A factor of integer polynomials with minimal integrals

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A FACTOR OF INTEGER POLYNOMIALS WITH MINIMAL INTEGRALS

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ABSTRACT. For each positive integer N, let S_N be the set of all polynomials $P(x) \in \mathbb{Z}[x]$ with degree less than N and minimal positive integral over $[0, 1]$. These polynomials are related to the distribution of prime numbers since $\int_0^1 P(x) \mathrm{d}x = \exp(-\psi(N))$, where ψ is the second Chebyshev function. We prove that for any positive integer N there exists $P(x) \in S_N$ such that $(x(1-x))^{\lfloor N/3 \rfloor}$ divides $P(x)$ in $\mathbb{Z}[x]$. In fact, we show that the exponent $\lfloor N/3 \rfloor$ cannot be improved. This result is analog to a previous of Aparicio concerning polynomials in $\mathbb{Z}[x]$ with minimal positive L^∞ norm on $[0, 1]$. Also, it is in some way a strengthening of a result of Bazzanella, who considered $x^{\lfloor N/2 \rfloor}$ and $(1-x)^{\lfloor N/2 \rfloor}$ instead of $(x(1-x))^{\lfloor N/3 \rfloor}$.

1. Introduction

It is well-known that the celebrated Prime Number Theorem is equivalent to the assertion:

$$\psi(x) \sim x, \text{ as } x \to +\infty.$$

Here $\psi(x)$ is the second Chebyshev function, defined for $x \geq 0$ as

$$\psi(x) := \sum_{p^m \,\leq\, x} \log p,$$

where the sum is extended over all the prime numbers p and all the positive integers m such that $p^m \leq x$.

In 1936, Gelfond and Shnirelman proposed an elementary and clever method to obtain lower bounds for $\psi(x)$ (see Gelfond's comments in [5, pp. 285–288]). In 1982, the same method was rediscovered and developed by Nair [9, 10].

The main idea of the Gelfond–Shnirelman–Nair method is the following: Given a positive integer N, let $P_N(x)$ be a polynomial with integer coefficients and degree less than N, say

$$P_N(x) = \sum_{n\,=\,0}^{N-1} a_n x^n,$$

with $a_0, \ldots, a_{N-1} \in \mathbb{Z}$. Now consider the integral of $P_N(x)$ over $[0, 1]$, that is

$$I(P_N) := \int_0^1 P_N(x) \mathrm{d}x = \sum_{n\,=\,0}^{N-1} \frac{a_n}{n+1}.$$

Clearly, $I(P_N)$ is a rational number whose denominator divides

$$d_N := \operatorname{lcm}\{1, 2, \ldots, N\},$$

hence $d_N |I(P_N)|$ is an integer. In particular, if we suppose $I(P_N) \neq 0$, then $d_N|I(P_N)| \geq 1$. Now $d_N = \exp(\psi(N))$, so we get

$$\psi(N) \geq \log\left(\frac{1}{|I(P_N)|}\right). \tag{1.1}$$

Finally, from the trivial upper bound

$$|I(P_N)| = \left|\int_0^1 P_N(x) \mathrm{d}x\right| \leq \int_0^1 |P_N(x)| \,\mathrm{d}x \leq \max_{x \,\in\, [0,1]} |P_N(x)| =: \|P_N\|,$$

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we obtain

$$\psi(N) \geq \log\left(\frac{1}{\|P_N\|}\right). \tag{1.2}$$

At this point, if we choose P_N to have a sufficiently small norm $\|P_N\|$, then a lower bound for $\psi(x)$ follows from (1.2). For example, the choice

$$P_N(x) = (x(1-x))^{2\lfloor (N-1)/2 \rfloor}$$

gives the lower bound

$$\psi(N) \geq \log 2 \cdot (N-2) > 0.694 \cdot (N-2).$$

This motivates the study of the quantities

$$\ell_N := \min\{\|P\| : P(x) \in \mathbb{Z}[x],\ \deg(P) < N,\ \|P\| > 0\},$$

$$C_N := \frac{1}{N} \log\left(\frac{1}{\ell_N}\right),$$

and the set of polynomials

$$T_N := \{P(x) \in \mathbb{Z}[x] : \deg(P) < N,\ \|P\| = \ell_N\};$$

the so-called Integer Chebyshev Problem [4].

In particular, Aparicio [1] proved the following theorem about the structure of polynomials in T_N.

Theorem 1.1. *Given any sufficiently large positive integer N, for all $P \in T_N$ it holds*

$$(x(1-x))^{\lfloor \lambda_1 N \rfloor}(2x-1)^{\lfloor \lambda_2 N \rfloor}(5x^2-5x+1)^{\lfloor \lambda_3 N \rfloor} \mid P(x)$$

in $\mathbb{Z}[x]$, where

$$\lambda_1 \in [0.1456, 0.1495],\ \lambda_2 \in [0.0166, 0.0187],\ \lambda_3 \in [0.0037, 0.0053]$$

are some constants.

It is known that C_N converges to a limit C, as $N \to +\infty$ (see [8, Chapter 10]). Furthermore, Pritsker [11, Theorem 3.1] showed that

$$C \in \left]0.85991, 0.86441\right[,$$

and this is the best estimate of C known to date.

As a consequence of Pritsker's result, the Gelfond–Shnirelman–Nair method cannot lead to a lower bound better than

$$\psi(x) \geq 0.86441 \cdot x,$$

which is quite far from what is expected by the Prime Number Theorem.

To deal with this problem, Bazzanella [2, 3] suggested to study the polynomials P_N such that $|I(P_N)|$ is nonzero and minimal, or, without loss of generality, such that $I(P_N)$ is positive and minimal.

We recall the following elementary lemma about the existence of solutions of some linear diophantine equations.

Lemma 1.2. *Fix some integers $c_1, \dots, c_k$. Then the diophantine equation*

$$\sum_{i=1}^{k} c_i x_i = 1$$

has a solution $x_1, \dots, x_k \in \mathbb{Z}$ if and only if $\gcd\{c_1, \dots, c_k\} = 1$. Moreover, if a solution exists, then there exist infinitely many solutions.

On the one hand, because of the above considerations, we known that if $I(P_N) > 0$ then $I(P_N) \geq 1/d_N$. On the other hand, $I(P_N) = 1/d_N$ if and only if

$$\sum_{n=0}^{N-1} \frac{d_N}{n+1} \cdot a_n = 1,$$

and it is easy to see that each of the coefficients $d_N/(n+1)$ is an integer and

$$\gcd\left\{\frac{d_N}{n+1} : n = 0, \dots, N-1\right\} = 1.$$

Hence, by Lemma 1.2, there exist infinitely many polynomials P_N such that $I(P_N) = 1/d_N$, so that (1.1) holds with the equality.

This leads to define the following set of polynomials

$$S_N := \{P(x) \in \mathbb{Z}[x] : \deg(P) < N,\ I(P) = 1/d_N\}.$$

Bazzanella proved some results about the roots of the polynomials in S_N. In particular, regarding the multiplicity of the roots $x = 0$ and $x = 1$, he gave the following theorem [2, Theorem 1], which is vaguely similar to Theorem 1.1.

Theorem 1.3. *For each positive integer N, there exists $P(x) \in S_N$ such that*

$$x^{\lfloor N/2 \rfloor} \mid P(x)$$

in $\mathbb{Z}[x]$. Moreover, the exponent $\lfloor N/2 \rfloor$ cannot be improved, i.e., there exist infinitely many positive integers N such that

$$x^{\lfloor N/2 \rfloor + 1} \nmid P(x)$$

for all $P(x) \in S_N$. The same results hold if the polynomial $x^{\lfloor N/2 \rfloor}$ is replaced by $(1-x)^{\lfloor N/2 \rfloor}$.

Actually, what Bazzanella proved is that the maximum nonnegative integer $K(N)$ such that there exists a polynomial $P(x) \in S_N$ divisible by $x^{K(N)}$, respectively by $(1-x)^{K(N)}$, is given by

$$K(N) = \min\{p^m - 1 : p \text{ prime},\ m \geq 1,\ p^m > N/2\},$$

so that Theorem 1.3 follows quickly.

Despite the similarity between Theorems 1.1 and 1.3, note that the statement of Theorem 1.1 holds "for all $P(x) \in T_N$", while Theorem 1.3 only says that "there exists $P(x) \in S_N$". However, this distinction is unavoidable, indeed: On the one hand, T_N is a finite set, even conjectured to be a singleton for any sufficiently large N [4, Sec. 5, Q2]. On the other hand, S_N is an infinite set and if $P(x) \in S_N$ then $(d_N + 1)P(x) - 1 \in S_N$, hence the elements of S_N have no common nontrivial factor in $\mathbb{Z}[x]$.

The purpose of this paper is to move another step further in the direction of a stronger analog of Theorem 1.1 for the set of polynomials S_N. For we prove the following theorem.

Theorem 1.4. *For each positive integer N, there exist infinitely many $P(x) \in S_N$ such that*

$$(x(1-x))^{\lfloor N/3 \rfloor} \mid P(x)$$

in $\mathbb{Z}[x]$. Moreover, the exponent $\lfloor N/3 \rfloor$ cannot be improved, i.e., there exist infinitely many positive integers N such that

$$(x(1-x))^{\lfloor N/3 \rfloor + 1} \nmid P(x),$$

for all $P(x) \in S_N$.

We leave the following informal question to the interested readers:

Question. Let $\{Q_N(x)\}_{N \geq 1}$ be a sequence of "explicit" integer polynomials such that for each positive integer N it holds $Q_N(x) \mid P(x)$ in $\mathbb{Z}[x]$, for some $P(x) \in S_N$. In light of Theorems 1.3 and 1.4, three examples of such sequences are given by $\{x^{\lfloor N/2 \rfloor}\}_{N \geq 1}$, $\{(1-x)^{\lfloor N/2 \rfloor}\}_{N \geq 1}$, and $\{(x(1-x))^{\lfloor N/3 \rfloor}\}_{N \geq 1}$.

How big can be

$$\delta := \liminf_{N \to +\infty} \frac{\deg(Q_N)}{N}\ ?$$

Can δ be arbitrary close to 1, or even equal to 1?

Note that the sequences of Theorem 1.3 give $\delta = 1/2$, while the sequence of Theorem 1.4 gives $\delta = 2/3$.

2. Preliminaries

In this section, we collect a number of preliminary results needed to prove Theorem 1.4. The first is a classic theorem of Kummer [7] concerning the p-adic valuation of binomial coefficients.

Theorem 2.1. *For all integers $u, v \geq 0$ and any prime number p, the p-adic valuation of the binomial coefficient $\binom{u+v}{v}$ is equal to the number of carries that occur when u and v are added in the base p.*

Now we can prove the following lemma.

Lemma 2.2. *For any positive integer N, and for all integers $u, v \geq 0$ with $u + v < N$, we have that*

$$\frac{d_N}{(u+v+1)\binom{u+v}{u}} \tag{2.1}$$

is an integer.

Proof. We have to prove that for any prime number $p \leq N$ the p-adic valuation of the denominator of (2.1) does not exceed $\nu_p(d_N) = \lfloor \log_p N \rfloor$. Write $u + v + 1$ in base p, that is

$$u + v + 1 = \sum_{i=i_0}^{s} d_i p^i,$$

where $i_0 := \nu_p(u + v + 1)$ and $d_{i_0}, \dots, d_s \in \{0, \dots, p-1\}$, with $d_{i_0}, d_s > 0$. Hence, the expansion of $u + v$ in base p is

$$u + v = \sum_{i=i_0+1}^{s} d_i p^i + (d_{i_0} - 1)p^{i_0} + \sum_{i=0}^{i_0-1} (p-1)p^i. \tag{2.2}$$

In particular, by (2.2), we have that $u + v$ written in base p has exactly $s + 1$ digits, of which the i_0 least significant are all equal to $p - 1$. Therefore, in the sum of u and v in base p there occur at most $s - i_0$ carries. Since, thanks to Theorem 2.1, we know that $i_1 := \nu_p\big(\binom{u+v}{v}\big)$ is equal to the number of carries occurring in the sum of u and v in base p, it follows that $i_1 \leq s - i_0$.

In conclusion,

$$\nu_p\left((u+v+1)\binom{u+v}{v}\right) = i_0 + i_1 \leq s \leq \lfloor \log_p N \rfloor,$$

where the last inequality holds since $u + v + 1 \leq N$. ☐

We recall the value of a well-known integral (see, e.g., [6, Sec. 11.1.7.1, Eq. 2]).

Lemma 2.3. *For all integers $u, v \geq 0$, it holds*

$$\int_0^1 x^u (1-x)^v \mathrm{d}x = \frac{1}{(u+v+1)\binom{u+v}{v}}.$$

We conclude this section with a lemma that will be fundamental in the proof of Theorem 1.4.

Lemma 2.4. *Let N and m be integers such that $N \geq 1$ and $0 \leq m \leq (N-1)/2$. The following statements are equivalent:*

(i) There exist infinitely many $P(x) \in S_N$ such that $(x(1-x))^m \mid P(x)$ in $\mathbb{Z}[x]$.
(ii) For each prime number $p \leq N$, there exists an integer h_p such that $h_p \in [m, N-m-1]$ and

$$\nu_p\left((h_p+m+1)\binom{h_p+m}{m}\right) = \lfloor \log_p N \rfloor.$$

Proof. Let $P(x) \in \mathbb{Z}[x]$ be such that $\deg(P) < N$ and

$$(x(1-x))^m \mid P(x)$$

in $\mathbb{Z}[x]$. Hence,

$$P(x) = (x(1-x))^m \sum_{h=m}^{N-m-1} b_h x^{h-m},$$

for some $b_m, \dots, b_{N-m-1} \in \mathbb{Z}$. Then, by Lemma 2.3, it follows that

$$I(P) = \sum_{h=m}^{N-m-1} b_h \int_0^1 x^h (1-x)^m \mathrm{d}x = \sum_{h=m}^{N-m-1} \frac{b_h}{(h+m+1)\binom{h+m}{m}}.$$

Now we have $P(x) \in S_N$ if and only if $I(P) = 1/d_N$, i.e., if and only if

$$\sum_{h=m}^{N-m-1} \frac{d_N}{(h+m+1)\binom{h+m}{m}} \cdot b_h = 1.$$

Therefore, thanks to Lemma 2.2 and Lemma 1.2, we get infinitely many $P(x) \in S_N$ if and only if

$$\gcd\left\{\frac{d_N}{(h+m+1)\binom{h+m}{m}} : h = m, \dots, N-m-1\right\} = 1.$$

At this point, recalling that $\nu_p(d_N) = \lfloor \log_p N \rfloor$ for each prime number p, the equivalence of (i) and (ii) follows easily. □

3. Proof of Theorem 1.4

We are ready to prove Theorem 1.4. Put $m := \lfloor N/3 \rfloor$, $s := \lfloor \log_p N \rfloor$, and pick a prime number $p \leq N$. In light of Lemma 2.4, in order to prove the first part of Theorem 1.4 we have to show the existence of an integer $h_p \in [m, N - m - 1]$ such that

$$\nu_p\left((h_p + m + 1)\binom{h_p + m}{m}\right) = s. \tag{3.1}$$

Let us write $N = \ell p^s + r$, for some $\ell \in \{1, \ldots, p - 1\}$ and $r \in \{0, \ldots, p^s - 1\}$. We split the proof in three cases:

Case $\ell \geq 2$. It is enough to take $h_p := \ell p^s - m - 1$. In fact, on the one hand, it is straightforward that (3.1) holds. On the other hand, since $\ell \geq 2$, we have

$$h_p = \ell p^s - m - 1 \geq \frac{2}{3}(\ell + 1)p^s - m - 1 > \frac{2}{3}N - m - 1 \geq m - 1,$$

while clearly $h_p \leq N - m - 1$, hence $h_p \in [m, N - m - 1]$, as desired.

Case $m < p^{s-1}$. It holds

$$\frac{p^s}{3} \leq \frac{N}{3} < m + 1 \leq p^{s-1},$$

hence $p = 2$. Now it is enough to take $h_2 := 2^s - m - 1$. In fact, on the one hand, it is again straightforward that (3.1) holds. On the other hand, since $m < 2^{s-1}$, we have

$$h_2 = 2^s - m - 1 > 2^s - 2^{s-1} - 1 = 2^{s-1} - 1 \geq m,$$

while obviously $h_2 \leq N - m - 1$, hence $h_2 \in [m, N - m - 1]$, as desired.

Case $\ell = 1$ and $m \geq p^{s-1}$. This case requires more effort. We have

$$p^{s-1} \leq m \leq \frac{N}{3} = \frac{p^s + r}{3} < \frac{2p^s}{3} < p^s,$$

hence the expansion of m in base p is

$$m = \sum_{i=0}^{s-1} d_i p^i,$$

for some $d_0, \ldots, d_{s-1} \in \{0, \ldots, p - 1\}$, with $d_{s-1} > 0$.

Let i_1 be the least nonnegative integer not exceeding s such that

$$d_i \geq \frac{p-1}{2}, \quad \forall i \in \mathbb{Z},\ i_1 \leq i < s. \tag{3.2}$$

Moreover, let i_2 be the greatest integer such that $i_1 \leq i_2 \leq s$ and

$$d_i = \frac{p-1}{2}, \quad \forall i \in \mathbb{Z},\ i_1 \leq i < i_2.$$

Note that, by the definitions of i_1 and i_2, we have

$$d_i > \frac{p-1}{2}, \quad \forall i \in \mathbb{Z},\ i_2 \leq i < s. \tag{3.3}$$

Clearly, it holds

$$m = \sum_{i_2 \leq i < s} d_i p^i + \sum_{i_1 \leq i < i_2} \frac{p-1}{2} p^i + \sum_{0 \leq i < i_1} d_i p^i. \tag{3.4}$$

Define now

$$h_p := \sum_{i_2 \leq i < s} d_i p^i + \sum_{i_1 \leq i < i_2} \frac{p-1}{2} p^i + \sum_{0 \leq i < i_1} (p - d_i - 1) p^i. \tag{3.5}$$

Note that (3.5) is actually the expansion of h_p in base p, that is, all the coefficients of the powers p^i belong to the set of digits $\{0, \ldots, p - 1\}$. At this point, looking at (3.4) and (3.5), and taking into account (3.3), it follows easily that in the sum of h_p and m in base p there occur exactly $s - i_2$ carries. Therefore, by Theorem 2.1 we have

$$\nu_p\left(\binom{h_p + m}{m}\right) = s - i_2. \tag{3.6}$$

Furthermore, from (3.4) and (3.5) we get

$$h_p + m + 1 = 2 \sum_{i_2 \leq i < s} d_i p^i + \sum_{0 \leq i < i_2} (p-1)p^i + 1 = 2 \sum_{i_2 \leq i < s} d_i p^i + p^{i_2},$$

hence

$$\nu_p(h_p + m + 1) = i_2. \tag{3.7}$$

Therefore, putting together (3.6) and (3.7) we obtain (3.1).

It remains only to prove that $h_p \in [m, N-m-1]$. If $i_2 = s$, then from (3.7) it follows that

$$h_p + m + 1 = 0 + p^s \leq N,$$

hence $h_p \leq N - m - 1$. If $i_2 < s$, then from (3.2) it follows $d_{i_2} \geq (p-1)/2$, hence $d_{i_2} \geq 1$ and from (3.7) and (3.4) we obtain

$$h_p + m + 1 \leq 2 \sum_{i_2 \leq i < s} d_i p^i + d_{i_2} p^{i_2} \leq 2m + m = 3m \leq N,$$

so that again $h_p \leq N - m - 1$. If $i_1 = 0$, then by (3.4) and (3.5) we have immediately that $h_p = m$. If $i_1 > 0$, then by the definition of i_1, we have $d_{i_1 - 1} < (p-1)/2$, i.e., $d_{i_1-1} < p - d_{i_1-1} - 1$, thus looking at the expansions (3.4) and (3.5) we get that $h_p > m$. Hence, in conclusion we have $h_p \in [m, N-m-1]$, as desired.

Regarding the second part of Theorem 1.4, take $N := 3q$, where $q > 3$ is a prime number. Put $m := \lfloor N/3 \rfloor + 1 = q + 1$, and let $h \in [m, N-m-1]$ be an integer. On the one hand, it is straightforward that $q \nmid h + m + 1$. On the other, it is also easy to see that in the sum of h and m in base q there is no carry, hence, by Theorem 2.1, we have that $q \nmid \binom{h+m}{m}$. Therefore,

$$\nu_q\left((h+m+1)\binom{h+m}{m}\right) = 0 < 1 = \lfloor \log_q N \rfloor,$$

so that, thanks to Lemma 2.4, we have $(x(1-x))^m \nmid P(x)$ in $\mathbb{Z}[x]$, for all $P(x) \in S_N$. This completes the proof.

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