum. By (5), non-confidence and $RASQ_j \geq 1$, it follows that $q_j^0 \leq q^s$. This implies that EU_j is monotonically increasing in $[q^s, m]$. Hence, $q_j^* = m$. If $RASQ_j < 1$, and the above inequality is not satisfied, then j prefers unanimity.

For completeness, we look at an agent that is neither confident nor non-confident, $\mu_\alpha + w_j = \mu_\beta$. In this case, for any q the probability of winning equals the probability of losing. Thus, $f^\alpha(q-w_j) = f^\beta(q)$. By (4), the optimality condition becomes

$$-[u_j(\alpha)-u_j(\varsigma)] + [u_j(\varsigma)-u_j(\beta)] \gtrless 0$$

The expression in the LHS is larger (smaller) than zero, if $RASQ_j > 1$ ($RASQ_j < 1$). This in turn implies that EU_j is increasing (decreasing) for all q. Hence j wants unanimity (simple majority). ■

Proof. Proposition 2.

To prove part *i)*, we first introduce the relationship between $RASQ_j$ and risk aversion, and then we relate $RASQ_j$ to the stationary point. We do this in the following two lemmas.

Lemma A.2. *Given the monetary values of α, β and ς, $RASQ_j$ is positively related to* *agent j's degree of risk aversion.*

Proof of Lemma A.2. We prove this Lemma by comparing two agents who get the same monetary payoffs from the same policies, but differ in their risk attitudes. Let $g_i : \{\alpha, \beta, \varsigma\} \to \mathbb{R}$ be a function that assigns the monetary values of policy outcomes to any agent i, $(i = 1, .., n)$. Consider two α-type agents, r and s in N. Assume that $g_r = g_s$. Let us write, for simplicity, $u_r(g_r(\cdot)) = u_r(\cdot)$ and $u_s(g_s(\cdot)) = u_s(\cdot)$. Suppose that r is more risk averse than s. Thus, an increasing and concave function $t : \mathbb{R} \to \mathbb{R}$ exists such that $u_r(x) = t(u_s(x))$ for each $x \in X$, where X is a closed interval in $\mathbb{R}$. Therefore, we can write

$$RASQ_r = \frac{u_r(\varsigma) - u_r(\beta)}{u_r(\alpha) - u_r(\varsigma)} = \frac{t\left(u_s(\varsigma)\right) - t\left(u_s(\beta)\right)}{t\left(u_s(\alpha)\right) - t\left(u_s(\varsigma)\right)}$$

We want to prove that

$$RASQ_r > RASQ_s$$

Since $u_s(\varsigma) \in (u_s(\beta), u_s(\alpha))$, it can be rewritten as a convex linear combination of $u_s(\beta)$ and $u_s(\alpha)$, i.e. $u_s(\varsigma) = a \cdot u_s(\alpha) + (1-a) \cdot u_s(\beta)$, with $a \in (0,1)$. Thus we can write

$$RASQ_s = \frac{[a \cdot u_s(\alpha) + (1-a) \cdot u_s(\beta)] - u_s(\beta)}{u_s(\alpha) - [a \cdot u_s(\alpha) + (1-a) \cdot u_s(\beta)]} = \frac{a}{1-a}$$

and

$$RASQ_r = \frac{t\left([a \cdot u_s(\alpha) + (1-a) \cdot u_s(\beta)]\right) - t\left(u_s(\beta)\right)}{t\left(u_s(\alpha)\right) - t\left([a \cdot u_s(\alpha) + (1-a) \cdot u_s(\beta)]\right)}$$

By the concavity of $t(\cdot)$, we know that $t\left[a \cdot u_s(\alpha) + (1-a) \cdot u_s(\beta)\right] > a \cdot t(u_s(\alpha)) + (1-a) \cdot t(u_s(\beta))$. Thus we can write

$$RASQ_r > \frac{a \cdot t(u_s(\alpha)) + (1-a) \cdot t(u_s(\beta)) - t\left(u_s(\beta)\right)}{t\left(u_s(\alpha)\right) - \left[a \cdot t\left(u_s(\alpha)\right) + (1-a) \cdot t\left(u_s(\beta)\right)\right]} = \frac{a}{1-a}$$

or $RASQ_r > RASQ_s$. This proves Lemma A.2.

Lemma A.3. *q_j^0 in (5) is positively (negatively) related to $RASQ_j$ if and only if j is confident (non-confident).*

Proof of Lemma A.3.

By (4),

$$\frac{f^{\alpha}(q_j^0 - w_j)}{f^{\beta}(q_j^0)} = RASQ_j$$

where

$$\frac{f^{\alpha}(q_j^0 - w_j)}{f^{\beta}(q_j^0)} = e^{\frac{\left(q_j^0-\mu_\beta\right)^2 - \left(q_j^0 - w_j - \mu_\alpha\right)^2}{2\sigma^2}}$$

The derivative wrt q is

$$\frac{\partial \left[\frac{f^{\alpha}(q-w_j)}{f^{\beta}(q)}\right]}{\partial q} = \frac{\partial \left[RASQ_j\right]}{\partial q} = \frac{1}{\sigma^2}(w_j + \mu_\alpha - \mu_\beta) \cdot e^{\frac{\left(q_j^0-\mu_\beta\right)^2 - \left(q_j^0 - w_j - \mu_\alpha\right)^2}{2\sigma^2}}$$

Since $\sigma^2 > 0$ and $e(\cdot) > 0$, the above derivative is positive (negative) if and only if j is confident (non-confident).

We can now prove part *i)* of Proposition 2.

Let us distinguish between two sub-cases: *i.a)* agent j is confident; *i.b)* agent j is non-confident.

i.a) By Lemmas A.2 and A.3, if agent j is confident then q_j^0 increases in her degree of risk aversion. Moreover, by part *i)* of Proposition 1, if $RASQ_j > 1$, then she prefers either a supermajority or unanimity. Therefore, as a consequence of an increase in risk aversion, we can have three cases: Case 1. $RASQ_j$ increases, but it is still not larger than one; Case 2. $RASQ_j$ increases and becomes larger than one; Case 3. $RASQ_j$ is already larger than one, and it increases. In Case 1, agent j keeps preferring simple majority. In Case 2, she stops preferring simple majority in favor of a supermajority (or unanimity). In Case 3, she prefers a higher majority threshold (or unanimity). Thus, the preferred majority threshold never decreases.

i.b) The proof works in the opposite way. By Lemmas A.2 and A.3, if agent j is non-confident then a higher degree of risk aversion negatively affects q_j^0. From Lemma A.1 we know that she can only prefer either simple majority or unanimity; part *ii.1)* Proposition 1 states that she prefers simple majority only if $RASQ_j < \frac{y_\alpha}{1-y_\alpha} < 1$. Therefore, if $RASQ_j \geq 1$ and risk aversion increases, she keeps preferring unanimity. If $RASQ_j < 1$ and risk aversion increases, we can have three cases. Case 1. If $RASQ_j$ increases and $RASQ_j < \frac{y_\alpha}{1-y_\alpha}$ is still satisfied, then she keeps preferring simple majority. Case 2. If $RASQ_j$ increases and $RASQ_j < \frac{y_\alpha}{1-y_\alpha}$ is no longer satisfied, she shifts from simple majority to unanimity. Case 3. If $RASQ_j > \frac{y_\alpha}{1-y_\alpha}$ and $RASQ_j$ increases, she keeps preferring unanimity. Therefore, the preferred threshold cannot decrease in risk aversion. This proves part *i)* of Proposition 2.

We now prove part *ii)*. Suppose w_j increases from w_j^1 to w_j^2. We can have three cases. Case 1. $\mu_\alpha + w_j^1 > \mu_\beta$ and $\mu_\alpha + w_j^2 > \mu_\beta$ (j is confident both before and after the increase in her voting weight). If $RASQ_j \leq 1$, then by Proposition 1, j keeps preferring simple majority. If

$RASQ_j > 1$, she prefers a lower majority threshold. Case 2. $\mu_\alpha + w_j^1 < \mu_\beta$ and $\mu_\alpha + w_j^2 > \mu_\beta$ (j is non-confident before the increase in her voting weight and becomes confident afterwards). If $RASQ_j < 1$, then by Proposition 1, there are two possibilities. If, before the increase in w_j, $RASQ_j < \frac{y_\alpha}{1-y_\alpha}$ then j keeps preferring simple majority. However, if $RASQ_j > \frac{y_\alpha}{1-y_\alpha}$ before the increase in w_j, she switches from unanimity to simple majority. Similarly, if $RASQ_j = 1$, then by Proposition 1, j switches from unanimity to simple majority. If $RASQ_j > 1$, she switches from unanimity to a supermajority. Case 3. $\mu_\alpha + w_j^1 < \mu_\beta$ and $\mu_\alpha + w_j^2 < \mu_\beta$ (j is non-confident both before and after the increase in weight). If $RASQ_j \geq 1$ then by Proposition 1 agent j keeps preferring unanimity. If $RASQ_j < 1$ and $RASQ_j > \frac{y_\alpha}{1-y_\alpha}$ before the increase in weight, then by Proposition 1 there are again two options. If after the increase in w_j, $RASQ_j > \frac{y_\alpha}{1-y_\alpha}$ still holds, agent j keeps preferring unanimity. If, after the increase in w_j, $RASQ_j < \frac{y_\alpha}{1-y_\alpha}$, then she switches from unanimity to simple majority. If before the increase in w_j, $RASQ_j < \frac{y_\alpha}{1-y_\alpha}$, then j keeps preferring simple majority after the increase in w_j. Hence, in all cases the most preferred threshold cannot increase in the voting weight. This proves part *ii)* of Proposition 2. Finally, we prove part *iii)*. First observe that p is positively related to μ_α and negatively related to μ_β. Then, by (5) an increase in p causes a decrease of the stationary point. Provided that an increase in p has the same effect on confidence as an increase in w_j, the rest of the proof parallels the proof of part *ii)* of this Proposition. Thus we omit it. ■

Proof. Proposition 3.

i) Since $\mu_\alpha + w_j > \mu_\beta$, by Lemma A.1, the stationary point $q_{j\lambda}^0$ in (8) maximizes (7). Following the same steps as in Proposition 1, if $(1+\lambda)RASQ_j \leq 1$ then j prefers simple majority; if $(1+\lambda)RASQ_j > 1$, then j prefers either a supermajority or unanimity. *i.1)* Consider the case $RASQ_j \leq 1$ and $(1+\lambda)RASQ_j > 1$. In this case, j would choose simple majority if she has no loss aversion, while loss aversion leads her to choose either a supermajority or unanimity. *i.2)* If $(1+\lambda)RASQ_j > 1$ and $RASQ_j > 1$, then $q_{j\lambda}^0 > q_j^0$. Thus, loss aversion leads j to prefer a higher supermajority.

ii) Since $\mu_\alpha + w_j < \mu_\beta$, $q_{j\lambda}^0$ is a minimum by Lemma A.1. By the proof of Proposition 1 (part *ii*), if $y_\alpha > (1+\lambda)RASQ_j \cdot (1-y_\alpha)$ then j prefers simple majority; otherwise she prefers unanimity. If $y_\alpha \geq RASQ_j \cdot (1-y_\alpha)$ and $y_\alpha \leq (1+\lambda)RASQ_j \cdot (1-y_\alpha)$, then j prefers unanimity under loss aversion. Without loss aversion she would have chosen simple majority.

iii) If j is confident, by (5) and (8), it is easy to see that

$$\frac{\partial q_{j\lambda}^0}{\partial w_j} < \frac{\partial q_j^0}{\partial w_j}$$

If j is non-confident we have a corner solution. By following the same steps as in the proof of Proposition 2 (part *ii* - Case 3), the optimum jumps discontinuously from unanimity to simple majority if inequality $RASQ_j < \frac{y_\alpha}{(1-y_\alpha)(1+\lambda)}$ is satisfied after the increase in w_j. This does not occur if the inequality is not satisfied before the increase in w_j. Observe that, all other things equal, the larger λ, the larger the shift in y_α that is needed to change the sign of the inequality. Since y_α positively depends on w_j, if λ is larger, then the jump from unanimity to simple majority requires a larger change in w_j. ■

Proof. Proposition 4. Let $q_j^{0|s,o}$ be j's optimal threshold if she is overprecise and $q_j^{0|s}$ be her optimal threshold if she is not overprecise. The effect of overprecision is given by the following difference:

$$q_j^{0|s,o} - q_j^{0|s} = \ln RASQ_j \left(\frac{\sigma_s^{o2}}{w_j + \mu_{\alpha|s}^o - \mu_{\beta|s}^o} - \frac{\sigma_s^2}{w_j + \mu_{\alpha|s} - \mu_{\beta|s}} \right)$$

Recall that $RASQ_j > 0$. Then, $q_j^{0|s,o} - q_j^{0|s}$ has the same sign as

$$\Delta^o = \left[\frac{(\mu_{\alpha|s}^o - \mu_{\beta|s}^o) - (\mu_{\alpha|s} - \mu_{\beta|s})}{\sigma_s^2} \right] - \phi \left[\frac{w_j + \mu_{\alpha|s} - \mu_{\beta|s}}{\bar{\sigma}^2/s} \right]$$

If ϕ increases, both the first term and the absolute value of the second term increase. The first term within square brackets is the impact of overprecision on expectations. It shows that the higher j's degree of overconfidence the larger the expectation adjustment. The second term within square brackets is the effect of overprecision on variance. The higher the degree of overprecision the lower the level of uncertainty after the reception of news. Differentiating Δ^o with respect to ϕ yields

$$\frac{\partial \Delta^o}{\partial \phi} = \frac{s}{\bar{\sigma}^2} \left\{ \frac{\bar{\sigma}^2 \left(\bar{\sigma}^2 + \sigma^2 s\right)}{\left(\bar{\sigma}^2 + \sigma^2(1+\phi)s\right)^2} \left[(\bar{\mu}_\alpha - \mu_\alpha) - (\bar{\mu}_\beta - \mu_\beta) \right] - \left(w_j + \mu_{\alpha|s} - \mu_{\beta|s} \right) \right\}$$

The term within square brackets is positive (negative) in the case of good (bad) news. Let $\underline{\phi}$ be the value of ϕ such that $(\partial \Delta^o / \partial \phi) = 0$.

Suppose j is confident before receiving the signal (i.e., $w_j + \mu_\alpha - \mu_\beta > 0$). If the news is good, the last term in round brackets, $\left(w_j + \mu_{\alpha|s} - \mu_{\beta|s}\right)$, is positive. If ϕ is sufficiently large, the first term within curly brackets is small, then $\frac{\partial \Delta^o}{\partial \phi} < 0$. More precisely, if $\phi > \underline{\phi}$, then $\partial \Delta^o / \partial \phi < 0$. In the case of bad news, if $\phi > \underline{\phi}$ and the news is such that $w_j + \mu_{\alpha|s} - \mu_{\beta|s} < 0$, then $\partial \Delta^o / \partial \phi > 0$.

Following the same steps, suppose j is non-confident before receiving the signal. Hence, in the case of bad news, if $\phi > \underline{\phi}$, then $\partial \Delta^o / \partial \phi > 0$. In the case of good news, if $\phi > \underline{\phi}$ and $w_j + \mu_{\alpha|s} - \mu_{\beta|s} > 0$, then $\partial \Delta^o / \partial \phi < 0$.

Therefore, for any $\phi > \underline{\phi}$, it is more likely that: a) $q_j^{0|s,o} < q_j^{0|s}$ if the signal contains good news; b) $q_j^{0|s,o} > q_j^{0|s}$ if the signal contains bad news. The greater ϕ, the bigger the difference between $q_j^{0|s,o}$and $q_j^{0|s}$. ■

Proof. Lemma 2. Statements *i)* and *ii)* in the Lemma follow from simple comparative statics on EU_j in (3). Let us normalize $u_j(\varsigma) = 0$ (hence $u_j(\alpha) > 0 > u_j(\beta)$) and rewrite $EU_j(L_j(q))$ in its explicit form, i.e. by substituting $\Pr_j\{\alpha, q\}$ and $\Pr_j\{\beta, q\}$ from (2) into (3):

$$EU_j(L_j(q)) = u_j(\alpha) \int\limits_{q-w_j}^{m-w_j} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu_\alpha)^2}{2\sigma^2}\right) dx + u_j(\beta) \int\limits_{q}^{m-w_j} \exp\left(-\frac{(x-\mu_\beta)^2}{2\sigma^2}\right) dx$$

If j's degree of risk aversion increases, then, by Lemma A.2, $RASQ_j$ increases. By setting $u_j(\varsigma) = 0$, this means that in (6) the absolute value of $u_j(\beta)$ increases more than $u_j(\alpha)$. This increases the negative weight of the value of the second integral relative to the positive weight of the value of the first integral, thereby leading to a downward shift in EU_j. If j is confident, then EU_j also shifts rightward: the new unique stationary point is $\tilde{q}_j^0 = \frac{m}{2} + \frac{\sigma^2 \ln \widetilde{RASQ}_j}{w_j + \mu_\alpha - \mu_\beta} > q_j^0$, where the inequality follows from $\widetilde{RASQ}_j > RASQ_j$ and $w_j + \mu_\alpha - \mu_\beta > 0$. The opposite (a leftward shift) occurs if j is non-confident, $\tilde{q}_j^0 < q_j^0$. Due to the downward shift, $\widetilde{EU}_j(L_j((\tilde{q}_j^0)) \leq EU_j(L_j(q_j^0))$. If p decreases, then μ_α decreases, while μ_β and σ increase. Therefore, the value of the first integral, $\Pr_j\{\alpha, q\}$, decreases, while the value of the second integral, $\Pr_j\{\beta, q\}$, increases. Since $u_j(\beta) < 0$, then EU_j shifts downward. If j is confident, then EU_j also shifts rightward: the new unique stationary point is $\tilde{q}_j^0 = \frac{m}{2} + \frac{(\sigma')^2 \ln RASQ_j}{w_j + \tilde{\mu}_\alpha - \tilde{\mu}_\beta} > q_j^0$, where the inequality follows from the fact that $\sigma' > \sigma$ and $w_j + \mu_\alpha - \mu_\beta > w_j + \tilde{\mu}_\alpha - \tilde{\mu}_\beta > 0$. The opposite (a leftward shift) occurs if j is non-confident: $\tilde{q}_j^0 < q_j^0$. Due to the downward shift, $\widetilde{EU}_j(L_j((\tilde{q}_j^0)) \leq EU_j(L_j(q_j^0))$. If w_j decreases, both the upper and the lower limits of the first integral increase of the same amount. This lowers the value of the integral, since a rightward shift in the set of possible thresholds reduces the probability of winning, $\Pr_j\{\alpha, q\}$. As for the second integral, only the upper limit increases, while the lower limit is not affected: this increases the set of possible thresholds and so the value of the integral. Since $u_j(\beta) < 0$, then $EU_j(L_j(q))$ shifts downwards. Furthermore, if j is confident, then EU_j shifts rightward: the new unique stationary point is $\tilde{q}_j^0 = \frac{m}{2} + \frac{\sigma^2 \ln RASQ_j}{\tilde{w}_j + \mu_\alpha - \mu_\beta} > q_j^0$. The inquality follows from the fact that $\tilde{w}_j < w_j$ and $\tilde{w}_j + \mu_\alpha - \mu_\beta > 0$. The opposite (a leftward shift) occurs if j is non-confident: $\tilde{q}_j^0 < q_j^0$. Notice that, in both cases, due to the downward shift, it is $\widetilde{EU}_j(L_j((\tilde{q}_j^0)) \leq EU_j(L_j(q_j^0))$.

Let us now analyze separately the impact of a vertical and a horizontal shift on q^N.

The magnitude of a *downward shift* in the payoff function, EU_j, can be measured by an increase in a positive parameter b that enters EU_j in the following way: $EU_j = h_j(q) - b$, with $h_j : [q^s, m] \to \mathbb{R}^+$. Let us simplify notation by letting $\Pi_{-j} = \prod\limits_{i \in N \setminus j} EU_i$, and $\Pi'_{-j} = \frac{\partial \Pi_{-j}}{\partial q}$. Moreover, unless it is differently specified, hereafter EU_j, Π_{-j} and their derivatives are evaluated at q^N. Recall that q^N satisfies the FOC and the SOC of problem (13). In our simplified notation, the FOC is:

$$EU'_j \cdot \Pi_{-j} + EU_j \cdot \Pi'_{-j} = 0 \tag{A.7}$$

and the SOC is:

$$EU''_j \cdot \Pi_{-j} + 2EU'_j \cdot \Pi'_{-j} + EU_j \cdot \Pi''_{-j} < 0 \tag{A.8}$$

Let us study the sign of $\frac{\partial q^N}{\partial b}$. By implicitly differentiating (A.7) at the point q^N, we get:

$$\frac{\partial q^N}{\partial b} = -\frac{-\Pi'_{-j}}{EU''_j \cdot \Pi_{-j} + 2EU'_j \cdot \Pi'_{-j} + EU_j \cdot \Pi''_{-j}} \tag{A.9}$$

The denominator of (A.9) is the SOC in (A.8), and it is negative by assumption. As for the numerator, from (A.7) it is easy to see that Π'_{-j} has the opposite sign of EU'_j. If $EU'_j > 0$, then $-\Pi'_{-j} > 0$ and $\frac{\partial q^N}{\partial b} > 0$. If $EU'_j < 0$, then $-\Pi'_{-j} < 0$ and $\frac{\partial q^N}{\partial b} < 0$.

As for horizontal shifts, let us first consider the case j is confident. The magnitude of a *rightward shift* in j's expected utility function can be measured by a positive parameter c that enters j's expected utility in the following way: $EU_j = l_j(q - c)$, with $l_j : \mathbb{R} \to \mathbb{R}^+$. Observe that $\frac{\partial EU_j}{\partial c} = -\frac{\partial EU_j}{\partial q} = -EU'_j$.

Let us study the sign of $\frac{\partial q^N}{\partial c}$. Implicit differentiation of (A.7) at the point q^N yields:

$$\frac{\partial q^N}{\partial c} = -\frac{-EU''_j \cdot \Pi_{-j} - EU'_j \cdot \Pi'_{-j}}{EU''_j \cdot \Pi_{-j} + 2EU'_j \cdot \Pi'_{-j} + EU_j \cdot \Pi''_{-j}} \tag{A.10}$$

The denominator in (A.10) is the SOC in (A.8), which is negative. Observe that EU'_j and Π'_{-j} have opposite signs, otherwise (A.7) would not be satisfied; therefore, $EU'_j \cdot \Pi'_{-j} < 0$. Given that

j is confident, EU_j is concave at q_j^0. Hereafter we make the simplifying assumption that it is also concave at q^N. Thus, the numerator in (A.10) is always positive. Therefore, independently of the sign of EU'_j, $\frac{\partial q^N}{\partial c} > 0$.

Let us now consider the case of a non-confident j. We measure the magnitude of a *leftward shift* in j's expected utility function by a positive parameter d such that $EU_j = l_j(q+d)$, with $l_j : \mathbb{R} \to \mathbb{R}^+$. Observe that $\frac{\partial EU_j}{\partial d} = \frac{\partial EU_j}{\partial q} = EU'_j$.

Let us study the sign of $\frac{\partial q^N}{\partial d}$. Implicit differentiation of (A.7) at q^N yields:

$$\frac{\partial q^N}{\partial d} = -\frac{EU''_j \cdot \Pi_{-j} + EU'_j \cdot \Pi'_{-j}}{EU''_j \cdot \Pi_{-j} + 2EU'_j \cdot \Pi'_{-j} + EU_j \cdot \Pi''_{-j}} \tag{A.11}$$

The denominator is negative. Hence, $\frac{\partial q^N}{\partial d} > 0$ iff $EU''_j \cdot \Pi_{-j} + EU'_j \cdot \Pi'_{-j} > 0$, which in turn yields

$$\frac{EU''_j}{EU'_j} > -\frac{\Pi'_{-j}}{\Pi_{-j}} \tag{A.12}$$

iii) By (A.9-A.10), if j is confident and $EU'_j(q^N) > 0$ then $\frac{\partial q^N}{\partial b} > 0$ and $\frac{\partial q^N}{\partial c} > 0$. Thus q^N increases.

iv) If j is non-confident and $EU'_j(q^N) > 0$, the NBS increases if $\frac{\partial q^N}{\partial b} + \frac{\partial q^N}{\partial d} > 0$. By (A.9) and (A.11), this condition holds if $-\Pi'_{-j} + EU''_j \cdot \Pi_{-j} + EU'_j \cdot \Pi'_{-j} > 0$. If $EU'_j > 1$, it yields $\frac{EU''_j}{EU'_j - 1} > -\frac{\Pi'_{-j}}{\Pi_{-j}}$. If $EU'_j \in (0,1)$, the inequality becomes $-\frac{EU''_j}{EU'_j - 1} > \frac{\Pi'_{-j}}{\Pi_{-j}}$. A sufficient condition such that the last two inequalities hold is:

$$\left| \frac{EU''_j}{EU'_j} \right| > \left| \frac{\Pi'_{-j}}{\Pi_{-j}} \right|$$

Thus, "large enough" appearing in statement *iv)* of the Lemma is a sufficient condition that should be read as "higher than $\left|\frac{\Pi'_{-j}}{\Pi_{-j}}\right| > 0$". ■

Proof. Proposition 5.

Statement *i)* immediately follows from Lemma 2.

As for statement *ii)*, assume j is confident. Since $EU'_j < 0$ then $\Pi'_{-j} > 0$. By (A.9-A.10), $\frac{\partial q^N}{\partial b} < 0$ and $\frac{\partial q^N}{\partial c} > 0$. Now j wants a lower threshold. Thus the NBS decreases if $\frac{\partial q^N}{\partial b} + \frac{\partial q^N}{\partial c} < 0$. This inequality holds if $-\Pi'_{-j} - EU''_j \cdot \Pi_{-j} - EU'_j \cdot \Pi'_{-j} < 0$. After some algebraic manipulation, if $EU'_j \in (-1, 0)$, we have $\frac{EU''_j}{1+EU'_j} > -\frac{\Pi'_{-j}}{\Pi_{-j}}$, and if $EU'_j < -1$, $\frac{EU''_j}{1+EU'_j} < -\frac{\Pi'_{-j}}{\Pi_{-j}}$. The latter inequality is never satisfied. Thus "unlikely" in the Proposition's statement means that q^N can decrease only if $EU_j' \in (0,1)$ and $\frac{EU_j''}{1+EU_j'} > -\frac{\Pi_{-j}'}{\Pi_{-j}}$ holds.

Assume now j is non-confident. She wants a lower q^N, while $\frac{\partial q^N}{\partial b} < 0$ and $\frac{\partial q^N}{\partial d} > 0$. Repeating the same steps as above, by (A.9-A.11), $\frac{\partial q^N}{\partial b} + \frac{\partial q^N}{\partial d} < 0$ holds if $-\Pi_{-j}' + EU_j'' \cdot \Pi_{-j} + EU_j' \cdot \Pi_{-j}' < 0$, or

$$\frac{EU_j''}{1 - EU_j'} < \frac{\Pi_{-j}'}{\Pi_{-j}}$$

■

Appendix B
A discrete model for a small number of agents

In this Section we present the "discrete" version of the model in Section 2 of the main text. We start with the case of unweighted votes, then we extend it to consider weighted votes.

Unweighted votes

Assume n agents have one vote each. For simplicity, let us normalize the status quo utility to zero, and assume that n is even. Thus, $u_j(\beta) < 0 < u_j(\alpha)$, and $RASQ_j = -\frac{u_j(\beta)}{u_j(\alpha)} > 0$. The exact probabilities of winning and losing, $\Pr_j\{\alpha, q\}$ and $\Pr_j\{\beta, q\}$, respectively, are the following:

$$\Pr_j\{\alpha, q\} = \sum_{x=q-1}^{n-1} \binom{n-1}{x} p^x (1-p)^{n-1-x} \tag{B.1}$$

$$\Pr_j\{\beta, q\} = \sum_{x=q}^{n-1} \binom{n-1}{x} (1-p)^x p^{n-1-x} \tag{B.2}$$

The idea is the same as in the continuous model: the chance of winning is given by the probability that the sum of votes (distributed as a binomial with parameters (n, p)) is strictly larger than $q-1$ and lower than $n-1$; the chance of losing is given by probability that the sum of votes (a binomial with parameters $(n, 1-p)$) is strictly larger than q and lower than $n-1$. Following the same steps as in Section 2 of the main paper and using the fact that $u_j(\varsigma) = 0$,

$$EU_j(q) = \Pr_j\{\alpha, q\} \cdot u_j(\alpha) + \Pr_j\{\beta, q\} \cdot u_j(\beta)$$

or

$$EU_j(q) = \sum_{x=q-1}^{n-1} \binom{n-1}{x} p^x (1-p)^{n-1-x} \cdot u_j(\alpha) + \sum_{x=q}^{n-1} \binom{n-1}{x} (1-p)^x p^{n-1-x} \cdot u_j(\beta) \tag{B.3}$$

$EU_j(q)$ is a highly non-monotonic function. In order to study the most preferred threshold, let us first compute the first-order difference of (B.3):

$$EU_j(q+1) - EU_j(q) =$$
$$= -\tbinom{n-1}{q-1} p^{q-1} (1-p)^{n-1-(q-1)} \cdot u_j(\alpha) + RASQ_j \tbinom{n-1}{q} (1-p)^q p^{n-1-q} u_j(\alpha)$$

After some algebraic manipulation, the above expression can be written as:

$$EU_j(q+1) - EU_j(q) = C \cdot [RASQ_j - \Delta(q)] \tag{B.4}$$

where $C \equiv u_j(\alpha)\frac{(n-1)!}{(q-1)!(n-q-1)!}\frac{(1-p)^q p^{n-1-q}}{q}$, $\Delta(q) \equiv \frac{q}{n-q}\left(\frac{p}{1-p}\right)^{2q-n}$. Observe that C is a positive coefficient. Thus the first-order difference of $EU_j(q)$ in (B.4) has the same sign as $RASQ_j - \Delta(q)$:

$$\begin{aligned} EU_j(q+1) - EU_j(q) > 0 \quad &\text{iff} \quad RASQ_j > \Delta(q) \\ EU_j(q+1) - EU_j(q) < 0 \quad &\text{iff} \quad RASQ_j < \Delta(q) \end{aligned} \tag{B.5}$$

Inequalities in (B.5) define the conditions for $EU_j(q)$ to increase or decrease as q increases by one unit, respectively. Namely, the conditions depend on the relative size of $RASQ_j$ and $\Delta(q)$. Since $RASQ_j$ is independent of q, we can study these conditions only by looking at the behavior of $\Delta(q)$. Let us first consider the values of $\Delta(q)$ at the endpoints, i.e. simple majority, $q = \frac{n}{2} + 1$, and unanimity, $q = n - 1$.

At the simple majority point, $\Delta(\frac{n}{2}+1) = \frac{n+2}{n-2}\left(\frac{p}{1-p}\right)^2$. If $p > \frac{1}{2}$, then $\Delta(\frac{n}{2}+1) > 1$. Since with unweighted votes $p > \frac{1}{2}$ implies confidence, this is the case in which j is confident. Therefore, if $RASQ_j < 1$, the condition in the second line of (B.5) holds. This means that we have a local maximum at the simple majority point. If $p < \frac{1}{2}$, then j is non-confident, and $\Delta(\frac{n}{2}+1) \lesseqgtr 1$. A local maximum occurs only if $RASQ_j$ is sufficiently small.

At the unanimity point, $\Delta(n-1) = (n-1)\left(\frac{p}{1-p}\right)^{n-2}$. If j is confident, then $\Delta(n-1) > 1$. A local maximum at the unanimity point occurs only if the condition in the first line of (B.5) holds. Thus, $RASQ_j$ must be sufficiently larger than one. If j is non-confident, $\Delta(n-1) \gtreqless 1$. A local maximum might occur even if $RASQ_j$ is smaller than one.

Of course, the above conditions apply only at the endpoints. Since $EU_j(q)$ is highly non-monotonic, several other local maxima may occur for intermediate values of q. By (B.5), we can study the shape of $EU_j(q)$ by studying how $\Delta(q)$ behaves at intermediate values as q increases. By $\Delta(q) \equiv \frac{q}{n-q}\left(\frac{p}{1-p}\right)^{2q-n}$, the first-order difference of $\Delta(q)$ is

$$\Delta(q+1) - \Delta(q) = \left(\frac{p}{1-p}\right)^{2q-n}\left[\frac{q+1}{n-q-1}\left(\frac{p}{1-p}\right)^2 - \frac{q}{n-q}\right] \tag{B.6}$$

Since the first term of the RHS is positive, the sign of the above expression is given by the sign of the term in the square brackets. If p is sufficiently large (small), the sign is negative (positive) for any q. Thus $\Delta(q)$ is decreasing (increasing) in q.

Suppose j is confident and p is sufficiently large. In this case $\Delta(q)$ decreases for any q. We pointed out earlier that if $RASQ_j < 1$ then simple majority is a local maximum. Since $\Delta(q)$ is decreasing, simple majority is also a global maximum because the condition in the second line of (B.5) cannot be satisfied for any value of q larger than simple majority. This result parallels point *i.1)* of Proposition 1 in the main text. Consider now the unanimity point. Suppose $RASQ_j$ is sufficiently large to have a local maximum at this point. Since $\Delta(q)$ is decreasing, it is perfectly plausible that it is not a global maximum. The latter can eventually be a super-majority. This result parallels point *i.2)* of Proposition 1.

Suppose p is sufficiently small. In this case, $\Delta(q)$ is increasing for any q. We pointed out earlier that if $RASQ_j$ is sufficiently large, then unanimity is a local maximum. In this case, it is also a global maximum, since $\Delta(q)$ is increasing. This parallels point *ii.2)* of Proposition 1. We have a local maximum at the simple majority point only if $RASQ_j$ is sufficiently small. Since $\Delta(q)$ is increasing in q, simple majority is also a global maximum only if $RASQ_j$ is larger than the highest value of $\Delta(q)$, which does occur at the unanimity point. This parallels point *ii.1)* of Proposition 1 (see also the proof of this point in part A of the present Appendix).

The proof of the other results in the main paper (about the effect of loss aversion and overconfidence) is straightforward, since it works in the same way as in the continuous model. Finally, notice that if $RASQ_j = 1$ for all j, then this model is similar to the model used by Rae (1969) and subsequent literature (e.g. Badger, 1972, Curtis, 1972, and Coelho, 2005). There are only two differences. First, in our model any player j knows in advance how she will vote. Second, p is the same for all other voters. In particular, if $RASQ_j = 1$ for all j and $p > 1/2$, then all want simple majority. This is the case of "radical voters" studied by Coelho (2005). However, the maximin solution of his constitutional game requires no less than simple majority.

Weighted votes

Intuitively, voting power is not given by the number of voters an individual j can command per se. It is rather given by j's number of votes *relative* to the number of votes other individuals can command on average. For instance, suppose j's number of votes is w_j and, say, on average other individuals have exactly w_j votes each. In this situation j has the same exact power as in the unweighted vote case presented above. The reason is that q/w_j voters are needed to form a majority. With her vote, j can swing on average only coalitions in which exactly one more voter is needed to reach the majority threshold. This example shows that w_j yields more (less) power than other voters, but only if w_j is larger (smaller) than their average number of votes. Starting from this intuition, we simplify things by assuming that all other individuals have only one vote each, while w_j can be any integer larger than one. This assumption rules out all burdensome cases in which the per capita number of votes collected by any coalition which does not include j is not an integer. This simplification allows us to re-write (B.1-B.2) as

$$\Pr_j\{\alpha, q\} = \sum_{x=q-w_j}^{n-w_j} \binom{n-w_j}{x} p^x (1-p)^{n-w_j-x} \tag{B.7}$$

$$\Pr_j\{\beta, q\} = \sum_{x=q}^{n-w_j} \binom{n-w_j}{x} (1-p)^x p^{n-w_j-x} \tag{B.8}$$

Then, by (B.7-B.8),

$$EU_j^W(q) = \sum_{x=q-w_j}^{n-w_j} \binom{n-w_j}{x} p^x (1-p)^{n-w_j-x} \cdot u_j(\alpha) + \sum_{x=q}^{n-w_j} \binom{n-w_j}{x} (1-p)^x p^{n-w_j-x} \cdot u_j(\beta)$$

where the superscript W denotes the case where votes are weighted. Following the same steps as in the unweighted voted case, the first-order difference of $EU_j^W(q)$ is

$$EU_j^W(q+1) - EU_j^W(q) =$$
$$= -\tbinom{n-w_j}{q-w_j} p^{q-w_j} (1-p)^{n-w_j-(q-w_j)} \cdot u_j(\alpha) + RASQ_j \tbinom{n-w_j}{q} (1-p)^q p^{n-w_j-q} u_j(\alpha)$$

After some algebraic manipulation, the above expression can be written as:

$$EU_j^W(q+1) - EU_j^W(q) = D \cdot \left[RASQ_j - \Delta^W(q)\right]$$

where

$$D \equiv u_j(\alpha) \frac{(n-w_j)!}{(q-w_j)!(n-q-w_j)!} \frac{(1-p)^q p^{n-w_j-q}}{q(q-1)...(q-w_j+1)}$$

and

$$\Delta^W(q, w_j) \equiv \frac{q}{n-q} \frac{q-1}{n-q-1} ... \frac{q-w_j+1}{n-q-w_j+1} \left(\frac{p}{1-p}\right)^{2q-n}$$

Therefore, the conditions for a positive or a negative first-order difference in EU_j^W are

$$\begin{aligned} EU_j^W(q+1) - EU_j^W(q) > 0 \quad \text{iff} \quad RASQ_j > \Delta^W(q, w_j) \\ EU_j^W(q+1) - EU_j^W(q) < 0 \quad \text{iff} \quad RASQ_j < \Delta^W(q, w_j) \end{aligned} \tag{B.9}$$

By comparing unweighted vote conditions (B.5) with weighted vote conditions (B.9), one can say whether EU_j^W is more or less likely to be, say, increasing in q, with respect to EU_j.

This comparison actually amounts to comparing $\Delta(q)$ in (B.5) with $\Delta^W(q, w_j)$ in (B.9). It is easy to see that $\Delta^W(q, w_j) = \gamma(q, w_j) \cdot \Delta(q)$, where

$$\gamma(w_j) = \frac{q-1}{n-q-1} \cdot \frac{q-2}{n-q-2} \cdot ... \cdot \frac{q-w_j+1}{n-q-w_j+1}$$

Since $q \geq \frac{n}{2} + 1$, all terms in the RHS of the above expression are larger than one. Thus $\gamma(q, w_j) > 1$ for any w_j, and it is increasing in w_j. Thus $\Delta^W(q, w_j) - \Delta(q)$ is strictly positive, and it is increasing in w_j. This implies that, for any q, the first-order difference of the expected utility function under weighted votes, $EU_j^W(q+1) - EU_j^W(q)$ is less likely to be positive than $EU_j(q+1) - EU_j(q)$. Moreover, as w_j increases, $EU_j^W(q+1) - EU_j^W(q)$ is less likely to be positive. This means that, all else equal, if w_j increases, then voter j is more likely to choose a lower threshold, and she will never choose a higher one. This result parallels the result in Proposition 2 (*part ii*) in the main text.