Hydroassets Portfolio Management for Intraday Electricity Trading from a Discrete Time Stochastic Optimization Perspective

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Hydroassets Portfolio Management for Intraday Electricity Trading from a Discrete Time Stochastic Optimization Perspective

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Abstract Hydro storage system optimization is becoming one of the most challenging tasks in Energy Finance. While currently the state-of-the-art of the commercial software in the industry implements mainly linear models, we would like to introduce risk aversion and a generic utility function. At the same time, we aim to develop and implement a computational efficient algorithm, which is not affected by the curse of dimensionality and does not utilize subjective heuristics to prevent it. For the short term power market we propose a simultaneous solution for both dispatch and bidding problems.

Following the Blomvall and Lindberg (2002) interior point model, we set up a stochastic multiperiod optimization procedure by means of a "bushy" recombining tree that provides fast computational results. Inequality constraints are packed into the objective function by the logarithmic barrier approach and the utility function is approximated by its second order Taylor polynomial. The optimal solution for the original problem is obtained as a diagonal sequence where the first diagonal dimension is the parameter controlling the logarithmic penalty and the second one is the parameter for the Newton step in the construction of the approximated solution. Optimal intraday electricity trading and water values for hydroassets as shadow prices are computed. The algorithm is implemented in Mathematica.

Keywords Stochastic multiperiod optimization · Stochastic market · Blomvall and Lindberg interior point model · Logarithmic barrier approach · Energy markets · Spot and intraday prices

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1 Introduction

The liberalised electricity market poses new challenges to power generating companies for the electrical grid. A key driver to set up economically efficient grids is the capacity to store electricity through hydro storage systems and thereby decouple electricity generation from electricity consumption. So, the hydro storage system optimization is becoming one of the most challenging tasks in Energy Finance, as highlighted in [31] and in [17]. While the current industrial standard for hydro optimization covers linear models, recently risk aversion optimizations, which are very common in financial portfolio optimization, have been introduced into the energy sector, see f.i. [1] and [36].

The aim of this research work is to set up a computational efficiently implementable concave stochastic dynamic program in order to optimize intraday electricity trading under risk aversion, and model at the same time water values for hydro assets. It extends the previous work of the authors ([20]) by presenting the complete algorithm and constructing numerical examples. Its two main contributions are:

- The implementation of the optimization algorithm of Blomvall and Lindberg on a lattice guaranteeing computational efficiency. To our knowledge this approach is new and can be utilized for the discretization of virtually any intertemporal portfolio optimization.
- The introduction of deterministic water values of an hydro infrastructutre as certainty equivalents of optimal stochastic Lagrangian multipliers corresponding to the basin level equations.

The optimization of electricity trading under risk aversion is formulated as a stochastic multiperiod optimization problem in discrete time for a generic utility function. More exactly, the objective function is the weighted sum of the expected utility of the wealth generated by the electricity trading during each subinterval. The optimization problem is subject to equality restrictions, such as the equations for the levels of all basins and to inequality restrictions, such as the lower and upper bounds for the levels of all basins or the limits for the turbined or pumped water. For linear restrictions and a generic concave utility function this optimization problem is known to have always a unique solution, an optimal (stochastic) dynamic dispatch plan. However, in general, an explicit solution cannot be computed directly but can only be approximated by a sequence of suboptimal dispatch plans. These can be obtained following the seminal Blomvall and Lindberg's ideas (see [9], [10], [11], [12] and [13]), where inequality constraints are packed into the objective function by means of an additive logarithmic penalty - a technique known as logarithmic barrier approach. The optimization problem with the barrier approximates the original one and can be solved by a Newton's scheme, where the utility function is approximated by its second order Taylor polynomial. This newly obtained quadratic optimization problem, approximates again the original one, and has an explicit closed formula solution, which depends on two parameters: the first one is the parameter controlling the logarithmic penalty and the second one is the step parameter in Newton's scheme. Finally, the optimal solution for the original problem is obtained as a diagonal sequence over this two parameters.

We provide generic formulae in terms of conditional expectations and thus not depending on the way the underlying stochastic processes are modelled for the original deterministic equivalent formulation as in Blomvall and Lindberg. In the practical implementation intraday prices and water inflows are discretized in the space dimensions by means of a "bushy" recombining tree (meaning by this a k-dimensional lattice with $k >> 1$), so that we are not worried by the dimensionality curse nor we have to deal with heuristic arguments concerning the choice of representative branches in a non recombining "sparse" tree, as Blomvall and Lindberg implicitly have to deal with in their original work. For a more recent treatment of scenario reduction techniques in stochastic programming we refer to [37] and [42].

The obtained algorithm is implemented in Mathematica and applied to optimize intraday electricity trading and model at the same time stochastic water values for hydro assets. These are defined as shadow prices, that is the optimal Lagrangian multipliers associated with the equality restrictions given by the equations for the basin levels. Deterministic water values are obtained by passing to the certainty equivalents.

This paper is structured as follows. Section 2 introduces the set up for discrete intertemporal expected utility optimization of portfolio subject to constraints, solved by means of an algorithm developed in Section 3, where Remark 3.1 highlights the differences between Blomvall and Lindberg's work and our proposed approach. Section 4 deals with the implementation of the solution method on a lattice, seen as recombing tree. This is applied in Section 5 to the intraday electricity trading to find an optimal strategy and to determine water values of hydro electric infrastructures to be used for market bids. Section 6 presents a numerical example. Section 7 concludes.

1.1 A Short Review of the Literature

Energy trading methods have been widely studied in the technical literature in the past 20 years. References [21], [28] and [46] are some of the few reviews about different algorithms applied to hydro power planning. Some of these techniques became standard in solving of medium-term hydro power planning problems. The pioneering research of R. Bellman ([6]) introduced and made popular the framework of dynamic programming, which was very soon extended to stochastic dynamic programming to account for the uncertainties of the underlying processes. With randomly variable inflows and consumption (electricity prices were liberalized only in the 1990s) hydro power scheduling was therefore used as an application example for stochastic dynamic programming from the beginning. But because of its computational challenging nature the problem was first solved for a single basin configuration only at the end of the 1960s (see [47]) and was an active field of research during the 1970s and the early 1980s as the comprehensive reviews [33] and [45] show. The basic algorithms were extended to better account for stochasticity, multi reservoirs, hydro thermal systems, reliability constraints, and improving the model for water inflows. During the 1990s, thanks to the increase of computing power, approximate dynamic programming and, in particular stochastic dual dynamic programming, was in the spotlight. For the techniques allowing to approximate some of the problem's elements and reducing the computational time we refer to the description of many of the algorithms in question, which can be found in [8] and [38].

Originally, risk aversion was introduced into hydro power production in order to achieve a certain reliability, which was mainly expressed in terms of constraints for the optimization problem (e.g. [4], [43] and [44]). With the liberalization of electricity markets the attention was focused on profit risk mitigation. In terms of modelling this was achieved first by similar methods, i.e. by setting target ranges for some variables (e. g. [18]). In more recent years, following the discussion on coherent risk measures ([3]) first and time consistency of risk measures ([41]) later, stochastic dynamic programming has considered risk measures in the objective function depending on the control rules and on the underlying stochastic processes. Applications to hydro power production can be found in [15], [40], [14] and [37].

We remark that risk aversion optimization can be formulated by choosing the objective function as a trade off between reward and risk, or, by setting the objective function equal to the expected utility for a concave utility function. The latter is the approach followed in this paper, where by means of risk averse stochastic dynamic programming applied to the intraday electricity market, we derive optimal short term dispatch plans and appropriate hydro infrastructure water values for the day ahead market bids. Of course this model can be extended to arbitrary long time horizons, for which the risk aversion plays an even more important role, if the whole dynamics of the hourly priced forward curve and not just the intraday prices are considered.

In [26] a mixed-integer linear program maximizes the expected profit of a hydro chain in the day-ahead market, avoiding unnecessary spillages and considering start-up costs. In [32] expected discounted cash flows of rewards are maximized without taking risk aversion into account. But, for computational efficiency, instead of linear programming, an approximated stochastic dynamic programming algorithm is utilized, which consists in a combination of temporal difference learning and least squares policy evaluation. In [22] and [23] a two stage mixed integer-linear program maximizes a trade off between the expected profit for the one-day operation and a penalty/reward for imbalances in the future production. Being the objective function linear, there is no explicit risk aversion. While the first stage determines the one-day production plan and involves the bidding process, the second stage evaluates the impact of the one-day production plan on future production. The output is an optimal bid for the day-ahead market in terms of volumes and prices and an optimal dispatch plan. For a similar problem set up [31] efficiently solve a stochastic mixed-integer quadratic program integrating stochastic dynamic programming with ideas of approximate dynamic programming.

Recent references giving a thorough overview of producer models for bidding in the auction market with and without a dispatch plan are [22] (mixed integer programming), [30] (mathematical programming, game theory and agent-based models), [5] (simulation, various forms of integer programming, various forms of dynamic programming, equilibrium models, evolutionary algorithms), and [24] (stochastic programming models in short term power generation scheduling and bidding). Similar problems in economic dispatch are solved in [2] by means of a oblivious routing economic dispatch algorithm.

How does our work fit into this model landscape? It has the following characteristics:

- It is a convex risk averse optimization problem.
- It is solved for a generic utility function.
- It utilizes stochastic dynamic programming and the Bellman recursion.
- It is implemented on fully recombining tree avoiding the curse of dimensionality.
- It solves the scheduling and the bidding problem simultaneously.

1.2 Overview of the Nomenclature and of the Document Structure

T : Final time horizon (2)
$t = 0, 1, 2, 3, \dots, T$: Time points (2)
$(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t=0,\dots T}, P)$: Filtered probability space (2)
$\mathbb{E}_0[\cdot]$: Statistical expectation (2)
$\mathbb{E}_t[\cdot]$: Statistical conditional expectation at time t (3.3)
$Z_0, Z_1, Z_2, Z_3, \dots, Z_T$: Risk drivers (2)
K : Dimension of risk drivers (2)
$X_0, X_1, X_2, X_3, \dots, X_T$: External states (or risk factors) (2)
N : Dimension of external states (2)
$u_0, u_1, u_2, u_3, \dots, u_{T-1}$: Control rules (2)
$Y_0, Y_1, Y_2, Y_3, \dots, Y_T$: Internal states (functions of external states and control rules) (2)
M : Dimension of internal states (2)
U : Utility function (2)
V_t : Portfolio value at time t (2)
$\mathcal{C}$: Set of linear equality and inequality constraints (2)
$\mathcal{C}_{\text{ineq}}$: Set of linear inequality constraints (3)
$(E_t)_t \subset \mathbf{R}^{L\times M}, (F_t)_t \subset \mathbf{R}^{L\times N}, (e_t)_t \subset \mathbf{R}^{L\times 1}$: Processes utilized to express linear inequality constraints (3)
$\mathcal{C}_{\text{eq}}$: Set of linear equality constraints (3)
$(A_t)_t \subset \mathbf{R}^{M\times M}, (B_t)_t \subset \mathbf{R}^{M\times N}, (b_t)_t \subset \mathbf{R}^{M\times 1}$: Processes utilized to express linear equality constraints (3)
$(\beta_t)_{t=1,\dots T} > 0$: Positive deterministic weights (2)
μ : Trade off parameter between expected utility and penalty function induced by the restrictions (3)
1 : Vector of ones (3.1)
Φ : Lagrange principal function (3.2)
$y_{\geqslant t} := (y_s)_{s\geqslant t}, u_{\geqslant t} := (u_s)_{s\geqslant t}$: Internal states and control rules from time t till the end (3.3)
h_t : Quadratic Taylor polynomial of objective function at time t (3.3)
q_t : Gradient of h_t with respect to internal states $y_{\geqslant t}$ (3.3)
r_t : Gradient of h_t with respect to control rules $u_{\geqslant t}$ (3.3)
Q_t, P_t, R_t : Submatrices of the Hessian of h_t with respect to internal states and control rules (3.3)
J_t : Value function at time t for the Bellman recursion of the optimization problem (3.4)
$\overline{q}_t$: Gradient of h_t with respect to internal states y_t (3.4)
$\overline{r}_t$: Gradient of h_t with respect to control rules u_t (3.4)
$\overline{Q}_t, \overline{P}_t, \overline{R}_t$: Submatrices of the Hessian of h_t with respect to internal states y_t and control rules u_t (3.4)
$(W_t)_t, (\alpha_t)_t, (w_t)_t, (\tilde{\alpha}_t)_t, (\tilde{r}_t)_t, (\tilde{R}_t)_t, (\tilde{q}_t), (\tilde{Q}_t)_t, (\tilde{P}_t)_t$: Adapted processes utilized in the inductive assumption for $(J_t)_t$ (3.4)
u_t^* : Optimal control rule (3.5)
$(\alpha_t)_t, (w_t)_t, (W_t)_t$: Adapted processes utilized in the Riccati equation (3.5)
$\mathcal{L}$: Lattice (4)
$\mathcal{L}_t$: Time t layer of lattice (4)
k : Number of branches for every node in the lattice (4)
$n_t(i)$: Node in lattice layer at time t (4)
$\text{Children}(n_t(i))$: Children of node $n_t(i)$ (4)
$\text{Parents}(n_s(j))$: Parents of node $n_s(j)$ (4)
N_t : Number of nodes in lattice layer at time t (4)
$\mathcal{N}_T$: Number of nodes in lattice (4)
$z_t^1, \dots, z_t^{N_t}$: Simulated values for the risk drivers on the lattice layer at time t (4)
ϵ_t : Contraction factor for Δu_t which guarantees feasibility in every Newton step (4)
$\mathcal{B}(n_t)$: Atom associated to the node n_t of the σ algebra $\mathcal{A}_t$ for the time t lattice layer (4)
S_t : Spot electricity price (5)
$\text{GP}_t^{\text{Bid}}, \text{GP}_t^{\text{Ask}}$: Electricity bid and ask prices in the day ahead market bidding (5)
$\Xi_t^{\text{Bid}}, \Xi_t^{\text{Ask}}$: Electricity bid and ask volumes in the day ahead market bidding (5)
$\Xi_t^{\text{Spot, Sell}}, \Xi_t^{\text{Spot, Buy}}$: Electricity sell and buy volumes in the day ahead market (5)
B : Number of basins (5)
gp_t^{Ask} : Stochastic water value (5)
F_t : Forward price (5)
Ψ_t : Energy volume for the forward market (5)
$\mathbb{E}_0[r]$: Reward measure (5)
$\mathbb{E}_0[\rho]$: Risk measure (5)
w : Risk aversion (5)

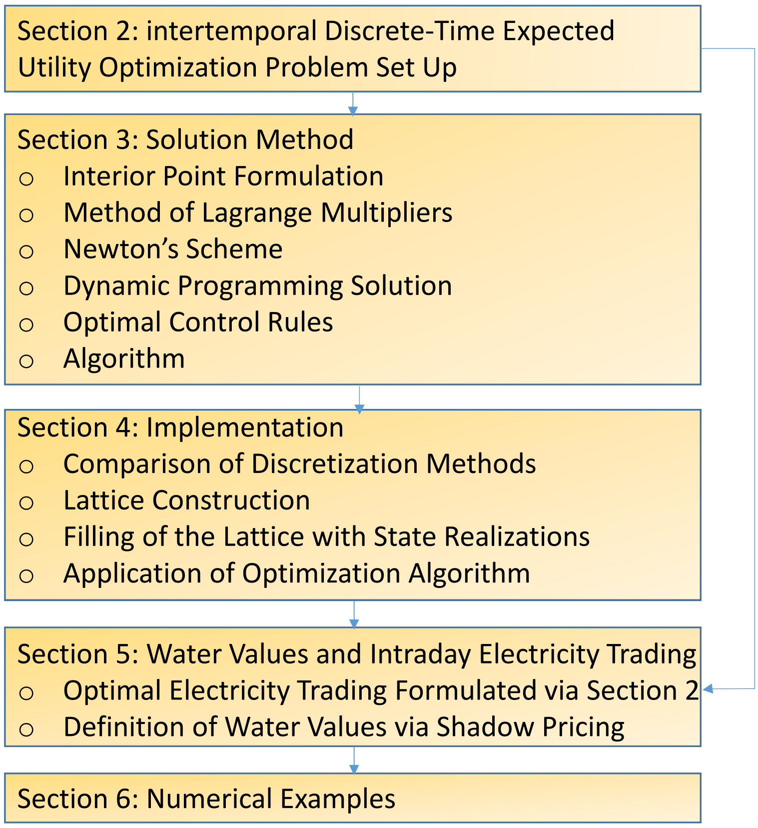

Fig. 1 Document Structure

2 Discrete Multiperiod Portfolio Expected Utility Maximization

The purpose of this section is to show how the intertemporal expected utility framework can be used to solve optimization problems for a portfolio of financial assets (Example 1) or for the power production of an hydro infrastructure (Example 2). We first introduce the necessary notation for the discrete time setting given a final time horizon T, time points $t = 0, 1, 2, 3, \dots, T$ and a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t=0,\dots T}, P)$ with statistical expectation $\mathbb{E}_0[\cdot]$:

- **Risk drivers:**
  $Z_0, Z_1, Z_2, Z_3, \dots, Z_T$, where $Z_t : \Omega \to \mathbf{R}^K$ is a t-measurable random variable. The random variables $(Z_t)_t$ are assumed to be i.i.d.
- **External states (or risk factors):**
  $X_0, X_1, X_2, X_3, \dots, X_T$, where $X_t : \Omega \to \mathbf{R}^M$ is a t-measurable random

variable. We assume that there exists deterministic functions $(f_t)_{t=1,\dots T}$ such that $X_t = f_t(X_{t-1}, Z_t)$.
  - **Ex1:** Asset values for the different asset classes.
  - **Ex2:** Hourly electricity price for the intraday market, hourly water inflows for the basins.
- **Control rules:**
  $u_0, u_1, u_2, u_3, \dots, u_{T-1}$, where $u_t : \Omega \to \mathbf{R}^N$ is a t-measurable random variable.
  - **Ex1:** Holdings in the different asset classes.
  - **Ex2:** Water processed by the different turbines and pumps of the hydro infrastructure.
- **Internal states (functions of external states and control rules):**
  $Y_0, Y_1, Y_2, Y_3, \dots, Y_T$, where $Y_t : \Omega \to \mathbf{R}^M$ is a t-measurable random variable. We assume that there exists deterministic functions $(g_t)_{t=1,\dots T}$ such that $Y_t = g_t(Y_{t-1}, u_{t-1}, X_t)$.
  - **Ex1:** Wealth level for the portfolio.
  - **Ex2:** Basin levels of the hydro infrastructure.
- **Utility function:**
  a concave, monotone increasing differentiable function $U : \mathbf{R} \hookrightarrow \mathbf{R}$
- **Portfolio value:**
  $V_t = V_t(X_t, u_{t-1})$. If we choose as risk factors X_t the values of base assets, then $N = M$ and $V_t(X_t, u_{t-1}) = u_{t-1}^{\dagger} X_t$.
  - **Ex1:** Total portfolio wealth level at time t.
  - **Ex2:** Wealth generated by the intraday trading during the period $[t-1, t]$.
- **Constraints $\mathcal{C}$:**
  linear equality and inequality constraints in the rules u_t.
  - **Ex1:** Self-financing constraint $V_t(X_t, u_{t-1}) = V_t(X_t, u_t)$ for all $t = 0, \dots, T-1$ (equality constraint), and lower and upper bounds in the portfolio holdings (inequality constraints).
  - **Ex2:** Basin level equations (equality constraints), and lower and upper bounds for water turbined or pumped as well as for basin levels (inequality constraints).
- **Optimization problem:**
  given positive deterministic weights $(\beta_t)_{t=1,\dots T} > 0$ modelling the relative importance assigned to the measurements in the different subintervals, the optimization problem (P) writes

$$\boxed{\max_{u \in \mathcal{C}} \mathbb{E}_0 \left[\sum_{t=1}^{T} \beta_t U(V_t(X_t, u_{t-1})) \right].} \tag{1}$$

Remark 2.1 *The role of internal states is to simplify the representation of constraints and the recursion formulae, but we could formulate and solve the optimization problem without introducing them. However, they typically are quantities of interest for the problem at hand.*

Remark 2.2 *The structure of the objective function in the optimization (1) allows for an application of Bellman's equation leading to a decomposition in one step equations with closed or semiclosed solution. This would not work for a generic utility function for the maximization of the expected utility of the cumulated values over the different time subperiods.*

3 Solution Method

To solve the optimization problem (P) formulated in (1) we modify the model of Blomvall and Lindberg described in [9], [10], [11], [12] and applied in [13] by adapting it to our needs. The constraint set $\mathcal{C}$ can be decomposed as union of inequality and equality constraints

$$\mathcal{C} = \mathcal{C}_{\text{ineq}} \cup \mathcal{C}_{\text{eq}}, \tag{2}$$

where we have set

- $\mathcal{C}_{\text{ineq}}$: inequality constraints. In our case they are linear inequalities, which reads
$$\mathcal{C}_{\text{ineq}} := \{E_t Y_t + F_t u_t - e_t \geqslant 0 \,|\, t = 0, \ldots, T-1\}. \tag{3}$$
Thereby, $(E_t)_{t=0,\ldots,T-1} \subset \mathbf{R}^{L\times M}$, $(F_t)_{t=0,\ldots,T-1} \subset \mathbf{R}^{L\times N}$ and $(e_t)_{t=0,\ldots,T-1} \subset \mathbf{R}^{L\times 1}$ are processes adapted to the filtration.
- $\mathcal{C}_{\text{eq}}$: equality constraints, like the selffinancing condition in Ex 1 or the basin level equation in Ex 2 given by the stochastic dynamics, which reads
$$\mathcal{C}_{\text{eq}} := \{Y_{t+1} = g_{t+1}(Y_t, u_t, X_{t+1}) \,|\, t = 0, \ldots, T-1\}. \tag{4}$$
for appropriate choices of the internal states $(Y_t)_t$ and of the functions $(g_t)_t$. The latter typically incorporate the dynamics. Note that u_{-1} denotes the deterministic rule in force just before the rule at time 0 is enforced.

The problem can (but must not) be further simplified by choosing a linear valuation function and a linear dynamics, that is $g_t(y, u, x) := A_{t+1}y + B_{t+1}u + b_{t+1}(x)$ and thus

$$\mathcal{C}_{\text{eq}} = \{Y_{t+1} = A_{t+1}Y_t + B_{t+1}u_t + b_{t+1} \,|\, t = 0, \ldots, T-1\}, \tag{5}$$

where $(A_t)_{t=1,\ldots,T} \subset \mathbf{R}^{M\times M}$, $(B_t)_{t=1,\ldots,T} \subset \mathbf{R}^{M\times N}$ and $(b_t)_{t=1,\ldots,T} \subset \mathbf{R}^{M\times 1}$ are processes adapted to the filtration.

Subsequently, the optimization problem undergoes the following transformations:

1. **(P):** Original problem (1) with a generic concave utility function u and inequality constraints among others.

2. **(P_μ):** Problem with objective function defined as trade-off between the expected utility and the logarithms of the functions defining the inequality constraints u. The trade-off parameter is denoted by $\mu > 0$. The optimization problem has equality constraints only.
3. **($\bar{P}_\mu$):** Approximation of problem P_μ by substituting the objective function with its quadratic Taylor polynomial.

More exactly, we mean that:

- We write out the expression for the objective function
$$\mathbb{E}_0\left[\sum_{t=1}^{T}\beta_t U(V_t(X_t, u_{t-1}))\right].$$
- We extend the objective function by packaging in it all the restrictions $\mathcal{C}$ utilizing the logarithmic barrier approach, which approximates the constraints. Thereby, the approximate solution for (P) is the solution for (P_μ) for $\mu > 0$ small enough.
- We approximate the extended objective function by its quadratic Taylor polynomial and the solution of (P_μ) is given by a Newton's scheme sequence of solutions of problems of the type $(\bar{P}_\mu)$.
- We find optimal rules for the approximated problem (approximated constraints and approximated objective function).
- There are two approximations schemes, one for the constraints and one for the extended objective function. We choose a diagonal sequence to obtain a sequence of rules converging towards the optimal rules of the original problem (P).

Remark 3.1 *The differences between this approach and the Blomvall-Lindberg original solution are both formal and substantial:*

- *Blomvall-Lindberg formulate directly the optimization problem on the nodes of a non recombining tree. We formulate it for a general filtration. This is a rather a formal distinction, because the formulae are essentially the same. But it has the advantage of being independent of the way we model the underlying external risk factors. To this aim, conditional expectations are introduced.*
- *The objective function in the Blomvall-Lindberg approach at time t is a function of the risk factors realizations at time t. The objective function in our approach at time t is the expectation at time t of the discounted sums of Blomvall-Lindberg's objective functions at times $s = t+1, \dots, T$. In other words, in the case of the hydro optimization of Ex 2, our model optimizes at every stage t the expected profit till the final horizon T while Blomvall-Lindberg's model optimizes at every stage t the expected profit for the subperiod $[t, t+1]$.*

The remainder of this chapter implements the transformation steps described above and culminates in the optimal control rules (26) for the problem $(\bar{P}_\mu)$. Readers not interested in the mathematical details can skip directly to subsection 3.6.

3.1 Interior Point Formulation

The problem (P_μ) is an approximation of problem (P) by means of the logarithmic approach, and reads as

$$\max_{u \in \mathcal{C}_{\mathrm{eq}}} \mathbb{E}_0 \left[\sum_{t=1}^{T} \beta_t U(V_t(X_t, u_{t-1})) + \mu 1^\dagger \sum_{t=0}^{T-1} \log(E_t Y_t + F_t u_t - e_t) \right], \qquad (6)$$

where $1 := [1, \ldots, 1]^\dagger \in \mathbf{R}^{L \times 1}$ and $\mu > 0$ is a real parameter.

As long as we move inside the interior of the feasible set $E_t Y_t + F_t u_t - e_t > 0$ for all $t = 0, \ldots, T-1$, the logarithm function is well defined. As soon as we approach to a boundary point, the logarithmic penalty function tends to $-\infty$. This means that, if the maximum is attained, it must be for an interior point, which depends on the parameter μ. For $\mu \to 0^+$ this interior point converges to a point in the feasible set (on the boundary or in the interior), which is the candidate for the solution to the original problem (1).

If we choose a linear dynamic and a convex utility function, then, by convex optimization theory ([35],[39] and [34]), the problem

$$\boxed{\begin{aligned} \max_{\substack{Y_{t+1} = A_{t+1} Y_t + B_{t+1} u_t + b_{t+1} \\ t = 0, 1, \ldots, T-1}} \mathbb{E}_0 \Bigg[& \sum_{t=1}^{T} \beta_t U(V_t(X_t, u_{t-1})) + \\ & + \mu 1^\dagger \sum_{t=0}^{T-1} \log(E_t Y_t + F_t u_t - e_t) \Bigg], \end{aligned}} \qquad (7)$$

has always a unique solution. As a matter of fact a convex function over a convex closed domain has always a global minimum. More exactly, if the sample space Ω is finite, then existence and uniqueness of the solution directly follows from Kuhn-Tucker's Theorem, see f.i. Theorem 5.6 in [35]. The general case is proved in Corollary 3.5.1 of [7].

3.2 Method of Lagrange Multipliers

The problem (P_μ) in (7) has only linear restrictions, and can therefore be solved by a closed expression by utilizing the method of Lagrange multipliers. The Lagrange principal function reads for the Lagrange multiplier $\lambda = (\lambda_t(\omega))$

$$\begin{aligned} \Phi(u; \lambda) := \mathbb{E}_0 \Bigg[& \sum_{t=1}^{T} \beta_t U(V_t(X_t, u_{t-1})) + \mu 1^\dagger \sum_{t=0}^{T-1} \log(E_t Y_t + F_t u_t - e_t) + \\ & - \sum_{t=1}^{T} \lambda_t \left(Y_{t+1} - A_{t+1} Y_t - B_{t+1} u_t - b_{t+1} \right) \Bigg], \end{aligned} \qquad (8)$$

and the corresponding Lagrange equations in the unknown optimal process $u = (u_t(\omega))_{t=0,\dots,T-1}$ and unknown optimal Lagrange multiplier $\lambda = (\lambda_t(\omega))$

$$\begin{cases} \frac{\partial \Phi}{\partial u_t}(u;\lambda) = 0 \quad (t = 0,\dots,T-1) \\ \\ \frac{\partial \Phi}{\partial \lambda}(u;\lambda) = 0. \end{cases} \tag{9}$$

3.3 Newton's Scheme

The second equation in (9) is equivalent to the dynamics (4) and the first equation of (9) can be solved pathwise in $\omega \in \Omega$ for all processes satisfying such dynamics as a restriction. If we want to find the zeros of the gradient of the objective function by means of Newton's method, then we have to consider its quadratic Taylor polynomial

$$\boxed{\begin{aligned} h_t(y_{\geqslant t}, u_{\geqslant t}) := \mathbb{E}_t\Bigg[&\sum_{s=t+1}^{T}\beta_s U(V_s(x_s, u_{s-1})) + \\ &+\mu 1^\dagger \sum_{s=t}^{T-1}\log(E_t y_t + F_t u_t - e_t)\Bigg], \end{aligned}} \tag{10}$$

and to express its gradient with respect to the variables $y_{\geqslant t} := (y_s)_{s\geqslant t}$ and $u_{\geqslant t} := (u_s)_{s\geqslant t}$ we introduce

$$\boxed{\begin{aligned} q_t^\dagger(y_{\geqslant t}, u_{\geqslant t}) := \nabla_{y_{\geqslant t}} h_t(y,u) = \mathbb{E}_t\Bigg[&\sum_{s=t}^{T}\beta_s \nabla_{x_{\geqslant t}} U(V_s(x_s, u_{s-1})) + \\ &+\mu \sum_{t=s}^{T-1}\left(\frac{1}{E_t y_t + F_t u_t - e_t}\right)^\dagger E_t\Bigg], \\ r_t^\dagger(y_{\geqslant t}, u_{\geqslant t}) := \nabla_{u_{\geqslant t}} h_t(y,u) = \mathbb{E}_t\Bigg[&\sum_{s=t+1}^{T}\beta_s \nabla_{u_{\geqslant t}} U(V_s(x_s, u_{s-1})) + \\ &+\mu \sum_{s=t}^{T-1}\left(\frac{1}{E_t y_t + F_t u_t - e_t}\right)^\dagger F_t\Bigg], \end{aligned}} \tag{11}$$

where the vector divisions are made componentwise. The Hessian of the objective function reads

$$
\boxed{
\begin{aligned}
Q_t(y_{\geqslant t}, u_{\geqslant t}) &:= \nabla^2_{y_{\geqslant t}} h_t(y,u) = \mathbb{E}_t \left[\sum_{s=t+1}^{T} \beta_s \nabla^2_{x_{\geqslant t}} U(V_s(x_s, u_{s-1})) + \right. \\
&\left. \qquad -\mu \sum_{s=t+1}^{T} E_t^\dagger \operatorname{diag}\left(\frac{1}{E_t y_t + F_t u_t - e_t} \right)^2 E_t \right], \\
P_t(y_{\geqslant t}, u_{\geqslant t}) &:= \nabla_{u_{\geqslant t}} \nabla_{y_{\geqslant t}} h_t(y,u) = \\
&= \mathbb{E}_t \left[\sum_{s=t+1}^{T} \beta_s \nabla_{u_{\geqslant t}} \nabla_{y_{\geqslant t}} U(V_s(x_s, u_{s-1})) + \right. \\
&\left. \qquad - \mu \sum_{s=t}^{T-1} E_t^\dagger \operatorname{diag}\left(\frac{1}{E_t y_t + F_t u_t - e_t} \right)^2 F_t \right], \\
R_t(y_{\geqslant t}, u_{\geqslant t}) &:= \nabla^2_{u_{\geqslant t}} h_t(y,u) = \mathbb{E}_t \left[\sum_{s=t+1}^{T} \beta_s \nabla^2_{u_{\geqslant t}} U(V_s(x_s, u_{s-1})) + \right. \\
&\left. \qquad -\mu \sum_{s=t}^{T-1} F_t^\dagger \operatorname{diag}\left(\frac{1}{E_t y_t + F_t u_t - e_t} \right)^2 F_t \right].
\end{aligned}
}
\tag{12}
$$

The second order approximation of $h(y,u)$ can be described as a function of the increment in the variables

$$
\begin{aligned}
\Delta h_t(y_{\geqslant t}, u_{\geqslant t}) :=& h_t(y_{\geqslant t} + \Delta y_{\geqslant t}, u_{\geqslant t} + \Delta u_{\geqslant t}) - h_t(y_{\geqslant t}, u_{\geqslant t}) = \\
&= q_t^\dagger(y_{\geqslant t}, u_{\geqslant t}) \Delta y_{\geqslant t} + \frac{1}{2} \Delta y_{\geqslant t}^\dagger Q_t(y_{\geqslant t}, u_{\geqslant t}) \Delta y_{\geqslant t} + \\
&+ r_t^\dagger(y_{\geqslant t}, u_{\geqslant t}) \Delta u_{\geqslant t} + \frac{1}{2} \Delta u_{\geqslant t}^\dagger R_t(y_{\geqslant t}, u_{\geqslant t}) \Delta u_{\geqslant t} + \\
&+ \Delta y_{\geqslant t}^\dagger P_t(y_{\geqslant t}, u_{\geqslant t}) \Delta u_{\geqslant t},
\end{aligned}
\tag{13}
$$

and the matrix

$$
\begin{bmatrix} Q_t(y,u) & P_t(y,u) \\ P_t^\dagger(y,u) & R_t(y,u) \end{bmatrix}
\tag{14}
$$

is positive definite for all t, y, u and ω, an so are the matrices $Q_t(y,u)$ and $R_t(y,u)$. The second order expansion of (P_μ) in (6) denoted as $(\bar{P}_\mu)$ is the following quadratic optimization on Ω

$$
\boxed{\max_{\substack{u=(u_t)_{t=0,\dots,T-1} \\ \Delta y_{t+1} = A_{t+1}\Delta y_t + B_{t+1}\Delta u_t}} \Delta h_0(y,u),}
\tag{15}
$$

that is

$$\max_{\substack{u=(u_t)_{t=0,\dots,T-1}\\ \Delta y_{t+1}=A_{t+1}\Delta y_t+B_{t+1}\Delta u_t}} \left(q_0^\dagger \Delta y + \frac{1}{2}\Delta y^\dagger Q_0 \Delta y + r_0^\dagger \Delta u + \frac{1}{2}\Delta u^\dagger R_0 \Delta u + \right. \\ \left. + \frac{1}{2}\Delta y^\dagger P_0 \Delta u\right). \tag{16}$$

3.4 Dynamic Programming Solution

We solve $(\bar{P}_\mu)$ by dynamic programming and, to this end, we introduce value functions

$$J_t(\Delta y_{\geqslant t}) := \max_{\substack{u=(u_s)_{s=t,\dots,T-1}\\ \Delta y_{s+1}=A_{s+1}\Delta x_s+B_{s+1}u_s\\ s=t,\dots,T-1}} \mathbb{E}_t\left[q_t^\dagger \Delta y_{\geqslant t} + \frac{1}{2}\Delta y_{\geqslant t}{}^\dagger Q_t \Delta y_{\geqslant t} + \right. \\ \left. + r_t^\dagger \Delta u_{\geqslant t} + \frac{1}{2}\Delta u_{\geqslant t}{}^\dagger R_t \Delta u_{\geqslant t} + \frac{1}{2}\Delta y_{\geqslant t}{}^\dagger P_t \Delta u_{\geqslant t}\right], \tag{17}$$

which allow to formulate Bellman's backward recursion as

$$J_t(\Delta y_{\geqslant t}) = \max_{\substack{u_t\\ \Delta y_{t+1}=A_{t+1}\Delta y_t+B_{t+1}u_t}} \left\{\overline{q}_t^\dagger \Delta y_t + \frac{1}{2}\Delta y_t{}^\dagger \overline{Q}_t \Delta y_t + \overline{r}_t^\dagger \Delta u_t + \right. \\ \left. + \frac{1}{2}\Delta u_t{}^\dagger \overline{R}_t \Delta u_t + \frac{1}{2}\Delta y_t{}^\dagger \overline{P}_t \Delta u_t + \mathbb{E}_t\left[J_{t+1}(\Delta y_{\geqslant t+1})\right]\right\}, \tag{18}$$

where

$$\boxed{\begin{aligned}
\overline{q}_t^\dagger(y_t,u_t) &:= \nabla_{y_t} h_t(y,u) = \mathbb{E}_t\Big[\beta_{t+1}\nabla_{y_t} U(V_{t+1}(x_{t+1},u_t)) + \\
&\qquad\qquad + \mu\left(\frac{1}{E_t y_t + F_t u_t - e_t}\right)^\dagger E_t\Big],\\
\overline{r}_t^\dagger(y_t,u_t) &:= \nabla_{u_t} h_t(y,u) = \mathbb{E}_t\Big[\beta_{t+1}\nabla_{u_t} U(V_{t+1}(x_{t+1},u_t)) + \\
&\qquad\qquad + \mu\left(\frac{1}{E_t y_t + F_t u_t - e_t}\right)^\dagger F_t\Big],\\
\overline{Q}_t(y_t,u_t) &:= \nabla^2_{y_t} h_t(y,u) = \mathbb{E}_t\Big[\beta_{t+1}\nabla^2_{y_t} U(V_{t+1}(x_{t+1},u_t))) \\
&\qquad\qquad - \mu E_t^\dagger \operatorname{diag}\left(\frac{1}{E_t y_t + F_t u_t - e_t}\right)^2 E_t\Big],\\
\overline{P}_t(y_t,u_t) &:= \nabla_{u_t}\nabla_{y_t} h_t(y,u) = \mathbb{E}_t\Big[\beta_{t+1}\nabla_{u_t}\nabla_{x_t} U(V_{t+1}(x_{t+1},u_t))) + \\
&\qquad\qquad - \mu E_t^\dagger \operatorname{diag}\left(\frac{1}{E_t y_t + F_t u_t - e_t}\right)^2 F_t\Big],\\
\overline{R}_t(y_t,u_t) &:= \nabla^2_{u_t} h_t(y,u) = \mathbb{E}_t\Big[\beta_t\nabla^2_{u_t} U(V_t(x_t,u_t))) + \\
&\qquad\qquad - \mu F_s^\dagger \operatorname{diag}\left(\frac{1}{E_t y_t + F_t u_t - e_t}\right)^2 F_t\Big],
\end{aligned}} \tag{19}$$

assuming that the matrices R_t and Q_t have the form

$$R_t = \begin{bmatrix} \overline{R}_t & 0 \\ 0 & R_{t+1} \end{bmatrix} \qquad Q_t = \begin{bmatrix} \overline{Q}_t & 0 \\ 0 & Q_{t+1} \end{bmatrix}. \tag{20}$$

This is equivalent with the

Inductive Assumption: J_t is a quadratic function in Δy_t:

$$J_t(\Delta y_{\geqslant t}) = J_t(\Delta y_t) = \alpha_t + w_t^\dagger \Delta y_t + \frac{1}{2}\Delta y_t{}^\dagger W_t \Delta y_t, \tag{21}$$

where $(W_t)_{t=0,\dots,T-1} \subset \mathbf{R}^{M\times M}$ is an adapted, definite matrix valued process and $(\alpha_t)_{t=0,\dots,T-1} \subset \mathbf{R}$, $(w_t)_{t=0,\dots,T-1} \subset \mathbf{R}^{M\times 1}$ are adapted processes.
Using the dynamics $\Delta y_{t+1} = A_{t+1}\Delta y_t + B_{t+1}u_t$ we can rewrite the value function (21) as

$$\begin{aligned}
J_{t+1}(\Delta y_{\geqslant t+1}) =& \alpha_{t+1} + w_{t+1}^\dagger A_{t+1}\Delta y_t + \frac{1}{2}\Delta y_t{}^\dagger A_{t+1}^\dagger W_{t+1} A_{t+1}\Delta y_t + \\
&+ \frac{1}{2}\Delta u_t{}^\dagger B_{t+1}^\dagger W_{t+1} B_{t+1}\Delta u_t + \\
&+ (w_{t+1}^\dagger + \Delta y_t^\dagger A_{t+1}^\dagger W_{t+1}) B_{t+1}\Delta u_t.
\end{aligned} \tag{22}$$

With the definitions

$$\boxed{\begin{aligned} \widetilde{a}_t &:= \sum_{s=t+1}^{T} \alpha_s & \widetilde{r}_t &:= r_t + \sum_{s=t+1}^{T} B_s^\dagger w_s^\dagger, \\ \widetilde{R}_t &:= R_t + \sum_{s=t+1}^{T} B_s^\dagger W_s B_s & \widetilde{q}_t &:= \overline{q}_t + \sum_{s=t+1}^{T} A_s^\dagger w_s, \\ \widetilde{Q}_t &:= \overline{Q}_t + \sum_{s=t+1}^{T} A_s^\dagger W_s A_s & \widetilde{P}_t &:= \overline{P}_t + \sum_{s=t+1}^{T} A_s^\dagger W_s B_s, \end{aligned}} \tag{23}$$

expression(21) for the value function becomes

$$\begin{aligned} J_t(\Delta y_t) = \max_{\Delta u_t} \Big[&\widetilde{a}_t + \widetilde{q}_t^\dagger \Delta y_t + \frac{1}{2}\Delta y_t{}^\dagger \widetilde{Q}_t \Delta y_t + \left(\widetilde{r}_t^\dagger + \Delta y_t{}^\dagger \widetilde{P}_t\right) \Delta u_t + \\ &+ \frac{1}{2}\Delta u_t{}^\dagger \widetilde{R}_t \Delta u_t \Big]. \end{aligned} \tag{24}$$

3.5 Optimal Control Rules

The optimum can be found by differentiating the expression maximized in (24) with respect to Δu_t:

$$\begin{aligned} 0 = \nabla_{\Delta u_t} \Big[&\widetilde{a}_t + \widetilde{q}_t^\dagger \Delta y_t + \frac{1}{2}\Delta y_t{}^\dagger \widetilde{Q}_t \Delta y_t + \left(\widetilde{r}_t^\dagger + \Delta y_t{}^\dagger \widetilde{P}_t\right) \Delta u_t + \\ &+ \frac{1}{2}\Delta u_t{}^\dagger \widetilde{R}_t \Delta u_t \Big] = \widetilde{r}_t^\dagger + \Delta y_t{}^\dagger \widetilde{P}_t + \Delta u_t{}^\dagger \widetilde{R}_t, \end{aligned} \tag{25}$$

which means, being R_t symmetric,

$$\boxed{\Delta u_t^* = -\widetilde{R}_t^{-1}(\widetilde{r}_t + \widetilde{P}_t^\dagger \Delta y_t).} \tag{26}$$

Inserting this optimal Δu_t^* in (24), the value function becomes

$$J_t(\Delta x_t) = \alpha_t + w_t^\dagger \Delta x_t + \frac{1}{2}\Delta x_t{}^\dagger W_t \Delta x_t, \tag{27}$$

where

$$\boxed{\begin{aligned} \alpha_t &:= \widetilde{a}_t - \frac{1}{2}\widetilde{r}_t^\dagger \widetilde{R}_t^{-1} \widetilde{r}_t, \\ w_t &:= \widetilde{q}_t - \widetilde{P}_t \widetilde{R}_t^{-1} \widetilde{r}_t, \\ W_t &:= \widetilde{Q}_t - \widetilde{P}_t \widetilde{R}_t^{-1} \widetilde{P}_t^\dagger, \end{aligned}} \tag{28}$$

The expression for W_t in the third equation of (28) together with (23) is known as the **discrete time Riccati equation** in control theory.

Remark 3.2 *If W_t is positive definite, if W_s is positive semidefinite for all $s = t+1, \ldots, T$.*

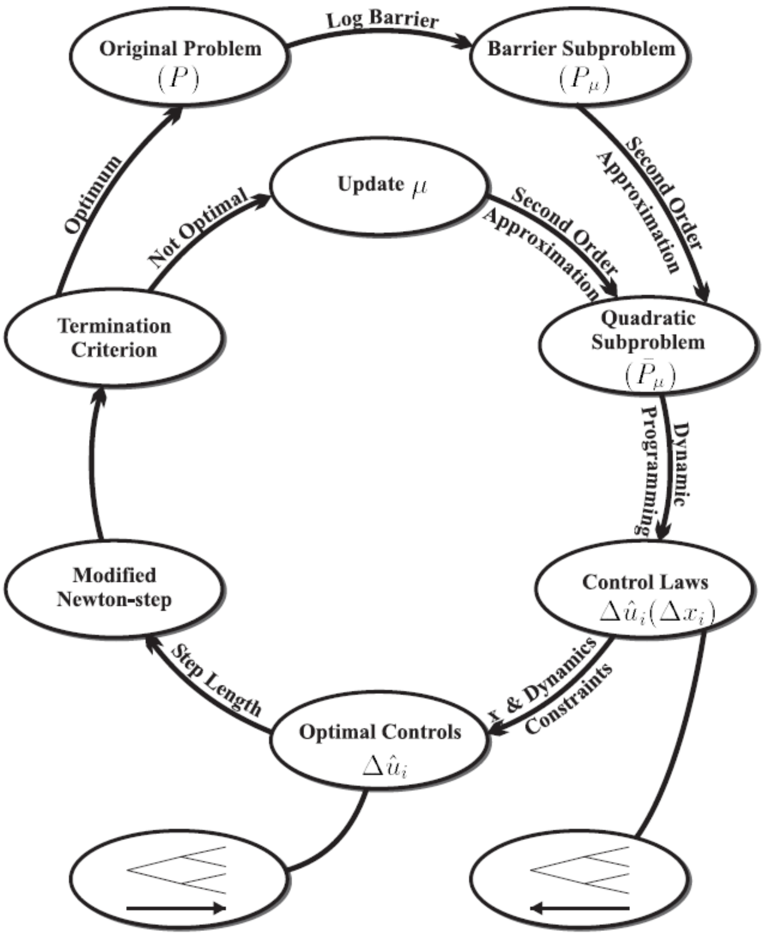

Fig. 2 Converging Sequence of Optimal Rules (Picture from [10])

3.6 The Algorithm

Newton's step determination problem $(\bar{P}_\mu)$ in (15) for the barrier subproblem (P_μ) is solved by (26), where matrices, vectors and constants are defined recursively by (23) and (28).

4 Implementation

The purpose of this section is to show that the solution algorithm shown in Figure 2 can be efficiently implemented by means of time and space discretization on a lattice, i.e. a particular kind of tree, where each node will host a realization of risk drivers, risk factors and optimal rules, comparing this modelling choice with other possible approaches. When we want to implement discrete time dynamic stochastic programming models, we have basically four possibilities:

1. **The (semi-)closed formula solution:**
   In some (seldom) cases one can find a set of (semi-)closed formulae representing the optimal control rules as a functional of conditional expectations of functions of risk factors. The optimal rules can be therefore explicitly determined given a probability model for the risk factors. But, in most of the cases, the computation can be only numerical, and we therefore have to switch to
2. **The graph solution:**
   There are several possibilities to choose a graph, and for all the nodes of the graph will have to correspond to the atoms of the sigma algebras of the filtration $(\mathcal{A}_t)_{t=0,\dots T}$:
   (a) **The full non recombining tree:**
   This is the most generic solution, which has the disadvantage of being implementable in its fully fledged version on high performing computer only, because the number of nodes in a time layer increases exponentially with time. The alternative is to reduce drastically the number of branches from every node when time increases. To do so, one has to develop criteria to generate representative branches. Those criteria are mostly heuristic.
   (b) **The grid:**
   Parallel paths for simulated external states are stored in the nodes. If we have a (semi-) closed formula for the optimal rules, these can be computed on every node. If not, then the optimal rule is computed on the node by solving the Bellman's backward optimization step by simulating jumps from that node to all nodes in the following time layer. This method is computationally effective and is therefore widespread.
   (c) **The lattice:**
   We see a lattice with many branches as a totally recombining tree. Therefore, being the number of nodes in a time layer a linear function of time, the full fledged model is implementable even on standard computers. Of course, the main challenge is to fill the nodes with state realizations in such a way that these are compatible with their dynamics on one hand, and with the full recombining property of the graph, on the other. To our knowledge this method is new, and is a generalization of binomial and trinomial trees' construction utilized for option pricing. This is the way we choose here. It has the advantage of being extensible

to the case where no (semi-)closed solution on the nodes exists, and is thus a viable implementation method for a numerical solution of Bellman's backward recursion. In contrast to the grid method one does not have to resimulate the jumps from one node into its children everytime the algorithm performs an approximation step.

The algorithm for the lattice construction, the simulation of states and the approximation of optimal control rules is structured into the following steps:

Step 1

We construct the lattice with k branches for every node and final horizon T. Let $e_s := (0, \dots, 1, 0, \dots, 0) \in \mathbf{R}^k$ is the sth standard basis vector. For $t = 0, \dots, T$ we set

$$\boxed{\begin{aligned}
&\mathcal{L} := \bigcup_{t=0}^{T} \mathcal{L}_t : \text{lattice}, \\
&\mathcal{L}_t := \left\{ (t, i) \in \{0 \dots, T\} \times \mathbf{N}_0^k \,\middle|\, \sum_{s=1}^{k} i_s = t \right\} : \text{lattice time } t \text{ layer}, \\
&n_t(i) = (t, i) \in \mathcal{L}_t : \text{node layer at time } t \\
&\text{Children}(n_t(i)) := \{(t+1, i + e_s) \,|\, s = 1, \dots, k\}, \\
&\text{Parents}(n_s(j)) := \{(s-1, i) \,|\, n_s(j) \in \text{Children}((s-1, i))\}, \\
&\text{Parents}(n_0(0)) := \{\},
\end{aligned}} \tag{29}$$

where the number of nodes at time t is

$$\boxed{N_t := |\mathcal{L}_t| = (k-1)t + 1 = O(t),} \tag{30}$$

and the total number of nodes is

$$\boxed{\mathcal{N}_T := |\mathcal{L}| = \sum_{t=0}^{T} N_t = \left((k-1)\frac{T}{2} + 1 \right) (T+1) = O(T^2).} \tag{31}$$

Geometrically speaking the (infinite) lattice consists in the points in the k-dimensional space with non negative integer coordinates. The time t layer of the lattice consists in the points laying on the hyperplane with orthogonal vector $(1, 1, \dots, 1)$ passing through the point $(k, 0, \dots, 0)$.

Step 2

We introduce the probability space $(\Omega, P, \mathcal{A})$, where the cartesian product

$$\boxed{\Omega := \mathrm{X}_{t=0}^{T}\,\mathrm{Children}^{t}(n_0(0)),} \tag{32}$$

corresponds to all possibilities of traveling across the lattice from left to right as times goes by,

$$\boxed{\mathcal{A} := \mathcal{P}(\Omega)} \tag{33}$$

is the sigma algebra of all measurable events, and, the sigma algebra generated by the lattice nodes in the time layer t, (that is having the nodes as basis) leads to a filtration $(\mathcal{A}_t)_{t=0,\dots,T}$, where

$$\boxed{\mathcal{A}_t := \sigma\left(\mathcal{L}_t\right).} \tag{34}$$

The probability of every node event is recursively defined as:

$$\boxed{\begin{aligned} &P[n_t] := \sum_{n_{t-1}\in\,\mathrm{Parents}(n_t)} \frac{P[n_{t-1}]}{k} \\ &P[n_0] := 1, \end{aligned}} \tag{35}$$

and that for the elementary event $\omega = (n_0(0), n_1(i_1), \dots, n_{T-1}(i_{T-1}), n_T(i_T))$ is

$$\boxed{P[\omega] := \frac{1}{k^T},} \tag{36}$$

Step 3

By means of simulations we fill the lattice nodes with realizations of the risk drivers $Z = (Z_t)_{t=1,\dots,T}$. Since these are multivariate i.i.d. over time, these simulations are straightforward: for every $t = 1, \dots, T$

1. simulate N_t values $z_t^1, \dots, z_t^{N_t}$ values with all the same probability.
2. set $Z_t(n_t^k) := z_t^k$ for all nodes in $\mathcal{L}_t$, the layer at time t.

These are simulated values for the risk drivers on the nodes.

Step 4

We compute the corresponding realizations of the external states (risk factors) $X = (X_t)_{t=1,\dots,T}$, by translating the dynamics at elementary event level $X_t(\omega) := f_t(X_{t-1}(\omega), Z_t(\omega))$ to the nodes as

$$X_{t+1}(\overline{n}) := \sum_{n\in\mathrm{Parents}(\bar{n})} \frac{p(n)}{\sum_{n\in\mathrm{parents}(\bar{n})} p(n)} f_{t+1}(X_t(n), Z_{t+1}(n)). \tag{37}$$

Step 5

We pick a $\mu = \mu_{\mathrm{Start}} > 0$ and a positive sequence $(\mu_j)_{j\geqslant 0}$ such that $\mu_0 = \mu_{\mathrm{Start}}$ and $\mu_j \to 0^+$ $(j \to +\infty)$.

Step 6

We pick initial values for the control variables u_t and the internal states y_t.

Step 7

For the value μ we compute all the realization of the processes in (23) and (28) by inserting the realizations of all control rules and both internal and external states.

Step 8

We compute Δu_t^* and check if it is approximatively very small. If not then, for $\Delta y_t(\Delta u_t^*)$ and do the increase step

$$\begin{aligned} u_t &\mapsto u_t + \Delta u_t^* \\ y_t &\mapsto y_t + \Delta y_t(\Delta u_t^*), \end{aligned} \tag{38}$$

update μ according to the sequence in (5) and jump to point (7). If Δu_t^* is too big, so that $u_t + \Delta u_t^*$ and $y_t + \Delta y_t(\Delta u_t^*)$ lie outside the feasible set, then Δu_t^* has to be substituted by $\epsilon_t \Delta u_t^*$ for an appropriate $\epsilon_t \in]0,1[$ small enough. Typically, ϵ_t depends on the node where it is computed.

Remark 4.1 *There are different possibilities to choose ϵ_t to guarantee feasibility. Blomvall and Lindberg propose to choose the same ϵ_t for all t and all nodes by looking at the largest $\bar{\epsilon}$ such that $u_t + \epsilon \Delta u_t^*$ is still feasible for all nodes and all times, and then set $\epsilon := \min(\xi\bar{\epsilon}, 1)$ for a $\xi \in]0,1[$. We, instead, proceed layerwise. Assuming that up to time layer $t-1$ the appropriate choice has already being made, in order to find a node dependent ϵ_t for all nodes in the time layer t we look for a node dependent $\bar{\epsilon}_t$ such that $u_t + \bar{\epsilon}_t \Delta u_t^*$ and u_s are feasible for all $s = t+1, \dots, T-1$. This can be efficiently achieved by a linear program, where the objective function is not really relevant. For a fixed $\xi \in]0,1[$ we then set $\epsilon_t := \min(\xi\bar{\epsilon}_t, 1)$ for all nodes in the layer, and repeat the procedure for the next time step. This refined procedure guarantees a faster convergence then Blomvall and Lindberg's when the maximizer lies on the boundary of the feasible set.*

Remark 4.2 *Why does the implementation on the lattice work? When implementing the dynamics, there is a fundamental difference between the non recombining tree and the lattice. The value of a process on a node depends on the values of the process on the parent nodes. In the non recombining tree case a node has only one parent, while in the lattice case a node can have several parents. But in both cases the process values on the nodes are expressed by conditional expectations. More exactly, we have the situation summarized in Table 1.*
An internal state variable defined as $Y_{t+1} = g_{t+1}(Y_t, u_t, X_{t+1})$ for deterministic functions g_t for $t = 1, \dots, T$, typically utilized to define constraints. On the nodes it is represented by $Y_t(n) = \mathbb{E}[Y_t|\mathcal{B}(n)]$, and at elementary event level it has the dynamics

$$Y_{t+1}(\omega) = g_{t+1}(Y_t(\omega), u_t(\omega), X_{t+1}(\omega)), \tag{39}$$

Symbol	Description	Mapped to
Ω	Space of all elementary events	All possibilities of travelling through the lattice from left to right
n_t	Node	An atom $\mathcal{B}(n_t)$ of the σ-algebra for the time layer t containing that node
$Y(n)$	Value on the node n of any random variable Y	$Y(n) = \mathbb{E}[Y\|\mathcal{B}(n)] \neq Y(\omega) \quad \text{for } \omega \in n$
$X_t(n)$	Value on the node n of the external state X_t	$X_t(n) = \mathbb{E}[X_t\|\mathcal{B}(n)]$

Table 1 Lattice Variables

which becomes the external state variable dynamics at node level

$$\begin{aligned} Y_{t+1}(\overline{n}) &:= \mathbb{E}[Y_{t+1}|\mathcal{B}(\bar{n})] = \sum_{n \in Parents(\bar{n})} \frac{p(n)}{p(\bar{n})} \mathbb{E}[Y_{t+1}|\mathcal{B}(n)] = \\ &= \sum_{n \in Parents(\bar{n})} \frac{p(n)}{\sum_{n \in parents(\bar{n})} p(n)} g_{t+1}(Y_t(n), u_t(n), X_{t+1}(n)). \end{aligned} \tag{40}$$

This holds for a generic dynamics of the internal states and hence for the implemented linear dynamics $g_t(y, u, x) := A_t y + B_t u + b_t(x)$.

5 Application: Water Values and Intraday Electricity Trading

The algorithm presented in the preceding section can be utilized to optimize intraday electricity trading and model at the same time water values for hydro assets.

Everyday by 11:00 CEST all the participants to the Swiss electricity spot market have to submit to the energy exchange their aggregated bids for the day-ahead both demand and supply. These, in the "ask"-case specify for every hour of the following day, from 00:00 till 24:00$^-$ CEST the quantity of energy Ξ_t^{Ask} in MWh that one participant is willing to deliver during that hour $t = 0, \dots, 23$ if the electricity price S_t then is greater than or equal to a certain value GP_t^{Ask}, called *generation water value*. In the "bid"-case the electric market participants specify for every hour of the following day the quantity of energy Ξ_t^{Bid} in MWh that the participant is willing to buy during that hour $t = 0, \dots, 23$ if the electricity price S_t then is smaller than or equal to a certain value GP_t^{Bid}, called *delivery water value*. For every hour the energy exchange aggregates all asks and all bids two monotone step functions, the ask curve and the bid curve, representing the quantity of energy deliverable (ask) or requested (bid) as a function of the price. The intersection point of the two curves, i.e. the market clearing price at time t is the spot price which will hold for the hour t of the next day. The 24 spot prices for the day-ahead are published at around 11:15 CEST of the current day. Note that all of the market participants are due to deliver or to buy the quantities of energy specified during the bidding process, but at the market clearing price determined

Restriction	Description
Hydro-infrastructure dynamics	Equations connecting basin levels and water inflows or outflows
Lower and upper bounds for the energy produced	Limits for turbines and pumps

Table 2 Restrictions

by the energy exchange for the day-ahead spot prices. However, the auction is not physically binding, that is, energy must not necessarily be produced but can be bought and delivered.

All the trades for the day ahead settled between 11:15 and 23:59 CEST, where energy quantities $\Xi_t^{\text{Spot, Sell}}$ and $\Xi_t^{\text{Spot, Buy}}$ will be sold and respectively bought at hour t of the next day at price S_t have to be taken into account by the trading strategy of the intraday - given what the spot desk has done. Given a certain utility function $U : \mathbf{R} \hookrightarrow \mathbf{R}$, the relevant optimization problem at 23:59 CEST of the day before ($t = 0$) reads for $T := 24$ and $t_0 := 1$

$$\max_{\substack{(u_t)_{t=t_0,\dots,T-1} \\ \text{Restrictions}}} \mathbb{E}_0 \left[\sum_{t=t_0}^{T} \beta_t U(V_t(X_t, u_{t-1})) \right] \tag{41}$$

and we make the choices needed to model the intraday-spot $P\&L$ in

- $\beta_t := 1$ for all t,
- X_t is the intraday price holding during $]t-1, t]$,
- We assume that we have B basins, labelled with $b = 1, \dots, B$. Basin b is connected with N_b turbines/pumps. Turbine/pump $j_b = 1, \dots, N_b$ processes u_t^{b,j_b} energy at time t. The aggregated processed energy quantity at time t for basin b is given by $u_t^b := \sum_{j_b=1}^{N_b} u_t^{b,j_b}$ and for the whole hydro infrastructure by $u_t := \sum_{b=1}^{B} u_t^b$.
- $V_t(X_t, u_{t-1}) := u_{t-1}X_t + (\Xi_t^{\text{Spot, Sell}} - \Xi_t^{\text{Spot, Buy}})S_t$ is the portfolio profit and loss for both spot and intraday desks.

The optimization problem reads after these choices

$$\max_{\substack{(u_t)_{t=t_0,\dots,T-1} \\ \text{Restrictions}}} \mathbb{E}_0 \left[\sum_{t=t_0}^{T} U(u_{t-1}X_t + (\Xi_t^{\text{Spot, Sell}} - \Xi_t^{\text{Spot, Buy}})S_t) \right]. \tag{42}$$

The restrictions are listed in Table 2 and explained in detail here below.

- **The dynamics of the hydro infrastructure** connecting:
  - the basins' volumes,
  - the water inflows,
  - the water outflows (turbined water, overspills).

  The basin b level dynamics $(Y_t^b)_{t=t_0,\dots,T-1}$ is given for all $t = t_0, \dots, T-1$ by

$$Y_{t+1}^b(\omega) = Y_t^b(\omega) - u_t^b(\omega) + i_{t+1}^b(\omega) \tag{43}$$

where the process $(i^b_t)_{t=t_0+1,\dots,T}$ denotes the exogenous dynamics of basin b inflow, and the level lower and upper constraints are given for all $t = t_0, \dots, T-1$ by

$$Y^{b,\mathrm{Min}} \leqslant \mathbb{E}_t[Y^b_{t+1}] \leqslant Y^{b,\mathrm{Max}}, \tag{44}$$

for specified constants $Y^{b,\mathrm{Max}} > Y^{b,\mathrm{Min}} > 0$ which are (flexible) basin characteristics. Remark, that, being the basins' inflows uncertain, we cannot express (44) as a predictable constraint for the water turbined or pumped, but the consequences on the optimal solution are typically not material, because the inflow volatility is small and we can assume for most applications that the inflow is deterministic and given as a table characterizing the basins' system.

In contrasts to financial applications we do not have here the self financing constraint, because we can decide to turbine/pump or not in a certain period independently of what has been done before or what will be done afterwards, as long as the basin constraints are not binding.

– **Lower and upper bounds for the energy produced by each turbine every hour**. Note that negative lower bounds account for *pumping.* These bounds capture expected potential market liquidity restrictions in the day ahead market and, for all $t = t_0, \dots, T-1$, $j_b = 1, \dots, N_b$, $b = 1, \dots, B$, read as:

$$u^{b,j_b,\mathrm{Min}} \leqslant u^{b,j_b}_t(\omega) \leqslant u^{b,j_b,\mathrm{Max}}. \tag{45}$$

Finally we make the following modeling choices for the intraday price stochastic dynamics:

$$dX_t = X_t[\mu_t(X_t)dt + \sigma_t(X_t)dW_t], \tag{46}$$

where $\mu_t : \mathbf{R} \to \mathbf{R}$ and $\sigma_t : \mathbf{R} \to \mathbf{R}^{K\times q}$ are functions with appropriate regularity and $(W_t)_{t\geqslant 0}$ is a K-dimensional standard Brownian motion with respect to the statistical measure P. We assume that, for the short future period, the intraday price dynamics is approximatively driftless, i.e. $\mu_t \doteq 0$. We can assume one risk driver (i.e. $K := 1$) and a deterministic volatility, that is

$$\sigma_t(X_t(\omega)) \equiv \sigma_t \in \mathbf{R}. \tag{47}$$

A better way to model intraday prices X_t is by modelling their spreads $Z_t := X_t - S_t$ to spot prices S_t

$$dZ_t = Z_t[\nu_t(Z_t)dt + \eta_t(Z_t)dW_t], \tag{48}$$

where $\nu_t : \mathbf{R} \to \mathbf{R}$ and $\eta_t : \mathbf{R} \to \mathbf{R}$ with appropriate regularity. Again, the spread dynamics is approximatively driftless, i.e. $\nu_t \doteq 0$ and we assume a deterministic volatility, that is

$$\eta_t(Z_t(\omega)) \equiv \eta_t = \sqrt{\sigma_t^2 + \chi_t^2 + 2\rho_t\sigma_t\chi_t} \in \mathbf{R}, \tag{49}$$

where χ_t denotes the instantaneous volatility for the log return of spot prices, and ρ_t the correlation between log return of spot and intraday prices. Note that to model intraday prices via their spread to spot one needs a spot price model first. In particular one has to model the expected spot prices in the day ahead market.

Now we proceed to model water values for the hydro infrastructure described so far. Before 11:00 CEST we can utilize (42) to determine the generation water values $\mathrm{GP}_t^{\mathrm{Ask}}$ for $t = 0, \dots, 23$ for the day ahead for the hydro infrastructure, whose bids we will aggregate in our bid for the energy exchange. We exclude for the moment the spot desk whose trades for the day ahead have not been established yet from (42). We define the water values as the shadow prices associated to the basin levels dynamics (43), that is the value of the Lagrangian multipliers associated to (43) for the optimal solution: they represents the instantaneous change per unit of constraints (43), in [MWh], in the objective function value of (42), in [EUR], for a variation of the constraints, i.e. the marginal utility of relaxing the basin level constraints. Therefore, after having expressed the basin level dynamics (43) with the equivalent expression

$$p(\omega)[Y_t^b(\omega) - Y_{t-1}^b(\omega) + u_{t-1}^b(\omega) - i_t^b(\omega)] = 0, \tag{50}$$

for all $t = 1, \dots, T$ and $b = 1, \dots B$, we obtain a Lagrangian principal function for the basin constraints

$$\begin{aligned} \Phi(u, \lambda) := \sum_{\substack{\omega\in\Omega \\ t=1,\dots,T \\ b=1,\dots,B}} & p(\omega) \left[U(u_{t-1}(\omega)X_t(\omega)) - \lambda_t^b(\omega)(Y_t^b(\omega) - Y_{t-1}^b(\omega) + \right. \\ & \left. + u_{t-1}^b(\omega) - i_t^b(\omega))\right], \end{aligned} \tag{51}$$

where $u = (u_t^{b,j_b}(\omega))$ is the energy corresponding to the water turbined or pumped and $\lambda = (\lambda_t^b(\omega))$ is the set of Lagrangian multipliers for the basin levels. The optimal solution satisfies the equations

$$\begin{cases} \frac{\partial\Phi(u,\lambda)}{\partial u_{t-1}^b(\omega)} = p(\omega)[X_t(\omega)U'(u_{t-1}(\omega)X_t(\omega)) - \lambda_t^b(\omega)] = 0 \\ \frac{\partial\Phi(u,\lambda)}{\partial\lambda_t^b(\omega)} = -(Y_{t+1}^b(\omega) - Y_t^b(\omega) + u_t^b(\omega) - i_t^b(\omega)) = 0, \end{cases} \tag{52}$$

which leads to

$$\lambda_t^b(\omega) = X_t(\omega)U'(u_{t-1}(\omega)X_t(\omega)). \tag{53}$$

The choice of the reformulation (50) takes the probability for the constraint to be binding into account and leads to the meaningful definition for the shadow price. Therefore, the stochastic water values are the same for all basins in the hydro infrastructure and read

$$\boxed{\mathrm{gp}_t^{\mathrm{Ask}}(\omega) := X_t(\omega)U'(u_{t-1}^*(\omega)X_t(\omega)),} \tag{54}$$

where u^* is the solution of the optimization problem (42) satisfying *all constraints, both equality and inequality ones*. We can use these stochastic water values to define production water values for the bid, by taking as possible definition the certainty equivalent of $\mathrm{gp}_t^{\mathrm{Ask}}$:

$$\boxed{\mathrm{GP}_t^{\mathrm{Ask}} := U^{-1}\mathbb{E}_0\left[U(\mathrm{gp}_t^{\mathrm{Ask}})\right].} \tag{55}$$

Being the utility function U monotone increasing and concave the risk add on $U^{-1}\mathbb{E}_0\left[U(\mathrm{gp}_t^{\mathrm{Ask}})\right] - \mathbb{E}_0\left[\mathrm{gp}_t^{\mathrm{Ask}}\right]$ is non negative and accounts for the risk aversion.

If the initial basin levels are distant enough from the lower and upper bounds, then we can assume that during the 24 hours of the optimization interval the basin level constraints are not binding and thus

$$u_t^{b,j_b,*}(\omega) \equiv u^{b,j_b,\mathrm{Max}}. \tag{56}$$

To our knowledge the expression "water value" was introduced for the first time by Larsson and Stage in [29]. For a treatment of water values defined by means of Lagrangian multipliers in a cost minimization problem see [16] and an approach consisting in a time dependent shadow pricing of water in profit maximization problem can be found in [27].

We can consider the joint intraday and spot desks in the determination of water values. The joint optimization problem at a certain hour before 11:00 CEST (t=0) reads for $T := 48$ and $t_0 := 24$

$$\max_{\substack{(u_t)_{t=t_0,\dots,T-1}\\ (\Xi_t)_{t=t_0,\dots,T-1}\\ \text{Restrictions}}} \mathbb{E}_0\left[\sum_{t=t_0}^{T} U(u_{t-1}X_t + \Xi_{t-1}S_t)\right], \tag{57}$$

where $(S_t)_{t=t_0+1,\dots,T}$ denote the (till 11:15 CEST) stochastic spot prices for the day ahead, and $(\Xi_t)_{t=t_0,T-1}$ the stochastic quantities of energy turbined for the spot market. The restrictions are those of (42), where u_t is substituted by $u_t + \Xi_t$. A computation analogous to the one for (54) leads to the following stochastic and deterministic water values for all basins in the hydro power plant:

$$\boxed{\begin{aligned} \mathrm{gp}_t^{\mathrm{Ask}}(\omega) &:= \frac{1}{2}(X_t(\omega) + S_t(\omega))U'(u_{t-1}^*X_t(\omega) + \Xi_{t-1}^*(\omega)S_t(\omega)) \\ \mathrm{GP}_t^{\mathrm{Ask}} &:= U^{-1}\mathbb{E}_0\left[U(\mathrm{gp}_t^{\mathrm{Ask}})\right], \end{aligned}} \tag{58}$$

where u^*, Ξ^* is the solution of the optimization problem (57) satisfying *all contraints, both equality and inequality ones.* As in the intraday case, if the initial basin levels are distant enough from the lower and upper bounds, then we can assume that during the 24 hours of the optimization interval the basin level constraints are not binding and thus

$$u_t^{b,j_b,*}(\omega) + \Xi_t^{b,j_b,*}(\omega) \equiv u^{b,j_b,\mathrm{Max}}, \tag{59}$$

for all basins and turbines.

Remark 5.1 (Strategy Extension: Accounting for Hourly Forward Trades) *The models (42) and (57) can be utilized at any hour* $0,\dots,11$ *of the current day to find stochastic and deterministic water values for the hours* $\{1,\dots,24\}+24$ *of the day ahead. Immediately after 11:15 CEST the day ahead spot prices* *are known. At any hour* $\{11, \dots 48\}$ *it is possible to initiate forward transactions with one hour in* $\{0, \dots, 23\} + 24$ *as delivery period. This means, at time* t *of the day ahead the (deterministic) energy quantity* Ψ_t *will be delivered for the price* F_t *established when the transaction was closed. In order for the allocation strategy to take this aspect into account, we choose* $T := 48$ *and* $t_0 := 24$ *and modify the optimization model (42) to*

$$\max_{\substack{(u_t)_{t=t_0,\dots,T-1} \\ \textit{Restrictions}}} \mathbb{E}_0 \left[\sum_{t=t_0}^{T} U(u_{t-1} X_t + \Psi_{t-1} F_t) \right], \tag{60}$$

where $(F_t)_{t=t_0,\dots,T}$ *denote the deterministic forward prices for the day ahead, and* $(\Psi_t)_{t=0,T-1}$ *the deterministic quantities of energy turbined for the forward market, established at a certain hour* $(t = 0)$ *of the day before. Of course one can add* $(\Psi_t)_{t=t_0,\dots,T-1}$ *to the optimization variables and run the at time* $\{12, \dots, 23\}$ *the algorithm solving (60) is to find both optimal rules for the turbined quantities in the intraday market in the day ahead and deterministic optimal forwards for the day ahead. From the equality*

$$\Psi_t^* = \Psi_t^{\textit{In Force}} + \Delta\Psi_t^{\textit{Forward},*}, \tag{61}$$

one reads off the energy quantity $\Delta\Psi_t^*$ *to be hedged with the new forward transaction at time* t_0 *with delivery period* $[t, t+1]$*. Model (60) can be further extended to account for intraday, forward and spot transactions, as well.*

Remark 5.2 *The model proposed is intrinsically balance-energy neutral for the balance group which the hydro infrastructure belongs to. A balance group is a set of electricity meters measuring* 15 *min consumption and production for net users. The transmission system operator makes sure that every balance group is in an equilibrium state, by adding or subtracting electric energy in such a way that the total sum of energies vanishes for every quarter of an hour. Of course this comes at a certain expensive price with which the TSO charges the balance group owner, which can be (but not necessarily is) the hydro infrastructure owner as well. Thus, there is an incentive not to generate or at least to reduce balance energy, in order to minimize costs.*

Remark 5.3 *If we assume that the utility function* $U : \mathbf{R} \hookrightarrow \mathbf{R}$ *can be written as as*

$$U = r - \frac{w}{2}\rho, \tag{62}$$

where r *is an increasing concave function,* ρ *is an increasing convex function and* $w > 0$ *the risk aversion parameter, then the optimization problems analyzed so far can be rewritten in terms of risk-reward optimization, as it is customary in financial portfolio theory.*

Definition 1 (Risk and Reward) *The functional*

- *Reward* : $L^2(\Omega, \mathcal{A}, P) :\to \mathbf{R}, R \mapsto \textit{Reward}(R) := \mathbb{E}_0[r(R)]$ *is termed as* ***reward measure****,*

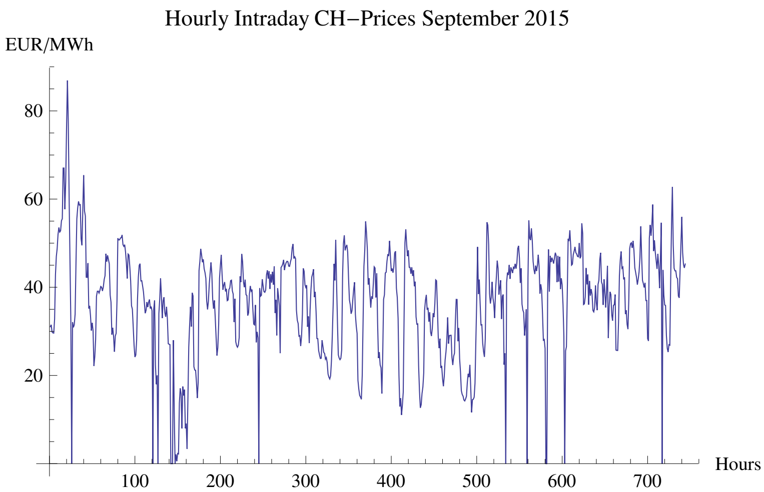

Fig. 3 Intraday Prices

Quantity	Value
Max	86.91 EUR/MWh
Min	0.56 EUR/MWh
Mean	37.40 EUR/MWh
Volatility	123.87 EUR/MWh
Volatility of hourly log returns	19.97%

Table 3 Intraday prices statistics

– *Risk* : $L^2(\Omega, \mathcal{A}, P) :\rightarrow \mathbf{R}, R \mapsto Risk(R) := \mathbb{E}_0[\rho(R)]$ *is termed as* ***risk measure****.*

The optimization problem (42) reads then as a trade-off between total risk and total reward

$$\max_{\substack{(u_t)_{t=0,\dots,T-1}\\ Restrictions}} \left[\sum_{t=1}^{T} \beta_t Reward(u_{t-1}X_t)\right] - \frac{w}{2}\left[\sum_{t=1}^{T} \beta_t Risk(u_{t-1}X_t)\right]. \qquad (63)$$

6 A Numerical Example

We utilize the weighted averaged September 2015 data from Epex Spot Intraday Continuous for the CH Market, downloaded from *www.epexspot.com/en/market-data/intradaycontinuous/intraday-table/2015-09-30/CH*. These weighted averaged intraday are plotted in Figure 3 and have descriptive statistics as in Table 3.

We construct a simple hydro infrastructure as described in Table 4 and test two possible intraday price dynamics as shown in Table 5.

Symbol	Description	Value
B	Number of basins	1
Y^{Max}	Basin level maximal capacity	160 GWh
Y^{Min}	Basin level minimal capacity	40 GWh
u^{Max}	Turbine maximal capacity	500 MW
u^{Min}	Turbine minimal capacity (no pumping)	0 MW
Y_0	Two possible basin starting level	80 GWh and 41.5 MWh
i_t	(No) Inflow	0 MWh

Table 4 Basin Parametrization

Model	Description	Parameters
Model 1	Driftless geometric Brownian motion as in (46)	$\sigma_t = 19.97\%$ $K = 1$
Model 2	Driftless spread to spot as in (48)	$\eta_t = 4.12$ EUR/MWh, $\mathbb{E}_0[S_t]$ =Sep 2015 means

Table 5 Intraday Dynamics X_t Models

Symbol	Description	Value
t	Valuation time	0 (08:00)
X_0	Intraday price starting value	weighted average price 08:00-09:00
t_0	Initial time day ahead	24
T	Final time day ahead	48
k	Number of branches out of a leaf in the lattice	15

Table 6 Lattice Parametrization

Utility Function	Definition	Parameters
Linear	$U(v) := v$	
Exponential	$U(v) := 1 - \exp(-\alpha v)$	$\alpha > 0$: Arrow-Pratt relative risk aversion
Logarithmic	$U(v) := \log(v)$	
Hyperbolic	$U(v) := \frac{1}{\gamma} v^{\gamma}$	$\gamma \in]0, 1[$

Table 7 Utility Functions

For every day in the sample we compute the optimal dynamic strategy and the water values for the day ahead market using the weighted average electricity price between 08:00 and 09:00 for the current day. More precisely, we make the choices for the lattice specified in Table 6.

As expected, when the chosen starting level is 80 GWh and thus the basin level constraints can never become binding, the optimal strategy is the same for all utility functions in Table 7 and reads

$$u_t^*(n) = 500\,\mathrm{MWh} \quad \text{for all times } t \text{ and nodes } n. \tag{64}$$

To back test the results for the optimal strategy we apply it to the historical realizations of the intraday prices. More exactly, we express the discretized optimal rules as function of the discretized intraday price of the preceding period and compute the optimal rule with the realized price by linear interpolation. Then, for every day in the back test, we pass through the different

Hours	Linear	Exp 0.0001	Exp 1.00	Log	Hyp 0.50	Hyp 0.75	Hyp 0.95
1	34.511	0.001	0.000	0.002	0.260	2.987	21.141
2	33.132	0.001	0.000	0.002	0.255	2.901	20.345
3	32.208	0.001	0.000	0.002	0.252	2.842	19.810
4	31.480	0.001	0.000	0.002	0.249	2.794	19.386
5	31.080	0.001	0.000	0.002	0.248	2.767	19.150
6	31.357	0.001	0.000	0.002	0.249	2.786	19.312
7	31.717	0.001	0.000	0.002	0.250	2.809	19.521
8	31.971	0.001	0.000	0.002	0.251	2.825	19.668
9	32.595	0.001	0.000	0.002	0.253	2.865	20.031
10	33.561	0.001	0.000	0.002	0.257	2.927	20.591
11	34.623	0.001	0.000	0.002	0.261	2.995	21.207
12	35.491	0.001	0.000	0.002	0.264	3.049	21.709
13	36.518	0.001	0.000	0.002	0.267	3.113	22.300
14	37.597	0.001	0.000	0.002	0.271	3.180	22.922
15	38.669	0.001	0.000	0.002	0.275	3.246	23.540
16	39.923	0.001	0.000	0.002	0.279	3.324	24.262
17	41.240	0.001	0.000	0.002	0.284	3.405	25.020
18	42.732	0.001	0.000	0.002	0.289	3.496	25.876
19	44.441	0.000	0.000	0.002	0.294	3.598	26.854
20	46.013	0.000	0.000	0.002	0.299	3.692	27.752
21	47.760	0.000	0.000	0.002	0.305	3.795	28.748
22	49.799	0.000	0.000	0.002	0.311	3.912	29.906
23	51.839	0.000	0.000	0.002	0.317	4.029	31.060
24	54.035	0.000	0.000	0.002	0.323	4.151	32.297

Table 8 Water Values $\mathrm{GP}_t^{\mathrm{Ask}}$ for September 2, 2015, Driftless GBM dynamics, initial basin level 80 GWh

hours choosing the optimal quantity of water to be turbined according to the dynamic control rule established before. The wealth generated for every hour for all days is depicted in Table 10.

If we set the initial basin level constraint near to the lower bound, the optimal strategy looks different: it becomes truly stochastic, tries to exploit the price dynamics and, of course, depends on the the utility function chosen. In the following toy examples with the parametrization specified by Table 11 we depict the realizations of prices, optimal turbined quantities and basin level with the hyperbolic utility function with $\gamma := 0.95$, once with the driftless geometric Brownian motion (Figures 4, 5, 6) and once with the spread to spot dynamics (Figures 7, 8, 9). In both cases we notice that the lower basin level bound becomes binding on some nodes on the final time layer $t = T$, which -due to the intertemporal nature of the optimization- has consequences on *all* earlier turbined quantities for $t = 0, \dots, T-1$ in some nodes, which do not reach their possible maximum even though there is still enough water in the basin. This phenomenon is the current “price” of future constraints.

Remark 6.1 (Algorithm Parameter Choices) *Following Blomvall and Lindberg we choose* $\mu_j := \mu_0 \exp(-j)$ *for* $mu_0 := 10^{-12}$*. Note that we take only one Newton step before reducing* μ_j*. As soon as* $\mu_j < \mu_{CP} := 10^{-16}$ *we assume that we have reached the close proximity to the so called central path and continue with Newton steps up to a maximum of* 100*.*

Hours	Linear	Exp 0.0001	Exp 1.00	Log	Hyp 0.50	Hyp 0.75	Hyp 0.95
1	25.729	0.001	0.000	0.002	0.225	2.403	16.007
2	22.450	0.001	0.000	0.002	0.211	2.169	14.061
3	22.671	0.001	0.000	0.002	0.212	2.186	14.196
4	25.975	0.001	0.000	0.002	0.227	2.425	16.160
5	34.464	0.001	0.000	0.002	0.262	3.002	21.149
6	39.993	0.001	0.000	0.002	0.282	3.358	24.365
7	41.709	0.001	0.000	0.002	0.288	3.466	25.358
8	41.365	0.001	0.000	0.002	0.287	3.445	25.159
9	40.437	0.001	0.000	0.002	0.284	3.386	24.622
10	40.747	0.001	0.000	0.002	0.285	3.406	24.802
11	39.535	0.001	0.000	0.002	0.281	3.329	24.100
12	38.179	0.001	0.000	0.002	0.276	3.243	23.313
13	36.985	0.001	0.000	0.002	0.271	3.166	22.619
14	36.079	0.001	0.000	0.002	0.268	3.107	22.091
15	35.939	0.001	0.000	0.002	0.268	3.098	22.010
16	38.180	0.001	0.000	0.002	0.276	3.243	23.313
17	41.036	0.001	0.000	0.002	0.286	3.424	24.968
18	43.912	0.000	0.000	0.002	0.296	3.603	26.629
19	43.347	0.000	0.000	0.002	0.294	3.568	26.303
20	40.053	0.001	0.000	0.002	0.283	3.362	24.400
21	38.274	0.001	0.000	0.002	0.276	3.249	23.368
22	35.537	0.001	0.000	0.002	0.266	3.072	21.776
23	31.893	0.001	0.000	0.002	0.252	2.832	19.646
24	28.504	0.001	0.000	0.002	0.238	2.601	17.655

Table 9 Water Values $\mathrm{GP}_t^{\mathrm{Ask}}$ for September 2, 2015, Spread to spot Dynamics, initial basin level 80 GWh

Remark 6.2 (Computational time of the Mathematica prototype) *We run the prototype on a Lenovo computer with Intel Core* $i7-3740QM$ *CPU* @2.70 *GHz. Typically, it takes between* 4 *and* 6 *minutes to compute the pedagogical examples of Figures 4, 5, 6 and 7, 8, 9 for the toy lattice parametrization specified by Table 11, and between* 7 *and* 8 *hours to compute the realistic example for the lattice parametrization specified by Table 6. We observe that, the more constraints are binding, the longer the computational time is. Since our Mathematica code is not optimized, we are confident that a reimplementation in a faster language (e.g. C) and the utilization of better hardware can drastically improve the performance.*

7 Conclusion and Further Research

A stochastic multiperiod portfolio optimization problem in discrete time for a generic utility function is discretized in the space dimensions by means of a lattice. Inequality constraints are packed into the objective function by means of a logarithmic penalty and the utility function is approximated by its second order Taylor polynomial. A sequence of solutions of the approximated problem converging to the optimal solution of the original problem is constructed and coded in an algorithm in Mathematica. We implement the algorithm on a lattice and apply it to intraday electricity trading. We obtain:

– a *novel, computationally efficient* implementation of a risk averse intertemporal portfolio optimization for the intraday market, and
– *deterministic water values* of an hydro infrastructure for the day ahead market bids as certainty equivalents of optimal stochastic Lagrangian multipliers corresponding to the basin level equations.

In a next work we will:

– compare the lattice implementation with the grid implementation, for both the semi-closed formula and the generic case.
– investigate the specific case of a quadratic utility function which needs no Newton-Scheme, being its second order Taylor polynomial the utility function itself, and, in particular, the dynamic mean variance case, for which in [19] a semi-closed solution was already provided.
– analyze the case of the maximization of the expected utility of the cumulated values over the different time subperiods, when the utility function is a trade-off between expectation and a dynamic risk measure, thus allowing for Bellman's recursive approach.
– construct an example where the pumping mode will occur in the optimal solution.
– analyze the present costs of future constraints.
– utilize the algorithm to compute opportunity costs to price ancillary services.

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References

[1] H. ABGOTTSPON, *Hydro power planning: Multi-Horizon Modeling and its Applications*, Diss. ETH No. 22729, 2015.
[2] M.H. AMINI, K.G. BOROOJENI, S.S. IYENGAR, P.M. PARDALOS and M. RAHMANI, *An Economic Dispatch Algorithm for Congestion Management of Smart Power Networks: An Oblivious Routing Approach*, Energy Systems, Vol. 8, (643667), 2017.
[3] P. ARTZNER, F. DELBAEN, J.-M. EBER, and D. HEATH, *Coherent Measures of Risk*, Mathematical Finance, Vol. 9, No. 3, (203-228), 1999.
[4] A. J. ASKEW, *Optimum Reservoir Operating Policies and the Imposition of a Reliability Constraint*, Water Resources Research, Vol. 10, No. 1, (51-56), 1974.
[5] L. BARROSO, S. REBENNACK and G. STEEGER, *Optimal Bidding Strategies for Hydro-electric Producers: A Literature Survey*, IEEE Transactions on Power Systems, Vol. 29, No. 4, (1758-1766), July 2014.
[6] R. BELLMAN, *Dynamic Programming*, Princeton University Press, 1957.
[7] D. BERTSEKAS and S. E. SHREVE, *Stochastic Optimal Control: The Discrete-Time Case*, Athena Scientific, 1996.
[8] D. BERTSEKAS, *Dynamic Programming and Optimal Control*, Optimization and Computation Series, Athena Scientific, Vol. 1, 2005.

[9] J. BLOMVALL and P. O. LINDBERG, *Validation of a Riccati-Based Primal Interior Point Solver for Multistage Stochastic Programming*, University of Linköping, Department of Mathematics, 2000.

[10] J. BLOMVALL and P. O. LINDBERG, *A Riccati-based Primal Interior Point Solver for Multistage Stochastic Programming*, European Journal of Operational Research 143 (2), (452-461), 2002.

[11] J. BLOMVALL and P. O. LINDBERG, *A Riccati-based Primal Interior Point Solver for Multistage Stochastic Programming – Extensions*, Optimization Methods and Software, Vol. 17, (383–407), 2002.

[12] J. BLOMVALL and P. O. LINDBERG, *A Multistage Stochastic Programming Algorithm Suitable for Parallel Computing*, Journal of Parallel Computing - Special issue: Parallel computing in numerical optimization, Volume 29 Issue 4, (431-445), 2003.

[13] J. BLOMVALL and P. O. LINDBERG, *Back-Testing the Performance of an Actively Managed Option Portfolio at the Swedish Stock Market, 1990-1999*, Journal of Economic Dynamics and Control, Vol. 27, No. 6, (1099-1112), 2003.

[14] J. D. COSTA, A. SHAPIRO, M. SOARES and W. TEKAYA, *Multistage Energy Planning - Risk Neutral and Risk Averse Approaches*, in XII SEPOPE: Symposium of Specialists in Electric Operational and Expansion Planning, 2012.

[15] M. DENSING, *Hydro-Electric Power Plant Dispatch-Planning: Multi-Stage Stochastic Programming with Time-Consistent Constraints on Risk*, Diss. ETH No. 17244, 2007.

[16] L. DUEHOLM and H. F. RAVN *Modelling of Short Term Electricity Prices, Hydro Inflow and Water Values*, Proceedings of the 6th IAEE European Conference, Zurich, 2004.

[17] J. EGERER, R. SCHARFF and L. SÖDER, *A Description of the Operative Decision-Making Process of a Power Generating Company on the Nordic Electricity Market*, Energy Systems, Vol. 5, No. 2, 349-369, 2014.

[18] A. GJELSVIK, A. GRUNDT and B. MO, *Integrated Risk Management of Hydro Power Scheduling and Contract Management*, IEEE Transactions on Power Systems, Vol. 16, No. 2, (216-221), May 2001.

[19] S. FARINELLI and P. VANINI, *Joint Interest Rate Risk Management of Balance Sheet and Hedge Portfolio*, The Icfai University Journal of Financial Risk Management, Vol. V, No. 3, 2008.

[20] S. FARINELLI and L. TIBILETTI, *Portfolio Management and Stochastic Optimization in Discrete Time: An Application to Intraday Electricity Trading and Water Values for Hydroassets*, Operations Research Proceedings 2015 Selected Papers of the International Conference of the German, Austrian and Swiss Operations Research Societies (GOR, ÖGOR, SVOR/ASRO), University of Vienna, Austria, September 1-4, 2015, Dörner, K.F., Ljubic, I., Pflug, G., Tragler, G. (Eds.), Springer Verlag. ISBN 978-3-319-42901-4. Forthcoming 2017.

[21] S.-E. FLETEN and S. W. WALLACE, *Stochastic Programming Models in Energy*, in Stochastic Programming, Handbooks in Operations Research and Management Science, A. Ruszczynski and A. Shapiro Eds., Elsevier, Vol. 10, (637-677), 2003.

[22] S.-E. FLETEN and T. K. KRISTOFFERSEN, *Stochastic Programming for Optimizing Bidding Strategies of a Nordic Hydropower Producer*, EJOR, Vol. 181, No. 2, (916-928), 2007.

[23] S.-E. FLETEN and T. K. KRISTOFFERSEN, *Short-Term Hydropower Production Planning by Stochastic Programming*, Computers and Operations Research, Vol. 35, No. 8, (2656-2671), 2008.

[24] S.-E. FLETEN and T. K. KRISTOFFERSEN, *Stochastic Programming Models for Short-Term Power Generation Scheduling and Bidding*, In: Bjørndal E., Bjørndal M., Pardalos P., Rönnqvist M. (eds.), Energy, Natural Resources and Environmental Economics, Energy Systems, Springer, Berlin, Heidelberg, 2010.

[25] D. FRANCES and R. H. KWON, *Optimization-Based Bidding in Day-Ahead Electricity Auction Markets: A Review of Models for Power Producers*, In: Sorokin A., Rebennack S., Pardalos P., Iliadis N., Pereira M. (eds.) Handbook of Networks in Power Systems I. Energy Systems. Springer, Berlin, Heidelberg, 2012.

[26] J. GARCÍA-GONZÁLEZ, A. MATEO and E. PARRILLA, *Risk-Averse Profit-Based Optimal Scheduling of a Hydro-Chain in the Day-Ahead Electricity Market*, EJOR, Vol. 181, No. 3, (1354-1369), 2007.

[27] A. HORSLEY and A. WROBEL, *Profit-Maximizing Operation and Valuation of Hydroelectric Plant: A new Solution to the Koopmans Problem*, Journal of Economic Dynamics and Control, Vol. 31 No. 3, (938-970), March 2007.

[28] J. W. LABADIE, *Optimal Operation of Multireservoir Systems: State-of-the-Art Review*, Journal of Water Resources Planning and Management, Vol. 130, No. 2, (93-111), March 2004.

[29] Y. LARSSON and S. STAGE, *Incremental Cost of Water Power*, Trans AIEE, (361-365), August 1961.

[30] G. LI, J. SHI and X. QU, *Modeling Methods for GENCO Bidding Strategy Optimization in the Liberalized Electricity Spot Market-A State of the Art Review*, Energy, Vol. 36, No. 8, (4686-4700), 2011.

[31] N. LÖHNDORF, S. MINNER and D. WOZABAL, *Optimizing Trading Decisions for Hydro Storage Systems using Approximate Dual Dynamic Programming*, Operations Research, Vol. 61, No. 4, 2013.

[32] N. LÖHNDORF and S. MINNER, *Optimal Day-Ahead Trading and Storage of Renewable Energies - An Approximate Dynamic Programming Approach*, Energy Systems, Vol. 1, (61-77), 2010.

[33] D. P. LOUCKS, J. R. STEDINGER and B. F. SULE, *Stochastic Dynamic Programming Models for Reservoir Operation Optimization*, Water Resources Research, Vol. 20, No. 11, (1499-1505), 1984.

[34] D. G. LUENBERGER, *Optimization by Vector Space Methods*, Wiley Professional Paperback Series, 1997.

[35] M. MINOUX, *Mathematical Programming: Theory and Algorithms*, Wiley, 1986.

[36] F. PARASCHIV, R. M. KOVACEVIC and M. SCHÜRLE, *Optimization of Hydro Storage Systems and Indifference Pricing of Power Contracts*, Presentation at Energy Finance Conference 2015, London, Cass Business School, 2015.

[37] G. PFLUG and A. PICHLER, *Multistage Stochastic Optimization*, Springer Series in Operations Research and Financial Engineering, Springer, 2014.

[38] W. B. POWELL, *Approximate Dynamic Programming*, Wiley, 2011.

[39] R. T. ROCKAFELLAR, *Convex Analysis*, Princeton Landmarks in Mathematics and Physics, 1996.

[40] A. RUSZCZYŃSKI, *Risk-Averse Dynamic Programming for Markov Decision Processes*, Mathematical Programming, Vol. 125, No. 2, (235-261), 2010.

[41] F. RIEDEL, *Dynamic Coherent Risk Measures*, Stochastic Processes and their Applications, Vol. 112, No. 2, (185-200), 2004.

[42] W. RÖMISCH, *Scenario Reduction Techniques in Stochastic Programming*, in Stochastic Algorithms: Foundations and Applica- tions, Lecture Notes in Computer Science, Eds. O. Watanabe and T. Zeugmann, Vol. 5792, (1-14) Springer, 2009.

[43] L. A. ROSSMAN, *Reliability-Constrained Dynamic Programing and Randomized Release Rules in Reservoir Management*, Water Resources Research, Vol. 13, No. 2, (247-255), 1977.

[44] M. SNIEDOVICH, *Reliability-Constrained Reservoir Control Problems: 1. Methodological Issues*, Water Resources Research, Vol. 15, No. 6, (1574-1582), 1979.

[45] S. YAKOWITZ, *Dynamic Programming Applications in Water Resources*, Water Resources Research, Vol. 18, No. 4, (673-696), 1982.

[46] H. YAMIN, *Review on Methods of Generation Scheduling in Electric Power Systems*, Electric Power Systems Research, Vol. 69, No. 2-3, (227-248), 2004.

[47] G. YOUNG, *Finding Reservoir Operating Rules*, Journal of the Hydraulic Division, American Society of Civil Engineering, Vol. 93, No. 6, (297-321), 1967.

Date\Hours	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	Tot
01.09.2015	14'300	15'310	16'050	15'455	15'745	16'955	22'680	27'750	29'000	29'735	29'325	29'375	25'490	24'755	28'795	32'710	28'645	28'080	21'140	21'120	22'740	17'600	17'890	16'775	547'420
02.09.2015	15'095	15'960	15'085	11'085	12'105	14'080	18'980	19'545	19'565	19'335	19'645	20'090	20'005	19'585	20'025	20'870	21'305	23'805	22'920	23'615	23'140	22'655	19'135	18'315	455'945
03.09.2015	14'630	15'410	14'450	12'690	14'465	14'775	18'460	25'560	25'370	25'375	25'580	25'625	25'925	24'840	24'640	24'830	24'010	22'750	21'000	23'765	23'650	22'330	20'710	17'905	508'745
04.09.2015	17'520	15'500	13'410	12'090	12'315	14'000	18'615	21'175	22'485	22'675	20'720	20'740	20'145	19'470	18'310	17'055	17'370	18'320	17'760	18'295	18'610	17'595	17'810	17'335	429'320
05.09.2015	14'680	17'500	18'500	11'180	9'000	10'000	12'245	17'000	18'610	21'000	19'000	16'500	18'460	18'875	17'120	16'655	17'780	16'375	13'600	13'540	13'500	14'605	15'850	15'255	376'830
06.09.2015	14'000	5'880	10'405	1'045	280	1'155	1'105	6'875	8'535	7'930	4'000	8'750	7'725	8'405	4'010	4'545	1'680	7'310	13'230	15'685	17'775	15'540	19'380	19'025	204'270
07.09.2015	11'000	10'615	10'585	9'230	7'445	9'195	21'885	22'930	24'355	23'670	22'840	23'150	22'105	21'675	21'020	18'425	17'495	18'450	20'355	21'590	22'810	21'540	18'655	17'580	438'600
08.09.2015	18'505	16'500	14'320	12'255	13'150	15'750	21'015	22'465	23'660	21'430	19'715	20'260	20'590	19'535	19'830	18'480	18'555	19'750	19'905	20'750	20'805	19'605	19'220	16'060	452'110
09.09.2015	18'700	13'820	13'395	13'205	13'430	14'290	21'225	20'600	23'770	22'885	20'650	20'005	20'865	19'155	19'065	16'790	17'095	19'230	20'140	19'565	19'740	17'540	15'825	15'735	436'720
10.09.2015	17'990	13'755	14'935	14'270	12'545	19'905	18'985	19'090	20'915	20'165	19'505	19'430	19'660	21'415	21'910	20'985	21'825	19'815	21'480	20'540	21'765	20'295	22'365	17'090	460'635
11.09.2015	18'570	14'545	19'900	19'050	17'895	12'525	22'250	22'390	22'640	23'110	21'960	22'495	22'805	22'950	22'900	22'395	22'370	22'875	23'245	24'450	24'900	23'265	23'460	23'250	516'195
12.09.2015	17'415	16'265	15'900	14'450	14'505	13'340	14'440	16'935	22'140	21'825	20'275	19'980	20'245	15'520	16'690	13'690	17'440	19'735	21'180	21'520	22'045	18'885	19'475	17'200	431'095
13.09.2015	14'185	14'170	12'825	12'335	11'925	11'950	13'995	13'320	12'610	12'495	11'845	12'030	11'480	10'165	9'810	9'605	9'950	11'665	14'565	18'025	22'655	20'780	25'360	20'435	338'180
14.09.2015	16'115	12'340	11'980	11'790	11'945	12'880	21'115	24'940	25'880	24'185	24'580	24'810	23'985	21'055	18'585	18'020	16'995	17'325	18'585	19'755	20'570	16'935	15'610	14'765	444'745
15.09.2015	9'440	8'465	7'745	7'545	7'300	9'985	16'045	22'375	24'205	27'460	26'185	24'280	20'310	21'035	19'210	17'710	18'960	18'695	20'430	22'320	22'185	18'850	21'590	16'000	428'325
16.09.2015	11'975	12'645	11'185	10'875	7'945	12'330	18'630	19'150	21'620	21'700	22'970	23'755	23'550	25'255	23'500	21'835	22'125	21'670	22'305	23'895	24'080	19'915	18'865	16'450	458'225
17.09.2015	10'000	6'470	7'420	5'525	6'620	8'095	16'620	25'065	26'560	24'880	23'535	24'185	22'270	22'255	21'540	22'275	21'290	21'925	20'110	20'640	16'650	15'805	15'845	11'125	416'705
18.09.2015	8'260	6'335	6'675	7'960	8'690	10'285	17'385	18'525	19'125	17'450	17'675	16'095	17'170	14'920	14'485	15'320	17'060	16'615	18'110	20'835	20'735	17'645	14'280	13'095	354'730
19.09.2015	14'175	10'860	11'085	9'660	8'780	9'895	11'740	12'650	14'045	16'575	13'605	14'530	14'545	14'560	12'040	11'195	12'165	12'530	16'125	18'625	18'615	13'880	15'380	11'895	319'155
20.09.2015	10'395	8'275	7'850	7'660	7'220	7'110	7'335	7'650	9'195	10'190	9'805	11'190	8'575	5'820	7'255	7'300	7'535	9'120	13'900	18'570	24'570	19'230	20'870	18'185	264'805
21.09.2015	15'725	15'760	13'660	12'710	12'115	14'130	20'950	27'370	27'045	24'820	19'295	18'170	18'850	20'605	18'785	18'125	16'520	19'530	21'185	23'020	22'145	19'175	19'985	20'795	460'470
22.09.2015	16'410	18'985	19'175	11'220	12'500	12'710	20'530	21'665	21'395	22'525	20'955	22'250	21'960	21'625	22'270	22'150	22'660	22'125	22'740	24'395	24'695	23'300	20'500	22'160	490'900
23.09.2015	18'080	17'415	17'125	14'530	15'695	17'505	21'125	21'115	27'585	25'570	25'355	26'665	25'305	22'930	21'995	21'510	22'455	21'850	22'460	23'690	22'380	19'270	20'325	18'000	509'935
24.09.2015	19'645	17'005	14'000	14'080	13'060	13'620	16'235	19'320	24'285	19'520	23'555	22'855	23'360	23'055	22'700	23'330	23'920	23'585	20'585	23'565	20'655	23'465	19'800	21'015	486'215
25.09.2015	20'275	14'500	16'315	12'790	13'255	19'450	25'500	25'230	26'465	24'670	22'510	22'620	23'695	24'100	24'570	23'925	23'775	23'495	23'455	25'010	23'285	23'510	27'250	26'100	535'750
26.09.2015	18'005	18'045	18'120	18'900	22'180	20'330	21'445	18'025	20'710	19'060	19'790	19'195	17'370	17'210	17'705	20'200	18'250	17'400	20'335	20'790	23'130	23'910	20'605	20'320	471'030
27.09.2015	20'905	19'400	17'715	19'695	22'420	14'250	17'830	19'485	19'185	16'910	16'210	17'950	18'020	19'175	12'875	12'835	12'835	16'160	21'325	23'105	24'055	22'155	22'290	20'895	447'680
28.09.2015	21'115	16'955	17'385	15'785	15'245	18'650	24'365	24'935	25'045	24'470	25'255	24'180	22'225	21'685	21'125	20'300	20'925	21'725	23'360	26'905	23'550	20'785	20'425	19'985	516'380
29.09.2015	20'500	18'500	18'500	14'180	13'920	25'545	27'060	25'845	27'130	29'380	24'110	25'305	23'940	23'035	22'090	23'775	22'415	19'900	23'260	27'320	21'000	21'955	19'115	18'000	535'780
30.09.2015	17'935	14'530	13'075	12'665	13'495	13'435	20'855	27'165	31'375	24'780	22'160	21'865	21'865	21'125	20'975	19'005	18'805	21'880	24'835	27'980	23'940	22'575	22'200	22'640	501'160

Table 10 Back Test Optimal strategy, Driftless GBM dynamics, initial basin level 80 GWh

Symbol	Description	Value
t_0	Initial time day ahead	0
T	Final time day ahead	6
k	Number of branches out of a leaf in the lattice	4

Table 11 Lattice Parametrization

X[{0,1}]→{52.86}

X[{1,1}]→{175.96}
X[{1,2}]→{4.90}
X[{1,3}]→{0.33}
X[{1,4}]→{299.79}

X[{2,1}]→{190.65}
X[{2,2}]→{77.18}
X[{2,3}]→{43.14}
X[{2,4}]→{135.00}
X[{2,5}]→{111.73}
X[{2,6}]→{134.64}
X[{2,7}]→{310.46}

X[{3,1}]→{206.57}
X[{3,2}]→{98.15}
X[{3,3}]→{56.48}
X[{3,4}]→{113.89}
X[{3,5}]→{106.13}
X[{3,6}]→{95.25}
X[{3,7}]→{150.67}
X[{3,8}]→{163.00}
X[{3,9}]→{130.22}
X[{3,10}]→{275.24}

X[{4,1}]→{223.81}
X[{4,2}]→{114.61}
X[{4,3}]→{68.13}
X[{4,4}]→{115.39}
X[{4,5}]→{105.30}
X[{4,6}]→{85.570}
X[{4,7}]→{120.63}
X[{4,8}]→{135.20}
X[{4,9}]→{89.83}
X[{4,10}]→{144.28}
X[{4,11}]→{200.24}
X[{4,12}]→{207.17}
X[{4,13}]→{319.71}

X[{5,1}]→{242.49}
X[{5,2}]→{128.88}
X[{5,3}]→{78.26}
X[{5,4}]→{125.62}
X[{5,5}]→{111.27}
X[{5,6}]→{85.50}
X[{5,7}]→{110.52}
X[{5,8}]→{119.37}
X[{5,9}]→{72.64}
X[{5,10}]→{108.59}
X[{5,11}]→{162.42}
X[{5,12}]→{174.17}
X[{5,13}]→{222.08}
X[{5,14}]→{135.06}
X[{5,15}]→{194.19}
X[{5,16}]→{305.00}

X[{6,1}]→{262.74}
X[{6,2}]→{142.31}
X[{6,3}]→{87.51}
X[{6,4}]→{138.93}
X[{6,5}]→{121.73}
X[{6,6}]→{91.18}
X[{6,7}]→{112.08}
X[{6,8}]→{114.02}
X[{6,9}]→{65.37}
X[{6,10}]→{91.12}
X[{6,11}]→{135.72}
X[{6,12}]→{150.20}
X[{6,13}]→{193.77}
X[{6,14}]→{106.84}
X[{6,15}]→{145.20}
X[{6,16}]→{189.54}
X[{6,17}]→{150.95}
X[{6,18}]→{340.05}
X[{6,19}]→{283.07}

Fig. 4 Intraday Prices, Driftless GBM dynamics u[{0,1}]→{237.77}

u[{1,1}]→{33.21}
u[{1,2}]→{467.64}
u[{1,3}]→{362.46}
u[{1,4}]→{153.93}

u[{2,1}]→{23.08}
u[{2,2}]→{23.08}
u[{2,3}]→{499.99}
u[{2,4}]→{499.99}
u[{2,5}]→{499.99}
u[{2,6}]→{23.08}
u[{2,7}]→{23.08}

u[{3,1}]→{23.08}
u[{3,2}]→{23.08}
u[{3,3}]→{207.20}
u[{3,4}]→{499.99}
u[{3,5}]→{499.99}
u[{3,6}]→{499.99}
u[{3,7}]→{499.99}
u[{3,8}]→{145.59}
u[{3,9}]→{23.08}
u[{3,10}]→{23.08}

u[{4,1}]→{499.99}
u[{4,2}]→{499.99}
u[{4,3}]→{441.93}
u[{4,4}]→{483.79}
u[{4,5}]→{358.32}
u[{4,6}]→{181.38}
u[{4,7}]→{96.02}
u[{4,8}]→{109.22}
u[{4,9}]→{347.36}
u[{4,10}]→{499.99}
u[{4,11}]→{393.66}
u[{4,12}]→{499.99}
u[{4,13}]→{499.99}

u[{5,1}]→{499.99}
u[{5,2}]→{499.99}
u[{5,3}]→{499.99}
u[{5,4}]→{330.32}
u[{5,5}]→{201.56}
u[{5,6}]→{98.31}
u[{5,7}]→{23.08}
u[{5,8}]→{23.08}
u[{5,9}]→{23.08}
u[{5,10}]→{29.84}
u[{5,11}]→{110.01}
u[{5,12}]→{182.96}
u[{5,13}]→{294.06}
u[{5,14}]→{467.57}
u[{5,15}]→{499.99}
u[{5,16}]→{499.99}

Fig. 5 Optimal Turbined Water, Driftless GBM dynamics, initial basin level 41.5 MWh

V[{0,1}]→{41500.00}	V[{1,1}]→{41262.22}	V[{2,1}]→{41229.01}	V[{3,1}]→{41205.93}	V[{4,1}]→{41182.84}	V[{5,1}]→{40682.84}
	V[{1,2}]→{41262.22}	V[{2,2}]→{41011.79}	V[{3,2}]→{41061.12}	V[{4,2}]→{41086.30}	V[{5,2}]→{40618.48}
	V[{1,3}]→{41262.22}	V[{2,3}]→{40974.45}	V[{3,3}]→{40767.78}	V[{4,3}]→{40823.44}	V[{5,3}]→{40500.00}
	V[{1,4}]→{41262.22}	V[{2,4}]→{41007.91}	V[{3,4}]→{40663.83}	V[{4,4}]→{40559.60}	V[{5,4}]→{40330.32}
		V[{2,5}]→{40934.21}	V[{3,5}]→{40561.25}	V[{4,5}]→{40358.32}	V[{5,5}]→{40201.56}
		V[{2,6}]→{41004.03}	V[{3,6}]→{40559.96}	V[{4,6}]→{40188.13}	V[{5,6}]→{40098.31}
		V[{2,7}]→{41108.29}	V[{3,7}]→{40638.14}	V[{4,7}]→{40105.80}	V[{5,7}]→{40023.08}
			V[{3,8}]→{40724.95}	V[{4,8}]→{40185.03}	V[{5,8}]→{40023.08}
			V[{3,9}]→{41015.70}	V[{4,9}]→{40347.36}	V[{5,9}]→{40023.08}
			V[{3,10}]→{41085.20}	V[{4,10}]→{40533.80}	V[{5,10}]→{40029.84}
				V[{4,11}]→{40797.57}	V[{5,11}]→{40110.01}
				V[{4,12}]→{41015.78}	V[{5,12}]→{40182.96}
				V[{4,13}]→{41062.12}	V[{5,13}]→{40294.06}
					V[{5,14}]→{40467.57}
					V[{5,15}]→{40531.23}
					V[{5,16}]→{40562.12}

Fig. 6 Basin Level, Driftless GBM dynamics, initial basin level 41.5 MWh

X[{0,1}]→{33.50}

X[{1,1}]→{30.10} X[{1,2}]→{25.18} X[{1,3}]→{21.51} X[{1,4}]→{30.83}

X[{2,1}]→{28.01} X[{2,2}]→{23.09} X[{2,3}]→{19.42} X[{2,4}]→{28.74} X[{2,5}]→{28.30} X[{2,6}]→{24.12} X[{2,7}]→{27.08}

X[{3,1}]→{25.21} X[{3,2}]→{20.29} X[{3,3}]→{16.62} X[{3,4}]→{25.94} X[{3,5}]→{25.50} X[{3,6}]→{21.32} X[{3,7}]→{24.28} X[{3,8}]→{24.93} X[{3,9}]→{15.41} X[{3,10}]→{21.07}

X[{4,1}]→{25.01} X[{4,2}]→{20.09} X[{4,3}]→{16.42} X[{4,4}]→{25.74} X[{4,5}]→{25.30} X[{4,6}]→{21.12} X[{4,7}]→{24.08} X[{4,8}]→{24.73} X[{4,9}]→{15.21} X[{4,10}]→{20.87} X[{4,11}]→{26.64} X[{4,12}]→{26.42} X[{4,13}]→{26.44}

X[{5,1}]→{28.39} X[{5,2}]→{23.47} X[{5,3}]→{19.80} X[{5,4}]→{29.12} X[{5,5}]→{28.68} X[{5,6}]→{24.50} X[{5,7}]→{27.46} X[{5,8}]→{28.11} X[{5,9}]→{18.59} X[{5,10}]→{24.25} X[{5,11}]→{30.02} X[{5,12}]→{29.80} X[{5,13}]→{29.82} X[{5,14}]→{16.44} X[{5,15}]→{21.97} X[{5,16}]→{25.76}

X[{6,1}]→{36.62} X[{6,2}]→{31.70} X[{6,3}]→{28.03} X[{6,4}]→{37.35} X[{6,5}]→{36.91} X[{6,6}]→{32.73} X[{6,7}]→{35.69} X[{6,8}]→{36.34} X[{6,9}]→{26.82} X[{6,10}]→{32.48} X[{6,11}]→{38.25} X[{6,12}]→{38.03} X[{6,13}]→{38.05} X[{6,14}]→{24.67} X[{6,15}]→{30.20} X[{6,16}]→{33.99} X[{6,17}]→{30.98} X[{6,18}]→{42.93} X[{6,19}]→{33.43}

Fig. 7 Intraday Prices, Spread To Spot dynamics u[{0,1}]→{304.89}

u[{1,1}]→{23.08}
u[{1,2}]→{499.99}
u[{1,3}]→{499.99}
u[{1,4}]→{87.88}

u[{2,1}]→{23.08}
u[{2,2}]→{23.08}
u[{2,3}]→{499.99}
u[{2,4}]→{499.99}
u[{2,5}]→{499.99}
u[{2,6}]→{23.08}
u[{2,7}]→{23.08}

u[{3,1}]→{23.08}
u[{3,2}]→{23.08}
u[{3,3}]→{23.08}
u[{3,4}]→{289.80}
u[{3,5}]→{462.50}
u[{3,6}]→{457.10}
u[{3,7}]→{46.26}
u[{3,8}]→{23.08}
u[{3,9}]→{23.08}
u[{3,10}]→{23.08}

u[{4,1}]→{499.99}
u[{4,2}]→{499.99}
u[{4,3}]→{389.60}
u[{4,4}]→{112.78}
u[{4,5}]→{237.36}
u[{4,6}]→{139.11}
u[{4,7}]→{112.74}
u[{4,8}]→{79.13}
u[{4,9}]→{216.19}
u[{4,10}]→{177.58}
u[{4,11}]→{293.00}
u[{4,12}]→{499.99}
u[{4,13}]→{499.99}

u[{5,1}]→{499.99}
u[{5,2}]→{499.99}
u[{5,3}]→{499.99}
u[{5,4}]→{499.99}
u[{5,5}]→{381.82}
u[{5,6}]→{279.04}
u[{5,7}]→{199.07}
u[{5,8}]→{118.74}
u[{5,9}]→{137.74}
u[{5,10}]→{246.38}
u[{5,11}]→{340.55}
u[{5,12}]→{410.19}
u[{5,13}]→{499.99}
u[{5,14}]→{499.99}
u[{5,15}]→{499.99}
u[{5,16}]→{499.99}

Fig. 8 Optimal Turbined Water, Spread To Spot dynamics, initial basin level 41.5 MWh

V[{0,1}]→{41500.00}

V[{1,1}]→{41195.10}
V[{1,2}]→{41195.10}
V[{1,3}]→{41195.10}
V[{1,4}]→{41195.10}

V[{2,1}]→{41172.02}
V[{2,2}]→{40933.56}
V[{2,3}]→{40854.07}
V[{2,4}]→{40917.36}
V[{2,5}]→{40832.47}
V[{2,6}]→{40901.16}
V[{2,7}]→{41107.22}

V[{3,1}]→{41148.93}
V[{3,2}]→{40989.96}
V[{3,3}]→{40672.02}
V[{3,4}]→{40570.15}
V[{3,5}]→{40462.50}
V[{3,6}]→{40457.10}
V[{3,7}]→{40550.71}
V[{3,8}]→{40639.62}
V[{3,9}]→{40946.76}
V[{3,10}]→{41084.13}

V[{4,1}]→{41125.85}
V[{4,2}]→{41019.87}
V[{4,3}]→{40834.40}
V[{4,4}]→{40612.78}
V[{4,5}]→{40384.76}
V[{4,6}]→{40204.54}
V[{4,7}]→{40196.20}
V[{4,8}]→{40257.82}
V[{4,9}]→{40439.59}
V[{4,10}]→{40677.58}
V[{4,11}]→{40793.00}
V[{4,12}]→{40969.47}
V[{4,13}]→{41061.05}

V[{5,1}]→{40625.85}
V[{5,2}]→{40555.19}
V[{5,3}]→{40500.00}
V[{5,4}]→{40500.00}
V[{5,5}]→{40381.82}
V[{5,6}]→{40279.04}
V[{5,7}]→{40199.07}
V[{5,8}]→{40118.74}
V[{5,9}]→{40137.74}
V[{5,10}]→{40246.38}
V[{5,11}]→{40340.55}
V[{5,12}]→{40410.19}
V[{5,13}]→{40500.00}
V[{5,14}]→{40500.00}
V[{5,15}]→{40500.00}
V[{5,16}]→{40561.05}

Fig. 9 Basin Level, Spread To Spot dynamics, initial basin level 41.5 MWh