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# ON THE EXPONENTIAL SUM WITH THE SUM OF DIGITS OF HEREDITARY BASE $b$ NOTATION 

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#### Abstract

Let $b \geq 2$ be an integer and $w_{b}(n)$ be the sum of digits of the nonnegative integer $n$ written in hereditary base $b$ notation. We give optimal upper bounds for the exponential sum $\sum_{n=0}^{N-1} \exp \left(2 \pi i w_{b}(n) t\right)$, where $t$ is a real number. In particular, our results imply that for each positive integer $m$ the sequence $\left\{w_{b}(n)\right\}_{n=0}^{\infty}$ is uniformly distributed modulo $m$; and that for each irrational real $\alpha$ the sequence $\left\{w_{b}(n) \alpha\right\}_{n=1}^{\infty}$ is uniformly distributed modulo 1 .


## 1. Introduction

Let $b \geq 2$ be an integer. Exponential sums involving $s_{b}(\cdot)$, the base $b$ sum of digits function, has been studied extensively [3] [12] [6] [1], primarily in relation to problems of uniform distribution. Similar exponential sums have been also investigated with respect to alternative digital expansions. For example, digital expansions arising from linear recurrences [5] [9], complex base number systems [10] [2] and number systems for number fields [14] [11]. We study the exponential sum

$$
S_{b}(t, N):=\sum_{n=0}^{N-1} \mathbf{e}\left(w_{b}(n) t\right)
$$

where $N$ is a positive integer, $t \in \mathbf{R} \backslash \mathbf{Z}$ and $w_{b}(n)$ is the sum of digits of $n$ written in hereditary base $b$ notation.

The hereditary base $b$ notation of a nonnegative integer $n$ is obtained as follows: write $n$ in base $b$, then write all the exponents in base $b$, etc. until there appear only the numbers $0,1, \ldots, b$. For example, the hereditary base 3 notation of 4384 is

$$
1 \cdot 3^{0}+1 \cdot 3^{2 \cdot 3^{0}}+2 \cdot 3^{1 \cdot 3^{0}+2 \cdot 3^{1 \cdot 3^{0}}}
$$

The hereditary base $b$ notation was used to define Goodstein sequences and prove
the related Goodstein theorem [4], which can be considered the first simple example of a statement, true in ZFC set theory, that is unprovable in Peano arithmetic [7].

So we define $w_{b}(n)$ as the sum of digits of $n$ written in hereditary base $b$ notation, to continue the example:

$$
w_{3}(4384)=1+1+2+2+1+2+1=10
$$

More precisely, the function $w_{b}(\cdot)$ can be defined recursively as follows:

$$
\begin{aligned}
w_{b}(0) & :=0 \\
w_{b}(n) & :=\sum_{j=1}^{u}\left(w_{b}\left(k_{j}\right)+a_{j}\right) \quad \text { for } n=\sum_{j=1}^{u} a_{j} b^{k_{j}}
\end{aligned}
$$

where $u \geq 1,0 \leq k_{1}<\ldots<k_{u}$ are integers and $a_{1}, \ldots, a_{u} \in\{1,2, \ldots, b-1\}$.
We give upper bounds for $S_{b}(t, N)$, distinguishing between rational and irrational $t$.
Theorem 1.1. Let $t=\ell / m$, with $1 \leq|\ell|<m$ relatively prime integers.
(i). If $m \mid b$ then $S_{b}(t, N)=\mathbf{e}\left(w_{b}(N)\right) P_{b}(t, N)$ for $N \geq 1$, where $P_{b}(t, \cdot)$ is a periodic function of period $b$.
(ii). If $m$ is even and $m \mid(b-2)$, then $S_{b}(t, N)=\mathbf{e}\left(w_{b}(N)\right) Q_{b}(t, N)$ for $N \geq 1$, where $Q_{b}(t, \cdot)$ is a periodic function of period $b^{k^{\prime}+1}$ and $k^{\prime}$ is the least nonnegative integer such that $2\left(w_{b}\left(k^{\prime}\right)+1\right) / m$ is an odd integer.
(iii). If $m \mid(b-1)$ then $S_{b}(t, N)=O_{b}(\log N)$ for $N>1$.
(iv). If we are not in cases (i), (ii) or (iii), then for $\varepsilon>0$ it results

$$
S_{b}(t, N)=O_{b, t, \varepsilon}\left(N^{C_{b, t}+\varepsilon}\right)
$$

for $N \geq 1$, where the constant

$$
\left.C_{b, t}:=\frac{1}{m} \log _{b}\left|1-(-1)^{m+b \ell}\left(\frac{\sin (\pi(b-1) t)}{\sin (\pi t)}\right)^{m}\right| \in\right] 0,1[
$$

is the best possible.
Theorem 1.2. If $b \geq 3$ and $t$ is irrational, then for $\varepsilon>0$ it results

$$
S_{b}(t, N)=O_{b, t, \varepsilon}\left(N^{C_{b, t}+\varepsilon}\right)
$$

for $N \geq 1$, where the constant

$$
C_{b, t}:=\max \left(0, \log _{b}\left|\frac{\sin (\pi(b-1) t)}{\sin (\pi t)}\right|\right) \in[0,1[
$$

is the best possible.

Unfortunately, in the case in which $b=2$ and $t$ is irrational, we have not been able to prove an upper bound for $S_{b}(t, N)$ similar to that of Theorem 1.2 . We explain later why our arguments are not enough for that. However, for $b \geq 2$ and $t \in \mathbf{R} \backslash \mathbf{Z}$ we have at least the crude upper bound $S_{b}(t, N)=o(N)$ as $N \rightarrow \infty$ (see Lemma 2.4). Therefore, we obtain the following corollaries (see [8, Chap. 1, Theorem 2.1] and [8, Chap. 5, Corollary 1.1]).

Corollary 1.1. For each positive integer $m$, the sequence $\left\{w_{b}(n)\right\}_{n=0}^{\infty}$ is uniformly distributed modulo m, i.e.,

$$
\lim _{x \rightarrow \infty} \frac{\#\left\{n \leq x: w_{b}(n) \equiv r \bmod m\right\}}{x}=\frac{1}{m} \quad \text { for all } r=1,2, \ldots, m
$$

Corollary 1.2. For each irrational real $\alpha$, the sequence $\left\{w_{b}(n) \alpha\right\}_{n=0}^{\infty}$ is uniformly distributed modulo 1, i.e.,

$$
\lim _{x \rightarrow \infty} \frac{\#\left\{n \leq x:\left(w_{b}(n) \alpha \bmod 1\right) \in[a, b]\right\}}{x}=b-a \quad \text { for all } 0 \leq a \leq b \leq 1
$$

## Notations

Through the paper, we reserve the variables $\ell, m, n, M, N, u, h, r, k$ and $k_{j}$ for integers. We use the Bachmann-Landau symbols $O$ and $o$, as well as the Vinogradov symbols $\ll$ and $\gg$, with their usual meanings. We write $\mathscr{D}_{b}:=\{1,2, \ldots, b-1\}$ for the set of nonzero base $b$ digits and $\mathbf{e}(x):=e^{2 \pi i x}$ for the standard additive character. We adopt the usual convention that empty sums and empty products, e.g. $\sum_{n=x}^{y}$ and $\prod_{n=x}^{y}$ with $x>y$, have values 0 and 1 , respectively.

## 2. Preliminaries to proofs of Theorem 1.1 and 1.2

Proposition 2.1. If $k \geq 0, a \in \mathscr{D}_{b}$ and $0 \leq n<b^{k}$, then

$$
w_{b}\left(a b^{k}+n\right)=w_{b}(k)+w_{b}(n)+a .
$$

Proof. It is a straightforward consequence of the definition of $w_{b}(\cdot)$.
Proposition 2.2. If $k \geq 0, a \in \mathscr{D}_{b}$ and $0<N \leq b^{k}$, then

$$
S_{b}\left(t, a b^{k}+N\right)=S_{b}\left(t, a b^{k}\right)+\mathbf{e}\left(\left(w_{b}(k)+a\right) t\right) S_{b}(t, N)
$$

Proof. Consider that

$$
\begin{aligned}
S_{b}\left(t, a b^{k}+N\right) & =S_{b}\left(t, a b^{k}\right)+\sum_{n=0}^{N-1} \mathbf{e}\left(w_{b}\left(a b^{k}+n\right) t\right) \\
& =S_{b}\left(t, a b^{k}\right)+\mathbf{e}\left(\left(w_{b}(k)+a\right) t\right) \sum_{n=0}^{N-1} \mathbf{e}\left(w_{b}(n) t\right) \\
& =S_{b}\left(t, a b^{k}\right)+\mathbf{e}\left(\left(w_{b}(k)+a\right) t\right) S_{b}(t, N),
\end{aligned}
$$

where we have applied Proposition 2.1.
For the next lemma, we define

$$
\begin{aligned}
E_{b}(t, a, \theta) & :=1+\mathbf{e}\left(\theta+\frac{1}{2} a t\right) \cdot \frac{\sin (\pi(a-1) t)}{\sin (\pi t)} \\
F_{b}(t, a, r) & :=E_{b}(t, a, r t) \\
f_{b}(t, a, h) & :=F_{b}\left(t, a, w_{b}(h)\right)
\end{aligned}
$$

for $a \in \mathscr{D}_{b}, \theta \in \mathbf{R}$ and $r, h \geq 0$.
Lemma 2.3. If $k \geq 0$ and $a \in \mathscr{D}_{b}$ then

$$
S_{b}\left(t, a b^{k}\right)=f_{b}(t, a, k) \prod_{h=0}^{k-1} f_{b}(t, b, h)
$$

Moreover, if $N=\sum_{j=1}^{u} a_{j} b^{k_{j}}$, where $u \geq 1, a_{1}, \ldots, a_{u} \in \mathscr{D}_{b}$ and $0 \leq k_{1}<\cdots<k_{u}$, then

$$
S_{b}(t, N)=\mathbf{e}\left(w_{b}(N) t\right) \sum_{j=1}^{u} \mathbf{e}\left(-\sum_{v=1}^{j}\left(w_{b}\left(k_{v}\right)+a_{v}\right) t\right) S_{b}\left(t, a_{j} b^{k_{j}}\right)
$$

Proof. By Proposition 2.2 (with $N=b^{k}$ ), we see that

$$
S_{b}\left(t,(a+1) b^{k}\right)=S_{b}\left(t, a b^{k}\right)+\mathbf{e}\left(\left(w_{b}(k)+a\right) t\right) S_{b}\left(t, b^{k}\right)
$$

As a consequence, by induction, we obtain

$$
\begin{aligned}
S_{b}\left(t,(a+1) b^{k}\right) & =\left(1+\sum_{j=1}^{a} \mathbf{e}\left(\left(w_{b}(k)+j\right) t\right)\right) S_{b}\left(t, b^{k}\right) \\
& =\left(1+\mathbf{e}\left(\left(w_{b}(k)+1\right) t\right) \cdot \frac{\mathbf{e}(a t)-1}{\mathbf{e}(t)-1}\right) S_{b}\left(t, b^{k}\right) \\
& =f_{b}(t, a+1, k) S_{b}\left(t, b^{k}\right)
\end{aligned}
$$

But $f_{b}(t, 1, k)=1$, so that $S_{b}\left(t, c b^{k}\right)=f_{b}(t, c, k) S_{b}\left(t, b^{k}\right)$ for each $c \in \mathscr{D}_{b} \cup\{b\}$. In particular, $S_{b}\left(t, b^{k+1}\right)=f_{b}(t, b, k) S_{b}\left(t, b^{k}\right)$ for $c=b$. Thus, since $S_{b}(t, 1)=1$, we find that

$$
S_{b}\left(t, a b^{k}\right)=f_{b}(t, a, k) S_{b}\left(t, b^{k}\right)=f_{b}(t, a, k) \prod_{h=0}^{k-1} f_{b}(t, b, h)
$$

and the first part of the claim is proved.
Now, from Proposition 2.2, we get that

$$
\begin{align*}
S_{b}(t, N+1) & =S_{b}\left(t, a_{u} b^{k_{u}}\right)+\mathbf{e}\left(\left(w_{b}\left(k_{u}\right)+a_{u}\right) t\right) S_{b}\left(t, N+1-a_{u} b^{k_{u}}\right)  \tag{1}\\
& =S_{b}\left(t, a_{u} b^{k_{u}}\right)+\mathbf{e}\left(\left(w_{b}\left(k_{u}\right)+a_{u}\right) t\right) S_{b}\left(t, a_{u-1} b^{k_{u-1}}\right)+\cdots \\
& =\sum_{j=1}^{u}\left(\prod_{v=j+1}^{u} \mathbf{e}\left(\left(w_{b}\left(k_{v}\right)+a_{v}\right) t\right)\right) S_{b}\left(t, a_{j} b^{k_{j}}\right)+\mathbf{e}\left(w_{b}(N) t\right)
\end{align*}
$$

Subtracting $\mathbf{e}\left(w_{b}(N) t\right)$ from (1) and recalling that $w_{b}(N)=\sum_{j=1}^{u}\left(w_{b}\left(k_{j}\right)+a_{j}\right)$, the second part of the claim follows.

Lemma 2.4. $S_{b}(t, N)=o(N)$ as $N \rightarrow \infty$.
Proof. Let $a \in \mathscr{D}_{b}$ and $k \geq 0$. From Lemma 2.3, we know that

$$
\begin{equation*}
\frac{\left|S_{b}\left(t, a b^{k}\right)\right|}{a b^{k}}=\frac{\left|f_{b}(t, a, k)\right|}{a} \prod_{h=0}^{k-1} \frac{\left|f_{b}(t, b, h)\right|}{b} \tag{2}
\end{equation*}
$$

We claim that the right-hand side of (2) tends to 0 as $k \rightarrow \infty$. Observe that $\left|f_{b}(t, a, k)\right| \leq a$ and

$$
\begin{equation*}
\frac{\left|f_{b}(t, b, h)\right|}{b} \leq \frac{1}{b}\left(1+\left|\frac{\sin (\pi(b-1) t)}{\sin (\pi t)}\right|\right) \leq 1 \tag{3}
\end{equation*}
$$

for all $h \geq 0$, so that the product in (2) is a nonincreasing function of $k$. If $b \geq 3$ then the last inequality in (3) is strict, so the claim follows. If $b=2$ then, for all $h \geq 0$,

$$
\frac{\left|f_{b}(t, b, h)\right|}{b}=\left|\cos \left(\pi\left(w_{b}(h)+1\right) t\right)\right|
$$

On the one hand, if $t$ is rational, then the sequence $\{|\cos (\pi r t)|\}_{r=1}^{\infty}$ is periodic and, since $t$ is not an integer, it has infinitely many terms less than 1 . On the other hand, if $t$ is irrational, then the sequence $\{|\cos (\pi r t)|\}_{r=1}^{\infty}$ is dense in $[0,1]$. In any case, there exists $\delta<1$ and an infinitude of positive integers $r$ such that $|\cos (\pi r t)| \leq \delta$. Since $w_{b}(\cdot)$ is surjective, we get that $\left|f_{b}(t, b, h)\right| / b<\delta$ for infinitely many $h \geq 0$, and the claim follows again.

At this point, we know that $\left|S_{b}\left(t, a b^{k}\right)\right| /\left(a b^{k}\right) \rightarrow 0$ as $k \rightarrow \infty$. Let

$$
L:=\limsup _{N \rightarrow \infty} \frac{\left|S_{b}(t, N)\right|}{N}
$$

so that $L \in[0,1]$. Put $a=a(N), k=k(N)$ and $M=M(N)$ as functions of $N \geq 1$ such that $N=a b^{k}+M$ and $0 \leq M<b^{k}$. Then by Proposition 2.2 and the above, we have

$$
\begin{aligned}
L & =\limsup _{N \rightarrow \infty} \frac{\left|S_{b}\left(t, a b^{k}+M\right)\right|}{a b^{k}+M} \\
& \leq \limsup _{N \rightarrow \infty} \frac{\left|S_{b}\left(t, a b^{k}\right)\right|}{a b^{k}+M}+\limsup _{N \rightarrow \infty} \frac{\left|S_{b}(t, M)\right|}{a b^{k}+M} \\
& \leq \limsup _{k \rightarrow \infty} \frac{\left|S_{b}\left(t, a b^{k}\right)\right|}{a b^{k}}+\frac{1}{2} \limsup _{M \rightarrow \infty} \frac{\left|S_{b}(t, M)\right|}{M} \\
& =0+\frac{1}{2} L
\end{aligned}
$$

so that $L=0$, as desired.
Proposition 2.5. We have $F_{b}(t, b, r)=0$ for some $r \in \mathbf{Z}$ if and only if $t=\ell / m$, where $\ell$ and $m \neq 0$ are relatively prime integers such that:
(a) $m \mid b$ and $m \mid r$; or
(b) $m$ is even, $m\left|(b-2), \frac{1}{2} m\right|(r+1)$ and $2(r+1) / m$ is odd.

Proof. If cases (a) or (b) hold, then we quickly deduce that $F_{b}(t, b, r)=0$. On the other hand, if $F_{b}(t, b, r)=0$ then necessarily $|\sin (\pi(b-1) t)|=|\sin (\pi t)|$, so that at least one of $b t$ and $(b-2) t$ is an integer. If $b t$ is an integer then $F_{b}(t, b, r)=$ $1-\mathbf{e}(r t)=0$, so also $r t$ is an integer. Hence, $t=\ell / m$ for some relatively prime integers $\ell$ and $m \neq 0$ such that $m \mid b$ and $m \mid r$, this is case (a). If $(b-2) t$ is an integer then $F_{b}(t, b, r)=1+\mathbf{e}((r+1) t)=0$, so that $(r+1) t=u+\frac{1}{2}$ for some integer $u$. Hence, $t=(2 u+1) /(2 r+2)=\ell / m$ for some relatively prime integers $\ell$ and $m \neq 0$ such that $m$ is even, $m \mid(b-2)$ and $2(r+1) / m$ is odd, this is case (b).

Proposition 2.6. If $t$ is irrational and $b \geq 3$ then $E_{b}(t, b, \theta) \neq 0$ for all $\theta \in \mathbf{R}$.
Proof. As in the proof of Proposition 2.5, if $E_{b}(t, b, \theta)=0$ then $|\sin (\pi(b-1) t)|=$ $|\sin (\pi t)|$, so that at least one of $b t$ and $(b-2) t$ is an integer, but this is impossible, since $t$ is irrational and $b \geq 3$.

## 3. Proof of Theorem 1.1

First, suppose that $k^{\prime \prime}$ is a nonnegative integer such that $f_{b}\left(t, b, k^{\prime \prime}\right)=0$. Then, by Lemma 2.3, we have that $S_{b}\left(t, a b^{k}\right)=0$ for all $a \in \mathscr{D}_{b}$ and $k>k^{\prime \prime}$. Moreover, again by Lemma 2.3, it results that $S_{b}(t, N)=\mathbf{e}\left(w_{b}(N) t\right) R_{b}(t, N)$ for any $N \geq 1$, where $R_{b}(t, \cdot)$ is a function depending only on $\left(N \bmod b^{k^{\prime \prime}+1}\right.$ ), so periodic of (not
necessarily minimal) period $b^{k^{\prime \prime}+1}$. Therefore, by Proposition 2.5, the claim follows in case (i), taking $k^{\prime \prime}=0$, and in case (ii), taking $k^{\prime \prime}=k^{\prime}$.

Now suppose we are in case (iii), i.e., $m \mid(b-1)$. Then $f_{b}(t, b, h)=1$ for all $h \geq 0$, so that from Lemma 2.3 we get $S_{b}\left(t, b^{k}\right)=1$ for all $k \geq 0$. Taking $N$ as in Lemma 2.3, we find that
$\left|S_{b}(t, N)\right| \leq \sum_{j=1}^{u}\left|S_{b}\left(t, a_{j} b^{k_{j}}\right)\right| \leq(b-1) \sum_{j=1}^{u}\left|S_{b}\left(t, b^{k_{j}}\right)\right|=(b-1) u \leq(b-1)\left(\log _{b} N+1\right)$.
Hence, $S_{b}(t, N)=O_{b}(\log N)$ for any $N>1$, as claimed.
Finally, suppose we are in case (iv). So by Proposition $2.5, f_{b}(t, b, h) \neq 0$ for all $h \geq 0$. Thus, Lemma 2.3 yields

$$
\begin{aligned}
\lim _{k \rightarrow \infty} \frac{\log _{b}\left|S_{b}\left(t, b^{k}\right)\right|}{k} & =\lim _{k \rightarrow \infty} \frac{1}{k} \sum_{h=0}^{k-1} \log _{b}\left|f_{b}(t, b, h)\right| \\
& =\lim _{k \rightarrow \infty} \frac{1}{k} \sum_{h=0}^{k-1} \log _{b}\left|F_{b}\left(t, b, w_{b}(h)\right)\right|=\frac{1}{m} \sum_{r=1}^{m} \log _{b}\left|F_{b}(t, b, r)\right|
\end{aligned}
$$

where the last equality follows since, by Corollary 1.1, $\left\{w_{b}(n)\right\}_{n=1}^{\infty}$ is uniformly distributed modulo $m$ and $\log \left|F_{b}(t, b, \cdot)\right|$ is a function of period $m$. Observe that

$$
\prod_{r=1}^{m}|1+\mathbf{e}(r t) z|=\left|1-(-z)^{m}\right|
$$

for any complex number $z$, thus putting $z=\mathbf{e}\left(\frac{1}{2} b t\right) \sin (\pi(b-1) t) / \sin (\pi t)$ we have

$$
\frac{1}{m} \sum_{r=1}^{m} \log _{b}\left|F_{b}(t, b, r)\right|=\frac{1}{m} \log _{b}\left|1-(-1)^{m+b \ell}\left(\frac{\sin (\pi(b-1) t)}{\sin (\pi t)}\right)^{m}\right|=C_{b, t}
$$

From here, we get

$$
\begin{equation*}
\left|S_{b}\left(t, b^{k}\right)\right|=\left(b^{k}\right)^{C_{b, t}+o(1)}, \tag{4}
\end{equation*}
$$

as $k \rightarrow \infty$. Therefore, if $\varepsilon>0$ then $S_{b}\left(t, b^{k}\right)<_{b, t, \varepsilon}\left(b^{k}\right)^{C_{b, t}+\varepsilon}$ for any $k \geq 0$, and taking $N$ as in Lemma 2.3 yields that

$$
\begin{aligned}
&\left|S_{b}(t, N)\right| \leq \sum_{j=1}^{u}\left|S_{b}\left(t, a_{j} b^{k_{j}}\right)\right| \leq(b-1) \sum_{j=1}^{u}\left|S_{b}\left(t, b^{k_{j}}\right)\right|<_{b, t, \varepsilon} \sum_{j=1}^{u}\left(b^{k_{j}}\right)^{C_{b, t}+\varepsilon} \\
& \ll b, t, \varepsilon \\
& \sum_{k=0}^{k_{u}}\left(b^{k}\right)^{C_{b, t}+\varepsilon}<_{b, t, \varepsilon}\left(b^{k_{u}}\right)^{C_{b, t}+\varepsilon} \leq N^{C_{b, t}+\varepsilon}
\end{aligned}
$$

for any $N \geq 1$, as claimed.

Now we prove that $\left.C_{b, t} \in\right] 0,1\left[\right.$. On the one hand, it must be $C_{b, t} \geq 0$, otherwise $S_{b}(t, N) \rightarrow 0$ as $N \rightarrow \infty$, which is absurd. If $C_{b, t}=0$ then, since $\sin (\pi(b-1) t) \neq 0$, it results

$$
2=(-1)^{m+b \ell}\left(\frac{\sin (\pi(b-1) t)}{\sin (\pi t)}\right)^{m}=\left(-\frac{\mathbf{e}((b-1) t)-1}{\mathbf{e}(t)-1}\right)^{m}
$$

so that $\mathbf{Q}(\sqrt[m]{2})$ is a subfield of $\mathbf{Q}(\mathbf{e}(t))$, but

$$
[\mathbf{Q}(\sqrt[m]{2}): \mathbf{Q}]=m>\phi(m)=[\mathbf{Q}(\mathbf{e}(t)): \mathbf{Q}]
$$

where $\phi(\cdot)$ is the totient function, which is a contradiction, hence $C_{b, t}>0$. On the other hand, since $m \geq 2$, we see that

$$
C_{b, t} \leq \frac{1}{m} \log _{b}\left(1+\left|\frac{\sin (\pi(b-1) t)}{\sin (\pi t)}\right|^{m}\right) \leq \frac{1}{m} \log _{b}\left(1+(b-1)^{m}\right)<1
$$

It remains only to prove that $C_{b, t}$ is the best constant possible for the bound obtained, i.e., that for any $C<C_{b, t}$ there exists $\varepsilon>0$ such that $S_{b}(t, N) \neq$ $O_{b, t, \varepsilon}\left(N^{C+\varepsilon}\right)$, as $N \rightarrow+\infty$. This is straightforward since if $\left.\varepsilon \in\right] 0, C_{b, t}-C[$ then from (4) we get

$$
\left|S_{b}\left(t, b^{k}\right)\right|=\left(b^{k}\right)^{C_{b, t}-C-\varepsilon+o(1)} \cdot\left(b^{k}\right)^{C+\varepsilon} \neq O_{b, t, \varepsilon}\left(\left(b^{k}\right)^{C+\varepsilon}\right),
$$

as $k \rightarrow \infty$, so a fortiori $S_{b}(t, N) \neq O_{b, t, \varepsilon}\left(N^{C+\varepsilon}\right)$. This completes the proof.

## 4. Proof of Theorem 1.2

Observe that, since $b \geq 3$ and $t$ is irrational, by Proposition 2.6 it follows that $\log _{b}\left|E_{b}(t, b, \cdot)\right|$ is a well defined function. On the other hand, since $t$ is irrational, Proposition 2.5 shows that $f_{b}(t, b, h) \neq 0$ for all $h \geq 0$. Thus, by Lemma 2.3,

$$
\begin{align*}
\lim _{k \rightarrow \infty} \frac{\log _{b}\left|S_{b}\left(t, b^{k}\right)\right|}{k} & =\lim _{k \rightarrow \infty} \frac{1}{k} \sum_{h=0}^{k-1} \log _{b}\left|f_{b}(t, b, h)\right|  \tag{5}\\
& =\lim _{k \rightarrow \infty} \frac{1}{k} \sum_{h=0}^{k-1} \log _{b}\left|E_{b}\left(t, b, w_{b}(h) t\right)\right|=\int_{0}^{1} \log _{b}\left|E_{b}(t, b, \theta)\right| \mathrm{d} \theta
\end{align*}
$$

since $\log _{b}\left|E_{b}(t, b, \cdot)\right|$ is a continuous function of period 1 and, by Corollary 1.2, the sequence $\left\{w_{b}(h) t\right\}_{h=0}^{\infty}$ is uniformly distributed modulo 1 (see [8, Chap. 1, Corollary 1.2]). Now, from Jensen's formula [13, Theorem 15.18], we know that

$$
\int_{0}^{1} \log |1+\mathbf{e}(\theta) A| \mathrm{d} \theta= \begin{cases}\log |A| & \text { if }|A|>1 \\ 0 & \text { if }|A| \leq 1\end{cases}
$$

for all complex number $A$. Putting $A=\mathbf{e}\left(\frac{1}{2} b t\right) \sin (\pi(b-1) t) / \sin (\pi t)$ we can evaluate the integral in (5) and get

$$
\int_{0}^{1} \log _{b}\left|E_{b}(t, b, \theta)\right| \mathrm{d} \theta=\max \left(0, \log _{b}\left|\frac{\sin (\pi(b-1) t)}{\sin (\pi t)}\right|\right)=C_{b, t}
$$

Hence, $S_{b}\left(t, b^{k}\right)=\left(b^{k}\right)^{C_{b, t}+o(1)}$ as $k \rightarrow \infty$. At this point, the claim follows by the same arguments used at the end of the proof of Theorem 1.1.

## 5. Concluding remarks

It is perhaps interesting that, in the proof of Theorem 1.1 (respectively Theorem 1.2), we used a coarse upper bound on $S_{b}(t, N)$, namely Lemma 2.4, to get that $\left\{w_{b}(n)\right\}_{n=1}^{\infty}$ is uniformly distributed modulo $m$ (respectively $\left\{w_{b}(n) \alpha\right\}_{n=1}^{\infty}$ is uniformly distributed modulo 1), and then we used this result to get an improved upper bound on $S_{b}(t, N)$.

An open question is if it is possible to get an upper bound, similar to that of Theorem 1.2, for $S_{2}(t, N)$ when $t$ is irrational. Note that we cannot use the same arguments as in the proof of Theorem 1.2, because Proposition 2.6 breaks down for $b=2$. Precisely, the function $\log _{2}\left|E_{2}(t, 2, \cdot)\right|=\log _{2}|1+\mathbf{e}(\cdot+t)|$ has singularities on $\mathbf{Z}+\left(\frac{1}{2}-t\right)$ and although the integral

$$
\int_{0}^{1} \log _{2}\left|E_{2}(t, 2, \theta)\right| \mathrm{d} \theta
$$

exists finite (in fact, it is equal to zero), it is no longer true that

$$
\int_{0}^{1} \log _{2}\left|E_{2}(t, 2, \theta)\right| \mathrm{d} \theta=\lim _{k \rightarrow \infty} \frac{1}{k} \sum_{h=0}^{k-1} \log _{2}\left|E_{2}\left(t, 2, \theta_{h}\right)\right|
$$

for any sequence $\left\{\theta_{h}\right\}_{h=0}^{\infty}$ which is uniformly distributed modulo 1 . For example, for all $h \geq 0$ not a power of 2 take $\theta_{h}:=\theta_{h}^{\prime}$, where $\left\{\theta_{h}^{\prime}\right\}_{h=0}^{\infty}$ is some sequence uniformly distributed modulo 1 ; and for all $n \geq 0$ take $\theta_{2^{n}}$ such that $\sum_{h=0}^{2^{n}} \log _{2}\left|E_{2}\left(t, 2, \theta_{h}\right)\right|<$ $-\left(2^{n}+1\right)$. We left to the reader to prove that then $\left\{\theta_{h}\right\}_{h=0}^{\infty}$ is uniformly distributed modulo 1 and

$$
\liminf _{k \rightarrow \infty} \frac{1}{k} \sum_{h=0}^{k-1} \log _{2}\left|E_{2}\left(t, 2, \theta_{h}\right)\right| \leq-1
$$

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## References

[1] M. Drmota and G. Larcher. The sum-of-digits-function and uniform distribution modulo 1. J. Number Theory, 89(1):65-96, 2001.
[2] M. Drmota, J. Rivat, and T. Stoll. The sum of digits of primes in $\mathbb{Z}[i]$. Monatsh. Math., 155:317-348, 2008.
[3] A. Gelfond. Sur les nombres qui ont des propriétés additives et multiplicatives données. Acta Arith., 13(3):259-265, 1968.
[4] R. L. Goodstein. On the restricted ordinal theorem. J. Symbolic Logic, 9(2):3341, 1944.
[5] P. J. Grabner and R. F. Tichy. Contributions to digit expansions with respect to linear recurrences. J. Number Theory, 36(2):160-169, 1990.
[6] D.-H. Kim. On the joint distribution of q-additive functions in residue classes. J. Number Theory, 74(2):307-336, 1999.
[7] L. Kirby and J. Paris. Accessible independence results for Peano arithmetic. Bull. Lond. Math. Soc., 14:285-293, 1982.
[8] L. Kuipers and H. Niederreiter. Uniform Distribution of Sequences. Dover Publications, Incorporated, 2012.
[9] M. Lamberger and J. M. Thuswaldner. Distribution properties of digital expansions arising from linear recurrences. Math. Slovaca, 53:1-20, 2003.
[10] P. Liardet and P. Grabner. Harmonic properties of the sum-of-digits function for complex bases. Acta Arith., 91(4):329-349, 1999.
[11] M. G. Madritsch. The sum-of-digits function of canonical number systems: Distribution in residue classes. J. Number Theory, 132(12):2756-2772, 2012.
[12] M. Olivier. Répartition des valeurs de la fonction "somme des chiffres". Séminaire Delange-Pisot-Poitou. Théorie des nombres, 12:1-5, 1970-1971.
[13] W. Rudin. Real and Complex Analysis, 3rd Ed. Mathematics series. McGrawHill, Inc., New York, NY, USA, 1987.
[14] J. M. Thuswaldner. The sum of digits function in number fields: Distribution in residue classes. J. Number Theory, 74(1):111-125, 1999.

