Meta-cognitive unity in indirect proofs.

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META-COGNITIVE UNITY IN INDIRECT PROOFS

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In this paper we focus on the cognitive aspects of indirect argumentation and proving processes. Drawing on the Habermas model of rational behaviour in three components (epistemic, teleological, and communicative) and on the notion of cognitive unity as developed by Boero and his collaborators, we distinguish between two levels of argumentation, a 'ground level' and a 'meta-level'. On the base of a case study in early Calculus context (secondary school), we introduce the notion of 'meta-cognitive unity', which may give reason of success and difficulties in indirect proving processes. Furthermore, we use Peirce's account of abduction to shed light into some cognitive processes behind the production of indirect proofs.

Key-words: indirect proof, meta-cognitive unity, argumentation processes, diagrammatic reasoning, abduction.

INTRODUCTION AND THEORETICAL FRAMEWORK

Proving is one of the main activities in mathematics, and the investigation on proving processes has been one of the major subjects of research in mathematics education for thirty years at least (Balacheff, 1987; Hanna, 1989; Duval, 1991; Mariotti & Antonini, 2008; Boero, Douek, Morselli, & Pedemonte, 2010).

Many of these studies have focused on the nature of the relationships between conjecturing and proving, and have put forward different perspectives. Some of them point out the differences between argumentation and proving processes (Balacheff, 1987; Duval, 1991). According to others, "in order to bring about a smooth approach to theorems in school, it is necessary to consider the connections between conjecturing and proving, in spite of the undeniable differences between the two processes" (Garuti, Boero, Lemut, & Mariotti, 1996, p. 113). The possibility of a cognitive continuity between the phases of conjecture production and proof construction has been analysed with the notion of *cognitive unity* (*ibid.*). Within this perspective, Pedemonte (2007) has further distinguished between

- "the *referential system*, made up of the representation system (the language, heuristics, and drawings) and the knowledge system (conceptions and theorems) of argumentation and proof (Pedemonte, 2005). The analysis of *cognitive unity* takes into account the referential system.

- the *structure* intended to allow logical cognitive connection between statements (deduction, abduction, and induction structures) (Pedemonte, 2007).

There is continuity in the referential system between argumentation and proof if some expressions, drawings, or theorems used in the proof have been used in the argumentation

> supporting the conjecture. There is structural continuity between argumentation and proof when inferences in argumentation and proof are connected through the same structure (abduction, induction, or deduction)."
>
> (Boero, Douek, Morselli, & Pedemonte, 2010, p. 183, emphasis as in the original).

Recently Boero and his collaborators (Boero, Douek, Morselli, & Pedemonte, 2010) have integrated the cognitive unity analysis with Habermas' elaboration of *rational behaviour in discursive practices*. Adapting the three components of rational behaviour according to Habermas (teleologic, epistemic, and communicative) to the discursive practice of proving, they have identified:

> "A) an **epistemic aspect,** consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning (cf. the definition of "theorem" by Mariotti & al. (1997) as the system consisting of a statement, a proof, derived according to shared inference rules from axioms and other theorems, and a reference theory);
>
> B) a **teleological aspect**, inherent in the problem-solving character of proving, and the conscious choices to be made in order to obtain the desired product;
>
> C) a **communicative aspect**, consisting in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning and the conformity of the products (proofs) to standards in a given mathematical culture."
>
> (*ibid*., p. 188)

In this model, the expert's behaviour in proving processes can be described in terms of (more or less) conscious constraints upon the three components of rationality: "constraints of epistemic validity, efficiency related to the goal to achieve, and communication according to shared rules" (*ibid.*, p. 192). As the authors point out, such constraints work at *two levels* of argumentation:

- a level (that we call *ground-level*) inherent in the specific nature of the three components of rational behaviour in proving;
- a *meta-level*, "inherent in the awareness of the constraints on the three components" (*ibid.*, p. 192).

In our research, we focus on the cognitive aspects involved in argumentation and proving processes, and use a Peircean *semiotic lens* to analyse them (see for instance Arzarello & Sabena, in print). Considering the description given by Peirce (Peirce, 1931-1958) of *diagrammatic reasoning* as a three-step process—constructing a representation, experimenting with it, observing the results—we can observe that the three components of Habermas and of Peirce have deep analogies (though not a one-to-one correspondence): the construction of a representation may be guided mainly by a teleological rationality, whereas the experimentation and observation with it assume an epistemic value.

In this paper we present the first results of an ongoing study on particular cases of argumentation and proving processes, i.e. those related to *indirect proofs*, namely proofs by contraposition or proof by contradiction.

Analysing indirect proofs and argumentations both from a mathematical and a cognitive point of view, Antonini and Mariotti (2008) provide a model to interpret students' difficulties. Essentially, the model splits any indirect proof of a sentence S (principal statement) in a pair (*s,m*), where *s* is a direct proof (within a theory T, for example Euclidean Geometry) of a secondary statement S* and *m* is a meta-proof (within a meta-theory MT, generally coinciding with classical logic) of the statement S* → S. An example given by the authors is the following. Consider the (principal) statement S: 'Let *a* and *b* be two real numbers. If $ab = 0$ then $a = 0$ or $b = 0$'; and the following indirect proof: 'Assume that $ab = 0$, $a \neq 0$, and $b \neq 0$. Since $a \neq 0$ and $b \neq 0$ one can divide both sides of the equality $ab = 0$ by *a* and by *b*, obtaining $1 = 0$'. In this proof, the secondary statement S* is: 'let *a* and *b* be two real numbers; if $ab = 0$, $a \neq 0$, and $b \neq 0$ then $1 = 0$'. A direct proof is given. The hypothesis of this new statement is the negation of the original statement and the thesis is a false proposition ("1 = 0").

Through this model, Antonini and Mariotti point out that the main difficulties for students who face indirect proofs consist in switching from *s* to *m*. On the contrary the difficulties seem less strong for statements that require a proof by contraposition, that is to prove ¬B → ¬A (secondary statement) in order to prove A → B (principal statement).

Let us observe that the meta-proof *m* does not coincide with the meta-level we considered above; rather, it is at the meta-mathematical level (based on logic). Integrating the two models, we can say that switching from *s* to *m* requires a well-established epistemic and teleological rationality in the students. To better disentangle this issue, we make the hypothesis regarding the importance of a *meta-cognitive unity* in argumentation and proving processes.

META-COGNITIVE UNITY

The distinction between the ground level and the meta-level drawn from the Boero *et al.* (2010) model may be very useful to investigate the proving processes related to indirect proof. Basing on such a distinction, we introduce the notion of *meta-cognitive unity*, as *a cognitive unity between the two levels of argumentation described above, specifically between the teleological component at the meta-level and the epistemic component at the ground-level*.

Differently from the structural and referential cognitive unity, which focuses on two diachronic moments in the discursive activities of students (namely the argumentation and the proving phases), the meta-cognitive unity refers to a *synchronic integration between ground- and meta- levels of argumentations*.

Our hypothesis is that the existence of such a meta-cognitive unity is an important condition for producing indirect proofs. In other words, a missing integration between the two levels of argumentations can block the students' proving processes, or produce cognitive breaks as those described in the literature on cognitive unity mentioned above. Furthermore, our ongoing data analysis suggests that meta-cognitive unity may entail also some kind of cognitive unity at the ground level (e.g. structural or referential).

We shall illustrate our claims through an emblematic example, in which meta-cognitive unity is accompanied by structural-cognitive unity, and develops through what we call the *logic of not* (see also Arzarello & Sabena, in print).

AN EXAMPLE

To illustrate the meta-cognitive unity and the logic of not, we discuss the case of Simone, related to the following Calculus problem in the graphical register.

> *The drawing [reported in Fig. 1] shows the graphs of: a function f, its derivative, one of its antiderivatives. Identify the graph of each function, and justify your answer.*

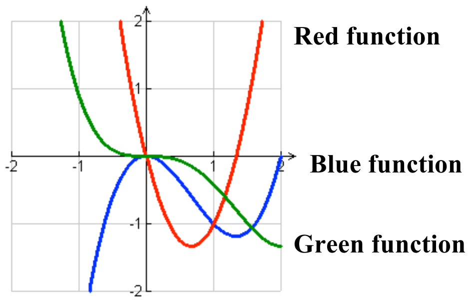

Fig. 1 The graphs of the problem

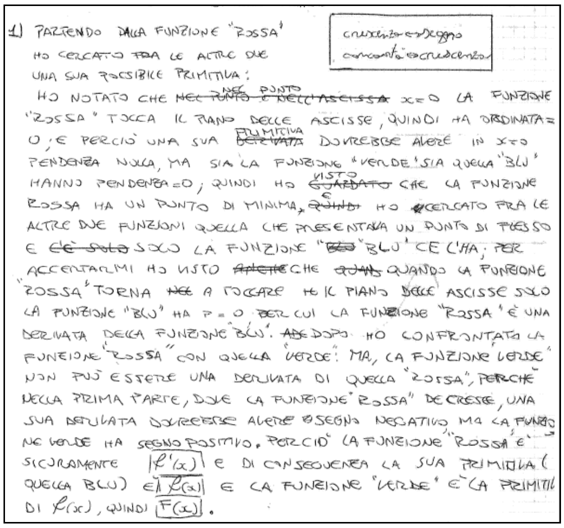

Fig. 2 Simone's protocol

The problem was given to grade 9 students in a scientifically oriented school (5 hours of mathematics per week). It worked as an assessment task at the end of a teaching sequence on the relationships between a function graph and the graphs of its derivative and of one of its primitives (introduced as 'antiderivatives', i.e. the function graphs whose derivative is the given graph). Though it may appear of simple combinatorial nature, the task indeed shows to be difficult for the students, who by "didactic contract" are asked to provide articulated arguments for their answers. A main source of difficulty consists in involving both direct and inverse problems mixed together. The antiderivative function has in fact been introduced as the inverse of the derivative function.

Simone solves the problem correctly, as we can see from his written production (shown in Fig. 2, and translated below). In doing so, he is able to develop a strategy that may be compared to the one of a chemist, who in the laboratory has to detect the nature of some substance. He knows that the substance must be of three different categories (a, b, c) and uses suitable reagents to accomplish his task. For example, he knows that if a substance reacts in a certain way to a certain reagent, it may be of type a or b but *not* c, and so on. This kind of strategy may be referred to the notion of *abduction* (for a discussion of abduction in mathematics education, see for instance Arzarello et al., 1998; and Hoffmann, 2005). According to Peirce (1931-1958, 2.623) abduction is a form of reasoning in which a *Case* is drawn from a *Rule* and a *Result*. It is well known his example about beans:

Rule:	All the beans from this bag are white
Result:	These beans are white
Case:	These beans are from this bag

As such, abduction is different from deduction, which would have the form: the Result is drawn from the Rule and the Case; and it is obviously different from an induction, which has the form: a Rule is drawn from a Case and many Results. Of course the conclusion of an abduction holds only with a certain probability; in fact Polya (1954) calls *heuristic syllogism* this form of reasoning. In our example of the chemist: if, as a *Rule*, the substance S makes blue the reagent r and if the *Result* of the experiment shows that the unknown substance X makes blue the reagent r, the *Case* of the abduction is that X=S.

Let us see how Simone develops abductive arguments in order to solve the task. We report the translation in English of the protocol shown in Fig. 2, parcelled and numbered for the sake of analysis:

0: Starting from the “red” function

1: I looked for a possible primitive among the other two:

2: I noticed that in the point x = 0 the “red” function touches the plane of abscissas, so it has ordinate = 0;

3: and therefore any of its primitives should have in x = 0 null slope,

4: but both the “green” function and the “blue” function have slope = 0;

5: so I saw that the red function has a point of minimum,

6: and I looked among the other two functions for the one with a point of inflection

7: and only the “blue” function has it;

8: to check [this] I saw that

9: when the “red” function comes to touch the plane of abscissas again,

10: only the “blue” function has s[slope]=0, 11: therefore the “red” function is a derivative of the “blue” function.

Simone first describes two phases of his inquiry (Part A: lines 1-4; Part B: lines 5-8). In both parts he develops what we could call an abductive attitude, namely how he has been looking for Results that allow him to state a Case because of a Rule. More precisely, Simone starts with f = red function (line 0), probably because it is the simplest graph, and wonders whether he can apply an abductive argument to the blue or to the green function. In both Parts the Rule is: “any primitive of f has property Q”; the Result is: “a specific function h has property Q”; the Case is “h is a primitive of f”.

In Part A, the Rule is in line 3, the Result is in lines 2 and 4, while the Case is contained implicitly in line 5, which states that the first inquiry has not been successful and starts a new inquiry. In Part B we have a new abductive process with a new Rule (implicitly contained in lines 5 and 6), a new Result (line 7), a judgment about the validity of the abduction (line 7) and a Case, which is not made explicit, but is implicitly stated in line 7.

Afterwards Simone checks his hypothesis (Part C: lines 8-11): he is successful with a fresh abductive argument. Recalling the metaphor with the chemist experiment, Simone has been able first to find a reagent that discriminates between the substances he is analysing, and then to confirm his hypothesis with a further discriminating experiment; that is, within the abductive frame, he has been able first to produce an hypothesis through an abduction and then to corroborate the hypothesis through a further abduction. The experiments of the chemists are here the practices with the graphs of functions. Such practices with graphs are examples of diagrammatic reasoning, according to the definition of Peirce: “by experimenting upon the diagram and observing the results thereof, it is possible to discover unnoticed and hidden relations among the parts” (Peirce, 1931-58, 3.363: quoted in Hoffmann, 2005, p. 48). Hence in Parts A-B-C Simone has produced and checked the Case of line 11.

Using Habermas model, some of Simone’s sentences can be considered *teleological* and at the *meta-level*, since they address the successive actions of Simone and his control of what is happening. On the other hand, other sentences show an *epistemic* character at *ground-level*, since they regard the specific mathematical notions and representations involved in the task. Coding the sentences of the protocol as $a \Rightarrow b$ to indicate that the sentence a is at the meta-level and controls the sentence b at the ground-level, it appears that the teleological component at the meta-level *intertwines* with the epistemic component at the ground level:

$$0, 1, 3 \Rightarrow 2, 4, 5$$

$$6, 8 \Rightarrow 7, 9, 10, 11$$

We interpret such intertwining as an index of *meta-cognitive unity* in Simone’s argumentation process. Besides, the presence of a meta-cognitive unity is signalled by the fact that the epistemic sentences can be understood only at the light of the teleological ones guiding them, and vice-versa. Even more than this: the sentences at the ground level have an epistemic component, e.g. they are logically linked each other, *because* of the influence of those at the meta-level. Let now see how this complex form of cognitive unity allows Simone to manage the problem using an argument by contraposition. It is the form of reasoning that logicians call "modus tollens" (from "A implies B" to "not B implies not A"). We have called such a process "the logic of not" (see also Arzarello & Sabena, in print). Let us explain it through what is written in Part D (lines 12-15):

12: Then I compared the "red" with the "green" function:

13: but, the "green" function cannot be a derivative of the "red" one,

14. a: because in the first part,

b: when the "red" function is decreasing,

c: its derivative should have a negative sign,

15: but the "green" function has a positive sign.

Here the structure of the sentence is more complex than before: Simone is thinking to a possible abductive argument, like the ones used before:

(a) Rule: "any derivative of a decreasing function is negative" (lines 14)
(b) Result: "the function h is negative";
(c) Case: "the function h is the derivative of f" } (ARG. 1)

But the argument is a refutation of this virtual abduction (line 13); namely it has the form of the following syllogism:

(a) Major premise:
"any derivative of a decreasing function is negative" (lines 14)
(b') Minor premise: "the "green" function has a positive sign" (line 15)
(c') Consequence: "the "green" function cannot be a derivative of
an increasing function" (line 13). } (ARG. 2)

In terms of the structure of the virtual abduction ARG 1, it has the form: *(a) and not (b); hence not (c)*. It is crucial here to observe that also the refutation of the usual Deduction (Rule, Case; hence Result) has the same structure, because of the contrapositive of an implication ("A implies B" is equivalent to "not B implies not A"); namely: *(a) and not (b); hence not (c)* is the same as *(a') and (b'); hence (c')*. In other words, the refutation of a virtual argument drawn through an abduction coincides with the refutation of a virtual argument drawn through a deduction. Simone produces in a very natural way this form of deductive argument within an abductive modality. This is remarkable from an epistemological point of view: whereas the abductive approach appears very natural for students in conjecturing phases (Arzarello et al., 1998), there is often a cognitive break with the deductive approach of the proving phase (Pedemonte, 2007 analyses it in terms of "structural discontinuity"). In fact, the transition from an abductive to a deductive modality requires a sort of "somersault", namely an inversion in the way things are seen and structured in the argument (the Case- and the Result-functions in the argument are exchanged). Such an inversion is not present in case of refutation of an abduction, insofar it coincides with the refutation of a deduction (expressed in a syllogistic form). Of course a greater cognitive load is required to manage the refutation of an abduction compared with that required to develop a simple direct abduction. But the coincidence between abduction and deduction in case they are refuted allows avoiding the somersault. In our case study, Simone has been able to lighten the cognitive load of the task through a transition to a new epistemological status of his statements.

Moreover, Simone is doing mental experiments with the graphs and observing their results. The ground level at which Simone epistemic arguments are drawn is very concrete; possibly this makes it possible for him to develop at the meta-level the teleological arguments we have underlined above, which support him in producing a correct proof by contraposition. Lines 16-18 below show the conclusion of Simone's argument:

16: Therefore the "red" function is surely f'(x)

17: and consequently its primitive (the blue one) is f(x)

18: and the "green" function is the primitive of f(x), thus F(x).

We shall now consider an example from the literature, in which the indirect argumentation process does not lead to a correct proof by contraposition. The example shows a typical difficulty, in which the students do not realise any meta-cognitive unity and shift from the problems they should solve to another problem, which allows them to skip the difficulties of the indirect proof. It is taken from Antonini & Mariotti (2008). A well known problem is considered in pencil and paper environment: "*Can two bisectors in a triangle form an angle of 90°?*". The students "formulate the conjecture that the angle…cannot be a right angle…*The argumentation produced can be summarized as follows: if the angle is right then the sum of two angles of the triangles is 180°, then the triangle becomes a quadrilateral.* After this argumentation, no proofs are generated by the students" (*ibid.*, p. 410). As stated in the paper, the shift to the quadrilateral can be considered an antidote to an "absurd world": "the theory is changed according to the validity of the theorems he [the student] knows" (*ibid.)*.

Here are the excerpts of the interview taken from that paper:

61 P: As far as 90°, it would be necessary that both K and H are 90°, then K/2 = 45, H/2 = 45...180°-90° and 90°.

62 I: In fact, it is sufficient that the sum is 90°, that K/2 + H/2 is 90°.

63 R: Yes, but it cannot be.

64 P: Yes, but it would mean that K + H is ... a square […]

65 R: It surely should be a square, or a parallelogram

66 P: (K - H)/2 would mean that […] K + H is 180°...

67 R: It would be impossible. Exactly, I would have with these two angles already 180°, that surely it is not a triangle. […]

71 R: We can exclude that [the angle] is p/2 [right] because it would become a quadrilateral.

The excerpts show that no teleological aspects are present in the students at the meta-level: on the contrary they are completely embedded in the situation at the ground (epistemic) level and their geometrical knowledge pushes them to shift from a triangle to a quadrilateral (# 64, 67, 71).

DISCUSSION

We presented an illustrative example of of how a *meta-cognitive unity* (i.e. a unity between a teleological control at meta-level and an epistemic knowledge at ground level) may contribute to the production of indirect arguments and proofs.

In the example, the graphical component is the core of the tasks, and it allows the student a visual approach to the problem, what in Peirce's words can be called 'diagrammatic reasoning'. This "concrete" component possibly lightens the cognitive load of the task, and facilitates the integration between the two levels to produce a meta-cognitive unity. Referring to the protocol, we see that the student is able to reduce the different cases to simple pass/not-pass 'experiments', like a chemist who checks the nature of an unknown substance, and we interpret his ability as an outcome of his meta-cognitive unity, which integrates the teleological meta-level control and the ground-level knowledge. As a consequence, the situation becomes a 'heuristic device' similar to cognitive mechanisms used naturally in everyday life, as pointed out by Freudenthal (1973, p. 629). The meta-cognitive unity construct brings about a unitary analysis of many students' difficulties as pointed out in the literature. We have also given a short example in which the absence of a teleological rationality at meta-level prevents the students from correctly carry out an indirect proof.

As far as concerns didactic consequences, our approach suggests a possible way for teaching indirect proofs, namely in making explicit the teleological dimension that can be developed in any reasoning made by the students. In particular it should be important to cultivate the idea of the rationality of impossible worlds related to the indirect arguments discussing with the students the intertwining between the teleological control and the epistemic knowledge starting from concrete examples produced by the same students. In a sense this was already suggested by Freudenthal, who wrote: "If the teacher would tell the student what is an indirect proof, he is advised not to contrive examples but to catch a student performing an indirect proof and let him understand consciously what he did unconsciously" (*ibid.*).

In fact in the teaching of both indirect and direct proofs, generally there is more attention to the communicative and possibly to the epistemic component, while the teleological one is not made explicit. This can start a sort of comedy of errors with students, who think that the communicative component is the more relevant and produce what Harel (2007) calls 'ritual schemes', which are not useful for understanding and possibly produce proofs, especially indirect ones.

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